Immobilization of convex bodies in $\mathbb{R}^n$

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Abstract. We extend to arbitrary finite n the notion of immobilization of a convex body O in $\mathbb{R}^n$ by a finite set of points $\mathcal{P}$ in the boundary of O . Because of its importance for this problem, necessary and sufficient conditions are found for the immobilization of an n -simplex. A fairly complete geometric description of these conditions is given: as n increases from $n = 2$, some qualitative difference in the nature of the sets $\mathcal{P}$ emerges.

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1. Introduction

Immobilization problems were introduced by [6] and were motivated by the need to understand the best position a machine (robot hand) can grasp an object, [7]. There is now an extensive literature in Robotics journals on immobilization, for example [1, 2, 4, 8, 10–12]. In [4] it was proved that four points suffice to immobilize any plane body and $2d$ points to immobilize a d -dimensional polytope. [2] answered Kuperberg's conjecture in the affirmative proving that apart from a circular disk, any plane convex object with smooth boundary could be immobilized with three points. It was [1] who first brought out both the geometrical and algebraic aspects of immobilization by treating the case of a tetrahedron. Their results have recently been used in [3] to prove a necessary condition on rotors in tetrahedra. An overview of the classical results on immobilization focusing on analysis, existence and synthesis is given in [13].

For obvious practical reasons, much of the literature focuses on the problem in $\mathbb{R}^3$, and for the actual grasping of objects, robots take full advantage of the effects of friction. The aim of this article is quite different. Following [1], we study the purely geometric problem of grasping a smooth convex body O at a finite set of points $\mathcal{P}$ in its boundary and seek conditions so that O is completely immobilized. Bracho et al. [1] examined this problem in $\mathbb{R}^3$ and found a set of necessary and sufficient conditions for immobilizing a 3-simplex, i.e a tetrahedron; for reasons explained below, simplices play a particularly

important role in the analysis. In this article, we reproduce their results, but generalise the problem to $\mathbb{R}^n$. We follow the approach of [1] by recasting the immobilization problem of the simplex Δ , now in $\mathbb{R}^n$, as an extremal problem, but thereafter our techniques are very different. Solution of the extremal problem leads to necessary and sufficient conditions for immobilization of Δ in terms of a matrix A which encodes the geometry of Δ and the points $\mathcal{P}$; these conditions, which are algebraic in character, agree with the results of [4] and [1] for $n = 2, 3$ respectively and reveal an interesting difference in behaviour between $n \leq 3$ and $n > 3$. We conclude the article by interpreting the conditions on A geometrically, and thus show that there is a particular set $\mathcal{P}$ which is optimal for immobilization in a sense described below. Some of the detailed proofs in the article have been relegated to the supplementary article [5].

2. Immobilization of convex bodies and simplices

Let O be a convex body in $\mathbb{R}^n$. We shall call a point p in the boundary of O a contact point, and a set $\mathcal{P}$ of points in the boundary of O will be called a contact set. A contact set $\mathcal{P}$ is said to immobilize O if, whenever O is held fixed, any rigid motion of the points of $\mathcal{P}$ causes one or more of its points to penetrate into the interior of O ; equivalently, penetration by at least one point $p \in \mathcal{P}$ occurs whenever O is moved and the set $\mathcal{P}$ is held fixed; in that case, we will call $\mathcal{P}$ an immobilizing contact set.

For a smooth convex body in $\mathbb{R}^n$, for each point of contact $p_i \in \mathcal{P}$, let $\mathbf{k}_i$ be an outward pointing normal to O at p_i and π_i the tangent hyperplane $\mathbf{k}_i \cdot (x - p_i) = 0$; then O lies in the intersection of the half spaces $\mathcal{H}_i : \mathbf{k}_i \cdot (x - p_i) \leq 0$.

For immobilization of O it is necessary that $\mathcal{P}$ contains at least $n+1$ points and that the intersection of all half-spaces $\mathcal{H}_i$, is a bounded polytope, completely enclosing O ; we assume this throughout below. Suppose that $\mathcal{P}$ contains precisely n points and the corresponding normals are $\mathbf{k}_1, \dots, \mathbf{k}_n$. A displacement $\underline{u}$ of O away from p_i satisfies $\mathbf{k}_i \cdot \underline{u} < 0$. If the n normals are linearly independent, then the system $\mathbf{k}_i \cdot \underline{u} = b_i, i = 1, \dots, n$ has a unique solution for all choices of the $\{b_i\}$, in particular for all $b_i < 0$. The displacement $\underline{u}$ moves O away from all the p_i , so no penetration occurs. If the $\{\mathbf{k}_i\}$ are linearly dependent, then they span W , a strict subspace of $\mathbb{R}^n$ and we simply choose $\underline{u}$ orthogonal to W . Then $\mathbf{k}_i \cdot \underline{u} = 0$ for all i , so O 'slides' along each hyperplane π_i and is again not immobilized by $\mathcal{P}$. Finally if $\mathcal{P}$ has less than n points, O is even less constrained than when $\mathcal{P}$ has n points.

If the polytope $\cap \mathcal{H}_i$ is unbounded, there is at least one direction $\underline{u}$ such that if any of the points of $\mathcal{P}$ is displaced by an arbitrary positive multiple of $\underline{u}$, it remains within the polytope; then if O is translated in this direction, it does not cut through any π_i , and so no penetration occurs.

Observe that if O is a smooth convex body and the set $\mathcal{P}$ contains precisely $n+1$ points $p_0, \dots, p_n$ the bounded polytope is a simplex Δ . Thus a necessary

condition for immobilization of O is immobilization of the bounding simplex Δ ; the converse is false, since owing to the curvature of the boundary of O into the interior of Δ , a displacement of $\mathcal{P}$ causing at least one or more p_i to penetrate Δ may not cause any of these p_i to penetrate O . In view of this necessity, understanding the immobilization of simplices is important for the study of immobilization of smooth convex bodies in general.

Let $\mathcal{K}$ denote the set $\{\mathbf{k}_0, \dots, \mathbf{k}_n\}$ of outward normals to O at the points p_i . In order to achieve the bounded simplex Δ , it is necessary and sufficient that

1. each subset $\mathcal{K} - \{\mathbf{k}_i\}$ of n outward normals is linearly independent;
2. in the unique (up to an overall scalar multiple) dependency relation $\sum_{i=0}^n \lambda_i \mathbf{k}_i = \mathbf{0}$ among the outward normals $\mathbf{k}_i$, all the coefficients λ_i are non-zero and of the same sign.

These conditions ensure that there is no direction $\underline{u}$ in $\mathbb{R}^n$ such that $\underline{u} \cdot \mathbf{k}_i \leq 0$ for $i = 0, 1, \dots, n$; then there is no translation $\underline{u}$ of O which does not cause O to cut into at least one of the π_i . By rescaling, so that each outward normal $\mathbf{k}_i$ is replaced by the outward normal $|\lambda_i| \mathbf{k}_i$, the conditions 1 and 2 above may equivalently be replaced by

1. $\mathcal{K} - \{\mathbf{k}_0\}$ is linearly independent
2. $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$.

It then follows that apart from an overall positive scaling factor (positive to ensure that normals remain outward pointing), $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$ is the unique dependency among the outward normals $\mathbf{k}_i$, and that apart from this overall scaling factor, all the $\mathbf{k}_i$ are now fixed.

We assume 1 and 2 below and in the next section construct a particular set of outward normals to Δ for which 1 and 2 are satisfied, and furthermore, the overall scaling factor is fixed. To do this, in Sect. 3, we initially change our view point: we start with the simplex Δ , defined by its vertices, and in Proposition 3.1 derive a neat algebraic relationship between the set of vertices and a set of outward normals which satisfy 1 and 2 and which have a particular overall scaling. Then, for consistency with the approach taken in Sect. 1, in Proposition 3.2, we show how starting with the convex body O and the contact set $\mathcal{P}$ with its associated outward normals, we can construct Δ and rescale the normals to obtain the same relationship between vertices and normals as was found in Proposition 3.1.

3. Matrix description of the simplex Δ and contact set $\mathcal{P}$

Let Δ be a simplex in $\mathbb{R}^n$ having vertex set $\mathcal{V} = \{v_0, v_1, \dots, v_n\}$. Suppose the vertices are oriented so that $\text{vol}(\Delta)$, the n -dimensional volume of Δ , is positive. Let $[a_1, \dots, a_n]$ denote the $n \times n$ matrix with columns $a_1, \dots, a_n$. Then $\text{vol}(\Delta) = \frac{1}{n!} \det[v_1 - v_0, \dots, v_n - v_0]$. To maintain symmetry, however, we regard each vertex u with coordinates $(u_1, \dots, u_n)$ as the point $\bar{u} = (1, u_1, \dots, u_n) =$

$(1, u)$ in the hyperplane $x_0 = 1$ in $\mathbb{R}^{n+1}$. Then letting V be the $(n+1) \times (n+1)$ matrix $[\bar{v}_0, \bar{v}_1, \dots, \bar{v}_n]$ it follows that $\text{vol}(\Delta) = \frac{1}{n!} \det(V)$.

Let the face $F_i, i = 0, 1, \dots, n$ of Δ be the $(n-1)$ dimensional simplex with vertex set $\mathcal{V} - \{v_i\}$, given a positive orientation. Let $\text{vol}(F_i)$ denote the $(n-1)$ dimensional volume of F_i . Then if h_i is the length of the altitude dropped from vertex v_i to face F_i , regarding Δ as a cone with base F_i , its volume is given by

$$\text{vol}(\Delta) = \int_0^{h_i} \left(\frac{x}{h_i} \right)^{n-1} \text{vol}(F_i) dx = \frac{1}{n} h_i \text{vol}(F_i). \quad (3.1)$$

Proposition 3.1. *Let Δ be a simplex in $\mathbb{R}^n$ having vertex set $\mathcal{V} = \{v_0, \dots, v_n\}$ with an orientation such that the n -dimensional volume of Δ is positive. There exists a set of outward pointing normals $\mathbf{k}_i$ to the faces F_i of Δ satisfying:*

1. $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$,
2. $|\mathbf{k}_i| = \text{vol}(F_i), i = 0, 1, \dots, n$.

Proof. Since $\text{vol}(\Delta) > 0$, the matrix V is invertible, so there exists the $(n+1) \times (n+1)$ matrix $K = [\bar{\mathbf{k}}_0, \bar{\mathbf{k}}_1, \dots, \bar{\mathbf{k}}_n]$ such that

$$K^T V = V K^T = -n \text{vol}(\Delta) I. \quad (3.2)$$

Analogous to the decomposition $\bar{v} = (1, v)$, we write each $\bar{\mathbf{k}}_i = (\kappa_i, \mathbf{k}_i)$; the n -vectors $\mathbf{k}_i, i = 0, 1, \dots, n$ are the normal vectors we seek as we now show. Then from $K^T V = -n \text{vol}(\Delta) I$ there follows

$$\kappa_i + \mathbf{k}_i \cdot v_i = -n \text{vol}(\Delta), \quad 0 \leq i \leq n, \quad (3.3)$$

$$\kappa_i + \mathbf{k}_i \cdot v_j = 0, \quad 0 \leq j \neq i \leq n, \quad (3.4)$$

while from the first row of $V K^T = -n \text{vol}(\Delta) I$, there follows

$$\sum_{i=0}^n \kappa_i = -n \text{vol}(\Delta), \quad (3.5)$$

$$\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}. \quad (3.6)$$

From (3.4) for each i and each $j, l \neq i$

$$\mathbf{k}_i \cdot (v_j - v_l) = 0. \quad (3.7)$$

Thus $\mathbf{k}_i$ is perpendicular to each edge of F_i and is thus normal to F_i .

For any point p_i of F_i , since $v_j \in F_i$, for all $j \neq i$, it follows that $p_i - v_j$ is parallel to F_i and hence

$$\mathbf{k}_i \cdot (p_i - v_j) = 0 \quad (3.8)$$

which, using (3.4), gives for each $0 \leq i \leq n$ and for all $j \neq i$

$$\kappa_i = -\mathbf{k}_i \cdot p_i = -\mathbf{k}_i \cdot v_j. \quad (3.9)$$

From (3.3) and (3.4), for each i and each $j \neq i$,

$$\mathbf{k}_i \cdot (v_j - v_i) = n \operatorname{vol}(\Delta). \quad (3.10)$$

Now since $\mathbf{k}_i$ is perpendicular to F_i , the projection of $(v_j - v_i)$ along $\mathbf{k}_i$ has the length h_i of the altitude from vertex v_i to the face F_i . Thus (3.10) implies $|\mathbf{k}_i| h_i = n \operatorname{vol}(\Delta)$ and comparison with (3.1) shows that $|\mathbf{k}_i| = \operatorname{vol}(F_i)$.

Finally by (3.10) since $n \operatorname{vol}(\Delta) > 0$, $\mathbf{k}_i$ is outward pointing. $\square$

In this section, we started with Δ , specified by $\mathcal{V}$; this led to the faces F_i and then, via equation (3.2), to the particular outward normals $\mathbf{k}_i$ of magnitude $\operatorname{vol}(F_i)$ encoded by the matrix K in which the first row comprised elements $\kappa_i = -\mathbf{k}_i \cdot p_i$ with p_i any point in F_i . We now show that we reach the same endpoint if we start, as in Sect. 2, with the points p_i and the outward normals $\mathbf{k}_i$ which satisfy $\mathbf{k}_0 + \mathbf{k}_1 + \dots + \mathbf{k}_n = \mathbf{0}$, $\mathbf{k}_1, \dots, \mathbf{k}_n$ linearly independent and with a normalisation to be specified below.

Proposition 3.2. *Let O be a convex body in $\mathbb{R}^n$ and $\mathcal{P} = \{p_0, p_1, \dots, p_n\}$ be a contact set of O . Let $\mathbf{k}_i$ be the outward pointing normals at p_i and let these normals satisfy conditions 1 and 2 of Sect. 2, namely $\mathbf{k}_1, \dots, \mathbf{k}_n$ are linearly independent and $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$. Then if for $i = 0, \dots, n$, π_i denotes the hyperplane $\mathbf{k}_i \cdot x = \mathbf{k}_i \cdot p_i$, there is a unique n -simplex Δ with vertices $v_j, j = 0, \dots, n$ where v_j is the intersection point of all the π_i with $i \neq j$; by orienting the set $\mathcal{P}$ suitably we can ensure that $\operatorname{vol}(\Delta) > 0$. Furthermore, if we write $\kappa_j = -\mathbf{k}_j \cdot p_j$ and denote, as before, $V = [\bar{v}_0, \dots, \bar{v}_n]$, $K = [\mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_n]$ where $\bar{v}_j = (1, v_j)$, $\mathbf{k}_j = (\kappa_j, \mathbf{k}_j)$, then there is an overall positive rescaling of the outward normals $\{\mathbf{k}_i\}$ so that K and V satisfy $K^T V = V K^T = -n \operatorname{vol}(\Delta) I$, from which follow all the relations between normals and vertices established in Proposition 3.1.*

Proof. The vertex v_j is the intersection point of the n hyperplanes π_i , where π_i is given by $\mathbf{k}_i \cdot x = \mathbf{k}_i \cdot p_i$, $0 \leq i \neq j \leq n$. Since the n normals $\mathbf{k}_i, i \neq j$ are linearly independent this defines v_j uniquely and hence the simplex Δ with vertex set $\mathcal{V} = \{v_0, \dots, v_n\}$. Note that Δ and $\mathcal{V}$ are independent of the overall scaling of the normals since $\mathbf{k}_i$ appears linearly in both sides of the equation for π_i . Thus for $j = 0, 1, \dots, n$, v_j satisfies

$$\mathbf{k}_i \cdot v_j = \mathbf{k}_i \cdot p_i, \quad 0 \leq i \neq j \leq n \quad (3.11)$$

and writing $\kappa_i = -\mathbf{k}_i \cdot p_i$, there follows

$$\kappa_i + \mathbf{k}_i \cdot v_j = 0, \quad 0 \leq i \neq j \leq n. \quad (3.12)$$

Now summing (3.12) over all $i \neq j$ and using $\mathbf{k}_j = -\sum_{i \neq j} \mathbf{k}_i$, we have

$$\kappa_i + \mathbf{k}_i \cdot v_i = \sum_{r=0}^n \kappa_r; \quad 0 \leq i \leq n \quad (3.13)$$

and we note that (3.12), (3.13) also hold independently of the overall scaling of the normals.

Now $\sum_{r=0}^n \kappa_r < 0$: to see this, let X be strictly inside O . Then since O is convex and the outward normal at each contact point p_r is $\mathbf{k}_r$, we must have $(p_r - X) \cdot \mathbf{k}_r > 0$, $r = 0, \dots, n$. Hence $\sum_{r=0}^n \kappa_r = -\sum_{r=0}^n \mathbf{k}_r \cdot p_r = -\sum_{r=0}^n \mathbf{k}_r \cdot (p_r - X) < 0$. Defining vectors $\bar{v}_j, \mathbf{k}_l$ and matrices V, K as in the statement of the Proposition, equations (3.12) and (3.13) then give $K^T V = (\sum_{r=0}^n \kappa_r) I$. Since $\sum_{r=0}^n \kappa_r < 0$, K and V are non-singular, so that $\text{vol}(\Delta) = \frac{1}{n!} \det V \neq 0$.

Now by re-orienting the p_i if necessary (and thus, by (3.11), re-orienting the v_i) we can take $\text{vol}(\Delta) > 0$. Noting that $\sum_{r=0}^n \kappa_r < 0$, we now do an overall positive scaling of the $\mathbf{k}_r$ (and hence all the $\kappa_r = -\mathbf{k}_r \cdot p_r$) so that

$$\sum_{r=0}^n \kappa_r = -n \text{vol}(\Delta).$$

We have thus now recovered (3.2)

$$K^T V = V K^T = -n \text{vol}(\Delta) I.$$

Hence with this orientation of contact points and overall scaling of the outward normals, all the relations between normals and vertices established in Proposition 3.1 follow. $\square$

Remark 3.3. In practice to use (3.2) to obtain V from K , it is first necessary to compute $-n \text{vol}(\Delta) = -n \det V / n!$ in terms of K rather than V . Taking determinants in (3.2) leads to $-n \text{vol}(\Delta) = (-(n-1)! \det K)^{\frac{1}{n}}$ from which it follows that $V = (-(n-1)! \det K)^{\frac{1}{n}} (K^T)^{-1}$, so giving the vertices.

We observe that

1) If all the faces are projected orthogonally onto a single face F_j , the sum of the projected volumes is zero (as is easily visualised in the cases $n = 2, 3$) which gives $\sum_{i=0}^n \mathbf{k}_i \cdot \mathbf{k}_j = 0$. Thus by projecting onto faces $F_1, F_2, \dots, F_n$ and using the linear independence of the corresponding normals, there follows $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$, thereby giving a geometric interpretation of (3.6).

2) From (3.4) and (3.10), for $j \neq i$,

$$\kappa_i = -\mathbf{k}_i \cdot v_j = -\mathbf{k}_i \cdot (v_j - 0)$$

is $-n \text{vol}(\Delta_i)$ where Δ_i is the simplex with vertices of F_i together with the origin. As the origin is not in general a point of Δ , the quantities κ_i do not play a significant role later.

The contact points p_i lie in the interior of each face F_i . Thus each p_i is a linear combination of the vertices of F_i with coefficients that are non-negative and sum to unity. Define $\bar{p}_i = (1, p_i)$ and $P = [\bar{p}_0, \bar{p}_1, \dots, \bar{p}_n]$; we then have

$$P = V \Lambda$$

where Λ is an $(n+1) \times (n+1)$ matrix whose columns list the coefficients of $\bar{p}_j$ in terms of $\bar{v}_0, \bar{v}_1, \dots, \bar{v}_n$. The matrix $\Lambda = (\lambda_{ij})$ then has the following properties: $\lambda_{ii} = 0$ since $v_i \notin F_i$; $\lambda_{ij} > 0$ for all $i \neq j$; $\sum_{i=0}^n \lambda_{ij} = 1$. Λ is thus a stochastic

matrix with the additional property that the diagonal entries are zero. These matrices enjoy certain properties which we exploit later. The matrix Λ thus provides an efficient encoding of the set of contact points.

4. The penetration function

The conditions for immobilization of Δ are now recast in terms of an extremal problem. Let $\mathbb{E}_n$ denote the group of all rigid motions of Euclidean space $\mathbb{R}^n$. Given the simplex described by the matrix V (and thus its normals K) and the contact points by the matrix P define the penetration function $\Phi : \mathbb{E}_n \rightarrow \mathbb{R}$ by

$$\Phi(g) = \sum_{i=0}^n (g(p_i) - p_i) \cdot \mathbf{k}_i. \quad (4.1)$$

Then Φ varies continuously with g and $-\Phi(g)$ measures the total amount of normal penetration into Δ by the points $p_i \in \mathcal{P}$ under the action of g in $\mathbb{E}_n$ (weighted by the volumes of the faces).

Now each $g \in \mathbb{E}_n$ has a unique decomposition $g = tr$ where $t \in T_n$, the group of translations of $\mathbb{R}^n$ and $r \in SO(n)$, the group of orientation preserving rotations of $\mathbb{R}^n$ about the origin, [9]. For each $t \in T_n$ and $r \in SO(n)$, $r^{-1}tr$ is also a translation from which it follows that $T_n r = r T_n$ for all $r \in SO(n)$, so that T_n is a normal subgroup of $\mathbb{E}_n$ and $SO(n)$ is the quotient group.

An element $g = tr \in \mathbb{E}_n$ has a convenient matrix representation. Let $t_{\underline{a}}$ denote the translation $x \mapsto x + \underline{a}$ and let r have matrix R . Then $g(x) = Rx + \underline{a}$. As in Sect. 3, let $\bar{x} = (1, x)$ and $\bar{a} = (1, \underline{a})$. Then the $(n+1) \times (n+1)$ matrix G with top row $(1, 0, 0, \dots, 0)$ and lower $n \times (n+1)$ array $[\underline{a}, R]$ satisfies

$$G\bar{x} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \underline{a} & R \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ g(x) \end{bmatrix} = \overline{g(x)}. \quad (4.2)$$

With this formulation of $g = t_{\underline{a}}r$, it follows that if g varies continuously so do $t_{\underline{a}}$ and r , and conversely. Thus the map from $\mathbb{E}_n$ to $T_n \times SO(n)$ given by $g = tr \mapsto (t, r)$ and its inverse are continuous.

Lemma 4.1. Φ is well defined on $\mathbb{E}_n/T_n = SO(n)$.

Proof. Let $r \in SO(n)$; we show that Φ is constant on $T_n r$. For $g = t_{\underline{a}}r$

$$\Phi(t_{\underline{a}}r) = \sum_{i=0}^n (t_{\underline{a}}rp_i - p_i) \cdot \mathbf{k}_i = \sum_{i=0}^n (rp_i + \underline{a} - p_i) \cdot \mathbf{k}_i = \Phi(r)$$

since $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$. □

We thus regard Φ as a continuous map from $SO(n)$ to $\mathbb{R}$.

Lemma 4.2. For each $r \in SO(n)$, there is a unique $t(r) \in T_n$ and hence $g(r) = t(r)r \in \mathbb{E}_n$ so that $\Phi(g(r)) = \Phi(r)$ and

$$(g(r)p_i - p_i) \cdot \mathbf{k}_i = \frac{\Phi(r)}{n+1}, \quad i = 0, 1, \dots, n. \quad (4.3)$$

In particular $g(r) = I \in \mathbb{E}_n$ if and only if $r = I \in SO(n)$. Furthermore $g(r)$ varies continuously with r .

Remark 4.3. Lemma 4.2 states that for each rotation in $SO(n)$ (and hence from the decomposition $g = tr$, for every rigid motion), there is a unique translation $t(r)$ so that the normal penetrations into Δ at the p_i caused by $t(r)r$ are all equal.

Proof. Equation (4.3) holds if and only if there is a unique $\underline{a}$ so that

$$(rp_i + \underline{a} - p_i) \cdot \mathbf{k}_i = \frac{\Phi(r)}{n+1}, \quad i = 0, 1, \dots, n. \quad (4.4)$$

Since $\{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n\}$ are linearly independent, $\underline{a}$ is uniquely specified by the n equations

$$(rp_i + \underline{a} - p_i) \cdot \mathbf{k}_i = \frac{\Phi(r)}{n+1}, \quad i = 1, \dots, n \quad (4.5)$$

But then from Lemma 4.1

$$\begin{aligned} (rp_0 + \underline{a} - p_0) \cdot \mathbf{k}_0 &= \sum_{i=0}^n (rp_i + \underline{a} - p_i) \cdot \mathbf{k}_i - \sum_{i=1}^n (rp_i + \underline{a} - p_i) \cdot \mathbf{k}_i \\ &= \Phi(r) - n \frac{\Phi(r)}{n+1} = \frac{\Phi(r)}{n+1} \end{aligned}$$

which gives (4.4).

When $r = I \in SO(n)$, then by (4.1), $\Phi(I) = 0$ so that $\underline{a} = \underline{0}$ by (4.4) and hence $g(r) = I \in \mathbb{E}_n$. Conversely if $r \neq I \in SO(n)$, then by (4.2), $t_{\underline{a}}r \neq I \in \mathbb{E}_n$ for any $\underline{a}$, so in particular $g(r) \neq I$. The system (4.4) for $\underline{a}$ can be written

$$\mathbf{k}_i \cdot \underline{a} = b_i, \quad i = 0, 1, \dots, n, \quad (4.6)$$

where $b_i = \frac{\Phi(r)}{n+1} - (rp_i - p_i) \cdot \mathbf{k}_i$. Using the notation of Sect. 3 and writing $\tilde{\underline{a}} = (0, \underline{a})$ and $\tilde{\underline{b}} = (b_0, b_1, \dots, b_n)$, the system (4.6) is

$$K^T \tilde{\underline{a}} = \tilde{\underline{b}}$$

which has solution

$$\tilde{\underline{a}} = -\frac{V\tilde{\underline{b}}}{n\text{vol}(\Delta)}$$

from which, invoking the continuity of $\Phi(r)$ with r , it follows that $\underline{a}$ and hence $t(r)$ and $g(r)$ all depend continuously on $r \in SO(n)$. $\square$

Lemma 4.4. For each $t \in T_n$, $\Phi(t) = 0$. If $t \neq I \in \mathbb{E}_n$, then under the action of t at least one point p_i penetrates Δ .

Proof. Since $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$,

$$\Phi(t_{\underline{a}}) = \sum_{i=0}^n (p_i + \underline{a} - p_i) \cdot \mathbf{k}_i = \underline{a} \cdot \sum_{i=0}^n \mathbf{k}_i = 0.$$

If $t_{\underline{a}} \neq I$, then $\underline{a} \neq 0$. Since $\{\mathbf{k}_1, \dots, \mathbf{k}_n\}$ are linearly independent at least one $\underline{a} \cdot \mathbf{k}_i$, $i = 1, \dots, n$ is non-zero and since $\sum_{i=0}^n \mathbf{k}_i = \mathbf{0}$ at least one $\underline{a} \cdot \mathbf{k}_i$, $i = 0, \dots, n$ is strictly negative. Hence $(t_{\underline{a}} p_i - p_i) \cdot \mathbf{k}_i = \underline{a} \cdot \mathbf{k}_i < 0$ for at least one i , so that p_i penetrates Δ under the action of $t_{\underline{a}}$. $\square$

Proposition 4.5. *The set $\mathcal{P}$ immobilizes Δ if and only if $\Phi : SO(n) \rightarrow \mathbb{R}$ has a strict local maximum at I .*

Proof. Let $\mathcal{P}$ immobilize Δ . Then each rigid motion $g \neq I$ in a sufficiently small neighbourhood N of I in $\mathbb{E}_n$ causes at least one p_i to penetrate Δ . By the continuity of $r \mapsto g(r)$, where g is now as in Lemma 4.2, there is a neighbourhood N' of $I \in SO(n)$ so that $r \in N'$ implies $g(r) \in N$. Now suppose that Φ does not have a strict local maximum at $I \in SO(n)$. Then each neighbourhood of $I \in SO(n)$, and in particular N' , contains a rotation $r \neq I$ such that $\Phi(r) \geq 0$. For this r , $g(r) \in N$, and by Lemma 4.2, $g(r) \neq I$ and $(g(r)p_i - p_i) \cdot \mathbf{k}_i = \frac{\Phi(r)}{n+1} \geq 0$ for $i = 0, 1, \dots, n$, so that $g(r)$ causes no p_i to penetrate Δ ; this gives the required contradiction.

Conversely let Φ have a strict local maximum at $I \in SO(n)$. Then $\Phi(r) < 0$ for all $r \neq I$ in a sufficiently small neighbourhood N' of I in $SO(n)$. Now using the continuity of $tr \mapsto (t, r)$, consider a rigid motion $g = tr \neq I$ in a sufficiently small neighbourhood of I in $\mathbb{E}_n$ so that $r \in N'$. If $r = I$, then $g = t \neq I$ and hence by Lemma 4.4, at least one p_i penetrates Δ ; if $r \neq I$, then $\Phi(g) = \Phi(r) < 0$ so that at least one term $(g(p_i) - p_i) \cdot \mathbf{k}_i < 0$ and hence p_i penetrates Δ . Thus $\mathcal{P}$ immobilizes Δ . $\square$

5. Conditions for a maximum of the penetration function

Given Δ and $\mathcal{P}$, we now examine conditions under which Φ has a strict local maximum at $I \in SO(n)$. Regarding $g \in SO(n)$ as an $n \times n$ matrix and noting that $p_i \cdot \mathbf{k}_i$ can be identified as the trace of the $n \times n$ matrix $\mathbf{k}_i p_i^T$, it follows that

$$\sum_{i=0}^n g(p_i) \cdot \mathbf{k}_i = \sum_{i=0}^n \text{tr}(\mathbf{k}_i p_i^T g^T) = \sum_{i=0}^n \text{tr}(g^T \mathbf{k}_i p_i^T) = \text{tr}(g^T A) = \text{tr}(A^T g), \quad (5.1)$$

where A is the matrix

$$A = \sum_{i=0}^n \mathbf{k}_i p_i^T \quad (5.2)$$

which depends solely on the geometry of Δ and the set of contact points $\mathcal{P}$. Hence

$$\Phi(g) = \text{tr}(A^T(g - I)) \quad (5.3)$$

and we now consider conditions on A so that Φ has a strict local maximum at I .

Now each $g \in SO(n)$ may be written as

$$g = \exp S = \sum_{k \geq 0} \frac{S^k}{k!}$$

where S is a skew symmetric matrix, and where $g = I$ corresponds to $S = 0$. Hence $\Phi(g) = \Psi(S)$ where

$$\Psi(S) = \text{tr}(A^T S) + \frac{1}{2!} \text{tr}(A^T S^2) + \dots \quad (5.4)$$

We regard (5.4) as a power series about $S = 0$ and examine it for a strict local maximum at $S = 0$. By definition Ψ has a strict local maximum at $S = 0$ if $\Psi(S) < 0$ for all $S \neq 0$ in some neighbourhood of $S = 0$. In the following we make repeated use of the result that there exists a sufficiently small neighbourhood N of $S = 0$ such that for all $S \neq 0$ in N , the sign of $\Psi(S)$ is the same as the sign of the first non-zero term in the power series (5.4).

Lemma 5.1. *For Ψ to have a local maximum at $S = 0$, it is necessary that for all skew symmetric matrices S , $\text{tr}(A^T S) = 0$.*

Proof. This is just the usual condition to make $S = 0$ a stationary point of Ψ , which is necessary for an extreme value at $S = 0$. $\square$

We now examine further conditions to ensure sufficiency of a strict local maximum at $S = 0$.

Lemma 5.2. *For Ψ to have a strict local maximum at $S = 0$, it is sufficient that for all skew-symmetric matrices $S \neq 0$:*

- (i) $\text{tr}(A^T S) = 0$;
- (ii) $\text{tr}(A^T S^2) < 0$.

Proof. Assuming (i) and (ii), it follows from (i) that

$$\Psi(S) = \frac{1}{2!} \text{tr}(A^T S^2) + \frac{1}{3!} \text{tr}(A^T S^3) + \dots \quad (5.5)$$

For all $S \neq 0$ in a neighbourhood N , the sign of $\Psi(S)$ is the same as that of $\text{tr}(A^T S^2)$. Since by (ii), $\text{tr}(A^T S^2) < 0$ for all $S \neq 0$ in N , it follows that $\Psi(S) < 0$ for all $S \neq 0$ in N and hence Ψ has a strict local maximum at $S = 0$. $\square$

We now examine the implications for A to ensure that conditions (i) and (ii) hold.

Lemma 5.3. *$\text{tr}(A^T S) = 0$ for all skew-symmetric matrices S if and only if A is symmetric.*

Proof. Let $S^{(ij)}$ be the skew symmetric matrix with a 1 in the position (i, j) , -1 in position (j, i) and 0 everywhere else. Then $\text{tr}(A^T S^{(ij)}) = a_{ji} - a_{ij} = 0$. The converse follows as each skew symmetric S can be expressed as $S = \sum_{i < j} c_{ij} S^{(ij)}$. $\square$

Definition 5.4. A real symmetric matrix A is almost positive definite if the sum of every pair of eigenvalues is positive.

Here, an eigenvalue can only be added to itself if it is a repeated eigenvalue. This condition is weaker than positive definiteness.

Lemma 5.5. *For a real symmetric matrix A , $\text{tr}(A^T S^2) < 0$ for all skew symmetric matrices $S \neq 0$ if and only if A is almost positive definite.*

Proof. We have $\text{tr}(A^T S^2) = \text{tr}(SAS) = -\text{tr}(S^T AS)$. A can be diagonalised as $P^T AP = D$ where P is orthogonal. Hence

$$\text{tr}(S^T AS) = \text{tr}(S^T PDP^T S) = \text{tr}(P^T S^T P D P^T SP).$$

Now S is skew-symmetric if and only if $P^T SP$ is skew-symmetric, hence $\text{tr}(S^T AS) > 0$ for all skew $S \neq 0$ if and only if $\text{tr}(S^T DS) > 0$ for all skew $S \neq 0$. Choosing now $S = S^{(ij)}$, then $S^{(ij)} S^{(ij)T}$ is diagonal with a 1 in position i and j and 0 elsewhere. Hence

$$\text{tr}(S^{(ij)T} D S^{(ij)}) = \text{tr}(S^{(ij)} S^{(ij)T} D) = \lambda_i + \lambda_j$$

so that if $S = \sum_{i < j} c_{ij} S^{(ij)}$, then

$$\text{tr}(S^T DS) = \sum_{i < j} c_{ij}^2 (\lambda_i + \lambda_j) > 0$$

if and only if each pair $\lambda_i + \lambda_j > 0$, which gives the result. $\square$

Combining Lemmas 5.2, 5.3 and 5.5 we thus have:

Corollary 5.6. *Ψ has a strict local maximum at $S = 0$ if (i) A is symmetric and (ii) A is almost positive definite.*

We want to examine the extent to which the conditions (i) and (ii) are necessary. Given that (i) holds, from (5.5) considering what happens with a real function of a real variable, one could envisage Ψ having a strict quartic maximum at $S = 0$, given by

$$\Psi(S) = \frac{1}{4!} \text{tr}(A^T S^4) + \dots$$

in which $\text{tr}(A^T S^2) = 0$, $\text{tr}(A^T S^3) = 0$ and $\text{tr}(A^T S^4) < 0$. This however does not happen; it transpires that Corollary 5.6 has a converse. Thus we have

Proposition 5.7. *Ψ has a strict local maximum at $S = 0$ if and only if:*

- (i) A is symmetric; and
- (ii) A is almost positive definite.

Proof. The if part is given by Corollary 5.6.

Now (i) is necessary by Lemmas 5.1 and 5.3. To establish the necessity of (ii) we now assume that Ψ has a strict local maximum at $S = 0$ and that A is symmetric, in which case $\Psi(S)$ is given by (5.5).

We complete the proof by contradiction, so we assume that A is not almost positive definite. Then by Lemma 5.5, there is a skew $S_0 \neq 0$ so that $\text{tr}(A^T S_0^2) \geq 0$. We now consider two cases:

(a) If $\text{tr}(A^T S_0^2) > 0$, then replacing S_0 by ϵS_0 where $\epsilon > 0$ is sufficiently small that $\epsilon S_0 \in N$, the neighbourhood of $S = 0$ defined after (5.4), it follows that $\Psi(\epsilon S_0) > 0$, giving the required contradiction;

(b) If there is no skew $S \neq 0$ such that $\text{tr}(A^T S^2) > 0$, then $\text{tr}(A^T S_0^2) = 0$. Using the same arguments as those in Lemma 5.5 we may take A to be diagonal; also, since there is no $S \neq 0$ such that $\text{tr}(A^T S^2) > 0$, there is no pair of eigenvalues λ_i, λ_j with $\lambda_i + \lambda_j < 0$; hence $\lambda_i + \lambda_j \geq 0$ for all pairs of eigenvalues. Next if $S_0 = \sum_{i < j} c_{ij} S^{(ij)}$ then from $\text{tr}(A^T S_0^2) = 0$ there follows $\sum_{i < j} c_{ij}^2 (\lambda_i + \lambda_j) = 0$ so that $\lambda_i + \lambda_j = 0$ for all pairs i, j for which $c_{ij} \neq 0$. We can thus assume $\lambda_1 + \lambda_2 = 0$. Then for $S = S^{(12)}$, we have that S^2 is diagonal with entries $-1, -1, 0, 0, \dots, 0$ from which follows $\text{tr}(A^T S^{2r}) = (-1)^r (\lambda_1 + \lambda_2) = 0$ for all $r \geq 1$ and since S^{2r+1} is skew $\text{tr}(A^T S^{2r+1}) = 0$ for all $r \geq 0$. Hence for $S = S^{(12)} \neq 0$, we have $\Psi(S) = 0$, again contradicting the existence of a strict local maximum at $S = 0$. $\square$

Proposition 5.7 gives necessary and sufficient conditions for the immobilization of Δ . These are couched as algebraic conditions on the matrix A : we recall that A depends solely on the geometry of Δ and the contact set $\mathcal{P}$.

We now establish a relation between the matrix A and the matrix Λ , introduced in Sect. 3, and which like A also only depends on the geometry of Δ and $\mathcal{P}$; we then use the special properties of Λ to show that symmetry of A alone suffices for immobilization of Δ in the cases $n = 2, 3$, but not for $n \geq 4$.

Lemma 5.8. *The eigenvalues of the $(n+1) \times (n+1)$ matrix $-n\text{vol}(\Delta)\Lambda$ are precisely those of the $n \times n$ matrix A together with the value $-n\text{vol}(\Delta)$.*

Proof. The matrix $A = \sum_{i=0}^n \mathbf{k}_i p_i^T$ is the product of the $n \times (n+1)$ matrix $[\mathbf{k}_0, \mathbf{k}_1, \dots, \mathbf{k}_n]$ with the transpose of the $n \times (n+1)$ matrix $[p_0, p_1, \dots, p_n]$. Reintroducing the $(n+1) \times (n+1)$ matrices $K = [\bar{\mathbf{k}}_0, \bar{\mathbf{k}}_1, \dots, \bar{\mathbf{k}}_n]$ and $P = [\bar{p}_0, \bar{p}_1, \dots, \bar{p}_n]$ we have

$$KP^T = \begin{bmatrix} \kappa_0 & \kappa_1 & \cdots & \kappa_n \\ \mathbf{k}_0 & \mathbf{k}_1 & \cdots & \mathbf{k}_n \end{bmatrix} \begin{bmatrix} 1 & p_0^T \\ 1 & p_1^T \\ \vdots & \vdots \\ 1 & p_n^T \end{bmatrix} = \begin{bmatrix} -n\text{vol}(\Delta) & \underline{b}^T \\ \mathbf{0} & A \end{bmatrix}, \quad (5.6)$$

where we have used $\sum \kappa_j = -n\text{vol}(\Delta)$, $\sum \mathbf{k}_j = \mathbf{0}$ and where $\underline{b} = \sum_j \kappa_j p_j$ plays no further role in the discussion. Thus the eigenvalues of KP^T are precisely those of A together with a further eigenvalue $-n\text{vol}(\Delta)$.

But from Sect. 3, $P = V\Lambda$ and $KV^T = -n\text{vol}(\Delta)I$ so that

$$KP^T = K\Lambda^T V^T = -n\text{vol}(\Delta)K\Lambda^T K^{-1}.$$

Hence KP^T is similar to $-n\text{vol}(\Delta)\Lambda^T$ which gives the result. $\square$

Now from Sect. 3, Λ is a stochastic matrix with diagonal entries all zero and all off-diagonal entries positive. The following proposition lists the possibilities

for the eigenvalues of such a matrix (the proof is actually for Λ^T which has the same eigenvalues as Λ).

Proposition 5.9. *Let $\Lambda = (\lambda_{ij})$ be a $m \times m$ matrix such that:*

- (i) $\sum_{j=1}^m \lambda_{ij} = 1$ for $1 \leq i \leq m$;
- (ii) $\lambda_{ii} = 0$ for all $1 \leq i \leq m$;
- (iii) $\lambda_{ij} > 0$ for all $1 \leq i \neq j \leq m$.

Then for all $m \geq 3$:

- *1 is an eigenvalue of Λ with eigenvector $(1, 1, \dots, 1)^T$*
- *all other eigenvalues lie strictly within the unit circle.*

Proof. We consider vectors in $\mathbb{C}^m$ with the supremum norm $\|\underline{z}\| = \max\{|z_k| : 1 \leq k \leq m\}$. Then for $k = 1, 2, \dots, m$

$$|(\Lambda \underline{z})_k| = \left| \sum_{j=1}^m \lambda_{kj} z_j \right| \leq \sum_{j=1}^m |\lambda_{kj}| |z_j| \leq \sum_{j=1}^m \lambda_{kj} \|\underline{z}\| = \|\underline{z}\|.$$

Hence if λ is an eigenvalue of Λ with eigenvector $\underline{z}$

$$|\lambda| \|\underline{z}\| = \|\lambda \underline{z}\| = \|\Lambda \underline{z}\| = \max\{|(\Lambda \underline{z})_k| : 1 \leq k \leq m\} \leq \|\underline{z}\|,$$

so that $|\lambda| \leq 1$.

If $\underline{z} = (1, 1, \dots, 1)^T$, then $(\Lambda \underline{z})_k = \sum_{j=1}^m \lambda_{kj} = 1$ so that $\Lambda \underline{z} = (1, 1, \dots, 1)^T$ and hence $\underline{z} = (1, 1, \dots, 1)^T$ is an eigenvector corresponding to $\lambda = 1$.

Now let $\Lambda \underline{z} = \mu \underline{z}$ where $|\mu| = 1$ and let $|z_{k^*}|$ be maximal among $|z_k|$, $1 \leq k \leq m$. Then

$$|z_{k^*}| = |\mu z_{k^*}| = |(\Lambda \underline{z})_{k^*}| = \left| \sum_{j=1}^m \lambda_{k^*j} z_j \right| \leq \sum_{j=1}^m \lambda_{k^*j} |z_j| \quad (5.7)$$

and using $\sum_j \lambda_{k^*j} = 1$, it follows that

$$0 \leq \sum_{j=1}^m \lambda_{k^*j} (|z_j| - |z_{k^*}|).$$

But since $\lambda_{k^*j} > 0$ and $|z_j| - |z_{k^*}| \leq 0$ for all $j \neq k^*$, it follows $|z_j| = |z_{k^*}|$ for all $j \neq k^*$. Thus the components z_j of eigenvector $\underline{z}$ all lie on a circle of radius $|z_{k^*}|$ in $\mathbb{C}$, and for $m \geq 3$, unless all the z_k are the same, the point $\sum_{j=1}^m \lambda_{k^*j} z_j$ lies strictly inside the convex hull of the $\{z_j\}$ and so has modulus strictly less than $|z_{k^*}|$ contradicting the equality $|z_{k^*}| = |\sum_{j=1}^m \lambda_{k^*j} z_j|$ in (5.7) above. Hence the eigenvector $\underline{z}$ is a multiple of $(1, 1, \dots, 1)^T$ and thus corresponds to $\lambda = 1$; thus all other eigenvalues λ satisfy $|\lambda| < 1$. $\square$

Remark 5.10. The final argument relies on the fact that for $m \geq 3$, Λ has elements satisfying $0 < \lambda_{ij} < 1$; this is not true when $m = 2$ in which case

$$\Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and then Λ has an eigenvalue of -1 corresponding to eigenvector $(-1, 1)^T$.

This proposition leads to a difference in the sufficiency conditions for immobilization between the cases $n \leq 3$ and $n > 3$.

Proposition 5.11. *When $n \leq 3$ a sufficient condition for immobilization is simply the symmetry of A .*

Proof. We show that for $n \leq 3$, the fact that the points p_i are internal to the faces F_i together with symmetry of A implies that A is almost positive definite. The result then follows from Corollary 5.6.

If A is symmetric its eigenvalues are real and following Lemma 5.8, we write them as $-n\text{vol}(\Delta)\lambda_1, \dots, -n\text{vol}(\Delta)\lambda_n$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Λ other than unity. Since Λ is tracefree, $\lambda_1 + \lambda_2 + \dots + \lambda_n + 1 = 0$. For $n = 2$, $\lambda_1 + \lambda_2 = -1$ so that $-n\text{vol}(\Delta)\lambda_1 - n\text{vol}(\Delta)\lambda_2 = n\text{vol}(\Delta)$ and so A is almost positive definite. For $n = 3$, $\lambda_1 + \lambda_2 + \lambda_3 + 1 = 0$, so, for example $\lambda_1 + \lambda_2 = -(1 + \lambda_3) < 0$ since $|\lambda_3| < 1$ by Proposition 5.9 and λ_3 is real, similarly for each pair of eigenvalues and hence A is almost positive definite. $\square$

This method clearly fails for $n \geq 4$ and indeed the reduced sufficiency condition does not hold for $n \geq 4$ as the following example shows.

Consider the 4-simplex having vertices $v_0, \dots, v_4$ where $v_0 = (-\frac{5}{12}, -1, 0, -3)$, $v_1 = (-\frac{83}{36}, 0, 0, 1)$, $v_2 = (1, 1, 0, -3)$, $v_3 = (\frac{35}{18}, 0, -1, 1)$ and $v_4 = (\frac{35}{18}, 0, 1, 1)$. The vectors $\mathbf{k}_0 = (0, 34, 0, \frac{17}{2})$, $\mathbf{k}_1 = (16, -\frac{34}{3}, 0, -\frac{119}{18})$, $\mathbf{k}_2 = (0, -34, 0, \frac{17}{2})$, $\mathbf{k}_3 = (-8, \frac{17}{3}, 34, -\frac{187}{36})$ and $\mathbf{k}_4 = (-8, \frac{17}{3}, -34, -\frac{187}{36})$ are the standard outward normal vectors to this simplex. The points

$$p_0 = \frac{3}{10}v_1 + \frac{2}{5}v_2 + \frac{3}{20}v_3 + \frac{3}{20}v_4 \quad (5.8)$$

$$p_1 = \frac{1}{10}v_0 + \frac{1}{10}v_2 + \frac{2}{5}v_3 + \frac{2}{5}v_4 \quad (5.9)$$

$$p_2 = \frac{2}{5}v_0 + \frac{2}{5}v_1 + \frac{1}{10}v_3 + \frac{1}{10}v_4 \quad (5.10)$$

$$p_3 = \frac{1}{10}v_0 + \frac{7}{10}v_1 + \frac{1}{10}v_2 + \frac{1}{10}v_4 \quad (5.11)$$

$$p_4 = \frac{1}{10}v_0 + \frac{7}{10}v_1 + \frac{1}{10}v_2 + \frac{1}{10}v_3 \quad (5.12)$$

are interior to their faces and satisfy the symmetry condition since

$$\sum_{i=0}^4 \mathbf{k}_i p_i^T = \begin{bmatrix} \frac{238}{5} & 0 & 0 & 0 \\ 0 & \frac{136}{5} & 0 & 0 \\ 0 & 0 & \frac{34}{5} & 0 \\ 0 & 0 & 0 & -\frac{68}{5} \end{bmatrix}.$$

However, a pair of eigenvalues of this matrix has a negative sum.

For the case $n = 3$ Proposition 5.11 was implied by the work of [1] and for the case $n = 2$, it was implied by the work of [4]. In both cases different methods were used. The use here of stochastic matrices which are widely studied seems advantageous.

6. Geometric interpretation of conditions for immobilization

We finally attempt to give a geometric description of the conditions for immobilization of Δ . The ideal result would be to give a complete geometric description of the elements of $\mathcal{S}$, the set of all contact sets $\mathcal{P}$ which immobilize Δ . This is possible when $n = 2$, (see [4]) but already becomes awkward for $n = 3$. Instead, we settle for something less. We are able to give a good geometric description of some of the elements of $\mathcal{S}$ and one which we call $\mathcal{G}$ stands out as being in some sense optimal. We will also describe a geometric process, which starting with $\mathcal{P} = \mathcal{G}$ displaces the point p_i , each within its face F_i , to reach other elements of $\mathcal{S}$. In this process, symmetry of A is maintained, but it is harder to keep track of almost positive definiteness.

Our procedure is as follows: we initially construct some special $n \times (n + 1)$ matrices $[p_0, p_1, \dots, p_n]$ of contact points which ensure that A is symmetric, but where we temporarily relax the condition $p_i \in F_i$. From the linearity of A with respect to the p_i in $A = \sum \mathbf{k}_i p_i^T$, it follows that any linear combination of these special matrices also yields a matrix A which is symmetric, and we choose the linear combinations to ensure that $p_i \in F_i$; we will also examine almost positive definiteness of such matrices A .

The following special sets $[p_0, p_1, \dots, p_n]$ guarantee symmetry of A :

- (i) $[t_0 \mathbf{k}_0, t_1 \mathbf{k}_1, \dots, t_n \mathbf{k}_n]$ where $t_i, i = 0, 1, \dots, n$ are arbitrary. Then $A = \sum_{i=0}^n t_i \mathbf{k}_i \mathbf{k}_i^T$ is clearly symmetric and positive definite if all $t_i > 0$.
- (ii) $[\underline{z}, \underline{z}, \dots, \underline{z}]$ where $\underline{z} \in \mathbb{R}^n$ is arbitrary. Then $A = \sum \mathbf{k}_i \underline{z} \underline{z}^T$ which is the zero matrix.
- (iii) $[v_0, \dots, v_n]$. This corresponds to the $(n + 1) \times (n + 1)$ matrix $V = [\bar{v}_0, \bar{v}_1, \dots, \bar{v}_n]$ for which $KV^T = -n \text{vol}(\Delta) I_{n+1}$ and then by (5.6), $A = -n \text{vol}(\Delta) I_n$.
- (iv) $[\mathbf{k}_1, \mathbf{k}_0, 0, \dots, 0]$ gives $A = \mathbf{k}_0 \mathbf{k}_1^T + \mathbf{k}_1 \mathbf{k}_0^T$ which is clearly symmetric: similarly any interchange of $\mathbf{k}_i$ and $\mathbf{k}_j$, $i \neq j$ with all other entries 0.

We now use these special sets to construct some immobilizing sets.

6.1. Centroids of the faces

Let $g = \frac{1}{n} \sum_{i=0}^n v_i$. Then $g_i = g - \frac{v_i}{n}$ is the centroid of the face F_i . Let $G = [\bar{g}_0, \bar{g}_1, \dots, \bar{g}_n]$ be the $(n + 1) \times (n + 1)$ matrix of the contact set of centroids. Then for this set

$$KG^T = K[\bar{g}, \bar{g}, \dots, \bar{g}]^T - \frac{1}{n}KV^T = 0 + \text{vol}(\Delta)I_{n+1}$$

so that by (5.6) $A = \text{vol}(\Delta)I_n$ which is symmetric and positive definite and hence the set of centroids is immobilizing. We can also see this by noting that for G the corresponding matrix Λ is $\frac{1}{n}(J - I)$ where J is the $(n+1) \times (n+1)$ matrix all of whose entries are 1. J has one eigenvalue of $n+1$ and n eigenvalues of 0, so that Λ has one eigenvalue of 1 and n eigenvalues of $-\frac{1}{n}$, whence by Lemma 5.8, A has n eigenvalues of $\text{vol}(\Delta)$.

6.2. Centred contact sets

A set of contact points $\mathcal{P} = \{p_0, p_1, \dots, p_n\}$ where $p_i \in F_i$ is centred at $z \in \mathbb{R}^n$ if the normal lines $l_i : x = p_i + t\mathbf{k}_i$, $t \in \mathbb{R}$ are concurrent, meeting in z . Given Δ , the set $\mathcal{Z}$ of such points $z \in \mathbb{R}^n$ may be constructed as the intersection of the $n+1$ cylinders C_i where $C_i = \{l_i : p_i \in F_i\}$. This set is a polytope in $\mathbb{R}^n$ which is shown in [5] to always include an open set of points interior to Δ , but not necessarily the whole of the interior points of Δ (for $n=2$, Δ a triangle, if Δ is acute angled, then the set is a hexagon completely enclosing Δ , of twice the area of Δ ; if Δ is obtuse angled, the set is a parallelogram only including part of Δ and which becomes unbounded as the obtuse angle approaches π).

Let z be a point of concurrency interior to Δ . Then there exists $t_i > 0$ so that $p_i = z + t_i\mathbf{k}_i \in F_i$ and for the contact set $\mathcal{P} = \{p_i\}$

$$A = \sum_{i=0}^n \mathbf{k}_i z^T + \sum_{i=0}^n t_i \mathbf{k}_i \mathbf{k}_i^T = \sum_{i=0}^n t_i \mathbf{k}_i \mathbf{k}_i^T$$

which is symmetric and positive definite, so that $\mathcal{P}$ immobilizes Δ .

Remark 6.1. For $n=2$, Δ a triangle, the collection of contact sets centred at z whether or not z lies within Δ , is precisely the collection of all immobilizing contact sets [4]. This is not true for $n \geq 3$. From the results in [5], the space of centred immobilizing contact sets has dimension n whereas the space of all immobilizing contact sets has dimension $\frac{1}{2}n(n+1) - 1$. These are equal when $n=2$ but not otherwise.

6.3. Displacements

A comparatively complete description of immobilizing contact sets of Δ can be given by examining, instead of the contact set itself, displacements Δp_i of the points p_i within the faces F_i from one immobilizing contact set to another. Leaving aside the condition of almost positive definiteness for the time being, the conditions p_i lying in the hyperplane containing F_i and A symmetric are:

$$\mathbf{k}_i \cdot p_i = \mathbf{k}_i \cdot v_j, \quad j \neq i, \quad i = 0, 1, \dots, n; \quad \sum_{i=0}^n \mathbf{k}_i \wedge p_i = \mathbf{0},$$

the latter merely stating that the antisymmetric part of A vanishes. Then a displacement Δp_i within F_i maintains the symmetry of A only if Δp_i satisfy

$$\mathbf{k}_i \cdot \Delta p_i = 0, \quad i = 0, 1, \dots, n, \quad \sum_{i=0}^n \mathbf{k}_i \wedge \Delta p_i = \mathbf{0}. \quad (6.1)$$

While the system (6.1) is necessary to keep each displaced point p_i in the face F_i and to maintain symmetry of A , it is not sufficient, because it places no constraint on the size of the displacements Δp_i : the system (6.1) achieves the more modest objective of keeping each point p_i in the hyperplane containing the face F_i (equivalently, the displacement Δp_i is parallel to F_i) together with maintaining symmetry of A .

We now exhibit a basis for the space Ω of displacements $\Delta P = [\Delta p_0, \Delta p_1, \dots, \Delta p_n]$ satisfying the system (6.1). There is no such displacement set where all but one of the Δp_i are zero, since, if for example, Δp_0 were the only non-zero displacement, we would have $\mathbf{k}_0 \cdot \Delta p_0 = 0$ and $\mathbf{k}_0 \wedge \Delta p_0 = \mathbf{0}$ which requires Δp_0 to be both perpendicular and parallel to $\mathbf{k}_0$.

There are, however, displacements where all but two of the Δp_i are zero. If the non-zero displacements are Δp_i and Δp_j , then (6.1) reduces to

$$\mathbf{k}_i \cdot \Delta p_i = 0, \mathbf{k}_j \cdot \Delta p_j = 0, \quad (6.2)$$

$$\mathbf{k}_i \wedge \Delta p_i + \mathbf{k}_j \wedge \Delta p_j = \mathbf{0}. \quad (6.3)$$

Wedging (6.3) with $\mathbf{k}_i$ gives $\Delta p_j \wedge (\mathbf{k}_i \wedge \mathbf{k}_j) = \mathbf{0}$ so that Δp_j is a linear combination of $\mathbf{k}_i$ and $\mathbf{k}_j$; likewise Δp_i . Then using (6.2), it follows that

$$\Delta p_i = t\mathbf{k}_{ij} \text{ and } \Delta p_j = t\mathbf{k}_{ji}, \quad (6.4)$$

where

$$\mathbf{k}_{ij} = \mathbf{k}_j - \frac{(\mathbf{k}_i \cdot \mathbf{k}_j)}{|\mathbf{k}_i|^2} \mathbf{k}_i \quad (6.5)$$

is the projection of $\mathbf{k}_j$ perpendicular to $\mathbf{k}_i$ and t is a parameter. We define a collection of displacement sets ΔP_{ij} , $0 \leq i < j \leq n$ by

$$\begin{aligned} \Delta P_{ij} &= [\Delta p_0, \Delta p_1, \dots, \Delta p_n] \text{ where } \Delta p_i = \mathbf{k}_{ij}; \quad \Delta p_j = \mathbf{k}_{ji} \text{ and} \\ \Delta p_m &= \mathbf{0} \text{ for all } m \neq i, j. \end{aligned} \quad (6.6)$$

We have thus shown that the displacements $\Delta P = [\Delta p_0, \Delta p_1, \dots, \Delta p_n]$ with each Δp_k parallel to F_k , which also maintains symmetry of A , and for which only Δp_i and Δp_j , $i \neq j$, are non-zero is necessarily given by $\Delta P = t\Delta P_{ij}$ for some t . In the displacement ΔP_{ij} , the points p_i and p_j move in a dependent way in directions parallel to the orthogonal projection of $\mathbf{k}_j$ onto F_i and the orthogonal projection of $\mathbf{k}_i$ onto F_j respectively; since these directions are both linear combinations of $\mathbf{k}_i$ and $\mathbf{k}_j$, they are both orthogonal to F_{ij} , the $n-2$ dimensional face in which F_i meets F_j .

It is shown in [5] that the collection $\mathcal{C}$ of ΔP_{ij} , $0 \leq i < j \leq n$ spans the space Ω of displacements which satisfy (6.1), but they are not independent: among the

$\frac{1}{2}n(n+1)$ displacement sets $\Delta P_{ij} = 0$, there is precisely one dependency relation, $\sum_{0 \leq i < j \leq n} \Delta P_{ij} = 0$, from which it follows that the space of displacement sets maintaining symmetry of A is $\frac{1}{2}n(n+1) - 1$ dimensional.

We can see the effect of the displacement set $t\Delta P_{01}$ on the immobilizing set of centroids $\mathcal{G}$. Since each g_i is strictly interior to F_i , for sufficiently small $|t|$, the displaced points remain in their faces and, by construction, A remains symmetric, so we examine almost positive definiteness. For the displaced set we have

$$[p_0, p_1, \dots, p_n] = [g_0, g_1, \dots, g_n] + t[\mathbf{k}_{01}, \mathbf{k}_{10}, 0, \dots, 0]$$

so that

$$A = \text{vol}(\Delta)I_n + tB_{01}$$

where B_{01} is the $n \times n$ -matrix

$$B_{01} = \mathbf{k}_0 \mathbf{k}_{01}^T + \mathbf{k}_1 \mathbf{k}_{10}^T$$

Now take a basis of $\mathbb{R}^n$ given by $\{\mathbf{k}_0, \mathbf{k}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\}$ where each $\mathbf{u}_i$ is orthogonal to span $\{\mathbf{k}_0, \mathbf{k}_1\}$. Then

$$\begin{aligned} B_{01}\mathbf{k}_0 &= \mathbf{k}_1 |\mathbf{k}_{10}|^2 \\ B_{01}\mathbf{k}_1 &= \mathbf{k}_0 |\mathbf{k}_{01}|^2 \\ B_{01}\mathbf{u}_j &= \mathbf{0}, \quad j = 2, \dots, n \end{aligned}$$

so that B_{01} is similar to a matrix (b_{ij}) where $b_{10} = |\mathbf{k}_{10}|^2$, $b_{01} = |\mathbf{k}_{01}|^2$ and all other entries are zero. Thus B_{01} has eigenvalues $\pm |\mathbf{k}_{01}||\mathbf{k}_{10}|$ and all other eigenvalues are 0, so that A has eigenvalues $\text{vol}(\Delta) \pm t|\mathbf{k}_{01}||\mathbf{k}_{10}|$ and the remaining $n-2$ eigenvalues are all $\text{vol}(\Delta)$, hence for the displaced contact set, the smallest sum of any pair of eigenvalues is

$$2\text{vol}(\Delta) - |t||\mathbf{k}_{01}||\mathbf{k}_{10}|$$

so that any displacement $t\Delta P_{01}$, $t \neq 0$, reduces the minimal sum of a pair of eigenvalues and hence for $|t|$ sufficiently large the almost positive definite condition fails.

We extend this to an arbitrary non-zero displacement $\Delta P = \sum_{0 \leq i < j \leq n} t_{ij} \Delta P_{ij}$ as follows. Following from above, we now have

$$A = \text{vol}(\Delta)I_n + \sum_{0 \leq i < j \leq n} t_{ij} B_{ij}$$

where

$$B_{ij} = \mathbf{k}_i \mathbf{k}_{ij}^T + \mathbf{k}_j \mathbf{k}_{ji}^T.$$

Now from before B_{01} is similar to (b_{ij}) which is trace-free. Thus B_{01} itself and similarly all B_{ij} are trace-free. Hence the sum of the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A satisfies

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{tr}(A) = \text{tr}(\text{vol}(\Delta)I_n + \sum t_{ij} B_{ij}) = n\text{vol}(\Delta).$$

Now $\lambda_1 = \lambda_2 = \dots = \lambda_n = \text{vol}(\Delta)$ if and only if $A = \text{vol}(\Delta)I_n$ (only if requires us to use symmetry of A so that A is diagonalizable). So for any non-zero perturbation $\sum t_{ij}B_{ij}$ of A , since the sum of eigenvalues is fixed, at least one eigenvalue increases strictly and at least one eigenvalue decreases strictly, so that the smallest sum of any pair of eigenvalues of A has to decrease for any non-zero displacement $\sum t_{ij}\Delta P_{ij}$ from $\mathcal{G}$. Because the eigenvalues of A change continuously with ΔP , using the dimensionality results in [5], we can summarise the results as follows:

1. in the $\frac{1}{2}n(n+1) - 1$ dimensional space of contact sets $[p_0, p_1, \dots, p_n]$ for which A is symmetric and where each p_i lies in the $(n-1)$ -dimensional hyperplane containing F_i , there is an open neighbourhood N of $\mathcal{G}$, the centroids, which immobilizes Δ ;
2. the contact set $\mathcal{G}$ is optimal in that any displacement within the neighbourhood N causes the smallest sum of a pair of eigenvalues of A to decrease from $2\text{vol}(\Delta)$, and if such a displacement is large enough to cause this sum to vanish, then immobilization is lost.

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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