Solitons of the midpoint mapping and affine curvature

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Abstract. For a polygon $x = (x_j)_{j \in \mathbb{Z}}$ in $\mathbb{R}^n$ we consider the midpoints polygon $(M(x))_j = (x_j + x_{j+1})/2$. We call a polygon a *soliton* of the midpoints mapping M if its midpoints polygon is the image of the polygon under an invertible affine map. We show that a large class of these polygons lie on an orbit of a one-parameter subgroup of the affine group acting on $\mathbb{R}^n$. These smooth curves are also characterized as solutions of the differential equation $\dot{c}(t) = Bc(t) + d$ for a matrix B and a vector d . For $n = 2$ these curves are curves of constant generalized-affine curvature $k_{ga} = k_{ga}(B)$ depending on B parametrized by generalized-affine arc length unless they are parametrizations of a parabola, an ellipse, or a hyperbola.

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1. Introduction

We consider an infinite polygon $(x_j)_{j \in \mathbb{Z}}$ given by its vertices $x_j \in \mathbb{R}^n$ in an n -dimensional real vector space $\mathbb{R}^n$ resp. an n -dimensional affine space $\mathbb{A}^n$ modelled after $\mathbb{R}^n$. For a parameter $\alpha \in (0, 1)$ we introduce the polygon $M_\alpha(x)$ whose vertices are given by

$$(M_\alpha(x))_j := (1 - \alpha)x_j + \alpha x_{j+1}.$$

For $\alpha = 1/2$ this defines the *midpoints polygon* $M(x) = M_{1/2}(x)$. On the space $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ of polygons in $\mathbb{R}^n$ this defines a discrete curve shortening process $M_\alpha : \mathcal{P} \rightarrow \mathcal{P}$, already considered by Darboux [4] in the case of a *closed* resp.

FIGURE 1 The parabola $c(t) = (t^2/2, t)$ as soliton of the mid-points map M

periodic polygon. For a discussion of this elementary geometric construction see Berlekamp et al. [1].

The mapping M_α is invariant under the canonical action of the affine group. The *affine group* $\text{Aff}(n)$ in dimension n is the set of *affine maps* $(A, b) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, x \longmapsto Ax + b$. Here $A \in \text{Gl}(n)$ is an invertible matrix and $b \in \mathbb{R}^n$ a vector. The translations $x \longmapsto x + b$ determined by a vector b form a subgroup isomorphic to $\mathbb{R}^n$. Let $\alpha \in (0, 1)$. We call a polygon x_j a *soliton* for the process M_α (or affinely invariant under M_α) if there is an affine map $(A, b) \in \text{Aff}(n)$ such that

$$(M_\alpha(x))_j = Ax_j + b \quad (1)$$

for all $j \in \mathbb{Z}$. In Theorem 1 we describe these solitons explicitly and discuss under which assumptions they lie on the orbit of a one-parameter subgroup of the affine group acting canonically on $\mathbb{R}^n$. We call a smooth curve $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ a *soliton* of the mapping M_α resp. *invariant* under the mapping M_α if there is for some $\epsilon > 0$ a smooth mapping $s \in (-\epsilon, \epsilon) \longmapsto (A(s), b(s)) \in \text{Aff}(n)$ such that for all $s \in (-\epsilon, \epsilon)$ and $t \in \mathbb{R}$:

$$\tilde{c}_s(t) := (1 - \alpha)c(t) + \alpha c(t + s) = A(s)c(t) + b(s). \quad (2)$$

Then for some $t_0 \in \mathbb{R}$ and $s \in (-\epsilon, \epsilon)$ the polygon $x_j = c(js + t_0), j \in \mathbb{Z}$ is a soliton of M_α . The parabola is an example of a soliton of $M = M_{1/2}$, cf. Fig. 1 and Example 1, Case (e). We show in Theorem 2 that the smooth curves invariant under M_α coincide with the orbits of a one-parameter subgroup of the affine group $\text{Aff}(n)$ acting canonically on $\mathbb{R}^n$. For $n = 2$ we give a characterization of these curves in terms of the *general-affine curvature* in Sect. 5.

The authors discussed *solitons*, i.e. curves affinely invariant under the curve shortening process $T : \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathcal{P}(\mathbb{R}^n)$ with

$$(T(x))_j = \frac{1}{4} \{x_{j-1} + 2x_j + x_{j+1}\} \quad (3)$$

in [9]. The solitons of $M = M_{1/2}$ form a subclass of the solitons of T , since $(T(x))_j = (M^2(x))_{j-1}$. Instead of the discrete evolution of polygons one can also investigate the evolution of polygons under a linear flow, cf. Viera and Garcia [11] and [9, Sec. 4] or a non-linear flow, cf. Glickenstein and Liang [5].

2. The affine group and systems of linear differential equations of first order

The affine group $\text{Aff}(n)$ is a semidirect product of the general linear group $\text{Gl}(n)$ and the group $\mathbb{R}^n$ of translations. There is a linear representation

$$(A, b) \in \text{Aff}(n) \longrightarrow \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \text{Gl}(n+1),$$

of the affine group in the general linear group $\text{Gl}(n+1)$, cf. [8, Sec. 5.1]. We use the following identification

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}. \quad (4)$$

Hence we can identify the image of a vector $x \in \mathbb{R}^n$ under the affine map $x \longmapsto Ax + b$ with the image $\begin{pmatrix} Ax + b \\ 1 \end{pmatrix}$ of the extended vector $\begin{pmatrix} x \\ 1 \end{pmatrix}$. Using this identification we can write down the solution of an inhomogeneous system of linear differential equations with constant coefficients using the power series $F_B(t)$ which we introduce now:

Proposition 1. *For a real (n, n) -matrix $B \in M_{\mathbb{R}}(n)$ we denote by $F_B(t) \in M_{\mathbb{R}}(n)$ the following power series:*

$$F_B(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} B^{k-1}. \quad (5)$$

(a) *We obtain for its derivative:*

$$\frac{d}{dt} F_B(t) = \exp(Bt) = B F_B(t) + \mathbb{1}. \quad (6)$$

The function $F_B(t)$ satisfies the following functional equation:

$$F_B(t+s) = F_B(s) + \exp(Bs) F_B(t), \quad (7)$$

resp. for $j \in \mathbb{Z}, j \geq 1$:

$$\begin{aligned} F_B(j) &= \{\mathbb{1} + \exp(B) + \exp(2B) + \cdots + \exp((j-1)B)\} F_B(1) \\ &= (\exp(B) - \mathbb{1})^{-1} (\exp(jB) - \mathbb{1}) F_B(1). \end{aligned}$$

(b) The solution $c(t)$ of the inhomogeneous system of linear differential equations

$$\dot{c}(t) = Bc(t) + d \quad (8)$$

with constant coefficients (i.e. $B \in M_{\mathbb{R}}(n, n), d \in \mathbb{R}^n$) and with initial condition $v = c(0)$ is given by:

$$c(t) = v + F_B(t)(Bv + d) = \exp(Bt)(v) + F_B(t)(d). \quad (9)$$

Proof. (a) Equation (6) follows immediately from Eq. (5). Then we compute

$$\frac{d}{dt}(F_B(t+s) - \exp(Bs)F_B(t)) = \exp(B(t+s)) - \exp(Bs)\exp(Bt) = 0.$$

Since $F_B(0) = 0$ Eq. (7) follows. And this implies Eq. (8).

(b) We can write the solution of the differential equation (8)

$$\frac{d}{dt} \left(\frac{c(t)}{1} \right) = \left(\frac{B|d}{0|0} \right) \left(\frac{c(t)}{1} \right)$$

as follows:

$$\begin{aligned} \left(\frac{c(t)}{1} \right) &= \exp \left(\left(\frac{B|d}{0|0} \right) t \right) \left(\frac{v}{1} \right) \\ &= \left(\frac{\exp(Bt)|F_B(t)(d)}{0|1} \right) \left(\frac{v}{1} \right) = \left(\frac{\exp(Bt)(v) + F_B(t)(d)}{1} \right) \end{aligned} \quad (10)$$

which is Eq. (9). One could also differentiate Eq. (9) and use Eq. (6) $\square$

Remark 1. Equation (2) shows that $c(t)$ is the orbit

$$t \in \mathbb{R} \mapsto c(t) = \exp \left(\left(\frac{B|d}{0|0} \right) t \right) \left(\frac{v}{1} \right) \in \mathbb{R}^n.$$

of the one-parameter subgroup

$$t \in \mathbb{R} \mapsto \exp \left(\left(\frac{B|d}{0|0} \right) t \right) \in \text{Aff}(n)$$

of the affine group $\text{Aff}(n)$ acting canonically on $\mathbb{R}^n$.

3. Polygons invariant under M_α

Theorem 1. Let $(A, b) : x \in \mathbb{R}^n \mapsto Ax + b \in \mathbb{R}^n$ be an affine map and $v \in \mathbb{R}^n$. Assume that for $\alpha \in (0, 1)$ the value $1 - \alpha$ is not an eigenvalue of A , i.e. the matrix $A_\alpha := \alpha^{-1}(A + (\alpha - 1)\mathbb{1})$ is invertible. Then the following statements hold:

(a) There is a unique polygon $x \in \mathcal{P}(\mathbb{R}^n)$ with $x_0 = v$ which is a soliton for M_α resp. affinely invariant under the mapping M_α with respect to the affine map (A, b) , cf. Eq. (1). If $b_\alpha = \alpha^{-1}b$, then for $j > 0$:

$$\begin{aligned} x_j &= A_\alpha^j(v) + A_\alpha^{j-1}(b_\alpha) + \cdots + A_\alpha(b_\alpha) + b_\alpha \\ &= v + (A_\alpha^j - \mathbb{1}) \left(v + (A_\alpha - \mathbb{1})^{-1}(b_\alpha) \right). \end{aligned} \quad (11)$$

and for $j < 0$:

$$\begin{aligned} x_j &= A_\alpha^j(v) - A_\alpha^j(b_\alpha) + \cdots + A_\alpha^{-1}(b_\alpha) \\ &= v + (A_\alpha^j - \mathbb{1}) \left(v - (A_\alpha^{-1} - \mathbb{1})^{-1} (A_\alpha^{-1}(b_\alpha)) \right). \end{aligned} \quad (12)$$

(b) If $A_\alpha = \exp(B_\alpha)$ for a (n, n) -matrix B_α and if $b_\alpha = F_{B_\alpha}(1)(d_\alpha)$ for a vector $d_\alpha \in \mathbb{R}^n$ then the polygon x_j lies on the smooth curve

$$c(t) = v + F_{B_\alpha}(t)(B_\alpha v + d_\alpha)$$

i.e. $x_j = c(j)$ for all $j \in \mathbb{Z}$.

Proof. (a) By Eq. (1) we have

$$(1 - \alpha)x_j + \alpha x_{j+1} = Ax_j + b$$

for all $j \in \mathbb{Z}$. Hence the polygon is given by $x_0 = v$ and the recursion formulae

$$x_{j+1} = A_\alpha(x_j) + b_\alpha; \quad x_j = A_\alpha^{-1}(x_{j+1} - b_\alpha).$$

for all $j \in \mathbb{Z}$. Then Eqs. (11) and (12) follow.

(b) For $A_\alpha = \exp B_\alpha; b_\alpha = F_{B_\alpha}(d_\alpha)$ we obtain from Eq. (6) for all $j \in \mathbb{Z}$: $A_\alpha - \mathbb{1} = B_\alpha F_{B_\alpha}(1)$ and $A_\alpha^j - \mathbb{1} = B_\alpha F_{B_\alpha}(j)$. Hence for $j > 0$:

$$\begin{aligned} x_j &= v + (A_\alpha^j - \mathbb{1}) \left(v + (A_\alpha - \mathbb{1})^{-1} b_\alpha \right) \\ &= v + B_\alpha F_{B_\alpha}(j) \left(v + (B_\alpha F_{B_\alpha}(1))^{-1} (b_\alpha) \right) \\ &= v + F_{B_\alpha}(j)(B_\alpha v + d_\alpha) = c(j). \end{aligned}$$

The functional Eq. (7) for $F_B(t)$ implies $0 = F_B(0) = F_B(-1+1) = F_B(-1) + \exp(-B)F_B(1)$, hence

$$F_B(-1) = -\exp(-B)F_B(1); \quad F_B(-1)^{-1} = -\exp(B)F_B(1)^{-1}.$$

Note that the matrices $B, F_B(t), F_B(t)^{-1}$ commute. With this identity we obtain for $j < 0$:

$$\begin{aligned} x_j &= v + (A_\alpha^j - \mathbb{1}) \left(v - (A_\alpha^{-1} - \mathbb{1})^{-1} (A_\alpha^{-1}b_\alpha) \right) \\ &= v + B_\alpha F_{B_\alpha}(j) \left(v - (B_\alpha F_{B_\alpha}(-1))^{-1} \exp(-B_\alpha)(b_\alpha) \right) \\ &= v + B_\alpha F_{B_\alpha}(j) \left(v - F_{B_\alpha}(-1)^{-1} B_\alpha^{-1} \exp(-B_\alpha) F_{B_\alpha}(1)(d_\alpha) \right) \\ &= v + F_{B_\alpha}(j)(B_\alpha v + d_\alpha) = c(j). \end{aligned}$$

□

Remark 2. (a) Using the identification Eq. (4) we can write

$$\left(\frac{x_{j+1}}{1} \right) = \left(\frac{A_\alpha | b_\alpha}{0 | 1} \right) \left(\frac{x_j}{1} \right); \quad \left(\frac{x_j}{1} \right) = \left(\frac{A_\alpha | b_\alpha}{0 | 1} \right)^j \left(\frac{v}{1} \right) \quad (13)$$

for all $j \in \mathbb{Z}$.

(b) If $A_\alpha = \exp(B_\alpha)$ for a (n, n) -matrix B_α and if $b_\alpha = F_{B_\alpha}(1)(d_\alpha)$ for a vector $d_\alpha \in \mathbb{R}^n$ then we obtain from Eq. (10):

$$\begin{aligned} \left(\frac{c(t)}{1} \right) &= \exp \left(\left(\frac{B_\alpha | d_\alpha}{0 | 0} \right) t \right) \left(\frac{v}{1} \right) = \left(\frac{\exp(B_\alpha t) | F_{B_\alpha}(t)(d_\alpha)}{0 | 1} \right) \left(\frac{v}{1} \right) \\ &= \left(\frac{\exp(B_\alpha t)(v) + F_{B_\alpha}(t)(d_\alpha)}{1} \right) = \left(\frac{v + F_{B_\alpha}(t)(B_\alpha v + d_\alpha)}{1} \right) \end{aligned}$$

Hence $t \in \mathbb{R} \mapsto c(t) \in \mathbb{R}^n$ is the orbit of a one-parameter subgroup of the affine group applied to the vector v .

4. Smooth curves invariant under M_α

For a smooth curve $c : \mathbb{R} \rightarrow \mathbb{R}^n$ and a parameter $\alpha \in (0, 1)$ we define the one-parameter family $\tilde{c}_s : \mathbb{R} \rightarrow \mathbb{R}^n, s \in \mathbb{R}$ by Eq. (2). And we call a smooth curve $c : \mathbb{R} \rightarrow \mathbb{R}^n$ a *soliton* of the mapping M_α (resp. affinely invariant under M_α) if there is $\epsilon > 0$ and a smooth map $\in (-\epsilon, \epsilon) \rightarrow (A, b) \in \text{Aff}(n)$ such that

$$\tilde{c}_s(t) = (1 - \alpha)c(t) + \alpha c(t + s) = A(s)(c(t)) + b(s). \quad (14)$$

Then we obtain as an analogue of [9, Thm.1]:

Theorem 2. *Let $c : \mathbb{R} \rightarrow \mathbb{R}^n$ be a soliton of the mapping M_α satisfying Eq. (14). Assume in addition that for some $t_0 \in \mathbb{R}$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \dots, c^{(n)}(t_0)$ are linearly independent.*

Then the curve c is the unique solution of the differential equation

$$\dot{c}(t) = Bc(t) + d$$

for $B = \alpha^{-1}A'(0), d = \alpha^{-1}b'(0)$ with initial condition $v = c(0)$.

And $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), b(s) = \alpha F_B(s)(d)$.

Hence the curve $c(t)$ is the orbit of a one-parameter subgroup

$$t \in \mathbb{R} \mapsto B(t) := \exp \left(\left(\frac{B | d}{0 | 0} \right) t \right) = (\exp(Bt), F_B(t)(d)) \in \text{Aff}(n)$$

of the affine group, i.e.

$$c(t) = B(t) \left(\frac{v}{1} \right) = v + F_B(t)(Bv + d),$$

cf. Remark 1.

Remark 3. For an affine map $(A, b) \in \text{Gl}(n), b \in \mathbb{R}^n$ the linear isomorphism A is called the *linear part*. For $n = 2$ we discuss the possible normal forms of $A \in \text{Gl}(2)$ resp. the normal forms of the one-parameter subgroup $\exp(tB)$ and of the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \exp(Bs)$ introduced in Theorem 2. This will be used in Sect. 5.

1. $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ for $\lambda, \mu \in \mathbb{R} - \{0\}$, i.e. A is diagonalizable (over $\mathbb{R}$), then A is called *scaling*, for $\lambda = \mu$ it is called *homothety*. For an endomorphism B which is diagonalizable over $\mathbb{R}$ the one-parameter subgroup $B(t) = \exp(Bt)$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs)$ consists of scalings.
2. $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for $a, b \in \mathbb{R}, b \neq 0$, i.e. A has no real eigenvalues. Then A is called a *similarity*, i.e. a composition of a *rotation* and a *homothety*. For an endomorphism B with no real eigenvalues the one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), s \neq 0$ consist of affine mappings without real eigenvalues, i.e. compositions of non-trivial rotations and homotheties.
3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is called *shear transformation*. Hence the matrix A has only one eigenvalue 1 and is not diagonalizable. If B is of the form $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, i.e. B is nilpotent, then the one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), s \neq 0$ consist of shear transformations.
4. $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda \in \mathbb{R} - \{0, 1\}$. Then A is invertible with only one eigenvalue $\lambda \neq 1$ and not diagonalizable. This linear map is a composition of a homothety and a shear transformation. The one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), s \neq 0$ consist of linear mappings with only one eigenvalue different from 1 which are not diagonalizable. Hence they are compositions of non-trivial homotheties and shear transformations, too.

We use the following convention: For a one-parameter family $s \mapsto c_s$ of curves or a one-parameter family $s \mapsto A(s), s \mapsto b(s)$ of affine maps we denote the differentiation with respect to the parameter s by $'$. On the other hand we use for the differentiation with respect to the curve parameter t of the curves $t \mapsto c(t), t \mapsto c_s(t)$ the notation $\dot{c}, \dot{c}_s$.

Proof. The proof is similar to the Proof of Theorem [9, Thm.1]: Let

$$c_s(t) = A(s)c(t) + b(s) = (1 - \alpha)c(t) + \alpha c(t + s). \quad (15)$$

For $s = 0$ we obtain $c(t) = c_0(t) = A(0)c(t) + b(0)$ for all $t \in \mathbb{R}$, resp. $(A(0) - \mathbb{1})(c(t)) = -b(0)$ for all t . We conclude that

$$(A(0) - \mathbb{1})\left(c^{(k)}(t)\right) = 0 \quad (16)$$

for all $k \geq 1$. Since for some t_0 the vectors $\dot{c}(t_0), \ddot{c}(t_0), \dots, c^{(n)}(t_0)$ are linearly independent by assumption we conclude from Eq. (16): $A(0) = \mathbb{1}, b(0) = 0$.

Eq. (15) implies for $k \geq 1$:

$$A(s)c^{(k)}(t) = (1 - \alpha)c^{(k)}(t) + \alpha c^{(k)}(t + s)$$

and hence

$$A'(s)c^{(k)}(t) = \alpha c^{(k+1)}(t + s).$$

We conclude from Eq. (15):

$$\begin{aligned} \frac{\partial c_s(t)}{\partial s} &= A'(s)c(t) + b'(s) \\ &= \frac{\partial c_s(t)}{\partial t} - (1 - \alpha)\dot{c}(t) = (A(s) - (1 - \alpha)\mathbb{1})\dot{c}(t). \end{aligned}$$

Since $A(0) = \mathbb{1}$ the endomorphisms $A(s) + (\alpha - 1)\mathbb{1}$ are isomorphisms for all $s \in (0, \epsilon)$ for a sufficiently small $\epsilon > 0$. Hence we obtain for $s \in (0, \epsilon)$:

$$\dot{c}(t) = (A(s) + (\alpha - 1)\mathbb{1})^{-1} A'(s)c(t) + (A(s) + (\alpha - 1)\mathbb{1})^{-1} b'(s). \quad (17)$$

Differentiating with respect to s :

$$\left((A(s) + (\alpha - 1)\mathbb{1})^{-1} A'(s) \right)' (c(t)) + \left((A(s) + (\alpha - 1)\mathbb{1})^{-1} b'(s) \right)' = 0$$

and differentiating with respect to t :

$$\left((A(s) + (\alpha - 1)\mathbb{1})^{-1} A'(s) \right)' c^{(k)}(t) = 0; \quad k = 1, 2, \dots, n.$$

By assumption the vectors $\dot{c}(t_0), \ddot{c}(t_0), \dots, c^{(n)}(t_0)$ are linearly independent. Therefore we obtain $\left((A(s) + (\alpha - 1)\mathbb{1})^{-1} A'(s) \right)' = 0$. Let $B = \alpha^{-1} A'(0)$, $d = \alpha^{-1} b'(0)$. Then we conclude

$$A'(s) = (A(s) + (\alpha - 1)\mathbb{1}) B; \quad b'(s) = (A(s) + (\alpha - 1)\mathbb{1}) (d), \quad (18)$$

We obtain from Eq. (17):

$$\dot{c}(t) = Bc(t) + d.$$

Equation (18) with $A(0) = \mathbb{1}$ implies $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs)$. And we obtain $b'(s) = \alpha \exp(Bs)(d) = \alpha F'_B(s)(d)$. Hence $b(s) = \alpha F_B(s)(d)$ since $b(0) = 0$. $\square$

As a consequence we obtain the following

Theorem 3. *For a (n, n) -matrix B and a vector d any solution of the inhomogeneous linear differential equation $\dot{c}(t) = Bc(t) + d$ with constant coefficients is a soliton of the mapping M_α . These solitons are orbits of a one-parameter subgroup of the affine group, i.e. they are of the form given in Eq. (9).*

Proof. Any solution of the equation $\dot{c}(t) = Bc(t) + d$ has the form

$$c(t) = v + F_B(t)(Bv + d)$$

with $v = c(0)$, cf. Proposition 1. Then with $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs)$ and $b(s) = \alpha F_B(s)(d)$ we conclude from Eqs. (6) and (7):

$$\begin{aligned}\tilde{c}_s(t) &= (1 - \alpha)c(t) + \alpha c(t + s) \\ &= v + (1 - \alpha)F_B(t)(Bv + d) + \alpha F_B(t + s)(Bv + d) \\ &= v + (1 - \alpha)(c(t) - v) + \alpha(F_B(s) + \exp(Bs)F_B(t))(Bv + d) \\ &= (1 - \alpha)c(t) + \alpha \exp(Bs)(v) + \alpha F_B(s)(d) + \alpha \exp(Bs)(c(t) - v) \\ &= ((1 - \alpha)\mathbb{1} + \alpha \exp(Bs))(c(t)) + \alpha F_B(s)(d) \\ &= A(s)c(t) + b(s).\end{aligned}$$

Hence c is a soliton of the mapping M_α , cf. Eq. (14). $\square$

Remark 4. In [9] the authors consider the curve shortening process $T : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{P}(\mathbb{R}^n)$, $T(x)_j = M^2(x)_{j-1}$ satisfying Eq. (3). Hence the midpoints mapping M is applied twice followed by an index shift. The smooth curves $c = c(t)$ invariant under this process can be characterized as solutions of a inhomogeneous linear system of second order differential equations

$$\ddot{c}(t) = Bc(t) + d$$

for a (n, n) -matrix B and a vector d , cf. [9, Thm. 2]. These solutions can be reduced to a system of first order differential equations, cf. [9, Rem. 1]. Explicit formulas for these solutions can be written down in terms of power series in t whose coefficients are expressed in terms of powers of B . If $B = B_1^2$, $d = B_1(d_1)$ for a (n, n) -matrix B_1 and a vector $d_1 \in \mathbb{R}^n$, then the orbits of one-parameter subgroups of the affine group $\text{Aff}(\mathbb{R}^n)$ acting on $\mathbb{R}^n$ satisfying

$$\dot{c}(t) = B_1 c(t) + d_1$$

are particular solutions. In the next section we will see that for $n = 2$ the solitons of the midpoints mapping have constant affine curvature. On the other hand not all solitons of the process T have constant affine curvature, cf. [9, Sec. 5].

5. Curves with constant affine curvature

The orbits of one-parameter subgroups of the affine group $\text{Aff}(2)$ acting on $\mathbb{R}^2$ can also be characterized as curves of constant general-affine curvature parametrized proportional to general-affine arc length unless they are parametrizations of a parabola, an ellipse or a hyperbola. This will be discussed in this section. The one-parameter subgroups are determined by an endomorphism B and a vector d . We describe in Proposition 2 how the general-affine curvature can be expressed in terms of the matrix B .

For certain subgroups of the affine group $\text{Aff}(2)$ one can introduce a corresponding *curvature* and *arc length*. One should be aware that sometimes in the literature the curvature related to the equi-affine subgroup $S\text{Aff}(2)$ generated by the special linear group $\text{SL}(2)$ of linear maps of determinant one and

the translations is also called *affine curvature*. We distinguish in the following between the *equi-affine curvature* k_{ea} and the *general-affine curvature* k_{ga} as well as between the *equi-affine length parameter* s_{ea} and the *general-affine length parameter* s_{ga} .

We recall the definition of the equi-affine and general-affine curvature of a smooth plane curve $c : I \rightarrow \mathbb{R}^2$ with $\det(\dot{c}(t) \ddot{c}(t)) = |\dot{c}(t) \ddot{c}(t)| \neq 0$ for all $t \in I$.

By eventually changing the orientation of the curve we can assume $|\dot{c}(t) \ddot{c}(t)| > 0$ for all $t \in I$. A reference is the book by P. and A. Schirokowitz [10, §10] or the recent article by Kobayashi and Sasaki [7]. Then $s_{ea}(t) := \int |\dot{c}(t) \ddot{c}(t)|^{1/3} dt$ is called *equi-affine arc length*. We denote by $t = t(s_{ea})$ the inverse function, then $\tilde{c}(s_{ea}) = c(t(s_{ea}))$ is the parametrization by *equi-affine arc length*. Then $\tilde{c}'''(s_{ea}), \tilde{c}'(s_{ea})$ are linearly dependent and the *equi-affine curvature* $k_{ea}(s_{ea})$ is defined by

$$\tilde{c}'''(s_{ea}) = -k_{ea}(s_{ea}) \tilde{c}'(s_{ea})$$

resp.

$$k_{ea}(s_{ea}) = |\tilde{c}''(s_{ea}) \tilde{c}'''(s_{ea})|.$$

Assume that $c = c(s_{ea})$, $s_{ea} \in I$ is a smooth curve parametrized by equi-affine arc length for which the sign $\epsilon = \text{sign}(k_{ea}(s_{ea})) \in \{0, \pm 1\}$ of the equi-affine curvature is constant. If $\epsilon = 0$ then the curve is up to an affine transformation a parabola (t, t^2) . Now assume $\epsilon \neq 0$ and let $K_{ea} = |k_{ea}| = \epsilon k_{ea}$. Then the *general-affine arc length* $s_{ga} = s_{ga}(s_{ea})$ is defined by

$$s_{ga} = \int \sqrt{K_{ea}(s_{ea})} ds_{ea}. \quad (19)$$

We call a curve $c = c(t)$ parametrized proportional to general-affine arc length if $t = \lambda_1 s_{ga} + \lambda_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq 0$. The *general-affine curvature* $k_{ga} = k_{ga}(s_{ea})$ is defined by

$$k_{ga}(s_{ea}) = K'_{ea}(s_{ea}) K_{ea}(s_{ea})^{-3/2} = -2 \left(K_{ea}^{-1/2}(s_{ea}) \right)'. \quad (20)$$

If the general-affine curvature k_{ga} (up to sign) and the sign ϵ is given with respect to the equi-affine arc length parametrization, then the equi-affine curvature $k_{ea} = k_{ea}(s_{ea})$ is determined up to a constant by Eq. (20). Hence the curve is determined up to an affine transformation. The invariant k_{ga} already occurs in Blaschke's book [2, §10, p.24]. Curves of constant general-affine curvature are orbits of a one-parameter subgroup of the affine group. These curves already were discussed by Klein and Lie [6] under the name *W-curves*.

Proposition 2. For a non-zero matrix $B \in M_{\mathbb{R}}(2, 2)$ and vectors $d, v \in \mathbb{R}^2$ where $Bv + d$ is not an eigenvector of B let $c : \mathbb{R} \rightarrow \mathbb{R}^2$ be the solution of the differential equation $c'(t) = Bc(t) + d$; $c(0) = v$, i.e. $c(t) = v + F_B(t)(Bv + d) = \exp(tB)(v) + F_B(t)(d)$. We assume that $\beta = |c'(0) c''(0)|^{1/3} = |Bv + d \ B(Bv + d)|^{1/3} > 0$. Define

$$k = k(B) = -2 + 9 \det(B) / \operatorname{tr}^2(B); \quad K = K(B) = |k(B)|^{-1/2}. \quad (21)$$

(a) If $\operatorname{tr}(B) = 0$ then the curve is parametrized proportional to equi-affine arc length and the equi-affine curvature is constant $k_{ea} = \det(B)/\beta^2$ and $\epsilon = \operatorname{sign}(\det(B))$, the curve is a parabola, if $\epsilon = 0$, an ellipse, if $\epsilon > 0$, or a hyperbola, if $\epsilon < 0$, cf. Remark 5.

(b) If $\operatorname{tr}(B) \neq 0$ then we can choose a parametrization by equi-affine arc length s_{ea} such that the equi-affine curvature k_{ea} is given by:

$$k_{ea}(s_{ea}) = k(B)s_{ea}^{-2}. \quad (22)$$

If $k(B) = 0$ the curve has vanishing equi-affine curvature and is a parametrization of a parabola, cf. the Remark 5. If $k(B) \neq 0$ then the general-affine curvature is defined and constant:

$$k_{ga}(s_{ga}) = -2K(B). \quad (23)$$

Up to an additive constant the general-affine arc length parameter s_{ga} is given by:

$$s_{ga} = \frac{\operatorname{tr} B}{3K(B)} t.$$

Hence the curve $c(t)$ is parametrized proportional to general-affine arc length.

Remark 5. It is well-known that the curves of constant equi-affine curvature are parabola, hyperbola or ellipses, cf. [2, §7]. For $k_{ea} = 0$ we obtain a parabola: $c(t) = c(0) + c'(0)s_{ea} + c''(0)s_{ea}^2/2$, for $k_{ea} > 0$ the ellipse $c(s_{ea}) = (a \cos(\sqrt{k_{ea}}s_{ea}), b \sin(\sqrt{k_{ea}}s_{ea}))$ with $k_{ea} = (ab)^{-2/3}$ and for $k_{ea} < 0$ the hyperbola $c(s_{ea}) = (a \cosh(\sqrt{-k_{ea}}s_{ea}), b \sinh(\sqrt{-k_{ea}}s_{ea}))$ with $k_{ea} = -(ab)^{-2/3}$. Here $a, b > 0$.

Proof. Following Proposition 1 we obtain as solution of the differential equation: $c(t) = v + F_B(t)(Bv + d)$, hence for the derivatives: $c^{(k)}(t) = B^{k-1} \exp(tB)(Bv + d)$. Then :

$$\begin{aligned} |\dot{c}(t) \ddot{c}(t)| &= |\exp(Bt)| |bv + d \ B(Bv + d)| \\ &= \exp(\operatorname{tr}(B)t) |Bv + d \ B(Bv + d)|. \end{aligned}$$

Let $\beta = (|Bv + d \ B(Bv + d)|)^{1/3}$ and $\tau = \operatorname{tr}(B)$. Then

$$|\dot{c}(t) \ddot{c}(t)| = \beta^3 \exp(\tau t).$$

(a) If $\tau = 0$ then $s_{ea} = t\beta$, i.e. the curve is parametrized proportional to equi-affine arc length and

$$\tilde{c}(s_{ea}) = c(t(s_{ea})) = c(s_{ea}/\beta) = v + F_B(s_{ea}/\beta)(Bv + d).$$

Then

$$\begin{aligned} \tilde{c}'(s_{ea}) &= \beta^{-1} \exp(Bs_{ea}/\beta)(Bv + d) \\ \tilde{c}'''(s_{ea}) &= \beta^{-3} B^2 \exp(Bs_{ea}/\beta)(Bv + d) = -\det(B)\beta^{-2} \tilde{c}'(s_{ea}). \end{aligned}$$

Here we use that by Cayley–Hamilton $B^2 - \tau B = B^2 = -\det(B) \cdot \mathbb{1}$. Hence we obtain $k_{ea}(s_{ea}) = \det(B)/\beta^2$ and $\epsilon = \text{sign}(\det(B))$. Then the claim follows from Remark 5.

(b) Assume $\tau \neq 0$. Then the equi-affine arc length $s_{ea} = s_{ea}(t)$ is given by

$$s_{ea}(t) = \beta \int \exp(\tau t/3) dt = \frac{3\beta}{\tau} \exp(\tau t/3). \quad (24)$$

Hence the equi-affine arc length parametrization of c is given by

$$\tilde{c}(s_{ea}) = v + F_B \left(\frac{3}{\tau} \ln \left(\frac{\tau}{3\beta} s_{ea} \right) \right) (Bv + d).$$

Then we can express the derivatives:

$$\begin{aligned} \tilde{c}'(s_{ea}) &= \frac{3}{\tau} \frac{1}{s_{ea}} \exp \left(\frac{3}{\tau} B \ln \left(\frac{\tau}{3\beta} s_{ea} \right) \right) (Bv + d) \\ \tilde{c}''(s_{ea}) &= \left(\frac{3}{\tau} B - \mathbb{1} \right) \frac{1}{s_{ea}} \tilde{c}'(s_{ea}) \\ \tilde{c}'''(s_{ea}) &= \left(\frac{3}{\tau} B - \mathbb{1} \right) \left(\frac{3}{\tau} B - 2\mathbb{1} \right) \frac{1}{s_{ea}^2} \tilde{c}'(s_{ea}) \\ &= - \left(\frac{9 \det(B)}{\tau^2} - 2 \right) \frac{1}{s_{ea}^2} \tilde{c}'(s_{ea}). \end{aligned}$$

Here we used that by Cayley–Hamilton $B^2 - \tau B = -\det(B) \cdot \mathbb{1}$. Hence we obtain for the equi-affine curvature

$$k_{ea}(s_{ea}) = \frac{k(B)}{s_{ea}^2}. \quad (25)$$

Then $\epsilon = \text{sign } k(B)$ and for $k(B) \neq 0$ we obtain from Eqs. (20) and (25):

$$k_{ga}(s_{ea}) = -\frac{2}{K(B)}.$$

And for the general-affine arc length we obtain

$$s_{ga} = \ln(|s_{ea}|)/K(B),$$

resp. up to an additive constant:

$$s_{ga} = \frac{\tau t}{3K(B)}$$

using Eq. (24).

The parametrization by general-affine arc length is given by

$$c^*(s_{ga}) = v + F_B (3K(B)s_{ga}/\tau) (Bv + d).$$

□

Example 1. Depending on the real Jordan normal forms of the endomorphism B we investigate the solitons $c(t)$, their special and general affine curvature. The normal forms of the corresponding one-parameter subgroup $B(t) = \exp(Bt)$ as well of the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \exp(Bs)$ follow from

Remark 3. Since $c(\mu t) = \exp(\mu Bt)$ the multiplication of B with a non-zero real μ corresponds to a linear reparametrization of the curve. If B has a non-zero real eigenvalue we can assume without loss of generality that it is 1 and in the case of a non-real eigenvalue we can assume that it has modulus 1.

(a) Let $B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $d = (0, 0)$, $c(0) = (1, 1)$ and $\lambda \neq 0, 1$. Then $\beta = (\lambda(1 - \lambda))^{1/3} \neq 0$, $\text{tr} B = 1 + \lambda$ and $c(t) = (\exp(t), \exp(\lambda t))$. Up to parametrization we have $c(u) = (u, u^\lambda)$.

If $\lambda = -1$ then c is a parametrization of a hyperbola, $\text{tr} B = 0$ and $k_{ea} = -2^{-2/3}$, cf. Remark 5.

If $\lambda \neq -1$ we obtain for the equi-affine curvature with respect to a equi-affine parametrization s_{ea} from Eq. (22):

$$k_{ea}(s_{ea}) = \left(9 \frac{\det B}{\text{tr}^2 B} - 2 \right) \frac{1}{s_{ea}^2} = - \frac{(\lambda - 2)(2\lambda - 1)}{(\lambda + 1)^2} \frac{1}{s_{ea}^2}.$$

For $\lambda = 1/2, 2$ we obtain a parametrization of a parabola with vanishing equi-affine curvature, cf. Remark 5. Now we assume $\lambda \neq 1/2, 2$. Hence $\epsilon = 1$ if and only if $1/2 < \lambda < 2$. The affine curvature k_{ga} is constant:

$$k_{ga} = -2 \frac{|\lambda + 1|}{\sqrt{(\lambda - 2)(2\lambda - 1)}},$$

cf. [7, Ex.2.14]. We have $\epsilon = 1$ if and only if $1/2 < \lambda < 2$, then $k_{ga} \in (-\infty, -4)$. And $\epsilon = -1$ if and only if $\lambda < 1/2$, $\lambda \neq 0$ or $\lambda > 2$, then $k_{ga} \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, 0)$.

Hence in this case the corresponding one-parameter subgroup

$$B(t) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(\lambda t) \end{pmatrix}$$

as well as the one-parameter family

$$A(s) = \begin{pmatrix} 1 - \alpha + \alpha \exp(s) & 0 \\ 0 & 1 - \alpha + \alpha \exp(\lambda s) \end{pmatrix}$$

consist of scalings.

(b) $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $d = (1, 0)$. Then the solution of the Equation $\dot{c}(t) = Bc(t) + d$ with $c(0) = (0, 1)$ is of the form $c(t) = (t, \exp(t))$. Then we obtain $\epsilon = -1$ and $k_{ga} = -\sqrt{2}$. The corresponding one-parameter subgroup $B(t)$ as well as the one-parameter family $A(s)$ consist of scalings, the affine transformation $(A(s), b(s))$ is given by $(A(s), b(s)) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 - \alpha + \alpha \exp(s) \end{pmatrix}, \alpha \begin{pmatrix} s \\ 0 \end{pmatrix} \right)$, i.e. a composition of scalings and translations.

FIGURE 2 The soliton $c(t) = ((t+1)\exp(t), \exp(t))$ with the family $c_s(t) = A(s)c(t)$

- (c) If $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $d = (0, 0)$, $c(0) = (1, 1)$ then $c(t) = ((t+1)\exp(t), \exp(t))$ i.e. up to an affine transformation and a reparametrization the curve is of the form $c(u) = (u, u \ln(u))$. Then $\epsilon = 1$ and $k_{ga} = -4$. The corresponding one-parameter subgroup $B(t)$ as well as the one-parameter family $A(s)$ consist of compositions of a homothety and a shear transformations;

$$B(t) = \exp(t) \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \quad A(s) = \begin{pmatrix} 1 - \alpha + \alpha \exp(s) & \alpha s \exp(s) \\ 0 & 1 - \alpha + \alpha \exp(s) \end{pmatrix},$$

cf. Fig. 2.

- (d) If $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $b \neq 0$, $a^2 + b^2 = 1$, $d = 0$, $c(0) = (1, 0)$ then $c(t) = \exp(at) (\cos(bt), \sin(bt))$. For $a = 0$, this is a circle with $k_{ea} = 1$. Now we assume $a \neq 0$: Then $\epsilon = \text{sign}(9 \det(B) - 2 \text{tr}^2(B)) = \text{sign}(a^2 + 9b^2) = 1$ and we obtain for the general-affine curvature $k_{ga} = -4|a|/\sqrt{a^2 + 9b^2} = -4|a|/\sqrt{9 - 8a^2}$, i.e. $k_{ga} \in (-4, 0)$. The corresponding one-parameter subgroup $B(t)$ as well as the one-parameter family $A(s)$ consist of similarities.

- (e) If $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ one can choose $c(0) = (0, 0)$, $d = (0, 1)$ and obtain $c(t) = (t^2/2, t)$, i.e. a parabola. In this case the one parameter subgroup $B(t) = \exp(tB)$ consists of shear transformations. The one-parameter family $(A(s), b(s)) = \left(\begin{pmatrix} 1 & \alpha s \\ 0 & 1 \end{pmatrix}, \alpha \begin{pmatrix} s^2/2 \\ s \end{pmatrix} \right)$ consists of a composition of a shear transformation and a translation, cf. Fig. 1. For the *affine curve shortening flow* the parabola is a *translational soliton*. Therefore it is also called the *affine analogue of the grim reaper*, cf. [3, p. 192]. For the curve

shortening process T defined by Eq. (3) the parabola is also a translational soliton, cf. [9, Sec. 5, Case (5)].

Note that the parabola occurs twice, in Case (a) it occurs with the parametrization $c(t) = (\exp(t), \exp(2t))$, in Case (e) it occurs with a parametrization proportional to equi-affine arc length. Summarizing we obtain from Theorems 2 and 3 together with Proposition 2 resp. Example 1 the following

Theorem 4. *Let $c : \mathbb{R} \rightarrow \mathbb{R}^2$ be a smooth curve for which $\dot{c}(0), \ddot{c}(0)$ are linearly independent. Then c is a soliton of the mappings $M_\alpha, \alpha \in (0, 1)$, in particular of the midpoints mapping $M = M_{1/2}$, if it is a curve of constant equi-affine curvature parametrized proportional to equi-affine arc length, or a parabola with the parametrization $c(t) = (\exp(t), \exp(2t))$ up to an affine transformation, or if it is a curve of constant general-affine curvature parametrized proportional to general-affine arc length.*

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