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The geometry of billiards in ellipses and their poncelet grids

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Abstract. The goal of this paper is an analysis of the geometry of billiards in ellipses, based on properties of confocal central conics. The extended sides of the billiards meet at points which are located on confocal ellipses and hyperbolas. They define the associated Poncelet grid. If a billiard is periodic then it closes for any choice of the initial vertex on the ellipse. This gives rise to a continuous variation of billiards which is called billiard motion though it is neither a Euclidean nor a projective motion. The extension of this motion to the associated Poncelet grid leads to new insights and invariants.

Mathematics Subject Classification. Primary 51N35, Secondary 51N20, 52C30, 37D50.

Keywords. Billiard in ellipse, Caustic, Poncelet grid, Confocal conics, Billiard motion, Canonical parametrization.

1. Introduction

A *billiard* is the trajectory of a mass point within a domain with ideal physical reflections in the boundary. Already for two centuries, billiards in ellipses have attracted the attention of mathematicians, beginning with J.-V. Poncelet, C.G.J. Jacobi and A. Cayley. One basis for the investigations was the theory of confocal conics. In 2005 S. Tabachnikov published a book on various aspects of billiards, including their role as completely integrable systems [29]. In several publications and in the book [13], V. Dragović and M. Radnović studied billiards, also in higher dimensions, from the viewpoint of dynamical systems.

Computer animations of billiards in ellipses, which were carried out by Reznik [24], stimulated a new vivid interest on this well studied topic, where algebraic and analytic methods are meeting (see, e.g., [2,3,11,21–23] and many further references in [24]). These papers focus on invariants of periodic billiards when the vertices vary on the ellipse while the caustic remains fixed. This variation is called billiard motion though neither angles nor side lengths remain fixed; and it is not a projective motion preserving the circumscribed ellipse.

The goal of this paper is a geometric analysis of billiards in ellipses and their associated Poncelet grid, starting from properties of confocal conics. We concentrate on a certain symmetry between the vertices of any billiard and the contact points with the caustic, which can be an ellipse or hyperbola. Billiard motions induce motions of associated billiards with the same caustic and circumscribed confocal ellipses.

2. Metric properties of confocal conics

A family of *confocal* central conics (Fig. 1) is given by

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}=1,\ \text{ where } k\in\mathbb{R}\setminus\{-a^2,-b^2\} \tag{2.1}$$

serves as a parameter in the family. All these conics share the focal points

$$F_{1,2}=(\pm d,0),\ \text{ where } d^2:=a^2-b^2. \tag{2.2}$$

The confocal family sends through each point P outside the common axes of symmetry two orthogonally intersecting conics, one ellipse and one hyperbola [15, p. 38]. The parameters (k_e,k_h) of these two conics define the *elliptic coordinates* of P with

$$-a^2<k_h<-b^2<k_e\,.$$

If (x,y) are the cartesian coordinates of P, then (k_e,k_h) are the roots of the quadratic equation

$$k^2+(a^2+b^2-x^2-y^2)k+(a^2b^2-b^2x^2-a^2y^2)=0, \tag{2.3}$$

while conversely

$$x^2=\frac{(a^2+k_e)(a^2+k_h)}{d^2}\,,\quad y^2=-\frac{(b^2+k_e)(b^2+k_h)}{d^2}\,. \tag{2.4}$$

Let $(a,b)=(a_c,b_c)$ be the semiaxes of the ellipse c with $k=0$. Then, for points P on a confocal ellipse e with semiaxes (a_e,b_e) and $k=k_e>0$, i.e., exterior to c, the standard parametrization yields

$$\begin{gathered}P=(x,y)=(a_e\cos t,\,b_e\sin t),\ 0\le t<2\pi,\\ \text{with } a_e^2=a_c^2+k_e,\ b_e^2=b_c^2+k_e\,.\end{gathered} \tag{2.5}$$

For the elliptic coordinates (k_e,k_h) of P follows from (2.3) that

$$k_e+k_h=a_e^2\cos^2 t+b_e^2\sin^2 t-a_c^2-b_c^2.$$

After introducing the respective tangent vectors of e and c, namely

$$\begin{aligned}\mathbf{t}_e(t)&:=(-a_e\sin t,\,b_e\cos t),\\ \mathbf{t}_c(t)&:=(-a_c\sin t,\,b_c\cos t),\end{aligned}\ \text{ where } \|\mathbf{t}_e\|^2=\|\mathbf{t}_c\|^2+k_e\,, \tag{2.6}$$

we obtain[1]

$$k_h = k_h(t) = -(a_c^2 \sin^2 t + b_c^2 \cos^2 t) = -\|\mathbf{t}_c(t)\|^2 = -\|\mathbf{t}_e(t)\|^2 + k_e \tag{2.7}$$

and

$$\|\mathbf{t}_e(t)\|^2 = k_e - k_h(t)\,. \tag{2.8}$$

Note that points on the confocal ellipses e and c with the same parameter t have the same coordinate k_h. Consequently, they belong to the same confocal hyperbola (Figs. 5 and 8). Conversely, points of e or c on this hyperbola have a parameter out of $\{t, -t, \pi + t, \pi - t\}$.

Normal vectors of e and c can be defined respectively as

$$\begin{aligned} \mathbf{n}_e(t) &:= \Big(\frac{\cos t}{a_e}, \frac{\sin t}{b_e}\Big), \\ \mathbf{n}_c(t) &:= \Big(\frac{\cos t}{a_c}, \frac{\sin t}{b_c}\Big), \end{aligned} \quad \text{where } \|\mathbf{n}_c(t)\| = \frac{\|\mathbf{t}_c(t)\|}{a_c b_c}\,. \tag{2.9}$$

We complete with two useful relations between the parameter t and the second elliptic coordinate $k_h(t)$:

$$\tan^2 t = -\frac{b_c^2 + k_h(t)}{a_c^2 + k_h(t)} \quad \text{and} \quad \sin t \cos t = \frac{a_h b_h}{d^2} \tag{2.10}$$

with a_h and b_h as semiaxes of the hyperbola corresponding to the parameter t, i.e., $a_h^2 = a_c^2 + k_h$ and $b_h^2 = -(b_c^2 + k_h)$.

Proof. From (2.7) follows

$$k_h = -\frac{a_c^2 \tan^2 t + b_c^2}{1 + \tan^2 t}, \quad \text{hence} \quad \tan^2 t(a_c^2 + k_h) = -b_c^2 - k_h$$

and

$$\sin t \cos t = \frac{\tan t}{1 + \tan^2 t} = \frac{\sqrt{-(b_c^2 + k_h)(a_c^2 + k_h)}}{a_c^2 - b_c^2} = \frac{a_h\, b_h}{d^2}\,.$$

□

Referring to Fig. 1, the following lemma addresses an important property of confocal conics (note, e.g., [15, pp. 38 and 309]).

Lemma 2.1. *The tangents drawn from any fixed point P to the conics of a confocal family share the axes of symmetry, which are tangent to the two conics passing through P.*

This means, if a ray is reflected at P in one of the conics passing through, then the incoming and the outgoing ray contact the same confocal ellipse or hyperbola.

Below, we report about results concerning a pair of confocal conics. Due to their meaning for billiards in ellipses, we restrict ourselves to pairs (e, c) of confocal ellipses with c in the interior e, and we call c the *caustic* (Fig. 2).

[1] The norm $\|\mathbf{t}_e\|$ equals half length of the diameter of e which is parallel to $\mathbf{t}_e$.

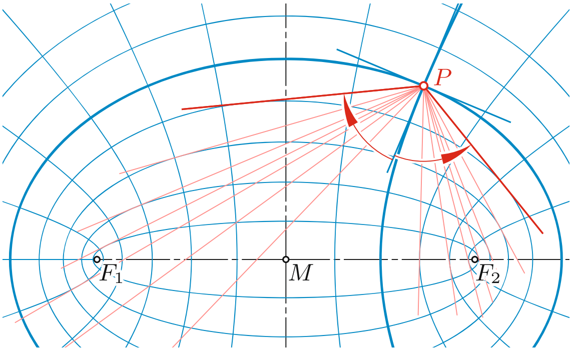

Figure 1 The tangents from the point P to the conics of a confocal family are symmetric w.r.t. the tangents at P to the confocal conics passing through P

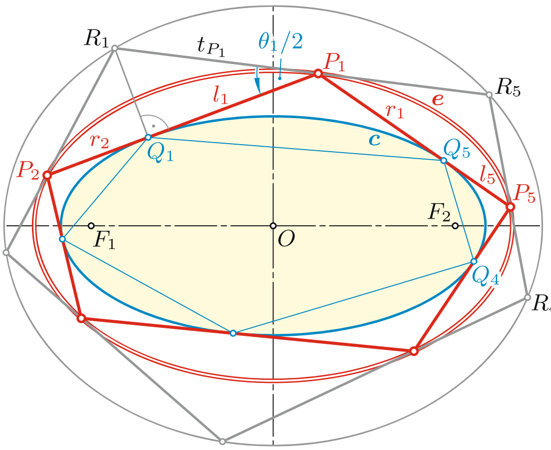

Figure 2 Periodic billiard $P_1P_2\dots P_5$ inscribed in the ellipse e with the caustic c

Lemma 2.2. *Let $P = (a_e \cos t,\, b_e \sin t)$ with elliptic coordinates (k_e, k_h) be a point on the ellipse e with $k_e > 0$ and c be the confocal ellipse with $k = 0$. Then, the angle $\theta(t)/2$ between the tangent at P to e and any tangent from P to c satisfies*

$$\sin^2\frac{\theta}{2} = \frac{k_e}{\|\boldsymbol{t}_e(t)\|^2} = \frac{k_e}{k_e - k_h}\,,\ \ \tan\frac{\theta}{2} = \pm\sqrt{-\frac{k_e}{k_h}}\,, \tag{2.11}$$

$$\cos\theta = 1 - \frac{2k_e}{\|\boldsymbol{t}_e(t)\|^2} = \frac{k_h + k_e}{k_h - k_e}\,, \quad \sin\theta = \pm\frac{2\sqrt{-k_e k_h}}{k_e - k_h}\,. \tag{2.12}$$

Proof. The tangent t_P to e at $P = (a_e \cos t,\ b_e \sin t)$ in direction of $\mathbf{t}_e$ has the slope

$$f := \tan\alpha_1 = \frac{-b_e \cos t}{a_e \sin t}\,.$$

If s_1 and s_2 denote the slopes of the tangents from P to c, then they satisfy

$$y - b_e \sin t = s_i(x - a_e \cos t), \quad i = 1, 2.$$

As tangents of c, their homogeneous line coordinates

$$(u_0 : u_1 : u_2) = ((b_e \sin t - s_i\, a_e \cos t) : s_i : -1)$$

must satisfy the tangential equation $-u_0^2 + a_c^2 u_1^2 + b_c^2 u_2^2 = 0$ of c. This results in a quadratic equation for the unknown s, namely

$$(a_e^2 \sin^2 t - k_e)s^2 + 2a_e b_e s \sin t \cos t + (b_e^2 \cos^2 t - k_e) = 0.$$

We conclude

$$s_1 + s_2 = \frac{-2a_e b_e \sin t \cos t}{a_e^2 \sin^2 t - k_e} \text{ and } s_1 s_2 = \frac{b_e^2 \cos^2 t - k_e}{a_e^2 \sin^2 t - k_e}.$$

The slopes $f = \tan\alpha_1$ of t_P and $\tan\alpha_2 = s_1$ or s_2 of the tangents to c imply for the enclosed signed angle $\theta(t)/2$ (for brevity, we often suppress the parameter t)

$$\tan\frac{\theta}{2} = \tan(\alpha_1 - \alpha_2) = \frac{s_1 - f}{1 + s_1 f} = \frac{f - s_2}{1 + s_2 f}\,,$$

hence

$$\tan^2\frac{\theta}{2} = \frac{(s_1 - f)(f - s_2)}{(1 + s_1 f)(1 + s_2 f)} = \frac{f(s_1 + s_2) - s_1 s_2 - f^2}{f(s_1 + s_2) + 1 + f^2 s_1 s_2}\,.$$

After some computations, we obtain

$$\tan^2\frac{\theta}{2} = \frac{k_e}{a_e^2 \sin^2 t + b_e^2 \cos^2 t - k_e} = \frac{k_e}{\|\mathbf{t}_e\|^2 - k_e} = \frac{k_e}{\|\mathbf{t}_c\|^2}\,,$$

therefore

$$\cot^2\frac{\theta}{2} = \frac{\|\mathbf{t}_e\|^2}{k_e} - 1 \text{ and } \sin^2\frac{\theta}{2} = \frac{1}{1 + \cot^2\frac{\theta}{2}} = \frac{k_e}{\|\mathbf{t}_e\|^2}\,,$$

where $k_e = a_e^2 - a_c^2 = b_e^2 - b_c^2$, and finally

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - \frac{2k_e}{\|\mathbf{t}_c\|^2 + k_e} = \frac{\|\mathbf{t}_c\|^2 - k_e}{\|\mathbf{t}_c\|^2 + k_e}\,.$$

□

Remark 2.3. A change of the origin $k = 0$ for the elliptic coordinates in a family of confocal conics corresponds to a shift of the coordinates. Hence, if in Lemma 2.2 the ellipse c is replaced by another confocal conic with the coordinate k, then the formulas (2.11) and (2.12) remain valid under the condition that we replace k_e by $k_e - k$ and k_h by $k_h - k$.

Lemma 2.4. *Let P_1P_2 be a chord of the ellipse e, which contacts the caustic c at the point Q_1. Then the signed distances of the line $[P_1, P_2]$ to the center O and to the pole R_1 w.r.t. e have the constant product $-k_e$. The lines $[P_1, P_2]$ and $[Q_1, R_1]$ are orthogonal* (Fig. 2).

Proof. Let the side P_1P_2 touch the caustic c at the point $Q_1 = (a_c \cos t'_1,\ b_c \sin t'_1)$. Then the Hessian normal form of the spanned line $t_Q = [P_1, P_2]$ reads

$$t_Q\colon\ \frac{b_c \cos t'_1\, x + a_c \sin t'_1\, y - a_c b_c}{\sqrt{b_c^2 \cos^2 t'_1 + a_c^2 \sin^2 t'_1}} = 0.$$

Its pole w.r.t. e has the coordinates

$$R_1 = \left(\frac{a_e^2 \cos t'_1}{a_c},\ \frac{b_e^2 \sin t'_1}{b_c} \right). \tag{2.13}$$

This yields for the signed distances to the line t_Q

$$\overline{Ot_Q} = \frac{-a_c b_c}{\|\mathbf{t}_c(t'_1)\|} \tag{2.14}$$

and

$$\overline{R_1t_Q} = \overline{R_1Q_1} = \frac{k_e(b_c^2 \cos^2 t'_1 + a_c^2 \sin^2 t'_1)}{a_c b_c \|\mathbf{t}_c(t'_1)\|} = \frac{k_e \|\mathbf{t}_c(t'_1)\|}{a_c b_c}\,. \tag{2.15}$$

Thus, we obtain a constant product $\overline{Ot_Q} \cdot \overline{R_1t_Q} = -k_e$, as stated in Lemma 2.4.

The last statement holds since R_1 and Q_1 are the poles of $[P_1, P_2]$ w.r.t. the confocal conics e and c. It is wellknown that the poles of any line ℓ w.r.t. confocal conics lie on a line ℓ^* orthogonal to ℓ (see, e.g., [15, p. 340]). □

3. Confocal conics and billiards

By virtue of Lemma 2.1, all sides of a billiard inscribed to the ellipse e with parameter $k = k_e$ are tangent to a fixed conic c confocal with e (Fig. 2). The caustic c with parameter k_c can be a smaller ellipse with $-b^2 < k_c < k_e$ or a hyperbola with $-a^2 < k_c < -b^2$ or, in the limiting case with $k_c = -b^2$, consist of the pencils of lines with the focal points F_1, F_2 of c as carriers. At the beginning, we confine ourselves to an ellipse with $k_c = 0$ (Fig. 2), and we speak of an *elliptic* billiard. Only the Figs. 10, 11 and 12 show (periodic) billiards in e with a hyperbola as caustic, called *hyperbolic* billards.

For billiards $\dots P_1P_2P_3 \dots$ in the ellipse e and with the ellipse c as caustic, we assume from now on a counter-clockwise order and signed exterior angles $\theta_1, \theta_2, \theta_3, \dots$ (see Fig. 2). The tangency points $Q_1, Q_2, \dots$ of the billiard's sides $P_1P_2, P_2P_3, \dots$ with the caustic c subdivide the sides into two segments. We denote the lengths of the segments adjacent to P_i as

$$l_i := \overline{P_iQ_i} \text{ and } r_i := \overline{P_iQ_{i-1}}. \tag{3.1}$$

Based on the parametrizations $(a_e \cos t,\, b_e \sin t)$ of e and $(a_c \cos t',\, b_c \sin t')$ of c, we denote the respective parameters of P_1, Q_1, P_2, Q_2, $P_3, \dots$ with t_1, t_1', t_2, t_2', $t_3, \dots$ in strictly increasing order.

The following two lemmas deal with sides of billiards in the ellipse e.

Lemma 3.1. *The connecting line $[P_i, P_{i+1}]$ of the vertices with respective parameters t_1, t_2 on e contacts the caustic c if and only if*

$$\frac{a_c^2}{a_e^2}\cos^2\frac{t_1+t_2}{2} + \frac{b_c^2}{b_e^2}\sin^2\frac{t_1+t_2}{2} = \cos^2\frac{t_1-t_2}{2}.$$

This is equivalent to

$$\sin^2\frac{t_1-t_2}{2} = \frac{k_e}{a_e b_e}\left\| \boldsymbol{t}_e\Big(\frac{t_1+t_2}{2}\Big)\right\|^2.$$

Proof. The line connecting the points $(a_e \cos t_i,\, b_e \sin t_i)$, $i = 1, 2$, has homogeneous line coordinates $(u_0 : u_1 : u_2)$ equal to

$$(a_e b_e(\cos t_1 \sin t_2 - \sin t_1 \cos t_2) : b_e(\sin t_1 - \sin t_2) : a_e(\cos t_2 - \cos t_1))\,.$$

It contacts the caustic c if $-u_0^2 + a_c^2 u_1^2 + b_c^2 u_2^2 = 0$, i.e.,

$$\begin{aligned} &a_c^2 b_e^2 \sin^2\frac{t_1-t_2}{2}\cos^2\frac{t_1+t_2}{2} + b_c^2 a_e^2 \sin^2\frac{t_1-t_2}{2}\sin^2\frac{t_1+t_2}{2} \\ &= a_e^2 b_e^2 \sin^2\frac{t_2-t_1}{2}\cos^2\frac{t_2-t_1}{2}\,. \end{aligned}$$

Under the condition $\sin[(t_1 - t_2)/2] \neq 0$ we obtain the first claimed equation. The second follows after the substitutions $a_c^2 = a_e^2 - k_e$ and $b_c^2 = b_e^2 - k_e$ from

$$1 - \frac{k_e}{a_e^2 b_e^2}\left(b_e^2\cos^2\frac{t_1+t_2}{2} + a_e^2\sin^2\frac{t_1+t_2}{2}\right) = \cos^2\frac{t_2-t_1}{2}\,.$$

by (2.9). □

Lemma 3.2. *Referring to the notation in Lemma 3.1, if the side P_iP_{i+1} contacts the caustic c at Q_i with parameter t_i', then*

$$\sin t_i' = \frac{b_c}{b_e}\,\frac{\sin\frac{t_i+t_{i+1}}{2}}{\cos\frac{t_i-t_{i+1}}{2}}\,,\quad \cos t_i' = \frac{a_c}{a_e}\,\frac{\cos\frac{t_i+t_{i+1}}{2}}{\cos\frac{t_i-t_{i+1}}{2}}\,,\quad \tan t_i' = \frac{b_c a_e}{a_c b_e}\tan\frac{t_i+t_{i+1}}{2}\,.$$

Proof. The tangent to c at Q_1 has the line coordinates

$$(u_0 : u_1 : u_2) = (-a_c b_c : b_c \cos t_1' : a_c \sin t_1')\,,$$

which must be proportional to those in the proof of Lemma 3.1. □

Remark 3.3. 1. The half-angle substitution

$$\tau_i := \tan\frac{t_i}{2} \quad \text{for } i = 1, 2$$

allows to express the equation of Lemma 3.1 (for $i = 1$) in projective coordinates on e. We obtain a symmetric biquadratic condition

$$b_e^2 k_e \tau_1^2 \tau_2^2 - b_c^2 a_e^2(\tau_1^2 + \tau_2^2) + 2(a_e^2 k_e + a_c^2 b_e^2)\tau_1\tau_2 + b_e^2 k_e = 0,$$

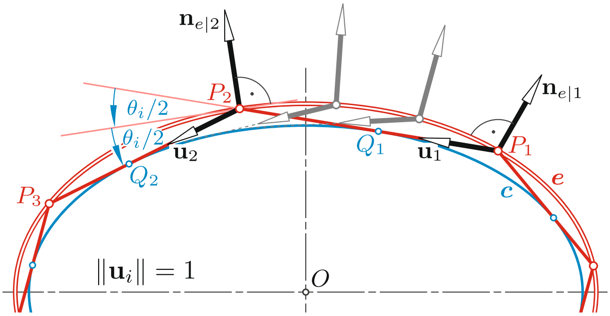

FIGURE 3 The Joachimsthal integral $J_e := -\langle \mathbf{u}_i,\, \mathbf{n}_{e|i}\rangle$ is constant along e

which defines a 2-2-correspondence on e between the endpoints P_1, P_2 of a chord which contacts c. This remains valid after iteration, i.e., between the initial point P_1 and the endpoint P_{N+1} of a billiard after N reflections in e.

Now, we recall a classical argument for the underlying Poncelet porism (see also [18] and the references there): A 2-2-correspondence different from the identity keeps fixed at most four points. However, four fixed points on e are already known as contact points between e and the common complex conjugate (isotropic) tangents[2] with the caustic c, since tangents of e remain fixed under the reflection in e. If therefore one N-sided billiard in e with caustic c closes, then the correspondence is the identity and all billiards close.

2. With the aid of Jacobi's arguments in [20], Lemma 3.2 paves already the way to a representation of the billiard's vertices in terms of Jacobian elliptic functions (note [27]).

Given any billiard $P_1P_2\dots$ in the ellipse e, let $\mathbf{p}_i = (x_i,\, y_i)$ denote the position vector of P_i for $i = 1, 2, \dots$, while $\mathbf{u}_1, \mathbf{u}_2, \dots$ denote the unit vectors of the oriented sides P_1P_2, P_2P_3, … (Fig. 3). By (2.9), the vector $\mathbf{n}_{e|i} := (x_i/a_e^2,\ y_i/b_e^2)$ is orthogonal to e at P_i. According to [2, Proposition 2.1], the scalar product

$$J_e := -\langle \mathbf{u}_i,\, \mathbf{n}_{e|i}\rangle \tag{3.2}$$

is invariant along the billiard in e and called *Joachimsthal integral* (note also [29, p. 54]).

The invariance of the Joachimsthal integral, which also holds in higher dimensions for billiards in quadrics, is the key result for the integrability of billiards, i.e., in the planar case for the existence of a caustic [2, p. 3]. In our approach, the invariance of J_e follows from Lemma 2.2.

Lemma 3.4. *The Joachimsthal integral $J_e := -\langle \boldsymbol{u}_i,\, \boldsymbol{n}_{e|i}\rangle$ equals*

$$J_e = \frac{\sqrt{k_e}}{a_e b_e}$$

[2] They follow from the second equation in Lemma 3.1 for $t_1 = t_2$.

with k_e *as elliptic coordinate of* e *w.r.t.* c*, i.e.,* $k_e = a_e^2 - a_c^2 = b_e^2 - b_c^2$.

Proof. From (2.11) follows for the points $(a_e \cos t,\ b_e \sin t)$ of e

$$J_e = -\langle \mathbf{u},\, \mathbf{n}_e\rangle = -\cos\Big(\frac{\pi}{2} + \frac{\theta}{2}\Big)\, \|\mathbf{n}_e\| = \sin\frac{\theta}{2}\, \|\mathbf{n}_e\| = \sin\frac{\theta}{2}\, \frac{\|\mathbf{t}_e\|}{a_e b_e}\,,$$

hence by (2.11), (2.7), (2.6), and (2.9)

$$J_e^2 = \sin^2\frac{\theta}{2}\, \|\mathbf{n}_e\|^2 = \frac{k_e}{\|\mathbf{t}_e\|^2}\, \|\mathbf{n}_e\|^2 = \frac{k_e}{a_e^2\, b_e^2}\,.$$

This confirms the claim. □

3.1. Poncelet grid

The following theorem is the basis for the *Poncelet grid* associated to each billiard. We formulate and prove a projective version. The special case dealing with confocal conics, has already been published by Chasles [9, p. 841] and later by Böhm in [8, p. 221]. The same theorem was studied in [21,25] and in [1]. In [19], the authors proved it in a differential-geometric way.

In the theorem and proof below, the term *conic* stands for regular dual conics, i.e., conics seen as the set of tangent lines, but also for pairs of line pencils and for single line pencils with multiplicity two. Expressed in terms of homogeneous line coordinates, the corresponding quadratic forms have rank 3, 2 or 1, respectively. Moreover, we use the term *range* for a pencil of dual conics. The term *net* denotes a 2-parametric linear system of dual curves of degree 2. Obviously, conics and ranges included in a net play the role of points and lines of a projective plane within the 5-dimensional projective space of dual conics. Any two ranges in a net share a conic (compare with [10, Théorèmes I−IV]).

Theorem 3.5. *Let* c *be a regular conic and* A_1, B_1 *two points such that the tangents* $t_1, \dots, t_4$ *drawn from* A_1 *and* B_1 *to* c *form a quadrilateral. Its remaining pairs of opposite vertices are denoted by* (A_i, B_i)*,* $i = 2, 3$*. Then,*

1. *for each conic* c_1 *passing through* A_1 *and* B_1*, the range* $\mathcal{R}_c$ *spanned by* c *and* c_1 *contains conics* c_i *passing through* A_i *and* B_i*, simultaneously. The tangents at* A_j *and* B_j *to* c_j *for* $j = 1, 2, 3$ *meet at a common point* R*. If* c_i *has rank 2, then we obtain, as the limit of* c_i*, the diagonal* $[A_i, B_i]$ *of the quadrilateral* $t_1, \dots, t_4$*.*
2. *This result holds also in the limiting case* $t_1 = t_2$*, where the chord* A_1B_1 *of* c_1 *contacts* c *at* B_2*.*

In Fig. 4, the particular case is displayed where c and $c_1 = e$ span a range $\mathcal{R}_c$ of confocal conics (note also [6, Fig. 19]). Then by Lemma 2.1, the tangents at A_j and B_j to c_j are angle bisectors of the quadrilateral. In case of a rank deficiency of c_i, either one axis of symmetry of the confocal family or the line at infinity shows up as c_i.

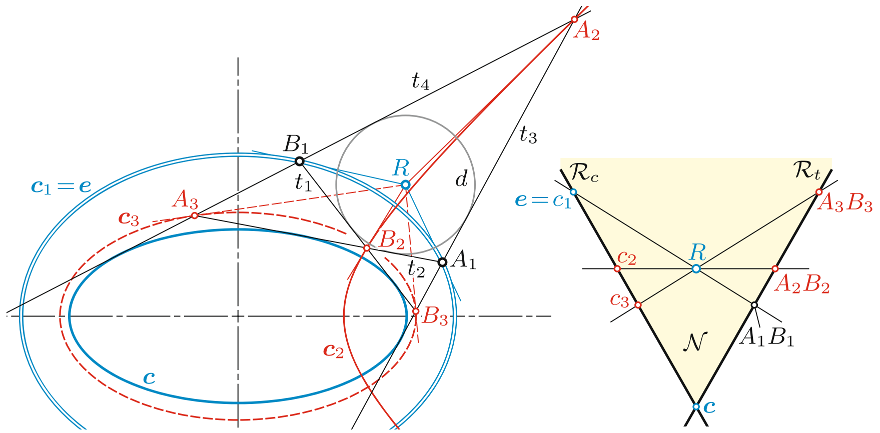

Figure 4 Left: Opposite vertices A_i, B_i of the quadrilateral $t_1 \dots t_4$ of tangents to c belong to a conic c_i out of the range $\mathcal{R}_c$ (Theorem 3.5). Right: The net $\mathcal{N}$ and the ranges $\mathcal{R}_t$, $\mathcal{R}_c$ in the projective space of dual conics

Proof. The conics tangent to $t_1, \dots, t_4$ define a range $\mathcal{R}_t$, which includes for $j = 1, 2, 3$ the pairs of line pencils (A_j, B_j) as well as the initial conic c. On the other hand, c and c_1 span a range $\mathcal{R}_c$. Since both ranges share the conic c, they span a net $\mathcal{N}$ of conics.

The pair (A_1, B_1) of line pencils spans together with c_1 the range of conics sharing the points A_1, B_1 and the tangents there, which meet at a point R. This range, which also belongs to $\mathcal{N}$, contains the rank-1 conic with carrier R. Each pair of line pencils (A_i, B_i), $i = 1, 2$, spans with the pencil R again a range within $\mathcal{N}$. This range shares with the range $\mathcal{R}_c$ a conic c_i passing through A_i and B_i with respective tangent lines through R.[3]

All these conclusions remain valid in the case, when $\mathcal{R}_t$ consists of conics which touch c at B_2 and are tangent to t_3 and t_4. □

As already indicated by the notation, we are interested in the particular case of Theorem 3.5 where the conics c and e in the range $\mathcal{R}_c$ are confocal. The following result follows directly from Theorem 3.5 and summarizes properties of the Poncelet grid. For the points of intersection between extended sides of a billiard $\dots P_0 P_1 P_2 \dots$ we use the notation

$$S_i^{(j)} := \begin{cases} [P_{i-k-1}, P_{i-k}] \cap [P_{i+k}, P_{i+k+1}] & \text{for} \quad j = 2k, \\ [P_{i-k}, P_{i-k+1}] \cap [P_{i+k}, P_{i+k+1}] & \text{for} \quad j = 2k-1 \end{cases} \tag{3.3}$$

where $i = \dots, 0, 1, 2, \dots$ and $j = 1, 2, \dots$ Note that there are j sides between those which intersect at $S_i^{(j)}$, and 'in the middle' of these j sides there is for

[3] An extended version of this theorem in [26] addresses the symmetry between the ranges $\mathcal{R}_c$ and $\mathcal{R}_t$. This generalizes the statement that in the case of confocal conics c and c_1 the quadrilateral $A_1 A_2 B_1 B_2$ has an incircle d (Figs. 4, 6 and 14, compare with [1,19]).

even j the vertex P_i and otherwise the point of contact Q_i. At the same token, the point $S_i^{(j)}$ is the pole of the diagonal $[Q_{i-k-1}, Q_{i+k}]$ or $[Q_{i-k}, Q_{i+k}]$ of the polygon $\dots Q_1Q_2Q_3\dots$ of contact points w.r.t. the caustic.

Theorem 3.6. *Let $\dots P_0P_1P_2\dots$ be a billiard in the ellipse e with sides P_iP_{i+1} contacting the ellipse c at the respective points Q_i for all $i \in \mathbb{Z}$. Then the vertices $S_i^{(j)}$ of the associated Poncelet grid are distributed on the following conics.*

1. *The points $S_i^{(1)}$, $S_i^{(3)}, \dots$ are located on the confocal hyperbola through Q_i, while the points $S_i^{(2)}$, $S_i^{(4)}, \dots$ are located on the confocal hyperbola through P_i.*
2. *For each $j \in \{1, 2, \dots\}$, the points $\dots S_i^{(j)} S_{i+(j+1)}^{(j)} S_{i+2(j+1)}^{(j)} \dots$ are vertices of another billiard with the caustic c inscribed in a confocal ellipse $e^{(j)}$, provided that $e^{(j)}$ is regular. Otherwise $e^{(j)}$ coincides with an axis of symmetry or with the line at infinity. The locus $e^{(j)}$ is independent of the position of the initial vertex $P_0 \in e$.*

Proof.

1. The side lines $[P_0, P_1]$ ($P_0 = P_7$ in Fig. 5) and $[P_2, P_3]$ meet at $S_1^{(1)}$, while $[P_1, P_2]$ contacts c at Q_1. By Theorem 3.5, 2. the points Q_1 and $S_1^{(1)}$ belong to the same confocal hyperbola.
   Now we go one step away from Q_1: the tangents from P_0 and P_3 to c intersect at $S_1^{(1)}$ and $S_1^{(3)} = [P_{-1}, P_0] \cap [P_3, P_4]$. The confocal conic through $S_1^{(1)}$ and $S_1^{(3)}$ must again be the hyperbola through Q_1. This follows by continuity after choosing Q_1 on one axis of symmetry. Iteration confirms the first claim in Theorem 3.6.
   The tangents to c from P_0 and P_2 form a quadrilateral with P_1 and $S_1^{(2)}$ as opposite vertices. Therefore, there exists a confocal conic passing through both points. This conic must be a hyperbola, as can be concluded by continuity: If P_1 is specified at a vertex of e, then due to symmetry the points P_1 and $S_1^{(2)}$ are located on an axis of symmetry.
   The tangents to c from P_{-1} and P_3 form a quadrilateral with $S_1^{(2)}$ and $S_1^{(4)}$ as opposite vertices. Theorem 3.5 and continuity guarantee that this is again the confocal hyperbola through P_1. Iteration shows the same of $S_1^{(6)}$ etc. However, the points P_1, $S_1^{(2)}$, $S_1^{(4)}, \dots$ need not belong to the same branch of the hyperbola.
2. The tangents through P_2 and $S_2^{(2)}$ (note Fig. 5) form a quadrilateral with $S_1^{(1)}$ and $S_2^{(1)}$ as opposite vertices. This time, continuity shows that the two points belong to the same confocal ellipse $e^{(1)}$. The same holds for the tangents through P_3 and $S_3^{(2)}$ etc.
   Similarily, starting with the points P_0 and P_3, we find the ellipse $e^{(2)}$ through $S_1^{(2)}$ and $S_2^{(2)}$, and so on.
   In order to prove that these ellipses $e^{(1)}, e^{(2)}, \dots$ are independent of the choice of the initial point $P_1 \in e$, we follow an argument from [2, proof of

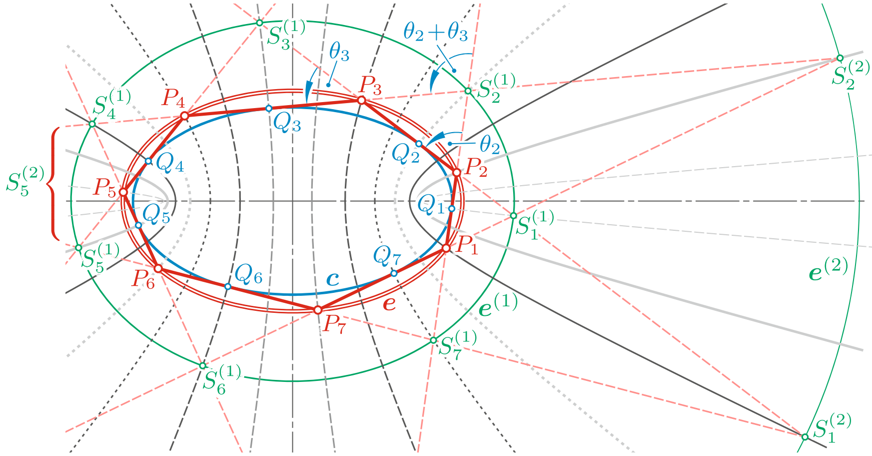

Figure 5 Periodic billiard ($N = 7$) with extended sides

Corollary 2.2]: The claim holds for all confocal ellipses e where billiards with the same caustic c are aperiodic and traverse e infinitely often. Since these ellipses form a dense set, the claim holds also for those with periodic billiards. The invariance of the ellipses $e^{(1)}, e^{(2)}, \dots$ is already mentioned in [2, Theorem 7].
An alternative proof consists in demonstrating that the elliptic coordinate $k_e^{(j)}$ of $e^{(j)}$ for all j does not depend on the parameter t. As one example, we present below in (3.7), (3.6) and (3.8) formulas for the semiaxes $a_{e|1}$, $b_{e|1}$ and the elliptic coordinate $k_{e|1}$ of $e^{(1)}$. □

Remark 3.7. Figure 5 reveals, that the polygons $P_1 S_1^{(1)} P_2 S_2^{(1)} \dots$ as well as $P_1 S_2^{(2)} P_3 S_4^{(2)} \dots$ and $S_1^{(2)} S_1^{(1)} S_2^{(2)} S_2^{(1)} \dots$ are zigzag billiards in rings bounded by two confocal ellipses. However, we find also zigzag billiards between two confocal hyperbolas, e.g., $\dots S_1^{(2)} P_2 P_1 S_2^{(2)} \dots$ or the twofold covered $\dots S_1^{(2)} S_1^{(1)} P_1 Q_1 P_1 S_1^{(1)} S_1^{(2)} \dots$. Billiards between other pairs of confocal conics can be found in [12].

The coming lemma addresses invariants related to the incircles of quadrilaterals built from the tangents to c from any two vertices P_i and P_j of a billiard in e (note circle d in Fig. 4 and [1,5,19]).

Lemma 3.8. *Referring to Fig. 6, the power w^2 of the point $S_i^{(1)}$ w.r.t. the incircle of the triangle $P_i P_{i+1} S_i^{(1)}$ is the same for all i. Similarly, the power w_1^2 of $S_i^{(2)}$ w.r.t. the incircle of the quadrangle $P_i S_{i-1}^{(1)} S_i^{(2)} S_i^{(1)}$ is constant.*

Proof. According to Graves's construction [15, p. 47], an ellipse e can be constructed from a smaller ellipse c in the following way: Let a closed piece of

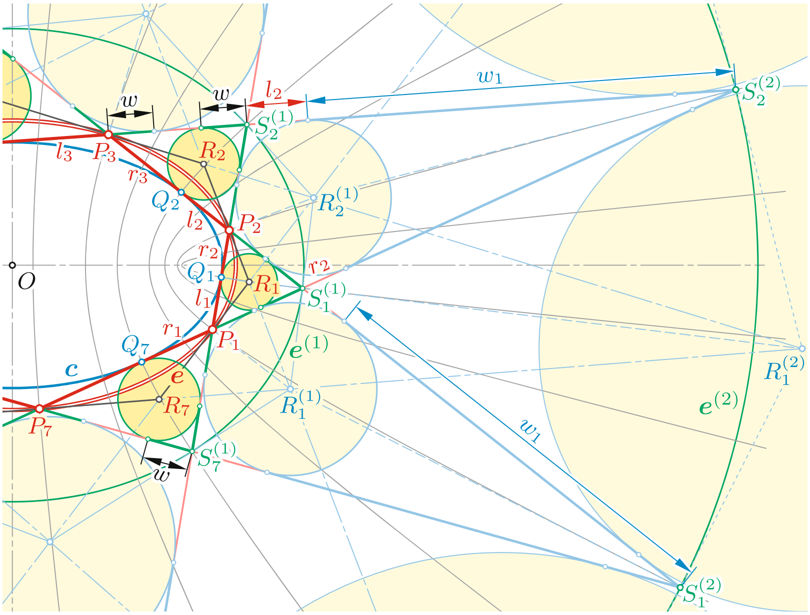

FIGURE 6 The power w^2 of $S_2^{(1)}$ w.r.t. the incircle of the triangle $S_2^{(1)}P_2P_3$ shows up at all $S_i^{(1)}$ and equals the power of P_i w.r.t. the incircle of the quadrangle $P_iS_{i-1}^{(1)}S_i^{(2)}S_i^{(1)}$

string strictly longer than the perimeter of c be posed around c. If point P is used to pull the string taut, then P traces a confocal ellipse e. Consequently, for each vertex P_i and neighboring tangency points Q_{i-1} and Q_i of a billiard in e with caustic c, the sum of the lengths $\overline{Q_{i-1}P_i}$ and $\overline{P_iQ_i}$ minus the length of the elliptic arc between Q_{i-1} and Q_i, i.e.,

$$D_e := \overline{Q_{i-1}P_i} + \overline{P_iQ_i} - \widehat{Q_{i-1}Q_i} \tag{3.4}$$

is constant (Fig. 6).

The incircle of $P_2P_3S_2^{(1)}$ has the center R_2 and the radius $\overline{Q_2R_2}$ by (2.15). The power of P_2 w.r.t. this circle is l_2^2, that of P_3 is r_3^2. From Graves' construction follows for the ellipse e that

$$D_e = r_2 + l_2 - \widehat{Q_1Q_2} = r_3 + l_3 - \widehat{Q_2Q_3} = \text{const.}$$

is the same for all P_i, provided that $\widehat{Q_iQ_j}$ denotes the length of the (shorter) arc along c between Q_i and Q_j. For the analogue invariant at $e^{(1)}$ follows (Fig. 6)

$$\begin{aligned} D_{e|1} &:= \overline{Q_1S_2^{(1)}} + \overline{S_2^{(1)}Q_3} - \widehat{Q_1Q_3} \\ &= (r_2 + l_2 + w) + (r_3 + l_3 + w) - \widehat{Q_1Q_3} = 2D_e + 2w = \text{const.}, \end{aligned} \tag{3.5}$$

hence $w = \text{const.}$, where w^2 is the power of $S_2^{(1)}$ w.r.t. the said incircle.

Since the incircle of the quadrangle $P_2 S_1^{(1)} S_2^{(2)} S_2^{(1)}$ is an excircle of the triangle $P_2 P_3 S_2^{(1)}$ (Fig. 6), the power of P_2 w.r.t. the excircle equals w^2, too. This follows from elementary geometry.

As an alternative, the constancy of w can also be concluded from the fact, that neighboring circles with centers R_i^1, R_i or R_i, R_{i+1}^1 share three tangents, and one circle is an incircle, the other an excircle of the triangle. Therefore, on the common side the same length w shows up twice and also at the adjacent pairs of neighboring circles. Since the distance w is constant for all aperiodic billiards, it reveals also for periodic billiards that w is independent of the choice of the initial vertex. In a similar way follows the invariance of the length w_1, as shown in Fig. 6. □

It needs to be noted that $S_2^{(1)}$ or $S_2^{(2)}$ can be located on the other branch of the related hyperbola. Then the said 'incircle' of the triangle $P_2 P_3 S_2^{(1)}$ has to be replaced by the excircle which contacts c at Q_2 and is tangent to the side lines $[P_1, P_2]$ and $[P_3, P_4]$. Similarly, the said 'incircle' of the quadrangle becomes an excircle. In all these cases, Lemma 3.8 and the proof given above have to be adapted.

Remark 3.9. The Theorems 3.5 and 3.6 as well as the constancy of the length w according to Lemma 3.8 are also valid in spherical geometry (note [26]) and in hyperbolic geometry. On the sphere (see Fig. 7), the caustic consists of a pair of opposite components, and for N-periodic billiards the confocal spherical ellipse $e^{(j)}$ coincides with $e^{(N-2-j)}$ w.r.t. the opposite caustic.

We obtain other families of incircles when we focus on pairs of consecutive sides of the billiards in $e^{(1)}$, $e^{(2)}$ and so on. However, these circles are not mutually disjoint. By the way, the centers $R_i^{(j)}$ of all these circles are the poles of diagonals of $\ldots P_1 P_2 P_3 \ldots$ w.r.t. the ellipse e.

For the sake of completeness, we express below in (3.9) the distance w in terms of the semiaxes of e and c. For this purpose, we compute first the semiaxes $a_{e|1}$ an $b_{e|1}$ of $e^{(1)}$, since we need the coordinates of $S_2^{(1)}$. From (2.13) and (2.15) follows

$$R_2 = \left(\frac{a_e^2 \cos t_2'}{a_c}, \ \frac{b_e^2 \sin t_2'}{b_c} \right), \quad \overline{Q_2 R_2} = \frac{k_e \|\mathbf{t}_c(t_2')\|}{a_c b_c}\,.$$

By virtue of Theorem 3.5, the tangents from R_2 to the confocal hyperbola through Q_2 contact at

$$Q_2 = (a_c \cos t_2', \ b_c \sin t_2') \in c \quad \text{and} \quad S_2^{(1)} = (a_{e|1} \cos t_2', \ b_{e|1} \sin t_2') \in e^{(1)}.$$

Both points lie on the polar of R_2 w.r.t. the hyperbola in question with the elliptic coordinate $k_h = -\|\mathbf{t}_e(t_2')\|^2$. This yields the condition

$$\frac{a_e^2 \cos t_2'}{a_c(a_c^2 + k_h(t_2'))}\, a_{e|1} \cos t_2' + \frac{b_e^2 \sin t_2'}{b_c(b_c^2 + k_h(t_2'))}\, b_{e|1} \sin t_2' = 1$$

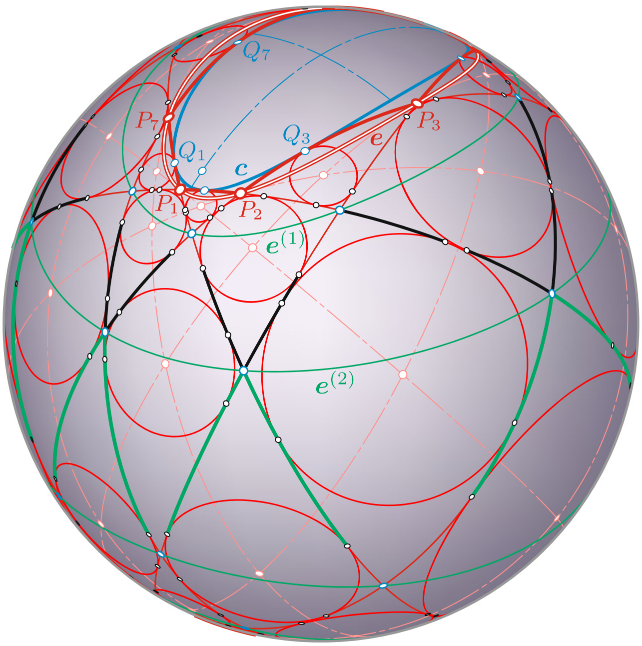

FIGURE 7 Periodic billiard $P_1P_2\dots P_7$ on the sphere with extended sides and their contact points with the incircles. All circular arcs marked in black have the same length, as well as those marked in green, and both lengths are invariant against changes of P_1 on e

or, by virtue of (2.7),

$$a_e^2 b_c a_{e|1} - b_e^2 a_c b_{e|1} = a_c b_c d^2, \quad \text{where} \quad a_{e|1}^2 - b_{e|1}^2 = d^2.$$

We eliminate $a_{e|1}$ and obtain after some computation the quadratic equation

$$(b_e^4 - 2b_c^2 b_e^2 - b_c^2 d^2)\, b_{e|1}^2 + 2a_c^2 b_c b_e^2\, b_{e|1} + b_c^2(a_c^2 d^2 - a_e^4) = 0.$$

The second solution besides $b_{e|1} = b_c$ is

$$b_{e|1} = \frac{b_c(a_c^2 d^2 - a_e^4)}{b_e^4 - 2b_c^2 b_e^2 - b_c^2 d^2} = \frac{b_c(a_e^2 b_e^2 + d^2 k_e)}{a_c^2 b_c^2 - k_e^2}. \tag{3.6}$$

This implies

$$a_{e|1} = \frac{a_c}{a_e^2 b_c}\left(b_c d^2 + b_e^2 b_{e|1}\right) = \frac{a_c(a_e^2 b_e^2 - d^2 k_e)}{a_c^2 b_c^2 - k_e^2} \tag{3.7}$$

and

$$k_{e|1} = b_{e|1}^2 - b_c^2 = k_e \left(\frac{2a_c b_c a_e b_e}{a_c^2 b_c^2 - k_e^2}\right)^2, \tag{3.8}$$

and yields finally

$$w := \frac{2a_e b_e \sqrt{k_e^3}}{a_c^2 b_c^2 - k_e^2}\,. \tag{3.9}$$

Negative semiaxes $a_{e|1}, b_{e|1}$ and a negative w in the formulas above mean that the points $S_i^{(1)}$ are located on the respectively second branches of the hyperbolas and the incircles of the triangles $P_i P_{i+1} S_i^{(1)}$ become excircles. In the case of a vanishing denominator for $k_e = a_c b_c$ (periodic four-sided billiard) the ellipse $e^{(1)}$ is the line at infinity.

If on the right-hand side of the formulas (3.7), (3.6) and (3.8) we replace a_e, b_e, k_e respectively by $a_{e|1}, b_{e|1}, k_{e|1}$, then we obtain expressions for $a_{e|3}, b_{e|3}$, $k_{e|3}$, i.e.,

$$k_{e|3} = k_{e|1}\left(\frac{2a_c b_c a_{e|1} b_{e|1}}{a_c^2 b_c^2 - k_{e|1}^2}\right)^2. \tag{3.10}$$

3.2. Conjugate billiards

For two confocal ellipses c and e, there exists an axial scaling

$$\alpha\colon\ (x,y) \mapsto \left(\frac{a_e}{a_c}x,\ \frac{b_e}{b_c}y\right) \quad \text{with} \quad c \to e\,. \tag{3.11}$$

Corresponding points share the parameter t. Hence, they belong to the same confocal hyperbola (Fig. 8). The affine transformation α maps the tangency point $Q_i \in c$ of the side $P_i P_{i+1}$ to a point $P_i' \in e$, while α^{-1} maps P_i to the tangency point Q_{i-1}' of $P_{i-1}' P_i'$, i.e.,

$$\alpha\colon\ Q_i \mapsto P_i', \quad Q_{i-1}' \mapsto P_i\,.$$

This results from the symmetry between t_i and t_i' in the equation

$$b_c a_e \cos t_i \cos t_i' + a_c b_e \sin t_i \sin t_i' = a_c b_c \tag{3.12}$$

which expresses that $P_i \in e$ with parameter t_i lies on the tangent to c at Q_i with parameter t_i'. Referring to Fig. 8, α sends the tangent $[P_{i-1}', P_i']$ to c at Q_{i-1}' to the tangent $[R_{i-1}, R_i]$ to e at P_i. Hence, by α the polygon $Q_1 Q_2 \dots$ is mapped to $P_1' P_2' \dots$ and futhermore to that of the poles $R_1 R_2 \dots$ of the billiard's sides $P_1 P_2$, $P_2 P_3, \dots$.

Definition 3.10. Referring to Fig. 8, the billiard $\dots P_0' P_1' P_2' \dots$ is called *conjugate* to the billiard $\dots P_0 P_1 P_2 \dots$ in the ellipse e with the ellipse c as caustic, when the axial scaling $\alpha\colon\ c \to e$ defined in (3.11) maps the tangency point Q_i of the side $P_i P_{i+1}$ to the vertex P_i'.

Lemma 3.11. *For each billiard $\dots P_0 P_1 P_2 \dots$ in the ellipse e with the ellipse c as caustic, there exists a unique conjugate billiard $\dots P_0' P_1' P_2' \dots$, and the relation between the two billiards in e is symmetric. Moreover,*

$$l_i = \overline{P_i Q_i} = \overline{P_i' Q_{i-1}'} = r_i' \quad \textit{and} \quad r_i = \overline{P_i Q_{i-1}} = \overline{P_{i-1}' Q_{i-1}'} = l_{i-1}'. \tag{3.13}$$

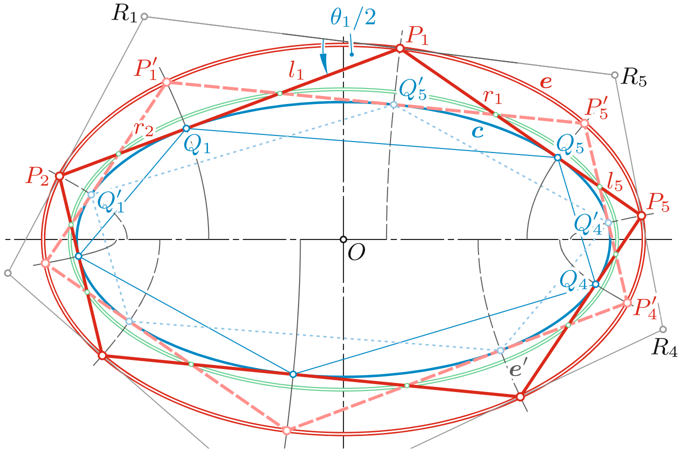

FIGURE 8 The periodic billiard $P_1P_2 \dots P_5$ in e with the caustic c and the conjugate billiard $P_1'P_2' \dots P_5'$

Proof. From the symmetry in (3.12) follows for $\alpha: c \to e$ that P_i is the preimage of P_i is the tangency point Q_{i-1} of $P_{i-1}'P_i'$. The congruences stated in (3.13) follow from Ivory's Theorem for the two diagonals in the curvilinear quadrangle $P_iP_i'Q_iQ_{i-1}'$. In view of the sequence of parameters $t_1, t_1', t_2, t_2', t_3, \dots$ of the vertices P_1, P_1', P_2, P_2', $P_3, \dots$ on e, the switch between the original billiard and its conjugate corresponds to the interchange of t_i with t_i' for $i = 1, 2, \dots$. □

Finally we recall that, based on the Arnold-Liouville theorem from the theory of completely integrable systems, it is proved in [19,21] that there exist *canonical coordinates* u on the ellipses e and c such that for any billiard the transitions from $P_i \to P_{i+1}$ and $Q_i \to Q_{i+1}$ correspond to shifts of the respective canonical coordinates u_i and u_{i+1} by $2\varDelta u$. Explicit formulas for the parameter transformation $t \mapsto u$ are provided in [27].

Figure 9 shows how on c such coordinates can be constructed by iterated subdivision, provided that Q_1 and Q_3 get the respective canonical coordinates $u = 0$ and 1. A comparison with Fig. 8 reveals that, in the sense of a canonical parametrization, the contact point Q_i is exactly halfway from P_i to P_{i+1}, i.e.,

$$u_i' = u_i + \varDelta u, \quad u_{i+1} = u_i + 2\varDelta u\,. \tag{3.14}$$

Hence, the transition from a billiard to its conjugate is equivalent to a shift of canonical coordinates by $\varDelta u$. An equivalent result can be found in [21, Sect. 4].

3.3. Billiards with a hyperbola as caustic

As illustrated in Fig. 10, billiards in ellipses e with a confocal hyperbola c as caustic are zig-zags between an upper and lower subarc of e. If the initial point P_1 is chosen at any point of intersection between e and the hyperbola c, then the billiard is twofold covered, and the first side P_1P_2 is tangent to c at P_1.

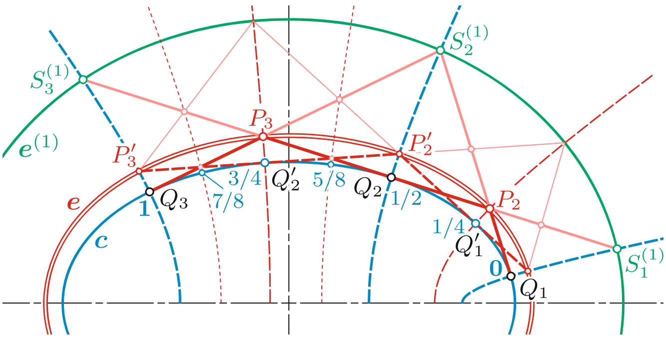

FIGURE 9 An example of canonical coordinates on c, e and $e^{(1)}$, this time with origin Q_1 and unit point Q_3

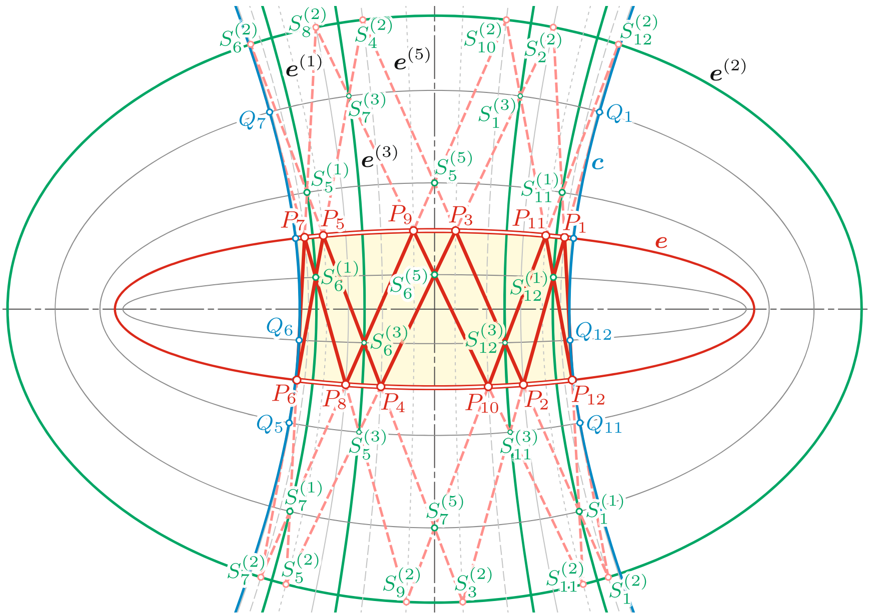

FIGURE 10 Periodic billiard $P_1P_2\dots P_{12}$ in the ellipse e with the hyperbola c as caustic, together with the hyperbolas $e^{(1)}$, $e^{(3)}$ and the ellipse $e^{(2)}$ with the inscribed billiard consisting of three quadrangles $S_i^{(2)}S_{i+3}^{(2)}S_{i+6}^{(2)}S_{i+9}^{(2)}$

Here we report briefly, in which way these billiard differ from those with an elliptic caustic. Proofs are left to the readers. In view of the associated Poncelet grid, we start with the analogue to Theorem 3.6 (see Figs. 10, 11 and 12).

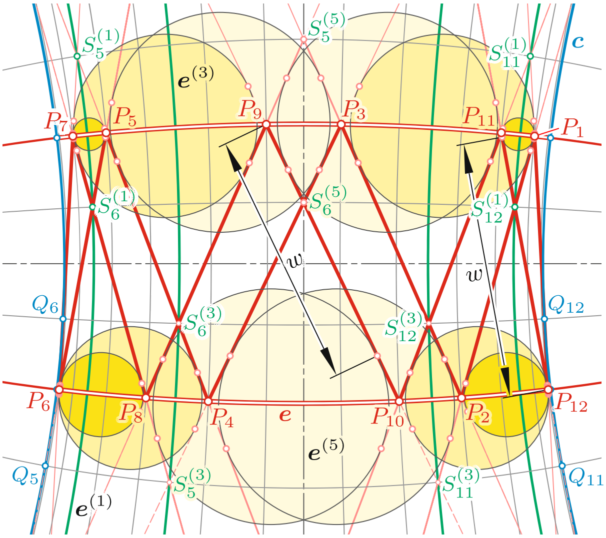

FIGURE 11 The power w^2 of P_9 w.r.t. the circle tangent to the sides $[P_7, P_8]$, $[P_8, P_9]$,$[P_9, P_{10}]$ equals that of P_{12} w.r.t. the circle tangent to $[P_{10}, P_{11}]$, $[P_{11}, P_{12}]$, $[P_{12}, P_1]$

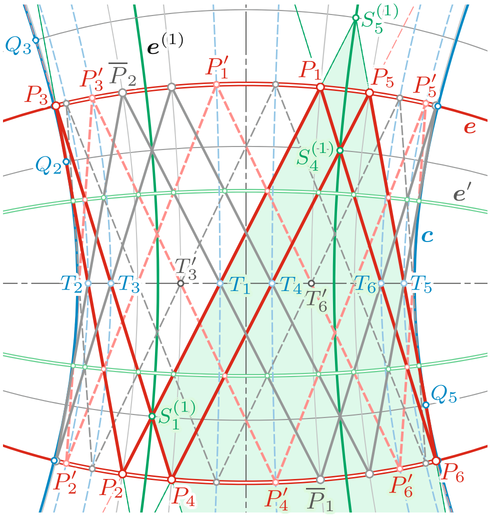

FIGURE 12 Periodic billiard $P_1 P \ldots P_6$ in the ellipse e with the hyperbola c as caustic, together with the conjugate billiard $P_1' P_2' \ldots P_6'$. The associated polygon with vertices on $e^{(1)}$ splits into two triangles $S_1^{(1)} S_3^{(1)} S_5^{(1)}$ (green shaded) and $S_2^{(1)} S_4^{(1)} S_6^{(1)}$

Theorem 3.12. *Let* $\dots P_0P_1P_2\dots$ *be a billiard in the ellipse* e *with the hyperbola* c *as caustic.*

1. *Then the points* $S_i^{(1)}$, $S_i^{(3)},\dots$ *are located on confocal ellipses through the contact point* Q_i *of* $[P_i,P_{i+1}]$ *with* c*, while the points* $S_i^{(2)}$, $S_i^{(4)},\dots$ *are located on the confocal hyperbola through* P_i*.*
2. *For even* j*, the points* $\dots S_i^{(j)}S_{i+(j+1)}^{(j)}S_{i+2(j+1)}^{(j)}\dots$ *are vertices of another billiard with the caustic* c *inscribed in a confocal ellipse* $e^{(j)}$*, provided that* $S_i^{(j)}$ *is finite.*
   For odd j*, the points* $S_i^{(j)}$ *are located on confocal hyperbolas* $e^{(j)}$ *or an axis of symmetry. At each vertex of* $\dots S_i^{(j)}S_{i+(j+1)}^{(j)}S_{i+2(j+1)}^{(j)}\dots$*, one angle bisector is tangent to* $e^{(j)}$*.*
   All conics $e^{(j)}$ *are independent of the position of the initial vertex* $P_1\in e$*.*

At the 12-periodic billiard depicted in Fig. 10, the billiard inscribed to $e^{(2)}$ splits into three quadrangles (dashed). Note that for odd j there are some points $S_i^{(j)}$ where the tangent to $e^{(j)}$ is the interior bisector of the angle $\angle S_{i-(j+1)}^{(j)}S_i^{(j)}S_{i+(j+1)}^{(j)}$. Hence, we obtain no billiards inscribed to hyperbolas $e^{(j)}$ with the hyperbola c as caustic. In Fig. 12, the 6-periodic billiard $P_1P_2\dots P_6$ yields two triangles $S_i^{(1)}S_{i+2}^{(1)}S_{i+4}^{(1)}$ inscribed to $e^{(1)}$; one of them is shaded in green.

Lemma 3.8 is also valid for hyperbolas as caustic. An example is depicted in Fig. 11: The power w^2 of P_9 w.r.t. the circle tangent to the four consecutive sides $[P_7,P_8]$, $[P_8,P_9]$,$[P_9,P_{10}]$, and $[P_{10},P_{11}]$ equals that of P_{12} w.r.t. the incircle of the quadrilateral with sides $[P_{10},P_{11}]$, $[P_{11},P_{12}]$, $[P_{12},P_1]$, and $[P_1,P_2]$.

Also for billiards $P_1P_2\dots$ in e with a hyperbola c as caustic, there exists a conjugate billiard $P_1'P_2'\dots$, and the relation is symmetric. However, the definition is different. It uses the singular affine transformation

$$\alpha_h\colon e\to F_1F_2 \text{ with } P_i\mapsto T_i=[P_i,P_{i+1}]\cap[F_1,F_2], \tag{3.15}$$

with F_1 and F_2 as the focal points of e and c (Fig. 12).

Definition 3.13. Referring to Fig. 12, the billiard $\dots P_0'P_1'P_2'\dots$ in the ellipse e with the hyperbola c as caustic is called *conjugate* to the billiard $\dots P_0P_1P_2\dots$ in e with the same caustic c if the axial scaling α_h defined in (3.15) maps the point P_i' to the intersection T_i of P_iP_{i+1} with the principal axis.

Lemma 3.14. *Let* $\dots P_0P_1P_2\dots$ *be a billiard in the ellipse* e *with the hyperbola* c *as caustic. Then to this billiard and to its mirror w.r.t. the principal axis exists a conjugate billiard* $\dots P_0'P_1'P_2'\dots$*, and it is unique up to a reflection in the principal axis. The relation between two conjugate billiards in* e *is symmetric. Moreover, if* T_i' *denotes the intersection of* $P_i'P_{i+1}'$ *with the principal axis,* *then*

$$\overline{P_iT_i} = \overline{P_i'T_{i-1}'} \text{ and } \overline{P_iT_{i-1}} = \overline{P_{i-1}'T_{i-1}'}. \tag{3.16}$$

Proof. The singular affine transformation α_h maps P_1' to T_1 and P_1 to a point T' ($=T_6'$ in Fig. 12). We assume that P_1 and P_1' lie on the same side of the principal axis, since otherwise we apply a reflection in the axis. Then we obtain a curvilinear Ivory quadrangle $P_1'T_1T'P_1$ with diagonals of equal lengths. On the other hand, the lines $[P_1', T']$ and $[P_1, T_1]$ must contact the same confocal conic (see, e.g., [7, p. 153] or [26, Lemma 1]). Hence, the billiard through the point $P_1' \in e$ with caustic c contains one side on the line $[P_1', T']$. Iteration confirms the claim. □

Let $\overline{P}_1$ and $\overline{P}_2$ be the images of P_1 and P_2 under reflection in the principal axis of e (Fig. 12). Then, a comparison with Fig. 9 reveals that the confocal hyperbola through the intersection $T_1 = [P_1, P_2] \cap [\overline{P}_1, \overline{P}_1]$ lies 'in the middle' between the hyperbolas through P_1 and P_2.

Remark 3.15. If $\ldots P_0P_1P_2\ldots$ and $\ldots P_0'P_1'P_2'\ldots$ is a pair of conjugate billiards in an ellipse e, then the points of intersection $[P_i, P_{i+1}] \cap [P_i', P_{i-1}']$ are located on a confocal ellipse e' inside e. This holds for ellipses and hyperbolas as caustics. If the billiards are N-periodic, then in the elliptical case, the restriction of the two billiards to the interior of e' is $2N$-periodic; conversely, e plays the role of $e^{(1)}$ w.r.t. e' (Fig. 8). In the hyperbolic case, the restriction to the interior of e' gives two symmetric $2N$-periodic billiards, provided that also the reflected billiards are involved (Fig. 12).

We conclude with citing a result from [28] about billiards in ellipses. It states that for each billiard $\ldots P_1P_2P_3\ldots$ in e with a hyperbola as caustic there exists a billiard $\ldots P_1^*P_2^*P_3^*\ldots$ in e^* with an ellipse as caustic such that corresponding sides P_iP_{i+1} and $P_i^*P_{i+1}^*$ are congruent.

4. Periodic N-sided billiards

Let the billiard $P_1P_2\ldots P_N$ in the ellipse e be periodic with an ellipse c as caustic. Then, the sequence of parameters $t_1, t_1', t_2, \ldots, t_N, t_N'$ of the vertices P_i and the intermediate contact points $Q_1, \ldots, Q_N$ with c is cyclic. Each side line intersects only a finite number of other side lines. Hence, the corresponding Poncelet grid contains a finite number of confocal ellipses $e^{(j)}$ through the points $S_i^{(j)}$, namely $[\frac{N-2}{2}]$ (including possibly the line at infinity), provided that $N \geq 5$ (Fig. 5). The sequence of ellipses $e, e^{(1)}, e^{(2)}, \ldots$ is cyclic, and

$$e^{(j)} = e^{(N-2-j)}. \tag{4.1}$$

For example, in the case $N = 7$ (Fig. 6), the ellipse $e^{(2)}$ coincides with $e^{(3)}$.

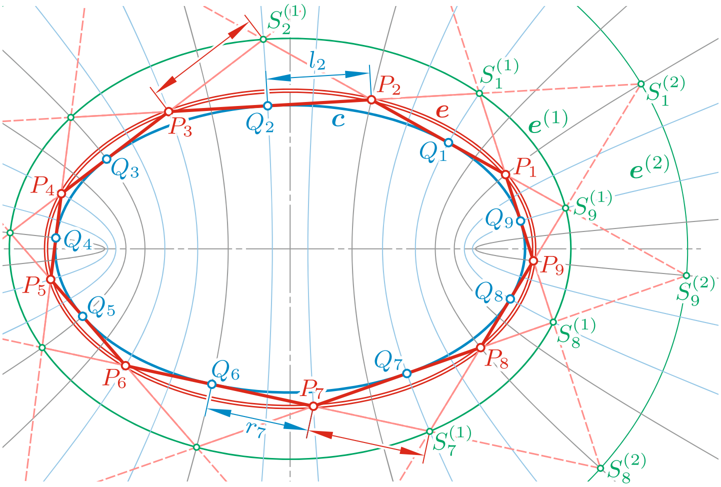

FIGURE 13 Periodic billiard $P_1P_2\ldots P_9$ with $\tau = 1$. Note that $l_2 = \overline{P_2Q_2} = \overline{Q_6P_7} = r_7$ and $\overline{S_2^{(1)}P_3} = \overline{P_7S_7^{(1)}}$. The associated billiard in $e^{(2)}$ splits into three triangles

Definition 4.1. The sum of the oriented exterior angles θ_i of a periodic billiard in an ellipse e is an integer multiple of 2π, namely $2\tau\pi$. We call $\tau \in \mathbb{N}$ the *turning number* of the billiard. It counts the loops of the billiard around the center O of e, anti-clockwise or clockwise.

If the periodic billiard $P_1P_2\ldots P_N$ has the turning number $\tau = 1$ (Fig. 13), then the billiard $S_1^{(1)}S_3^{(1)}S_5^{(1)}\ldots$ in $e^{(1)}$ has $\tau = 2$, that of $S_1^{(2)}S_4^{(2)}\ldots$ in $e^{(2)}$ the turning number $\tau = 3$, and so on. In cases with $g = \gcd(N, \tau) > 1$ the corresponding billiard splits into g $\frac{N}{g}$-sided billiards, each with turning number τ/g (note [25, Theorem 1.1]).

4.1. Symmetries of periodic billiards

The following is a corollary to Theorem 3.6.

Corollary 4.2. *Let $P_1P_2\ldots P_N$ be an N-sided periodic billiard in the ellipse e with the ellipse c as caustic.*

(i) For even N and odd τ, the billiard is centrally symmetric.
(ii) For odd $N = 2n+1$ and odd τ, the billiard is centrally symmetric to the conjugate billiard, where P_i corresponds to P'_{i+n}.[4]
(iii) If N is odd and τ is even, then the conjugate billiard coincides with the original one, and $P_i = P'_{i+n}$.

Proof. By virtue of Theorem 3.6, the lines $[P_{i-j-1}, P_{i-j},]$ and $[P_{i+j}, P_{i+j+1}]$ for $j = 1, 2, \ldots$ meet at the point $S_i(j)$ on the confocal hyperbola through P_i.

[4] All subscripts in this section are understood modulo N.

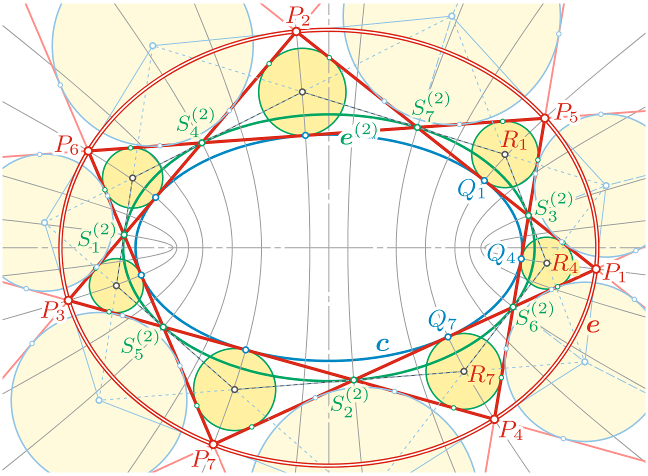

FIGURE 14 Periodic billiard with $N = 7$ and $\tau = 2$

(i) This means for even $N = 2n$, odd τ and $j = n-1$, that also the opposite vertex $S_i^{(n)} = P_{i-n} = P_{i+n}$ belongs to this hyperbola. If P_i is specified at a vertex on the minor axis of the ellipse e, then P_{i+n} is the opposite vertex. Continuity implies that the two points belong to different branches of the hyperbola and are symmetric w.r.t. the center O of e.

(ii) , (iii): If N is odd, say $N = 2n+1$, then for $j = n-1$ the sides $[P_{i-n+1}, P_{i-n}]$ and $[P_{i+n-1}, P_{i+n}]$ intersect at a point on the hyperbola through Q_{i+n} and P'_{i+n}. For odd τ (Figs. 8 and 13), the same continuity argument as before proves that P_i and P'_{i+n} are opposite w.r.t. O.
If τ is even (Fig. 14), then the choice of $P_i \in e$ on an axis of symmetry shows the coincidence with $P'_{i+n} \in e$, and this must be preserved, when P_i varies continuously on e. In the case of even τ and N the billiard splits.

□

The billiards with a hyperbola c as caustic (see Figs. 10 and 12) oscillate between the upper and lower section of e. Therefore, only billiards with an even N can be periodic. Also for billiards of this type, it possible to define a turning number τ which counts how often the points $P_1, \overline{P}_2, P_3, \ldots, P_N$ (Fig. 12) run to and fro along the upper component of e.[5] The symmetry properties of these periodic N-sided billiards differ from those in Corollary 4.2. They follow from Theorem 3.6, since opposite vertices P_i and $P_{i+N/2}$ belong to the same confocal hyperbola.

Corollary 4.3. *Let $P_1P_2 \ldots P_N$ be an N-sided periodic billiard in the ellipse e with the hyperbola c as caustic.*

[5] The turning number of hyperbolic billiards becomes more intuitive when the billiard is seen as the limit of a focal billiard in the sense of [28, Theorem 2].

(i) *For* $N \equiv 0 \pmod 4$*, the billiard is symmetric w.r.t. the secondary axis of* e *and* c*.*

(ii) *For* $N \equiv 2 \pmod 4$ *and odd turning number* τ*, the billiards are centrally symmetric. For even* τ*, each billiard is symmetric w.r.t. the principal axis of* e *and* c*.*

4.2. Some invariants

As a direct consequence of the results so far, we present new proofs for the invariants k101, k118 and k119 listed in [24, Table 2], though this table refers already to proofs for some of them in [2,11].

We begin with a result that has first been proved for a much more general setting in [29, p. 103].

Lemma 4.4. *The length* L_e *of a periodic* N*-sided billiard in the ellipse* e *with the ellipse* c *as caustic is independent of the position of the initial vertex* $P_1 \in e$*.*

Proof. We refer to Graves's construction [15, p. 47]. According to (3.4) holds

$$D_e := \overline{Q_{i-1}P_i} + \overline{P_iQ_i} - \widehat{Q_{i-1}Q_i}.$$

This yields for an N-sided billiard with turning number τ the total length

$$L_e = N \cdot D_e - \tau \cdot P_c\,, \tag{4.2}$$

where P_c denotes the perimeter of the caustic c. Thus, L_e does not depend of the choice of the initial vertex $P_1 \in e$. □

If the billiard in e has the turning number τ, then its extension in $e^{(1)}$ has the turning number 2τ, and from (3.5) and (3.9) follows

$$L_{e|1} = ND_{e|1} - 2\tau P_c = 2N(D_e + w) - 2\tau P_c = 2L_e + 2N\,\frac{2a_eb_e\sqrt{k_e^3}}{a_c^2b_c^2 - k_e^2}\,. \tag{4.3}$$

The following theorem on the invariant k118 in [24] deals with the lengths r_i and l_i of the segments $Q_{i-1}P_i$ and P_iQ_i, as defined in (3.1).

Theorem 4.5. *In each* N*-sided periodic billiard opposite segments are congruent, i.e., if* $N = 2n$*, then* $r_{i+n} = r_i$ *and* $l_{i+n} = l_i$ *, and if* $N = 2n+1$*, then* $r_{i+n} = l_{i-1}$ *and* $l_{i+n} = r_i$ *. Thus, for odd* N *holds*

$$\sum_{i=1}^{N} l_i = \sum_{i=1}^{N} r_i = \frac{L_e}{2}\,.$$

Proof. By Corollary 4.2, for even $N = 2n$ the central symmetry implies for opposite segments $r_i = r_{i+n}$ and $l_i = l_{i+n}$.

If $N = 2n+1$, then $l_i = \overline{P_iQ_i}$ shows up as $l'_{i+n} = \overline{P'_{i+n}Q'_{i+n}}$ at the conjugate billiard and, by virtue of (3.13), this equals $r_{i+n+1} = \overline{P_{i+n+1}Q_{i+n}}$ (Fig. 13). Similarly follows $r_i = l_{i+n}$. □

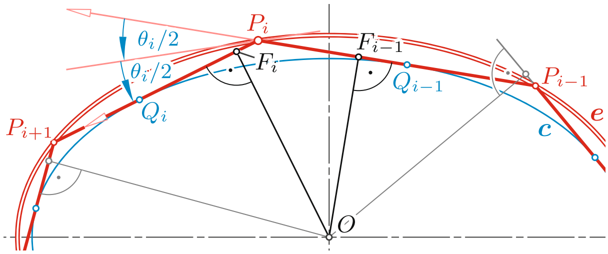

FIGURE 15 Billiard $P_1P_2\ldots$ with pedal points w.r.t. O

Remark 4.6. In the particular case $N=3$ the two segments adjacent to any side are congruent (note in Fig. 13 the triangular billiards in $e^{(2)}$). Therefore, the Cevians $[P_i, Q_{i+1}]$ are concurrent and meet at the Nagel point of the triangle. This has already been proved in [23] and agrees with the circles through Q_i and centered at R_i (see Figs. 6 and 14), which for $N=3$ are excircles of the triangle $P_1P_2P_3$.

The following theorem has first been proved in [2, p. 4]. Another proof can be found in [3, Cor. 3.2]. We give below a new proof.

Theorem 4.7. *For the exterior angles $\theta_1,\ldots,\theta_N$ of the periodic N-sided elliptic billiard in the ellipse e, the sum of cosines is independent of the initial vertex, namely*

$$\sum_{i=1}^{N}\cos\theta_i = N - J_eL_e = N - \frac{\sqrt{k_e}}{a_eb_e}\,L_e\,,$$

where L_e is the common perimeter of these billiards in e.

Proof. The pedal points F_i and F_{i-1} on the sides P_iP_{i+1} and $P_{i-1}P_i$ w.r.t. the center O (Fig. 15) have the position vectors

$$\mathbf{f}_{i,i-1} = \mathbf{p} + \tfrac{\lambda_{i,i-1}}{\|\mathbf{t}_e\|}\left(\cos\tfrac{\theta_i}{2}\,\mathbf{t} \pm \sin\tfrac{\theta_i}{2}\,\mathbf{t}^\perp\right),$$
$$\text{where}\quad 0 = \left\langle \mathbf{f}_{i,i-1},\, \left(\cos\tfrac{\theta_i}{2}\,\mathbf{t} \pm \sin\tfrac{\theta_i}{2}\,\mathbf{t}^\perp\right)\right\rangle.$$

Here, λ_i and λ_{i+1} denote the signed distances from the vertex P_i in one case towards P_{i+1}, in the other opposite to P_{i-1}. From

$$\langle \mathbf{p}_i,\, \mathbf{t}\rangle = (-a_e^2+b_e^2)\cos t\sin t \text{ and } \langle \mathbf{p}_i,\, \mathbf{t}^\perp\rangle = -a_eb_e$$

follows

$$\lambda_{i,i-1} = \frac{1}{\|\mathbf{t}_e\|}\left((-a_e^2+b_e^2)\cos t\sin t\cos\frac{\theta_i}{2} \pm a_eb_e\sin\frac{\theta_i}{2}\right).$$

This implies by virtue of (2.11) and after reversing the orientation for λ_{i-1},

$$\lambda_i - \lambda_{i-1} = \overline{P_iF_i} + \overline{P_iF_{i-1}} = \frac{2a_eb_e}{\|\mathbf{t}_e\|^2}\sqrt{k_e}\,.$$

Since the sum over all signed lengths between P_i and the adjacent pedal points gives the total perimeter L_e of the billiard, we obtain by (2.12)

$$L_e = \sum_{i=1}^{N} \left(\overline{P_iF_i} + \overline{P_iF_{i-1}} \right) = \frac{a_e b_e}{\sqrt{k_e}} \sum_{i=1}^{N} \frac{2k_e}{\|\mathbf{t}_e\|^2} = \frac{a_e b_e}{\sqrt{k_e}} \sum_{i=1}^{N} (1 - \cos\theta_i), \quad (4.4)$$

hence

$$\sum_{i=1}^{N} \frac{1}{\|\mathbf{t}_e\|^2} = \frac{L_e}{2a_e b_e \sqrt{k_e}} \text{ and also } \sum_{i=1}^{N} \cos\theta_i = N - \frac{\sqrt{k_e}}{a_e b_e} L_e\,, \quad (4.5)$$

as stated. □

Remark 4.8. Note that the result in [2] relates to the interior angles of the billiard. As already mentioned in [2, Theorem 7], the constant sum of cosines holds also for the 'extended' billiards in $e^{(j)}$, where the exterior angles are $\theta_i + \theta_{i+1} + \cdots + \theta_{i+j}$ (note Fig. 5).

The first equation in (4.5) gives rise to two invariants which are already known: Similar to (2.14), the distance of O to the tangent t_P to e at P equals

$$\overline{Ot_P} = \frac{a_e b_e}{\|\mathbf{t}_e\|} \quad (4.6)$$

and yields a result as stated in [3, Cor. 3.2, third equation]. The invariant k119, first proved by P. Roitmann, deals with the curvature of e, namely by [15, p. 79] with

$$\kappa_e(t) := \frac{a_e b_e}{\|\mathbf{t}_e(t)\|^3}\,.$$

Corollary 4.9. *The squared distances from the center O to the tangents t_{P_i} at the vertices P_i of the periodic N-sided elliptic billiard in the ellipse e have a constant sum, independent of the initial vertex, namely*

$$\sum_{i=1}^{N} \overline{Ot_P}^{\,2} = \frac{a_e b_e}{2\sqrt{k_e}} L_e\,.$$

The curvatures κ_i of e at the vertices P_i give rise to an invariant sum

$$\sum_{i=1}^{N} \kappa_i^{2/3} = \frac{L_e}{2\sqrt{k_e}} (a_e b_e)^{-1/3}\,.$$

Remark 4.10. It is remarkable that the quantity $\kappa^{2/3}$ appears in the billiard setting also at the Lazutkin parameter which coincides with the Poritsky string length, i.e., a kind of canonical parameter, up to additive and multiplicative constants. For details see [16].

We conclude this section with a comment on (4.1) in connection with (3.8) and (3.10). For example, we obtain $k_{e|1} = k_e$ for $N = 3$ and $k_e = a_c\, b_c$ for $N = 4$. The condition $k_{e|2} = k_{e|1}$ is valid for $N = 5$, and $k_{e|2} = \infty$ for $N = 6$. This yields algebraic conditions for the semiaxes of the ellipse e with an inscribed N-periodic billiard when the ellipse c is given as caustic. However, we need to recall that approximately 200 years ago N. Fuß and J. Steiner presented already equations for the projectively equivalent case of circles (see [20, pp. 378–380]), and A. Cayley published an explicit solution for general N in a projective setting ([17] or [15, Theorem 9.5.4]). Other approaches are provided in [4, Sect. VI] and [14, Sect. 11.2.3.9]. Equivalent conditions in terms of elliptic functions can be deduced from [27, Corollary 3].

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