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On quadruples of Griffiths points

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Abstract. Tabov (Math Mag 68:61–64, 1995) has proved the following theorem: if points A_1, A_2, A_3, A_4 are on a circle and a line l passes through the centre of the circle, then four Griffiths points G_1, G_2, G_3, G_4 corresponding to pairs (Δ_i, l) are on a line (Δ_i denotes the triangle $A_jA_kA_l$, $j, k, l \neq i$). In this paper we present a strong generalisation of the result of Tabov. An analogous property for four arbitrary points A_1, A_2, A_3, A_4, is proved, with the help of the computer program "Mathematica".

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1. Introduction

Tabov [1] has proved the following theorem: if points A_1, A_2, A_3, A_4 are on a circle and a line l passes through the centre of the circle, then four Griffiths points G_1, G_2, G_3, G_4 corresponding to pairs (Δ_i, l) are on a line (Δ_i denotes the triangle $A_jA_kA_l$, $j, k, l \neq i$).

Explanation. When a point P moves along a line through the circumcenter of a given triangle Δ, the circumcircle of the pedal triangle of P with respect to Δ passes through a fixed point, called Griffiths point, on the nine-point circle of Δ. The pedal triangle of P with respect to Δ is the triangle the vertices of which are feet of the perpendiculars from P to the sides of Δ. A very simple construction of the Griffiths point for a pair (Δ, l) is given in [2]. Namely, we project orthogonally the intersection points of l and the circumcircle of Δ onto the sides of Δ. The projections of each of these points are collinear and the common point of the two lines is the Griffiths point associated with (Δ, l).

2. Main results

In this paper we present a much stronger generalisation of the result of Tabov. We consider four arbitrary points A_1, A_2, A_3, A_4, no three of them collinear.

By Δ_i is denoted the triangle $A_j A_k A_l$, $j, k, l \neq i$. l_i is a line passing through the circumcenter of Δ_i $i = 1, 2, 3, 4$. Finally, G_i is the Griffiths point corresponding to (Δ_i, l_i) $i = 1, 2, 3, 4$.

Theorem. *If lines l_1, l_2, l_3, l_4 have a common point at infinity (every two of them are parallel), then points G_1, G_2, G_3, G_4 are collinear.*

Proof. As in [1] points A_1, A_2, A_3, A_4 are represented by complex numbers a, b, c, d, respectively. Without loss of generality, we may assume that points A_1, A_2, A_3 are on the circle of centre 0 and radius 1, i.e. $|a| = |b| = |c| = 1$. Similarly, we may assume that lines l_1, l_2, l_3, l_4 are parallel to the real axis. Hence [1] $g_4 = \frac{1}{2}(a+b+c-abc)$, where g_i is the complex number representing Griffiths point G_i, $i = 1, 2, 3, 4$. It is easy to check that if A_1, A_2, A_3, A_4 are on the circle of centre 0 and radius R instead of 1, then

$$g_4 = \frac{1}{2}\left(a + b + c - \frac{abc}{R^2}\right) \tag{2.1}$$

After short calculations we find the number $c_3 = \dfrac{abd(1-|d|^2)}{ad + bd - d^2 - ab|d|^2}$ representing the circumcenter C_3 of triangle $A_1 A_2 A_4$. Now we introduce a new coordinate system by the formula: $z = z' + c_3$. In the new system, according to (2.1),

$$g_3' = \frac{1}{2}\left(a' + b' + c' - \frac{a'b'c'}{|a - c_3|^2}\right).$$

Then in the former coordinate system we have

$$g_3 = \frac{1}{2}\left(a + b + d - c_3 - \frac{(a - c_3)(b - c_3)(d - c_3)}{|a - c_3|^2}\right).$$

In an analogous way we obtain

$$g_2 = \frac{1}{2}\left(a + c + d - c_2 - \frac{(a - c_2)(c - c_2)(d - c_2)}{|a - c_2|^2}\right),$$
$$g_1 = \frac{1}{2}\left(b + c + d - c_1 - \frac{(b - c_1)(c - c_1)(d - c_1)}{|a - c_1|^2}\right).$$

Points G_2, G_3, G_4 are collinear iff [1] the equality

$$\frac{g_3 - g_4}{g_2 - g_4} = \overline{\frac{g_3 - g_4}{g_2 - g_4}} \tag{2.2}$$

holds. In order to prove it, we use the computer program "Mathematica". The consecutive steps are as follows: First we write the complex numbers a, b, c, d in the form $a = \cos x + i \sin x$, $b = \cos y + i \sin y$, $c = \cos z + i \sin z$, $d = R(\cos u + i \sin u)$. Beginning from now, all formulae are obtained with the help of "Mathematica".

$$c_3 = \frac{e^{0.5i(x+y)}(1 - R^2)}{-2\cos\frac{x-y}{2} + 2\cos\left(u - \frac{x}{2} - \frac{y}{2}\right)}.$$

Similarly, $c_2 = \dfrac{e^{0.5i(x+y)}(1-R^2)}{-2\cos\frac{x-y}{2} + 2\cos\left(u - \frac{x}{2} - \frac{y}{2}\right)}$ (c_2 represents the circumcenter C_2 of triangle $A_1A_3A_4$).

$$g_4 = \frac{1}{2}(\cos x + \cos y + \cos z - \cos(x+y+z) \\ + i(\sin x + \sin y + \sin z - \sin(x+y+z))).$$

$$(a-c_2)(c-c_2)(d-c_2) = \\ = -\frac{e^{0.5i(4u+x+z)}(e^{ix} - Re^{iu})^2(e^{iz} - Re^{iu})^2(e^{iu} - Re^{ix})(e^{iu} - Re^{iz})}{8\left(\cos\frac{x-z}{2} - R\cos\left(u - \frac{x}{2} - \frac{z}{2}\right)\right)^3}.$$

$$a - c_2 = \cos x + \frac{\cos\frac{x+z}{2}}{-2\cos\frac{x-z}{2} + 2R\cos\left(u - \frac{x}{2} - \frac{z}{2}\right)} \\ - \frac{R^2\cos\frac{x+z}{2}}{-2\cos\frac{x-z}{2} + 2R\cos\left(u - \frac{x}{2} - \frac{z}{2}\right)} + \\ i\left(\times \sin x + \frac{\sin\frac{x+z}{2}}{-2\cos\frac{x-z}{2} + 2R\cos\left(u - \frac{x}{2} - \frac{z}{2}\right)} \right. \\ \left. - \frac{R^2\sin\frac{x+z}{2}}{-2\cos\frac{x-z}{2} + 2R\cos\left(u - \frac{x}{2} - \frac{z}{2}\times\right)}\right).$$

$$Abs^2[a-c_2] = \frac{(1+R^2-2R\cos(u-x))(1+R^2-2R\cos(u-z))}{4\left(\cos\frac{x-z}{2} - R\cos\left(u - \frac{x}{2} - \frac{z}{2}\right)\right)^2}.$$

$$\frac{(a-c_2)(c-c_2)(d-c_2)}{|a-c_2|^2} = \\ = \frac{R}{\cos(u+x+z) - i\sin(u+x+z) + \frac{(-1+R^2)e^{-iu}}{(e^{ix}-Re^{iu})(e^{iz}-Re^{iu})}}.$$

$$g_2 = \frac{1}{2}\left(\cos x + \cos z + i\sin x + i\sin z \right. \\ \left. + \frac{e^{iu}(e^{iz} - Re^{iu})(Re^{iu} - e^{ix})(-1 + \cos(x+z) + i\sin(x+z))}{-e^{i(u+x)} - e^{i(u+z)} + Re^{2iu} + Re^{i(x+z)}}\right).$$

In an analogous way we obtain

$$g_3 = \frac{1}{2}\left(\cos x + \cos y + i\sin x + i\sin y \right. \\ \left. + \frac{e^{iu}(e^{iy} - Re^{iu})(Re^{iu} - e^{ix})(-1 + \cos(x+y) + i\sin(x+y))}{-e^{i(u+x)} - e^{i(u+y)} + Re^{2iu} + Re^{i(x+y)}}\right).$$

$$L = \frac{g_3 - g_4}{g_2 - g_4} = \frac{\cos\frac{x-z}{2} - R\cos\left(u - \frac{x}{2} - \frac{z}{2}\right)\cos\frac{x+z}{2}\sin\frac{x+y}{2}}{\cos\frac{x-z}{2} - R\cos\left(u - \frac{x}{2} - \frac{y}{2}\right)}.$$

Since the above expression is real, the equality (2.2) holds. Obviously, in an identical way we prove that points G_1, G_2, G_4 colline and so on. This ends the proof. □

Remark. As we can observe, using of a computer program to obtain so complicated formulae, was necessary. It should be noticed that the results obtained by transforming symbolic expressions with the help of the program "Mathematica" are quite exact.

References

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