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On Baer cones in PG(3, q)

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Abstract. Recently, in Innamorati and Zuanni (J. Geom 111:45, 2020. https://doi.org/10.1007/s00022-020-00557-0) the authors give a characterization of a Baer cone of $\mathrm{PG}(3, q^2)$, q a prime power, as a subset of points of the projective space intersected by any line in at least one point and by every plane in $q^2 + 1$, $q^2 + q + 1$ or $q^3 + q^2 + 1$ points. In this paper, we show that a similar characterization holds even without assuming that the order of the projective space is a square, and weakening the assumptions on the three intersection numbers with respect to the planes.

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1. Introduction

Let p be a prime number and $q = p^h$. Let π and $V \notin \pi$ be a plane and a point of $\mathrm{PG}(3, q^2)$, respectively, and let $\mathcal{B}$ be a Baer subplane of π. A *Baer cone* of $\mathrm{PG}(3, q^2)$ with *vertex* V and *base* $\mathcal{B}$ is the set of points of the union of the lines through V and any point of $\mathcal{B}$ [3]. A Baer cone $\mathcal{C}$ has non–empty intersection with the lines of the projective space and if α is a plane of $\mathrm{PG}(3, q^2)$ then $|\alpha \cap \mathcal{C}| \in \{q^2 + 1, q^2 + q + 1, q^3 + q^2 + 1\}$.

In [3] one may find a characterization of Baer cones in terms of the sizes of their intersection with the planes of the projective space. To state this result, we need to recall the following definition.

Definition 1.1. Let d be a positive integer less than r, $\mathcal{P}_d$ be the family of all the d–dimensional subspaces of $\mathrm{PG}(r, q)$, $q = p^h$ and p a prime number, and

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$0 \leq m_1 \leq \cdots \leq m_s \leq q^2+q+1$ be s non–negative integers. A set $\mathcal{K}$ of points of $\mathrm{PG}(r,q)$ is of type $(m_1,\ldots,m_s)_d$ with respect to the dimension d if

(i) $|S_d \cap \mathcal{K}| \in \{m_1,\ldots,m_s\}$ for any $S_d \in \mathcal{P}_d$,
(ii) for every $m_i \in \{m_1,\ldots,m_s\}$ there exists a subspace $S_d \in \mathcal{P}_d$ such that $|S_d \cap \mathcal{K}| = m_i$.

The integers m_i are called the *intersection numbers* of $\mathcal{K}$ with respect to the d–dimensional subspaces of $\mathrm{PG}(r,q)$. When $d \in \{1,2\}$, the set $\mathcal{K}$ is of *line–type* $(m_1,\ldots,m_s)_1$ or *plane–type* $(m_1,\ldots,m_s)_2$.

The study of such sets is motivated not only since most of the classical objects of finite projective geometry (for example: quadrics, Hermitian varieties, subgeometries and linear sets) have few intersection numbers with respect to the hyperplanes of the projective space and/or with respect to other families of subspaces but also since they are connected with (projective) linear codes with s weights and with other combinatorial structures (cf e.g.[1,2,4–6]).

Theorem. ([3], Innamorati–Zuanni (2020))

Let $\mathcal{K}$ be a set of points of $\mathrm{PG}(3,q^2)$, $q=p^h$ a prime power, of plane–type $(q^2+1, q^2+q+1, q^3+q^2+1)_2$. If $\mathcal{K}$ has non–empty intersection with all the lines of $\mathrm{PG}(3,q^2)$, then it is a Baer cone.

In this paper, we generalize such a characterization by weakening the assumptions on the order of the space and on the values of the intersction numbers with respect to the planes proving the following result.

Theorem 1.2. *Let s be an integer such that $2 \leq s \leq q$ and let $\mathcal{K}$ be a set of points of $\mathrm{PG}(3,q)$ of plane–type $(q+1, s+q+1, sq+q+1)_2$. If any line of $\mathrm{PG}(3,q)$ intersects $\mathcal{K}$ in at least one point, then either $s=q$ and $\mathcal{K}$ is the set of points of the union of two planes or q is a square, $s=\sqrt{q}$ and $\mathcal{K}$ is a Baer cone of $\mathrm{PG}(3,q)$.*

Let us end this section by recalling the following result of Tallini [7] which will play a special role in a step of our proof.

Theorem 1.3. *In $\mathrm{PG}(2,q)$ a k–set of line–type $(1,n)_1$, $2 \leq n \leq q$, is either a Baer subplane or a Hermitian arc; so q is a square, $n = 1+\sqrt{q}$ and $k = q+\sqrt{q}+1$ or $q\sqrt{q}+1$.*

2. The proof

Let $1 \leq s \leq q$ be an integer, $\mathcal{K}$ be a k–set of points of $\mathrm{PG}(3,q)$ such that

(i) *Any line of $\mathrm{PG}(3,q)$ intersects $\mathcal{K}$ in at least one point*;
(ii) *$\mathcal{K}$ is of plane–type $(q+1, q+s+1, sq+q+1)_2$.*

As usual, a line (plane) intersecting $\mathcal{K}$ in exactly j points is called a *j–line* (*j–plane*), and when $j=1$ we also use *tangent* line (*tangent* plane). Clearly, a line contained in $\mathcal{K}$ is a $(q+1)$–line.

Proposition 2.1. *If π is a $(q+1)$–plane then $\pi \cap \mathcal{K}$ is a line.*

Proof. Since there is no external line to $\mathcal{K}$, any point of π not in $\mathcal{K}$ is on exactly $q+1$ tangent lines, thus all the $q+1$ points of $\pi \cap \mathcal{K}$ are collinear. $\square$

It follows that there are both lines contained in $\mathcal{K}$ and tangent lines. If there is no h–line, with $2 \le h \le q$, then $\mathcal{K}$ is of line–type $(1, q+1)_1$ and so it is a plane, which is not possible since in such a case $\mathcal{K}$ should be of plane–type $(q+1, q^2+q+1)_2$.

Proposition 2.2. *All the secant–lines not contained in $\mathcal{K}$ intersect $\mathcal{K}$ in a constant number $h \ge 2$ of points.*

Proof. Let ℓ and t be an i–secant and and h–secant line, respectively, with $i, h \ge 2$ and both not contained in $\mathcal{K}$. Let x and y denote the number of $(q+s+1)$–planes on ℓ and t, respectively. Then

$$\begin{aligned} k &= i + x(q+s+1-i) + (q+1-x)(sq+q+1-i) \\ &= sq(q+1) + (q+1)^2 - iq - sx(q-1), \\ k &= h + y(q+s+1-h) + (q+1-y)(sq+q+1-h) \\ &= sq(q+1) + (q+1)^2 - hq - sy(q-1); \end{aligned}$$

so

$$iq + sx(q-1) = hq + sy(q-1).$$

Hence

$$(h-i)q = s(q-1)(x-y).$$

If $x \ne y$ then $(q-1)|(h-i)$ and so $h=i$, since $h-i \le q-2$. If $x=y$ then again $h=i$. Thus, $\mathcal{K}$ is of line–type $(1, h, q+1)_1$. $\square$

Proposition 2.3. *If $s=q$, then $\mathcal{K}$ is the set of the points of the union of two planes.*

Proof. Since $sq+q+1 = q^2+q+1$, $\mathcal{K}$ contains at least one plane and any plane contains at least one line contained in $\mathcal{K}$. Let π be a $(2q+1)$–plane, ℓ be a $(q+1)$–line of π and let p be a point of $\mathcal{K}$ not in ℓ, if on p there is no $(q+1)$–line, then $2q+1 = 1+(q+1)(h-1)$ and so $(q+1)|2$ a contradiction. Hence on p there is a $(q+1)$–line, say t, and $\pi \cap \mathcal{K} = \ell \cup t$. Thus, any $(2q+1)$–plane intersects $\mathcal{K}$ in two $(q+1)$–lines and $h=2$. Since $\mathcal{K}$ is of plane–type $(q+1, 2q+1, q^2+q+1)$ and $h=2$ it contains at most two planes. If $\mathcal{K}$ contains exactly two planes, then it coincides with their union, otherwise there should be a line contained in $\mathcal{K}$ outside the union of these two planes and so there is no $(q+1)$–plane. Assume that $\mathcal{K}$ contains exactly one plane, say π. Since $\mathcal{K}$ has three disinct intersection numbers, there is a point p of $\mathcal{K}$ not in π. Since p lies on at least one $(2q+1)$–plane, it follows that that p belongs to at least one $(q+1)$–line not in π and so there is no $(q+1)$–plane, a contradiction. $\square$

So, we may assume that $s \leq q-1$.

Proposition 2.4. $h \leq s+1$.

Proof. Assume that $h \geq s+2$. A $(q+s+1)$–plane π contains no h–line, otherwise let P be a point of an h–line in π with $P \notin \mathcal{K}$, counting the number of points of $\pi \cap \mathcal{K}$ via the lines through P one gets $q+s+1 \geq h+q \geq s+2+q$, a contradiction. Thus, any $(q+s+1)$–plane is of line–type $(1, q+1)_1$ that is $\pi \cap \mathcal{K}$ is a $(q+1)$–line, which is not possible since s is positive. $\square$

Proposition 2.5. *If $s \geq 2$, a $(q+s+1)$–plane contains no $(q+1)$–line.*

Proof. Let π be a $(q+s+1)$–plane, and assume that π contains a $(q+1)$–line ℓ. Let P be a point of $\pi \cap \mathcal{K}$ outside ℓ. Since the lines through P are all secant ones and, being $s \leq q-1$ no one of them is contained in $\mathcal{K}$, it follows that $q+s+1=(q+1)(h-1)+1$ and so $h=2$ and $s=1$. $\square$

From now on, assume that $s \geq 2$.

Proposition 2.6. *Any two lines contained in $\mathcal{K}$ have non–empty intersection.*

Proof. Assume on the contrary that there are two $(q+1)$–lines ℓ and t which are skew. Then, by Proposition 2.5 all the planes passing through each of them are $(sq+q+1)$–planes, and so counting the number of points of $\mathcal{K}$ via the planes on ℓ gives $k=q+1+(q+1)\cdot sq$. On the other hand, if π is a $(q+1)$–plane, by Proposition 2.1 it intersects $\mathcal{K}$ in a $(q+1)$–line, say m. Counting the number of points of $\mathcal{K}$ via the planes though m gives $k \leq q+1+q\cdot sq$, a contradiction. $\square$

Also, from Proposition 2.5 it follows that *for every $(q+s+1)$–plane π the set $\pi \cap \mathcal{K}$ is of line–type $(1,h)_1$*. So, by Theorem 1.3, q is a square, $h=\sqrt{q}+1=s+1$, $s=\sqrt{q}$ and any $(q+\sqrt{q}+1)$–plane intersects $\mathcal{K}$ in a Baer subplane.

Now, let π be a $(q\sqrt{q}+q+1)$–plane, every point $p \in \pi \cap \mathcal{K}$ belongs to at least one line contained in $\mathcal{K}$, otherwise $q\sqrt{q}+q+1 \leq 1+(q+1)\sqrt{q}$. Thus, π contains at least two $(q+1)$–lines, and let V_π be their common point. Since there is at least one tangent line through V_π in π, then all the $(q+1)$–lines of π are concurrent in V_π. It follows that all the tangent lines of π pass through V_π and the lines of π on V_π are either $(q+1)$–lines or tangent ones. Indeed, if there was a $(\sqrt{q}+1)$–line ℓ through V_π, counting the number of points of $\pi \cap \mathcal{K}$ via the lines through a point $p \in \ell \setminus \mathcal{K}$ one gets $q\sqrt{q}+q+1=(q+1)(\sqrt{éq+1})$, a contradiction. Thus π, and so any other $(q\sqrt{q}+q+1)$–plane, contains exactly $\sqrt{q}+1$ lines contained in $\mathcal{K}$ which are concurrent in a point of $\pi \cap \mathcal{K}$.

Since any two $(q+1)$–lines are not disjoint and any $(q\sqrt{q}+q+1)$–plane contains at least two $(q+1)$–lines, it follows that there is a point V of PG$(3, q)$ such that every $(q+1)$–line contains V. From Proposition 2.5 it follows that the lines through V are either contained in $\mathcal{K}$ or tangent lines.

If π is a $(q+\sqrt{q}+1)$–plane, then it does not contain V and so $k = (q+\sqrt{q}+1)q+1$, that is $\mathcal{K}$ is the set of points of the union of the lines joining V with all the points of a Baer subplane of π, that is a Baer cone. So, Theorem 1.2 is completely proved.

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