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On the Local Existence of Solutions to the compressible Navier–Stokes-Wave System with a Free Interface

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Abstract. We address a system of equations modeling a compressible fluid interacting with an elastic body in dimension three. We prove the local existence and uniqueness of a strong solution when the initial velocity belongs to the space $H^{2+\epsilon}$ and the initial structure velocity is in $H^{1.5+\epsilon}$, where $\epsilon \in (0, 1/2)$.

Contents

1. Introduction

The objective of this paper is to establish the local-in-time existence of solutions for the free boundary fluid–structure interaction model under low regularity assumptions on the initial data. The model describes the interaction between a viscous compressible fluid and an elastic structure that is immersed in it. Mathematically, the dynamics of the fluid are governed by the compressible Navier–Stokes equations in the velocity and density variables (u, ρ), while the elastic dynamics are described by a second-order elasticity equation (which is replaced by a wave equation for the sake of simplicity) in the vector variables (w, w_t) representing the displacement and velocity of the structure.

The interaction between the structure and the fluid is mathematically characterized by velocity and stress matching boundary conditions at the moving interface that separates the solid and fluid regions. Since the interface position evolves with time and is unknown a priori, this is a free-boundary problem. The problem is challenging due to the mismatch between parabolic and hyperbolic regularity, as well as the complexity of the stress-matching condition on the free boundary.

The local-in-time existence and well-posedness results for the fluid–structure interaction model have been extensively studied in the literature. In 2005, the authors of [18,19] established the local-in-time existence and well-posedness for the incompressible model, using the Lagrangian coordinate system to fix the domain and the Tychonoff fixed point theorem to construct a solution, given an initial fluid velocity $u_0 \in H^5$ and structural velocity $w_1 \in H^3$. Subsequently, the papers [32,33] provided a priori estimates for the local existence of solutions using direct estimates for the initial data, namely $u_0 \in H^3$

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and $w_1 \in H^{5/2+r}$, where $r \in (0, (\sqrt{2}-1)/2)$. The authors relied on the hidden regularity trace theorem for wave equations, established in [12,38,39,42,51,53], as a key ingredient to obtain their result. Several works on wave-heat coupled systems on a non-moving domain have contributed to the understanding of the heat-wave interaction phenomena (cf. [2–4,9,10,21,35–37,40]). Recently, Raymond and Vanninathan [50] obtained a sharp regularity result for the case when the initial domain is a flat channel. They studied the system in the Lagrangian coordinate setting and obtained local-in-time solutions for the 3D model, with the initial velocity $u_0 \in H^{1+\alpha}$ and the initial structural velocity $w_1 \in H^{1/2+\alpha+\beta}$, where $\alpha \in (1/2, 1)$ and $\beta > 0$. In [11], Boulakia, Guerrero, and Takahashi obtained a unique local-in-time solution for the general domain case, given the initial data $u_0 \in H^2$ and $w_1 \in H^{9/8}$.

The compressible model under consideration was first treated in [6], where the authors obtained the existence and uniqueness for the initial density ρ_0 belonging to H^3, the velocity u_0 in H^4, and the structure displacement and velocity (w, w_t) in $H^3 \times H^2$. A similar result was later obtained by Kukavica and Tuffaha [34] with less regular initial data $(\rho_0, u_0, w_1) \in H^{3/2+r} \times H^3 \times H^{3/2+r}$, where $r \in (0, (\sqrt{2}-1)/2)$. In [7], the existence of a regular global solution is proved for small initial data. In a recent work [8], the authors proved the existence of a unique local-in-time strong solution of the interaction problem between a compressible fluid and elastic structure for initial data $(\rho_0, u_0, w_1) \in H^3 \times H^6 \times H^3$, where the elastic structure is modeled by the Saint-Venant Kirchhoff system. For some other works on fluid–structure models, cf. [1,5,13–17,20,22,24–31,40,41,44–49,52,54].

In this paper, we provide a natural proof of the existence of a unique local-in-time solution to the system under a low regularity assumptions $u_0 \in H^{2+\epsilon}$ and $w_1 \in H^{1.5+\epsilon}$, where $\epsilon \in (0, 1/2)$, in the case of the flat initial configuration. Our proof relies on a maximal regularity type theorem for the nonhomegeneous linear parabolic problem with Neumann type conditions on the fluid–structure interface, in addition to the hidden regularity theorems (cf. Lemmas 3.5–3.6) for the wave equation. The time regularity of the solution is obtained using the energy estimates, which, combined with the elliptic regularity, yield the spatial regularity of the solutions. An essential ingredient of the proof of the main results is a trace inequality

$$\|u\|_{H^\theta((-\infty,\infty),L^2(\Gamma_c))} \lesssim \|u\|_{L^2((-\infty,\infty),H^r(\Omega_f))}^{1/(2r+1)} \|u\|_{H^{2\theta r/(2r-1)}((-\infty,\infty),L^2(\Omega_f))}^{2r/(2r+1)} + \|u\|_{L^2((-\infty,\infty),H^r(\Omega_f))},$$

for functions which are Sobolev in the time variable and square integrable on the boundary (cf. Lemma 3.1 and (3.8) below). This is used essentially in the proof of the existence for the nonlinear parabolic-wave system, Theorem 5.4, and in the proof of the main result, Theorem 2.1. The construction of a unique solution for the fluid–structure problem is obtained via the Banach fixed point theorem. The scheme involves solving the nonlinear parabolic-wave system with the variable coefficients treated as a given forcing perturbations.

One of the essential difficulties in establishing the existence of solutions is that the constants in the inequality are inversely proportional to powers of time T, which poses a problem for establishing convergence of a fixed-point scheme for small time. The same issue with the growing constants also arises in the hidden regularity inequalities in Lemmas 3.5–3.6 for the wave equation. We overcome this difficulty by solving a modified system which is posed on the fixed time interval $(0, 1]$. As opposed to the velocity matching boundary condition (2.6) in the original fluid–structure interaction problem, we impose the integrated velocity matching boundary condition (5.13) on the unit time interval in the modified system. These two boundary conditions agree on a small time interval and thus the modified system agrees with the original system when restricted to a small time interval. In the integrated velocity matching boundary condition (5.13), an important ingredient is the cutoff function in time that depends on a variable time $\tilde{T}$, which is then chosen to be less than a fixed time T_0, allowing for contraction estimates on the solution map. Another major difficulty is the handling of the normal derivative of the elastic structure on the common boundary, which is estimated by appealing to the hidden trace regularity (see Lemma 3.6). The main issue with proving the fixed-point theorems (for the linear and nonlinear variants) is that time derivatives, which are frequently fractional, fall on the cutoff, showing that the constant dependence on $\tilde{T}$ needs to be treated carefully.

Similarly, for the nonlinear system, treated in Sect. 6, we also need to modify the definition of the Lagrangian map and the variable coefficient matrix using a cutoff in time function to ensure similar contraction-type estimates on the solution map for the system with given variable coefficients. The solution in each iteration step is used to prescribe new variable coefficients for the next iteration step. The contracting property of the Navier-Stokes-wave system is maintained by taking a sufficiently short time $\tilde{T}$ to ensure closeness of the Jacobian and the inverse matrix of the flow map to their initial states.

Note that the configuration we adopt, (2.8) with the periodic boundary conditions in the y_1 and y_2 directions, is needed only in Lemma 3.6. In these estimates, Sobolev time norms pose a particular challenge when the cutoff function is involved since they involve singular terms in $\tilde{T}$ that have to be compensated by taking sufficiently high L^p norms of time derivatives of v.

The paper is structured as follows. In Sect. 2, we introduce the fluid–structure model and state our main result. Next, in Sect. 3, we present the trace inequality, interpolation, and hidden regularity lemmas. Section 4 provides the maximal regularity for the nonhomogeneous parabolic problem, which is a crucial ingredient in the proof of local existence for the nonlinear parabolic-wave system, discussed in Sect. 5. Finally, in Sect. 6, we prove our main result, Theorem 2.1, using the local existence result established in Sect. 5 and constructing a unique solution via the Banach fixed point theorem.

2. The Model and Main Results

We consider the fluid–structure problem for a free boundary system involving the motion of an elastic body immersed in a compressible fluid. Let $\Omega_{\mathrm{f}}(t)$ and $\Omega_{\mathrm{e}}(t)$ be the domains occupied by the fluid and the solid body at time t in $\mathbb{R}^3$, whose common boundary is denoted by $\Gamma_{\mathrm{c}}(t)$. The fluid is modeled by the compressible Navier–Stokes equations, which in Eulerian coordinates reads

$$\rho_t + \operatorname{div}(\rho u) = 0 \text{ in } [0,T]\times\Omega_{\mathrm{f}}(t), \tag{2.1}$$

$$\rho u_t + \rho(u\cdot\nabla)u - \lambda \operatorname{div}(\nabla u + (\nabla u)^T) - \mu\nabla \operatorname{div} u + \nabla p = 0 \text{ in } [0,T]\times\Omega_{\mathrm{f}}(t), \tag{2.2}$$

where $\rho = \rho(t,x)\in\mathbb{R}_+$ is the density, $u = u(t,x)\in\mathbb{R}^3$ is the velocity, $p = p(\rho(t,x))\in\mathbb{R}_+$ is the pressure, and $\lambda,\mu>0$ are physical constants. (We remark that the condition for λ and μ can be relaxed to $\lambda>0$ and $3\lambda+2\mu>0$.) The system (2.1)–(2.2) is defined on $\Omega_{\mathrm{f}}(t)$ which set to $\Omega_{\mathrm{f}} = \Omega_{\mathrm{f}}(0)$ and evolves in time. The dynamics of the coupling between the compressible fluid and the elastic body are best described in the Lagrangian coordinates. Namely, we introduce the Lagrangian flow map $\eta(t,\cdot)\colon \Omega_{\mathrm{f}}\to\Omega_{\mathrm{f}}(t)$ and rewrite the system (2.1)–(2.2) as

$$R_t - Ra_{kj}\partial_k v_j = 0 \text{ in } [0,T]\times\Omega_{\mathrm{f}}, \tag{2.3}$$

$$\partial_t v_j - \lambda R a_{kl}\partial_k(a_{ml}\partial_m v_j + a_{mj}\partial_m v_l) - \mu R a_{kj}\partial_k(a_{mi}\partial_m v_i) + R a_{kj}\partial_k(q(R^{-1})) = 0 \text{ in } [0,T]\times\Omega_{\mathrm{f}}, \tag{2.4}$$

for $j = 1,2,3$, where $R(t,x) = \rho^{-1}(t,\eta(t,x))$ is the reciprocal of the Lagrangian density, $v(t,x) = u(t,\eta(t,x))$ is the Lagrangian velocity, $a(t,x) = (\nabla\eta(t,x))^{-1}$ is the inverse matrix of the flow map and q is a given function of the density. The system (2.3)–(2.4) is expressed in terms of Lagrangian coordinates and posed in a fixed domain Ω_{f}.

On the other hand, the elastic body is modeled by the wave equation in Lagrangian coordinates, which is posed in a fixed domain Ω_{e} as

$$w_{tt} - \Delta w = 0 \text{ in } [0,T]\times\Omega_{\mathrm{e}}, \tag{2.5}$$

where (w, w_t) are the displacement and the structure velocity. The interaction boundary conditions are the velocity and stress matching conditions, which are formulated in Lagrangian coordinates over the fixed common boundary $\Gamma_{\mathrm{c}} = \Gamma_{\mathrm{c}}(0)$ as

$$v_j = \partial_t w_j \text{ on } [0,T]\times\Gamma_{\mathrm{c}}, \tag{2.6}$$

$$\partial_k w_j \nu^k = \lambda J a_{kl}(a_{ml}\partial_m v_j + a_{mj}\partial_m v_l)\nu^k + \mu J a_{kj} a_{mi}\partial_m v_i \nu^k - J a_{kj} q \nu^k \text{ on } [0,T]\times\Gamma_{\mathrm{c}}, \tag{2.7}$$

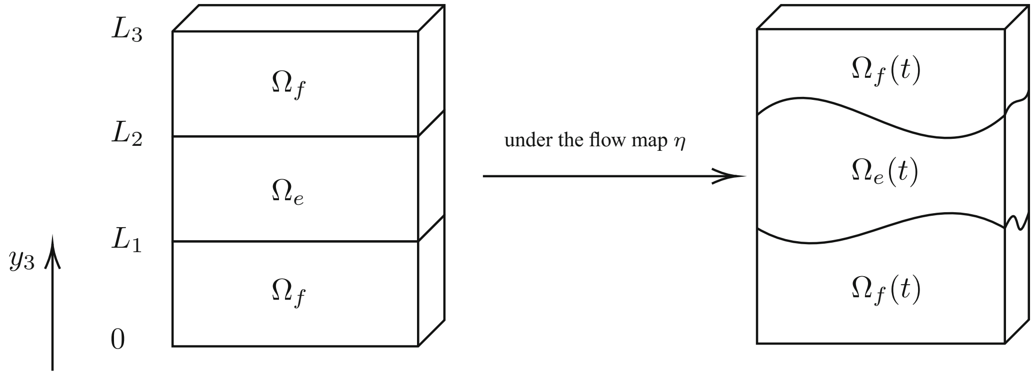

Fig. 1. Lagrangian domain to Eulerian domain

for $j = 1, 2, 3$, where $J(t, x) = \det(\nabla\eta(t, x))$ is the Jacobian and ν is the unit normal vector to $\Gamma_{\rm c}$, which is outward with respect to $\Omega_{\rm e}$. In the present paper, we consider the reference configurations $\Omega = \Omega_{\rm f}\cup\Omega_{\rm e}\cup\Gamma_{\rm c}$, $\Omega_{\rm f}$, and $\Omega_{\rm e}$ given by (see Fig. 1)

$$\begin{aligned}
&\Omega = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \mathbb{T}^2, 0 < y_3 < L_3\},\\
&\Omega_{\rm f} = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \mathbb{T}^2, 0 < y_3 < L_1 \ \text{ or } \ L_2 < y_3 < L_3\},\\
&\Omega_{\rm e} = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : (y_1, y_2) \in \mathbb{T}^2, L_1 < y_3 < L_2\},
\end{aligned} \tag{2.8}$$

where $0 < L_1 < L_2 < L_3$ and $\mathbb{T}^2$ is the two-dimensional torus with the side 2π. Thus, the common boundary is expressed as

$$\Gamma_{\rm c} = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1, y_2, y_3) \in \Omega, y_3 = L_1 \ \text{ or } \ y_3 = L_2\},$$

while the outer boundary is represented by

$$\Gamma_{\rm f} = \{y \in \bar{\Omega} : y_3 = 0\} \cup \{y \in \bar{\Omega} : y_3 = L_3\}.$$

To close the system, we impose the homogeneous Dirichlet boundary condition

$$v = 0 \text{ on } \ [0, T] \times \Gamma_{\rm f} \tag{2.9}$$

on the outer boundary $\Gamma_{\rm f}$ and the periodic boundary conditions for w, ρ, and u on the lateral boundary, i.e.,

$$w(t, \cdot), \rho(t, \eta(t, \cdot)), u(t, \eta(t, \cdot)) \ \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions.} \tag{2.10}$$

Note that the inverse matrix of the flow map a satisfies the ODE system

$$a_t(t, x) = -a(t, x)\nabla v(t, x)a(t, x) \text{ in } [0, T] \times \Omega_{\rm f}, \tag{2.11}$$

$$a(0) = \mathbb{I}_3 \text{ in } \Omega_{\rm f}, \tag{2.12}$$

where $\mathbb{I}_3$ is the three-dimensional identity matrix, while the Jacobian satisfies the ODE system

$$\begin{aligned}
&J_t(t, x) = J(t, x)a_{kj}(t, x)\partial_k v_j(t, x) \text{ in } [0, T] \times \Omega_{\rm f},\\
&J(0) = 1 \text{ in } \Omega_{\rm f}.
\end{aligned} \tag{2.13}$$

The initial data of the system (2.3)–(2.5) is given as

$$\begin{aligned}
&(R, v, w, w_t)(0) = (R_0, v_0, w_0, w_1) \text{ in } \Omega_{\rm f} \times \Omega_{\rm f} \times \Omega_{\rm e} \times \Omega_{\rm e},\\
&(R_0, v_0, w_0, w_1) \ \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions,}
\end{aligned} \tag{2.14}$$

where $w_0 = 0$. For $T > 0$, we denote

$$H^{r,s}((0, T) \times \Omega_{\rm f}) = H^r((0, T), L^2(\Omega_{\rm f})) \cap L^2((0, T), H^s(\Omega_{\rm f})),$$

with the corresponding norm

$$\|f\|^2_{H^{r,s}((0,T)\times\Omega_{\rm f})} = \|f\|^2_{H^r((0,T),L^2(\Omega_{\rm f}))} + \|f\|^2_{L^2((0,T),H^s(\Omega_{\rm f}))},$$

where $r, s \geq 0$ are constant parameters. In Sects. 4–6, we shall work on a modified system with $T = 1$ to avoid issue with dependence of constants on small time. For simplicity of notation, we write

$$\|f\|_{H^{r,s}} = \|f\|_{H^{r,s}((0,1)\times\Omega_{\rm f})} \quad \textit{and} \quad \|f\|_{H^r_t H^s_x} = \|f\|_{H^r((0,1),H^s(\Omega_{\rm f}))}.$$

It is also convenient to abbreviate

$$K^s = H^{s/2,s},$$

where the domain of integration is $(0,1)\times\Omega_{\rm f}$ unless stated otherwise. Similarly, for the analogous space of functions defined on the boundary $\Gamma_{\rm c}$, we write

$$\|f\|_{H^r_t H^s_x(\Gamma_{\rm c})} = \|f\|_{H^r((0,1),H^s(\Gamma_{\rm c}))}$$

and abbreviate

$$K^s_{\Gamma_{\rm c}} = H^{s/2,s}(\Gamma_{\rm c}),$$

where the domain of integration is $(0,1)\times\Gamma_{\rm c}$ unless stated otherwise. We emphasize that the time domain of integration in the norms is $(0,1)$ when not indicated.

Our main result states the local-in-time existence of solution to the system (2.3)–(2.5) with the mixed boundary conditions (2.6)–(2.10) and the initial data (2.14).

Theorem 2.1. *Let $s \in (2, 2+\epsilon_0]$ for $\epsilon_0 \in (0, 1/2)$. Assume that $R_0 \in H^s(\Omega_f)$, $R_0^{-1} \in H^s(\Omega_f)$, $w_1 \in H^{s-1/2}(\Omega_e)$, $v_0 \in H^s(\Omega_f)$, $v_0|_{\Gamma_c} \in H^{s+1/2}(\Gamma_c)$, $\partial_3 v_0|_{\Gamma_f} \in H^{s-1/2}(\Gamma_f)$, and $w_0 = 0$, with the compatibility conditions*

$$\begin{aligned}
&w_{1j} = v_{0j} \ \textit{on}\ \Gamma_c,\\
&v_{0j} = 0 \ \textit{on}\ \Gamma_f,\\
&\lambda(\partial_k v_{0j} + \partial_j v_{0k})\nu^k + \mu\partial_i v_{0i}\nu^j - q(R_0^{-1})\nu^j = 0 \ \textit{on}\ \Gamma_c,\\
&\lambda\partial_k(\partial_k v_{0j} + \partial_j v_{0k}) + \mu\partial_j\partial_k v_{0k} - \partial_k(q(R_0^{-1})) = 0 \ \textit{on}\ \Gamma_f,
\end{aligned}$$

for $j = 1, 2, 3$. Then the system (2.3)–(2.5) with the coupling conditions (2.6)–(2.7), boundary conditions (2.9)–(2.10), and the initial data (2.14) admits a unique solution

$$\begin{aligned}
&v \in K^{s+1}((0,T)\times\Omega_f)\\
&R \in H^1((0,T), H^s(\Omega_f))\\
&w \in C([0,T], H^{s+1/4-\epsilon_0}(\Omega_e))\\
&w_t \in C([0,T], H^{s-3/4-\epsilon_0}(\Omega_e)),
\end{aligned}$$

for some constant $T > 0$, where the corresponding norms are bounded by a function of the norms of the initial data.

Remark 2.2. We assume $v_0 \in H^s(\Omega_{\rm f})$ for $s \in (2, 2+\epsilon_0]$ where $\epsilon_0 > 0$, since the elliptic regularity for $\|v\|_{L^2_t H^4_x}$ in (4.29) requires that $R^{-1} \in L^\infty((0,T), H^2(\Omega_{\rm f}))$. From the density equation (2.3), we deduce that the regularity for the initial velocity must be at least in $H^2(\Omega_{\rm f})$, showing the optimality of the range $s \geq 2$. It would be interesting to find whether the statement of the theorem holds for the borderline case $s = 2$. □

The proof of the theorem is given in Sect. 6 below. For simplicity, we present the proof for the pressure law $q(R) = R$, noting that the case for smooth function $q(R)$ follows completely analogously using the Sobolev and Hölder's inequalities and (5.69). See Remark 5.5 below for necessary modifications.

3. Space-Time Trace, Interpolation, and Hidden Regularity Inequalities

In this section, we provide several auxiliary results needed in the fixed point arguments. The first lemma provides an estimate for the trace in a space-time norm and is an essential ingredient when constructing solutions to the nonlinear parabolic-wave system in Sect. 5 below.

Lemma 3.1. *Let $r > 1/2$ and $\theta \geq 0$. If $u \in L^2((-\infty,\infty),H^r(\Omega_f)) \cap H^{2\theta r/(2r-1)}((-\infty,\infty),L^2(\Omega_f))$, then $u \in H^\theta((-\infty,\infty),L^2(\Gamma_c))$, and for all $\epsilon \in (0,1]$, we have the inequality*

$$\|u\|_{H^\theta((-\infty,\infty),L^2(\Gamma_c))} \leq \epsilon \|u\|_{H^{2\theta r/(2r-1)}((-\infty,\infty),L^2(\Omega_f))} + C\epsilon^{1-2r}\|u\|_{L^2((-\infty,\infty),H^r(\Omega_f))}, \tag{3.1}$$

where $C > 0$ is a constant.

The above lemma was proven in [23], where moreover, the interpolation spaces were identified. Since in this paper, we only use the inequality (3.1), which allows a simpler proof, we provide an elementary argument below.

First, however, we point out a consequence when restricting the above result to a finite time interval.

Corollary 3.2. *Let $r > 1/2$, $\theta \geq 0$, and $T > 0$. If $u \in L^2((0,T),H^r(\Omega_f)) \cap H^{2\theta r/(2r-1)}((0,T),L^2(\Omega_f))$, then $u \in H^\theta((0,T),L^2(\Gamma_c))$, and for all $\epsilon \in (0,1]$, we have the inequality*

$$\|u\|_{H^\theta((0,T),L^2(\Gamma_c))} \leq \epsilon \|u\|_{H^{2\theta r/(2r-1)}((0,T),L^2(\Omega_f))} + C\epsilon^{1-2r}\|u\|_{L^2((0,T),H^r(\Omega_f))}, \tag{3.2}$$

where $C > 0$ is a constant, which depends on Ω_f and T.

The inequality (3.2) follows from Lemma 3.1 using the Sobolev extension operator. Clearly, the constant is uniform as $T \to \infty$, but may increase to infinity as $T \to 0$.

Proof of Lemma 3.1. It is sufficient to prove (3.2) for $u \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$ with the trace taken on the set

$$\Gamma = \big\{(t,x_1,x_2,x_3) \in \mathbb{R}\times\mathbb{R}^3 : x_3 = 0\big\};$$

the general case is settled by the partition of unity and straightening of the boundary. Since it should be clear from the context, we usually do not distinguish in notation between a function and its trace. Denoting by $\hat{u}$ the Fourier transform of u with respect to (t,x_1,x_2,x_3), we have

$$\|u\|^2_{H^\theta((-\infty,\infty),L^2(\Gamma))} \lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+\tau^2)^\theta \left|\int_{-\infty}^{\infty}\hat{u}(\xi_1,\xi_2,\xi_3,\tau)\,d\xi_3\right|^2 d\tau\,d\xi_1\,d\xi_2.$$

Denote by

$$\gamma = \frac{2r-1}{2\theta} \tag{3.3}$$

the quotient between the exponents r and $2\theta r/(2r-1)$ in (3.2). Then, with $\lambda > 0$ to be determined below, we have

$$\begin{aligned}
\|u\|^2_{H^\theta((-\infty,\infty),L^2(\Gamma))} &\lesssim \int_{\mathbb{R}^3}(1+\tau^2)^\theta\left|\int_{-\infty}^{\infty}\hat{u}(\xi_1,\xi_2,\xi_3,\tau)\,d\xi_3\right|^2 d\tau\,d\xi_1\,d\xi_2\\
&\lesssim \int_{\mathbb{R}^3}(1+\tau^2)^\theta\left(\int_{-\infty}^{\infty}\frac{(1+(\xi_1^2+\xi_2^2)^\gamma+\epsilon^{-2}\xi_3^{2\gamma}+\tau^2)^{\lambda/2}}{(1+(\xi_1^2+\xi_2^2)^\gamma+\epsilon^{-2}\xi_3^{2\gamma}+\tau^2)^{\lambda/2}}|\hat{u}|\,d\xi_3\right)^2 d\tau\,d\xi_1\,d\xi_2\\
&\lesssim \int_{\mathbb{R}^3}(1+\tau^2)^\theta\left(\int_{-\infty}^{\infty}\big(1+(\xi_1^2+\xi_2^2)^\gamma+\epsilon^{-2}\xi_3^{2\gamma}+\tau^2\big)^\lambda|\hat{u}|^2\,d\xi_3\right)\\
&\quad\times\left(\int_{-\infty}^{\infty}\frac{d\xi_3}{(1+(\xi_1^2+\xi_2^2)^\gamma+\epsilon^{-2}\xi_3^{2\gamma}+\tau^2)^\lambda}\right)d\tau\,d\xi_1\,d\xi_2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in ξ_3. Using a substitution, we have

$$\int_{-\infty}^{\infty}\frac{dx}{(A^2+\epsilon^{-2}x^{2\gamma})^\lambda} \lesssim \epsilon^{1/\gamma}A^{1/\gamma-2\lambda}, \qquad A,\epsilon > 0, \tag{3.4}$$

provided λ satisfies $2\gamma\lambda > 1$, which is by (3.3) equivalent to

$$\lambda > \frac{\theta}{2r-1}. \tag{3.5}$$

Note that $2\gamma\lambda > 1$ implies $1/\gamma - 2\lambda < 0$ for the exponent of A in (3.4). Now we use (3.4) for the integral in ξ_3 with $A = (1 + (\xi_1^2 + \xi_2^2)^\gamma + \tau^2)^{1/2}$, while noting that

$$\begin{aligned}(1+\tau^2)^\theta A^{1/\gamma - 2\lambda} &= \frac{(1+\tau^2)^\theta}{(1+(\xi_1^2+\xi_2^2)^\gamma + \tau^2)^{\lambda - 1/2\gamma}} \leq (1+\tau^2)^{\theta-\lambda+1/2\gamma} \\ &\leq (1+(\xi_1^2+\xi_2^2)^\gamma + \epsilon^{-2}\xi_3^{2\gamma} + \tau^2)^{\theta-\lambda+1/2\gamma},\end{aligned}$$

provided $\lambda - 1/2\gamma \leq \theta$, i.e.,

$$\lambda \leq \frac{2r\theta}{2r-1}. \tag{3.6}$$

Under the condition (3.6), we thus obtain

$$\begin{aligned}\|u\|^2_{H^\theta((-\infty,\infty),L^2(\Gamma_c))} &\lesssim \epsilon^{1/\gamma} \int_{\mathbb{R}\times\mathbb{R}^3} \big(1 + (\xi_1^2+\xi_2^2)^\gamma + \epsilon^{-2}\xi_3^{2\gamma} + \tau^2\big)^{\theta+1/2\gamma} |\hat{u}|^2 \, d\xi_3 \, d\xi_2 \, d\xi_1 \, d\tau \\ &\lesssim \epsilon^{1/\gamma} \int_{\mathbb{R}\times\mathbb{R}^3} \big(1 + \epsilon^{-2}(\xi_1^2+\xi_2^2+\xi_3^2)^\gamma + \tau^2\big)^{\theta+1/2\gamma} |\hat{u}|^2 \, d\xi_3 \, d\xi_2 \, d\xi_1 \, d\tau \\ &\lesssim \epsilon^{-2\theta} \|u\|^2_{L^2((-\infty,\infty),H^{\gamma\theta+1/2}(\Omega_f))} + \epsilon^{1/\gamma}\|u\|^2_{H^{\theta+1/2\gamma}((-\infty,\infty),L^2(\Omega_f))},\end{aligned}$$

for all $\epsilon \in (0, 1]$. Using (3.3), we get

$$\|u\|^2_{H^\theta((-\infty,\infty),L^2(\Gamma_c))} \lesssim \epsilon^{-2\theta} \|u\|^2_{L^2((-\infty,\infty),H^r(\Omega_f))} + \epsilon^{2\theta/(2r-1)} \|u\|^2_{H^{2\theta r/(2r-1)}((-\infty,\infty),L^2(\Omega_f))}, \tag{3.7}$$

for all $\epsilon \in (0, 1]$. Finally, note that $\lambda = 2r\theta/(2r-1)$ satisfies (3.5)–(3.6) under the condition $r > 1/2$.

Optimizing $\epsilon \in (0, 1]$ in (3.7) by using

$$\epsilon = \left(\frac{\|u\|_{L^2((-\infty,\infty),H^r(\Omega_f))}}{\|u\|_{L^2((-\infty,\infty),H^r(\Omega_f))} + \|u\|_{H^{2\theta r/(2r-1)}((-\infty,\infty),L^2(\Omega_f))}}\right)^{(2r-1)/2r\theta},$$

we obtain a trace inequality

$$\|u\|_{H^\theta((-\infty,\infty),L^2(\Gamma_c))} \lesssim \|u\|^{1/(2r+1)}_{L^2((-\infty,\infty),H^r(\Omega_f))} \|u\|^{2r/(2r+1)}_{H^{2\theta r/(2r-1)}((-\infty,\infty),L^2(\Omega_f))} + \|u\|_{L^2((-\infty,\infty),H^r(\Omega_f))}, \tag{3.8}$$

which is a more explicit version of (3.1). Note that from (3.8), one may obtain an inequality on the interval $(0, T)$ with a T-dependent constant.

The second lemma provides a space-time interpolation inequality which is needed in several places in Sects. 5 and 6 below.

Lemma 3.3. *Let $\alpha, \beta > 0$. If $u \in H^\alpha((-\infty,\infty), L^2(\Omega_f)) \cap L^2((-\infty,\infty), H^\beta(\Omega_f))$, then we have that $u \in H^\theta((-\infty,\infty), H^\lambda(\Omega_f))$ for all $\theta \in (0, \alpha)$ and $\lambda \in (0, \beta)$ such that*

$$\frac{\theta}{\alpha} + \frac{\lambda}{\beta} \leq 1.$$

In addition, for all $\epsilon \in (0, 1]$, we have the inequality

$$\|u\|_{H^\theta((-\infty,\infty),H^\lambda(\Omega_f))} \leq \epsilon \|u\|_{H^\alpha((-\infty,\infty),L^2(\Omega_f))} + C\epsilon^{-\frac{\theta\beta}{\alpha\lambda}} \|u\|_{L^2((-\infty,\infty),H^\beta(\Omega_f))},$$

where $C > 0$ is a constant.

This statement immediately implies the following one regarding the same type of interpolation inequality on a finite time interval.

Corollary 3.4. *Let $\alpha, \beta > 0$ and $T > 0$. If $u \in H^{\alpha}((0,T), L^2(\Omega_f)) \cap L^2((0,T), H^{\beta}(\Omega_f))$, then we have that $u \in H^{\theta}((0,T), H^{\lambda}(\Omega_f))$ for all $\theta \in (0,\alpha)$ and $\lambda \in (0,\beta)$ such that*

$$\frac{\theta}{\alpha} + \frac{\lambda}{\beta} \leq 1.$$

In addition, for all $\epsilon \in (0,1]$, we have the inequality

$$\|u\|_{H^{\theta}((0,T),H^{\lambda}(\Omega_f))} \leq \epsilon \|u\|_{H^{\alpha}((0,T),L^2(\Omega_f))} + C\epsilon^{-\frac{\theta\beta}{\alpha\lambda}} \|u\|_{L^2((0,T),H^{\beta}(\Omega_f))},$$

where $C > 0$ is a constant depending on Ω_f and T.

As above, Corollary 3.4 follows by employing a Sobolev extension operator in the t variable.

Proof of Lemma 3.3. Using a partition of unity, straightening of the boundary, and a Sobolev extension, it is sufficient to prove the inequality in the case $\Omega_{\mathrm{f}} = \mathbb{R}^3$ and $u \in C_0^{\infty}(\mathbb{R} \times \mathbb{R}^3)$. Then, using Parseval's identity and the definition of the Sobolev norms, we only need to prove

$$(1 + |\tau|^{2\theta})(1 + |\xi|^{2\lambda}) \leq \epsilon(1 + |\tau|^{2\alpha}) + C\epsilon^{-\frac{\theta\beta}{\alpha\lambda}}(1 + |\xi|^{2\beta}), \tag{3.9}$$

for $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, where $\epsilon \in (0,1]$. Finally, (3.9) follows from the Young's inequality.

In the last part of this section, we address the regularity for the wave equation. We first recall the hidden regularity result for the wave equation

$$w_{tt} - \Delta w = 0 \text{ in } [0,T] \times \Omega_{\mathrm{e}}, \tag{3.10}$$

$$w = \psi \text{ on } [0,T] \times \Gamma_{\mathrm{c}}, \tag{3.11}$$

$$w \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{3.12}$$

and the initial data

$$(w, w_t)(0, \cdot) = (w_0, w_1) \tag{3.13}$$

(cf. [42]).

Lemma 3.5. [42] *Assume that $(w_0, w_1) \in H^{\beta}(\Omega_e) \times H^{\beta-1}(\Omega_e)$, where $\beta \geq 1$, and*

$$\psi \in C([0,T], H^{\beta-1/2}(\Gamma_c)) \cap H^{\beta,\beta}((0,T) \times \Gamma_c),$$

with the compatibility conditions $\psi|_{t=0} = w_0|_{\Gamma_c}$ and $\partial_t \psi|_{t=0} = w_1|_{\Gamma_c}$. Then there exists a solution $(w, w_t) \in C([0,T], H^{\beta}(\Omega_e) \times H^{\beta-1}(\Omega_e))$ of (3.10)–(3.13), which satisfies the estimate

$$\|w\|_{C([0,T],H^{\beta}(\Omega_e))} + \|w_t\|_{C([0,T],H^{\beta-1}(\Omega_e))} + \left\|\frac{\partial w}{\partial \nu}\right\|_{H^{\beta-1,\beta-1}((0,T)\times\Gamma_c)}$$
$$\lesssim \|w_0\|_{H^{\beta}(\Omega_e)} + \|w_1\|_{H^{\beta-1}(\Omega_e)} + \|\psi\|_{H^{\beta,\beta}((0,T)\times\Gamma_c)},$$

where the implicit constant depends on Ω_e and T.

In the final lemma of this section, we recall an essential trace regularity result for the wave equation from [50].

Lemma 3.6. [50] *Assume that $(w_0, w_1) \in H^{\beta}(\Omega_e) \times H^{\beta+1}(\Omega_e)$, where $0 < \beta < 5/2$, and*

$$\psi \in L^2((0,T), H^{\beta+2}(\Gamma_c)) \cap H^{\beta/2+1}((0,T), H^{\beta/2+1}(\Gamma_c)),$$

with the compatibility condition $\partial_t \psi|_{t=0} = w_1|_{\Gamma_c}$. Then there exists a solution w of (3.10)–(3.13) such that

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{L^2((0,T),H^{\beta+1}(\Gamma_c))} \lesssim \|w_0\|_{H^{\beta+2}(\Omega_e)} + \|w_1\|_{H^{\beta+1}(\Omega_e)} + \|\psi\|_{L^2((0,T),H^{\beta+2}(\Gamma_c))}$$
$$+ \|\psi\|_{H^{\beta/2+1}((0,T),H^{\beta/2+1}(\Gamma_c))},$$

where the implicit constant depends on Ω_e and T.

4. The Nonhomogeneous Parabolic Problem

In this section, we consider the parabolic problem

$$u_t - \lambda R \text{ div } (\nabla u + (\nabla u)^T) - \mu R \nabla \text{ div } u = f \text{ in } [0,1] \times \Omega_{\text{f}}, \tag{4.1}$$

with the nonhomogeneous boundary conditions and the initial data

$$\lambda(\partial_k u_j + \partial_j u_k)\nu^k + \mu \partial_k u_k \nu^j = h_j \text{ on } [0,1] \times \Gamma_{\text{c}}, \tag{4.2}$$
$$u = 0 \text{ on } [0,1] \times \Gamma_{\text{f}}, \tag{4.3}$$
$$u \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{4.4}$$
$$u(0) = u_0 \text{ in } \Omega_{\text{f}}, \tag{4.5}$$

for $j = 1, 2, 3$. To state the maximal regularity for (4.1)–(4.5), we consider the homogeneous version when (4.2)–(4.5) is replaced by

$$\lambda(\partial_k u_j + \partial_j u_k)\nu^k + \mu \partial_k u_k \nu^j = 0 \text{ on } [0,1] \times \Gamma_{\text{c}}, \tag{4.6}$$
$$u = 0 \text{ on } [0,1] \times \Gamma_{\text{f}}, \tag{4.7}$$
$$u \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{4.8}$$
$$u(0) = 0 \text{ in } \Omega_{\text{f}}, \tag{4.9}$$

for $j = 1, 2, 3$.

Lemma 4.1. *Assume that $f \in K^2((0,1) \times \Omega_f)$ with $f(0,\cdot) = 0$ on Ω_f and*

$$(R, R^{-1}) \in (L^\infty((0,1), H^2(\Omega_f)) \cap H^1((0,1), L^\infty(\Omega_f)))^2. \tag{4.10}$$

Then the parabolic problem (4.1) with the boundary conditions and the initial data (4.6)–(4.9) admits a solution u satisfying

$$\|u\|_{K^2((0,1)\times\Omega_f)} \lesssim \|f\|_{K^0((0,1)\times\Omega_f)} \tag{4.11}$$

and

$$\|u\|_{K^4((0,1)\times\Omega_f)} \lesssim \|f\|_{K^2((0,1)\times\Omega_f)}, \tag{4.12}$$

where the implicit constants depend on the norms of R and R^{-1} in (4.10).

Proof. Analogously to [43, Theorem 3.2], the parabolic problem (4.1) admits a solution $u \in K^2((0,1)\times\Omega_{\text{f}})$ if $f \in K^0((0,1) \times \Omega_{\text{f}})$ and $u \in K^4((0,1) \times \Omega_{\text{f}})$ if $f \in K^2((0,1) \times \Omega_{\text{f}})$. Below, the norm of dependence on time and space are understood as $(0,1)$ and Ω_{f}, unless stated otherwise. In the reminder of the proof we shall prove the regularity. Taking the L^2-inner product of (4.1) with u, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\text{f}}} |u|^2 - \lambda \int_{\Omega_{\text{f}}} R u_j \partial_k(\partial_k u_j + \partial_j u_k) - \mu \int_{\Omega_{\text{f}}} R u_j \partial_j \partial_k u_k = \int_{\Omega_{\text{f}}} f u. \tag{4.13}$$

For the second and third terms on the left side of (4.13), we integrate by parts with respect to ∂_k and ∂_j respectively to get

$$\begin{aligned} -\lambda \int_{\Omega_{\text{f}}} R u_j \partial_k(\partial_k u_j + \partial_j u_k) = \lambda \int_{\Gamma_{\text{c}}} R u_j(\partial_k u_j + \partial_j u_k)\nu^k + \lambda \int_{\Omega_{\text{f}}} R \partial_k u_j(\partial_k u_j + \partial_j u_k) \\ + \lambda \int_{\Omega_{\text{f}}} u_j \partial_k R(\partial_k u_j + \partial_j u_k) \end{aligned} \tag{4.14}$$

and

$$-\mu \int_{\Omega_{\text{f}}} R u_j \partial_j \partial_k u_k = \mu \int_{\Gamma_{\text{c}}} R u_j \partial_k u_k \nu^j + \mu \int_{\Omega_{\text{f}}} R \partial_j u_j \partial_k u_k + \mu \int_{\Omega_{\text{f}}} u_j \partial_j R \partial_k u_k. \tag{4.15}$$

Inserting (4.14)–(4.15) into (4.13) and appealing to (4.6)–(4.7), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\rm f}}|u|^2+\lambda\int_{\Omega_{\rm f}}R\partial_k u_j(\partial_k u_j+\partial_j u_k)+\mu\int_{\Omega_{\rm f}}R\partial_j u_j\partial_k u_k\\
&\quad=\int_{\Omega_{\rm f}}fu-\lambda\int_{\Omega_{\rm f}}u_j\partial_k R(\partial_k u_j+\partial_j u_k)-\mu\int_{\Omega_{\rm f}}u_j\partial_j R\partial_k u_k\\
&\quad\lesssim\|f\|_{L^2}^2+\|u\|_{L^2}^2+\|u\|_{L^4}\|\nabla R\|_{L^4}\|\nabla u\|_{L^2},
\end{aligned}\tag{4.16}$$

where the last inequality follows from Hölder's and Young's inequalities. Note that for any $v\in H^1(\Omega_{\rm f})$, using the Sobolev and Young's inequalities, we have

$$\|v\|_{L^4(\Omega_{\rm f})}\lesssim\|v\|_{H^1(\Omega_{\rm f})}^{3/4}\|v\|_{L^2(\Omega_{\rm f})}^{1/4}\lesssim\epsilon\|v\|_{H^1(\Omega_{\rm f})}+C_\epsilon\|v\|_{L^2(\Omega_{\rm f})},\tag{4.17}$$

for any $\epsilon\in(0,1]$, where $C_\epsilon>0$ denotes a constant depending on ϵ. We integrate (4.16) in time from 0 to t and use

$$(\partial_k u_j+\partial_j u_k)\partial_k u_j=\frac{1}{2}\sum_{j,k=1}^{3}(\partial_k u_j+\partial_j u_k)^2,\tag{4.18}$$

obtaining

$$\begin{aligned}
&\|u(t)\|_{L^2}^2+\sum_{j,k=1}^{3}\int_0^t\int_{\Omega_{\rm f}}R(\partial_k u_j+\partial_j u_k)^2+\int_0^t\int_{\Omega_{\rm f}}R|\partial_k u_k|^2\\
&\quad\lesssim\|f\|_{L_t^2L_x^2}^2+\int_0^t\|u\|_{L^2}^2+\epsilon\int_0^t\|u\|_{H^1}^2+C_\epsilon\int_0^t\|u\|_{L^2}\|u\|_{H^1}\\
&\quad\lesssim\|f\|_{L_t^2L_x^2}^2+(\epsilon+C_\epsilon\bar\epsilon)\int_0^t\|u\|_{H^1}^2+C_{\epsilon,\bar\epsilon}\int_0^t\|u\|_{L^2}^2,
\end{aligned}\tag{4.19}$$

for any $\epsilon,\bar\epsilon\in(0,1]$, where we used (4.17) and the Young's inequality. For the second term on the left, we use Korn's inequality, which reads

$$\int_0^t\|u\|_{H^1}^2\lesssim\sum_{j,k=1}^{3}\int_0^t\int_{\Omega_{\rm f}}R(\partial_k u_j+\partial_j u_k)^2+\int_0^t\|u\|_{L^2}^2.\tag{4.20}$$

From (4.19)–(4.20) it follows that

$$\|u(t)\|_{L^2}^2+\|u\|_{L_t^2H_x^1}^2\lesssim\|f\|_{L_t^2L_x^2}^2+\int_0^t\|u\|_{L^2}^2,\tag{4.21}$$

by choosing suitable $\epsilon,\bar\epsilon>0$. By Gronwall's inequality, we obtain

$$\|u(t)\|_{L^2}^2\lesssim\|f\|_{L_t^2L_x^2}^2,\tag{4.22}$$

where we used $e^{Ct}\lesssim1$ for $t\le1$, and then, after using (4.22) in (4.21), we arrive at

$$\|u\|_{L_t^2H_x^1}^2\lesssim\|f\|_{L_t^2L_x^2}^2.\tag{4.23}$$

Next, we take the L^2-inner product of (4.1) with u_t, obtaining

$$\int_{\Omega_{\rm f}}|u_t|^2-\lambda\int_{\Omega_{\rm f}}Ru_{tj}\partial_k(\partial_k u_j+\partial_j u_k)-\mu\int_{\Omega_{\rm f}}Ru_{tj}\partial_j\partial_k u_k=\int_{\Omega_{\rm f}}fu_t.\tag{4.24}$$

Then, proceeding as in (4.14)–(4.15), we get

$$\begin{aligned}
-\lambda\int_{\Omega_{\rm f}}Ru_{tj}\partial_k(\partial_k u_j+\partial_j u_k)&=\lambda\int_{\Gamma_{\rm c}}Ru_{tj}(\partial_k u_j+\partial_j u_k)\nu^k+\lambda\int_{\Omega_{\rm f}}R\partial_k u_{tj}(\partial_k u_j+\partial_j u_k)\\
&\quad+\lambda\int_{\Omega_{\rm f}}u_{tj}\partial_k R(\partial_k u_j+\partial_j u_k)
\end{aligned}\tag{4.25}$$

and

$$-\mu\int_{\Omega_{\rm f}} Ru_{tj}\partial_j\partial_k u_k = \mu\int_{\Gamma_{\rm c}} Ru_{tj}\partial_k u_k\nu^j + \mu\int_{\Omega_{\rm f}} R\partial_j u_{tj}\partial_k u_k + \mu\int_{\Omega_{\rm f}} u_{tj}\partial_j R\partial_k u_k. \tag{4.26}$$

Inserting (4.25)–(4.26) into (4.24) appealing to (4.6)–(4.7), and using

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\rm f}} R\partial_k u_j(\partial_k u_j+\partial_j u_k) = \frac{1}{2}\int_{\Omega_{\rm f}} R_t\partial_k u_j(\partial_k u_j+\partial_j u_k) + \int_{\Omega_{\rm f}} R\partial_k u_{tj}(\partial_k u_j+\partial_j u_k), \tag{4.27}$$

we arrive at

$$\begin{aligned}\int_{\Omega_{\rm f}}|u_t|^2 &+ \frac{\lambda}{2}\frac{d}{dt}\int_{\Omega_{\rm f}} R\partial_k u_j(\partial_k u_j+\partial_j u_k) + \frac{\mu}{2}\frac{d}{dt}\int_{\Omega_{\rm f}} R\partial_j u_j\partial_k u_k\\ &= \int_{\Omega_{\rm f}} fu_t + \lambda\int_{\Omega_{\rm f}} u_{tj}\partial_k R(\partial_k u_j+\partial_j u_k) + \mu\int_{\Omega_{\rm f}} u_{tj}\partial_j R\partial_k u_k\\ &\quad+\frac{\lambda}{2}\int_{\Omega_{\rm f}} R_t\partial_k u_j(\partial_k u_j+\partial_j u_k) + \frac{\mu}{2}\int_{\Omega_{\rm f}} R_t\partial_j u_j\partial_k u_k\\ &\lesssim C_\epsilon\|f\|_{L^2}^2 + \epsilon\|u_t\|_{L^2}^2 + \|\nabla R\|_{L^4}\|\nabla u\|_{L^4}\|u_t\|_{L^2} + \|R_t\|_{L^\infty}\|\nabla u\|_{L^2}^2,\end{aligned}$$

for any $\epsilon\in(0,1]$, where we used Hölder's and Young's inequalities. Integrating in time from 0 to t and using the Young, Sobolev, and Korn's inequalities with (4.17)–(4.18), we get

$$\begin{aligned}&\|u_t\|_{L_t^2L_x^2}^2 + \|u(t)\|_{H^1}^2\\ &\lesssim C_\epsilon\|f\|_{L_t^2L_x^2}^2 + \epsilon\|u_t\|_{L_t^2L_x^2}^2 + \|u(t)\|_{L^2}^2 + \int_0^t(\bar\epsilon\|u\|_{H^2} + C_{\bar\epsilon}\|u\|_{H^1})\|u_t\|_{L^2} + \int_0^t\|R_t\|_{L^\infty}\|u\|_{H^1}^2\\ &\lesssim C_\epsilon\|f\|_{L_t^2L_x^2}^2 + (\epsilon+\bar\epsilon+\tilde\epsilon C_{\bar\epsilon})\|u_t\|_{L_t^2L_x^2}^2 + \|u(t)\|_{L^2}^2 + \bar\epsilon\|u\|_{L_t^2H_x^2}^2 + C_{\bar\epsilon,\tilde\epsilon}\|u\|_{L_t^2H_x^1}^2\\ &\quad+\int_0^t\|R_t\|_{L^\infty}\|u\|_{H^1}^2,\end{aligned} \tag{4.28}$$

for any $\epsilon,\bar\epsilon,\tilde\epsilon\in(0,1]$, where we used $\|u(0)\|_{H^1}=0$ in the last inequality by (4.9). For the space regularity, note that u is the solution of the elliptic problem

$$-\lambda\ \mathrm{div}\ (\nabla u+(\nabla u)^T) - \mu\nabla\ \mathrm{div}\ u = -\frac{u_t}{R} + \frac{f}{R}\ \text{ in } [0,T]\times\Omega_{\rm f}, \tag{4.29}$$

with the boundary conditions

$$\lambda(\partial_k u_j+\partial_j u_k)\nu^k + \mu\partial_k u_k\nu^j = 0 \text{ on } [0,1]\times\Gamma_{\rm c}, \tag{4.30}$$

$$u = 0 \text{ on } [0,1]\times\Gamma_{\rm f}, \tag{4.31}$$

for $j=1,2,3$. From the elliptic regularity for (4.29)–(4.31) it follows that

$$\|u\|_{H^2}\lesssim\|R^{-1}u_t\|_{L^2} + \|R^{-1}f\|_{L^2}\lesssim\|u_t\|_{L^2} + \|f\|_{L^2}, \tag{4.32}$$

from where

$$\|u\|_{L_t^2H_x^2}\lesssim\|u_t\|_{L_t^2L_x^2} + \|f\|_{L_t^2L_x^2}. \tag{4.33}$$

Combining (4.22)–(4.23), (4.28), and (4.33), we obtain

$$\|u_t\|_{L_t^2L_x^2}^2 + \|u(t)\|_{H^1}^2\lesssim\|f\|_{L_t^2L_x^2}^2 + \int_0^t\|R_t\|_{L^\infty}\|u\|_{H^1}^2, \tag{4.34}$$

by taking suitable $\epsilon,\bar\epsilon,\tilde\epsilon>0$. Using Gronwall's inequality, we arrive at

$$\|u(t)\|_{H^1}^2\le C\|f\|_{L_t^2L_x^2}^2\exp\left(C\int_\tau^t\|R_t(\tau)\|_{L^\infty}\,d\tau\right)\le C\|f\|_{L_t^2L_x^2}^2,$$

and thus (4.34) implies

$$\|u\|_{H_t^1L_x^2}^2\lesssim\|f\|_{L_t^2L_x^2}^2, \tag{4.35}$$

where we used $e^{Ct} \lesssim 1$ for $t \leq 1$. From (4.33) and (4.35) it follows that

$$\|u\|_{K^2} \lesssim \|u\|_{L_t^2 H_x^2} + \|u\|_{H_t^1 L_x^2} \lesssim \|f\|_{L_t^2 L_x^2},$$

completing the proof of (4.11).

Differentiating (4.1) in time and taking the L^2-inner product with u_t, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\mathrm{f}}} |u_t|^2 - \lambda \int_{\Omega_{\mathrm{f}}} R u_{tj}\partial_k(\partial_k u_{tj} + \partial_j u_{tk}) - \mu \int_{\Omega_{\mathrm{f}}} R u_{tj}\partial_j\partial_k u_{tk} \\ = \int_{\Omega_{\mathrm{f}}} f_t u_t + \lambda \int_{\Omega_{\mathrm{f}}} R_t u_{tj}\partial_k(\partial_k u_j + \partial_j u_k) + \mu \int_{\Omega_{\mathrm{f}}} R_t u_{tj}\partial_j\partial_k u_k.\end{aligned} \tag{4.36}$$

We proceed as in (4.14)–(4.15) to obtain

$$\begin{aligned}-\lambda \int_{\Omega_{\mathrm{f}}} R u_{tj}\partial_k(\partial_k u_{tj} + \partial_j u_{tk}) = \lambda \int_{\Gamma_{\mathrm{c}}} R u_{tj}(\partial_k u_{tj} + \partial_j u_{tk})\nu^k + \lambda \int_{\Omega_{\mathrm{f}}} u_{tj}\partial_k R(\partial_k u_{tj} + \partial_j u_{tk}) \\ + \lambda \int_{\Omega_{\mathrm{f}}} R\partial_k u_{tj}(\partial_k u_{tj} + \partial_j u_{tk})\end{aligned} \tag{4.37}$$

and

$$-\mu \int_{\Omega_{\mathrm{f}}} R u_{tj}\partial_j\partial_k u_{tk} = \mu \int_{\Gamma_{\mathrm{c}}} R u_{tj}\partial_k u_{tk}\nu^j + \mu \int_{\Omega_{\mathrm{f}}} u_{tj}\partial_j R \partial_k u_{tk} + \mu \int_{\Omega_{\mathrm{f}}} R\partial_j u_{tj}\partial_k u_{tk}. \tag{4.38}$$

Inserting (4.37)–(4.38) into (4.36), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_{\mathrm{f}}} |u_t|^2 + \lambda \int_{\Omega_{\mathrm{f}}} R\partial_k u_{tj}(\partial_k u_{tj} + \partial_j u_{tk}) + \mu \int_{\Omega_{\mathrm{f}}} R\partial_j u_{tj}\partial_k u_{tk} \\ &\quad \lesssim \|f_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|R_t\|_{L^\infty}\|u_t\|_{L^2}\|u\|_{H^2} + \|u_t\|_{L^4}\|\nabla R\|_{L^4}\|u_t\|_{H^1},\end{aligned}$$

where we used Young's, Hölder's, and Sobolev inequalities. Integrating in time from 0 to t and using the Young's and Korn's inequalities and (4.17)–(4.18), we obtain

$$\begin{aligned}\|u_t(t)\|_{L^2}^2 + \|u_t\|_{L_t^2 H_x^1}^2 \lesssim \|f\|_{H_t^1 L_x^2}^2 + \int_0^t \|R_t\|_{L^\infty}\|u\|_{H^2}^2 + \int_0^t \|R_t\|_{L^\infty}\|u_t\|_{L^2}^2 + (\epsilon + \bar{\epsilon}C_\epsilon)\|u_t\|_{L_t^2 H_x^1}^2 \\ + C_{\epsilon,\bar{\epsilon}}\|u_t\|_{L_t^2 L_x^2}^2,\end{aligned}$$

for any $\epsilon, \bar{\epsilon} \in (0, 1]$, since $\|u_t(0)\|_{L^2} = \|f(0)\|_{L^2} = 0$. From (4.32) and (4.35) it follows that

$$\begin{aligned}\|u_t(t)\|_{L^2}^2 + \|u_t\|_{L_t^2 H_x^1}^2 &\lesssim \|f\|_{H_t^1 L_x^2}^2 + \|u_t\|_{L_t^2 L_x^2}^2 + \int_0^t \|R_t\|_{L^\infty}\|u_t\|_{L^2}^2 + \int_0^t \|R_t\|_{L^\infty}\|f\|_{L^2}^2 \\ &\lesssim \|f\|_{H_t^1 L_x^2}^2 + \|u_t\|_{L_t^2 L_x^2}^2 + \int_0^t \|R_t\|_{L^\infty}\|u_t\|_{L^2}^2,\end{aligned} \tag{4.39}$$

by taking appropriate $\epsilon, \bar{\epsilon} > 0$, where we also used

$$\|f\|_{L_t^\infty L_x^2} \lesssim \|f\|_{H_t^1 L_x^2} \tag{4.40}$$

in the last inequality. Appealing to Gronwall's inequality, (4.39) implies

$$\|u_t(t)\|_{L^2}^2 \lesssim \|f\|_{K^2}^2, \tag{4.41}$$

and then, after using (4.41) in (4.39), we arrive at

$$\|u_t\|_{L_t^2 H_x^1}^2 \lesssim \|f\|_{K^2}^2.$$

Differentiating (4.1) in time and taking the L^2-inner product with u_{tt}, we obtain

$$\begin{aligned}&\int_{\Omega_{\rm f}} |u_{tt}|^2 + \frac{\lambda}{2}\frac{d}{dt}\int_{\Omega_{\rm f}} R\partial_k u_{tj}(\partial_k u_{tj} + \partial_j u_{tk}) + \frac{\mu}{2}\frac{d}{dt}\int_{\Omega_{\rm f}} R\partial_j u_{tj}\partial_k u_{tk}\\ &= \int_{\Omega_{\rm f}} f_t u_{tt} - \lambda\int_{\Omega_{\rm f}} u_{ttj}\partial_k R(\partial_k u_{tj} + \partial_j u_{tk}) - \mu\int_{\Omega_{\rm f}} u_{ttj}\partial_j R\partial_k u_{tk}\\ &\quad + \frac{\lambda}{2}\int_{\Omega_{\rm f}} R_t\partial_k u_{tj}(\partial_k u_{tj} + \partial_j u_{tk}) + \frac{\mu}{2}\int_{\Omega_{\rm f}} R_t\partial_j u_{tj}\partial_k u_{tk} + \lambda\int_{\Omega_{\rm f}} u_{ttj}R_t\partial_k(\partial_k u_j + \partial_j u_k)\\ &\quad + \mu\int_{\Omega_{\rm f}} u_{ttj}R_t\partial_{jk}u_k,\end{aligned}$$

where we integrated by parts in spatial variables. We proceed as in (4.36)–(4.39) to get

$$\begin{aligned}\|u_{tt}\|^2_{L^2_tL^2_x} + \|u_t(t)\|^2_{H^1} &\lesssim C_{\bar\epsilon}\|f\|^2_{H^1_tL^2_x} + \|u_t(t)\|^2_{L^2} + \epsilon\int_0^t \|u_t\|_{H^2}\|u_{tt}\|_{L^2} + C_{\tilde\epsilon}\int_0^t \|R_t\|^2_{L^\infty}\|f\|^2_{L^2}\\ &\quad + (\bar\epsilon C_\epsilon + \tilde\epsilon + \tilde\epsilon)\|u_{tt}\|^2_{L^2_tL^2_x} + C_{\epsilon,\bar\epsilon}\|u_t\|^2_{L^2_tH^1_x} + C_{\tilde\epsilon}\int_0^t (1 + \|R_t\|^2_{L^\infty})\|u_t\|^2_{H^1},\end{aligned} \tag{4.42}$$

for any $\epsilon, \bar\epsilon, \tilde\epsilon \in (0, 1]$, where we used the Young's, Hölder, Sobolev, and Korn's inequalities. Note that u_t is the solution of the elliptic problem

$$-\lambda \,\mathrm{div}\,(\nabla u_t + (\nabla u_t)^T) - \mu\nabla\,\mathrm{div}\,u_t = -R^{-1}u_{tt} + R^{-2}u_tR_t + R^{-1}f_t - R^{-2}R_tf \text{ in } [0, 1]\times\Omega_{\rm f},$$

with the boundary conditions

$$\begin{aligned}&\lambda(\partial_k u_{tj} + \partial_j u_{tk})\nu^k + \mu\partial_k u_{tk}\nu^j = 0 \text{ in } [0, 1]\times\Gamma_{\rm c},\\ &u_{tj} = 0 \text{ in } [0, 1]\times\Gamma_{\rm f},\end{aligned} \tag{4.43}$$

for $j = 1, 2, 3$. The elliptic regularity implies that

$$\begin{aligned}\|u_t\|_{H^2} &\lesssim \|u_{tt}\|_{L^2} + \|u_tR_t\|_{L^2} + \|f_t\|_{L^2} + \|R_tf\|_{L^2}\\ &\lesssim \|u_{tt}\|_{L^2} + \|u_t\|_{L^2}\|R_t\|_{L^\infty} + \|f_t\|_{L^2} + \|R_t\|_{L^\infty}\|f\|_{L^2},\end{aligned} \tag{4.44}$$

where we used Hölder's inequality. From (4.41)–(4.44), we obtain

$$\begin{aligned}\|u_{tt}\|^2_{L^2_tL^2_x} + \|u_t(t)\|^2_{H^1} &\lesssim \|f\|^2_{K^2} + \|u_t(t)\|^2_{L^2} + \int_0^t (1 + \|R_t\|^2_{L^\infty})\|u_t\|^2_{H^1} + \int_0^t \|R_t\|^2_{L^\infty}\|f\|^2_{L^2}\\ &\lesssim \|f\|^2_{K^2} + \int_0^t (1 + \|R_t\|^2_{L^\infty})\|u_t\|^2_{H^1},\end{aligned}$$

by taking $\epsilon, \bar\epsilon, \tilde\epsilon > 0$ sufficiently small, where we used (4.40). Appealing to Gronwall's inequality, we arrive at

$$\|u_t(t)\|^2_{H^1} \lesssim \|f\|^2_{K^2},$$

whence

$$\|u_{tt}\|^2_{L^2_tL^2_x} \lesssim \|f\|^2_{K^2}. \tag{4.45}$$

From the H^4 regularity of the elliptic problem (4.29)–(4.31) and (4.44) it follows that

$$\begin{aligned}\|u\|_{H^4} &\lesssim \|R^{-1}u_t\|_{H^2} + \|R^{-1}f\|_{H^2}\\ &\lesssim \|u_{tt}\|_{L^2} + \|R_t\|_{L^\infty}\|u_t\|_{L^2} + \|R_t\|_{L^\infty}\|f\|_{L^2} + \|f_t\|_{L^2} + \|f\|_{H^2},\end{aligned} \tag{4.46}$$

since H^2 is an algebra. We combine (4.41) and (4.45)–(4.46) to get

$$\begin{aligned}\|u\|_{K^4} &= \|u\|_{L^2_tH^4_x} + \|u\|_{H^2_tL^2_x}\\ &\lesssim \|u_{tt}\|_{L^2_tL^2_x} + \|R_t\|_{L^2_tL^\infty_x}\|u_t\|_{L^\infty_tL^2_x} + \|R_t\|_{L^2_tL^\infty_x}\|f\|_{H^1_tL^2_x} + \|f\|_{K^2} \lesssim \|f\|_{K^2},\end{aligned}$$

completing the proof of (4.12). □

The following lemma provides a maximal regularity for the parabolic system (4.1)–(4.5).

Lemma 4.2. *Let $s \in (2, 2+\epsilon_0]$, where $\epsilon_0 \in (0, 1/2)$ is arbitrary. Assume the compatibility conditions*

$$h_j(0) = \lambda(\partial_k u_{0j} + \partial_j u_{0k})\nu^k + \mu \partial_k u_{0k} \nu^j \text{ on } \Gamma_c, \tag{4.47}$$

$$u_{0j} = 0 \text{ on } \Gamma_f, \tag{4.48}$$

for $j = 1, 2, 3$. Suppose that

$$(R, R^{-1}) \in (L^\infty((0,1), H^2(\Omega_f)) \cap H^1((0,1), L^\infty(\Omega_f)))^2 \tag{4.49}$$

and

$$(u_0|_{\Gamma_c}, \partial_3 u_0|_{\Gamma_f}) \in H^{s+1/2}(\Gamma_c) \times H^{s-1/2}(\Gamma_f) \tag{4.50}$$

with the nonhomogeneous terms satisfying

$$(h, f, f(0)) \in K^{s-1/2}((0,1) \times \Gamma_c) \times K^{s-1}((0,1) \times \Omega_f) \times H^{s-2}(\Omega_f). \tag{4.51}$$

Then the system (4.1)–(4.5) admits a solution u satisfying

$$\begin{aligned} \|u\|_{K^{s+1}((0,1)\times\Omega_f)} \lesssim \|h\|_{K^{s-1/2}_{\Gamma_c}} + \|u_0|_{\Gamma_c}\|_{H^{s+1/2}(\Gamma_c)} + \|\partial_3 u_0|_{\Gamma_f}\|_{H^{s-1/2}(\Gamma_f)} \\ + \|u_0\|_{H^s} + \|f\|_{K^{s-1}((0,1)\times\Omega_f)} + \|f(0)\|_{H^{s-2}(\Omega_f)}, \end{aligned} \tag{4.52}$$

where the implicit constant depends on the norms of R and R^{-1} in (4.49).

Proof. In order to apply a lifting result in [43], we consider the boundary conditions

$$v = u_0|_{\Gamma_c} \text{ on } [0,1] \times \Gamma_c, \tag{4.53}$$

$$\lambda(\partial_k v_j + \partial_j v_k)\nu^k + \mu \partial_k v_k \nu^j = h_j \text{ on } [0,1] \times \Gamma_c, \tag{4.54}$$

$$\partial_k \partial_m v_j \nu^k \nu^m = 0 \text{ on } [0,1] \times \Gamma_c, \tag{4.55}$$

$$v = 0 \text{ on } [0,1] \times \Gamma_f, \tag{4.56}$$

$$\partial_k v_j \nu^k = \partial_k u_{0j} \nu^k \text{ on } [0,1] \times \Gamma_f, \tag{4.57}$$

$$\partial_m \partial_k v_j \nu^k \nu^m = 0 \text{ on } [0,1] \times \Gamma_f, \tag{4.58}$$

$$v \ \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{4.59}$$

for $j = 1, 2, 3$, and the initial data

$$v(0) = u_0 \text{ in } \Omega_f, \tag{4.60}$$

$$\partial_t v(0) = \lambda R_0 \text{ div } (\nabla u_0 + (\nabla u_0)^T) + \mu R_0 \nabla \text{ div } u_0 + f(0) \text{ in } \Omega_f. \tag{4.61}$$

Below, the norm of dependence on time and space are understood as $(0,1)$ and Ω_f, unless stated otherwise. From [43, Theorem 2.3] and the compatibility conditions (4.47)–(4.48) and since $s > 1/2$ it follows that there exists $v \in K^{s+1}((0,1) \times \Omega_f)$ satisfying the boundary conditions and initial conditions (4.53)–(4.61) with

$$\|v\|_{K^{s+1}} \lesssim \|h\|_{K^{s-1/2}_{\Gamma_c}} + \|u_0|_{\Gamma_c}\|_{K^{s+1/2}_{\Gamma_c}} + \left\| \frac{\partial u_0}{\partial \nu} \right\|_{K^{s-1/2}_{\Gamma_f}} + \|u_0\|_{H^s} + \|R_0 D^2 u_0\|_{H^{s-2}} + \|f(0)\|_{H^{s-2}},$$

from where

$$\|v\|_{K^{s+1}} \lesssim \|h\|_{K^{s-1/2}_{\Gamma_c}} + \|u_0|_{\Gamma_c}\|_{H^{s+1/2}(\Gamma_c)} + \|\partial_3 u_0|_{\Gamma_f}\|_{H^{s-1/2}(\Gamma_f)} + \|u_0\|_{H^s} + \|f(0)\|_{H^{s-2}}. \tag{4.62}$$

Now we consider the homogeneous parabolic problem

$$w_t - \lambda R \text{ div } (\nabla w + (\nabla w)^T) - \mu R \nabla \text{ div } w = F \text{ in } [0,1] \times \Omega_f, \tag{4.63}$$

with the homogeneous boundary conditions and the initial data

$$\lambda(\partial_k w_j + \partial_j w_k)\nu^k + \mu \partial_k w_k \nu^j = 0 \text{ on } [0,1] \times \Gamma_c, \tag{4.64}$$

$$w = 0 \text{ on } [0,1]\times\Gamma_{\mathrm{f}}, \tag{4.65}$$
$$w \ \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{4.66}$$
$$w(0,\cdot) = 0 \text{ in } \Omega_{\mathrm{f}}, \tag{4.67}$$

for $j = 1,2,3$, where

$$F = v_t - f - \lambda R \text{ div } (\nabla v + (\nabla v)^T) - \mu R \nabla \text{ div } v \text{ in } [0,1]\times\Omega_{\mathrm{f}}. \tag{4.68}$$

Note that (4.61) implies that

$$F(0,\cdot) = 0 \text{ in } \Omega_{\mathrm{f}}. \tag{4.69}$$

By (4.49), (4.69), and Lemma 4.1, there exists a solution w to the system (4.63)–(4.68) satisfying

$$\|w\|_{K^2} \lesssim \|F\|_{K^0} \tag{4.70}$$

and

$$\|w\|_{K^4} \lesssim \|F\|_{K^2}, \tag{4.71}$$

where the implicit constants depend on the norms of R and R^{-1} in (4.49). From [43, Theorem 6.2] and (4.70)–(4.71) it follows that

$$\|w\|_{K^{s+1}} \lesssim \|F\|_{K^{s-1}}, \tag{4.72}$$

since $s \notin 1/2 + \mathbb{Z}$ and $s/2 \notin \mathbb{Z}$. From (4.68), we get

$$\|F\|_{K^{s-1}} \lesssim \|f\|_{K^{s-1}} + \|v_t\|_{K^{s-1}} + \|RD_x^2 v\|_{K^{s-1}}. \tag{4.73}$$

For the second term on the right side of (4.73), we obtain

$$\|v_t\|_{K^{s-1}} \lesssim \|v_t\|_{L_t^2 H_x^{s-1}} + \|v_t\|_{H_t^{(s-1)/2} L_x^2} \lesssim \|v\|_{K^{s+1}},$$

where we used Corollary 3.4. To treat the last term on the right side of (4.73), we claim that

$$\|AB\|_{H_t^{(s-1)/2} L_x^2} \lesssim \|A\|_{H_t^1 L_x^\infty} \|B\|_{H_t^{(s-1)/2} L_x^2} + \|A\|_{L_t^\infty L_x^\infty} \|B\|_{H_t^{(s-1)/2} L_x^2} \tag{4.74}$$

on the domain $(0,1)\times\Omega_{\mathrm{f}}$. Using extensions, we may assume that the domain is actually $\mathbb{R}\times\mathbb{R}^3$. From the Hölder inequality it follows that

$$\begin{aligned}\|AB\|_{H_t^{(s-1)/2} L_x^2} &\lesssim \|A\|_{W_t^{(s-1)/2,4} L_x^\infty} \|B\|_{L_t^4 L_x^2} + \|A\|_{L_t^\infty L_x^\infty} \|B\|_{H_t^{(s-1)/2} L_x^2}. \\ &\lesssim \|A\|_{W_t^{3/4,4} L_x^\infty} \|B\|_{L_t^4 L_x^2} + \|A\|_{L_t^\infty L_x^\infty} \|B\|_{H_t^{(s-1)/2} L_x^2}.\end{aligned}$$

since $2 < s < 5/2$. The claim (4.74) is thus completed by appealing to the Sobolev inequality. For the last term on the right side of (4.73), we use the Hölder's inequality, yielding

$$\|RD_x^2 v\|_{L_t^2 H_x^{s-1}} \lesssim \|R\|_{L_t^\infty H_x^2} \|D_x^2 v\|_{L_t^2 H_x^{s-1}} \lesssim \|v\|_{K^{s+1}}$$

and

$$\|RD_x^2 v\|_{H_t^{(s-1)/2} L_x^2} \lesssim \|R\|_{H_t^1 L_x^\infty} \|D_x^2 v\|_{H_t^{(s-1)/2} L_x^2} + \|R\|_{L_t^\infty L_x^\infty} \|D_x^2 v\|_{H_t^{(s-1)/2} L_x^2} \lesssim \|v\|_{K^{s+1}}, \tag{4.75}$$

where we appealed to (4.74) and Corollary 3.4. Note that from (4.63)–(4.68), we infer that the difference $u = v - w$ is a solution of the system (4.1)–(4.5). From (4.62), (4.72)–(4.73), and (4.75) it follows that

$$\begin{aligned}\|u\|_{K^{s+1}} &\lesssim \|w\|_{K^{s+1}} + \|v\|_{K^{s+1}} \\ &\lesssim \|h\|_{K_{\Gamma_c}^{s-1/2}} + \|u_0|_{\Gamma_c}\|_{H^{s+1/2}(\Gamma_c)} + \|\partial_3 u_0|_{\Gamma_{\mathrm{f}}}\|_{H^{s-1/2}(\Gamma_{\mathrm{f}})} + \|u_0\|_{H^s} + \|f\|_{K^{s-1}} + \|f(0)\|_{H^{s-2}},\end{aligned}$$

concluding the proof of (4.52). □

5. Solution to a Parabolic-Wave System

In this section, we consider the coupled parabolic-wave system

$$v_t - \lambda R \operatorname{div} (\nabla v + (\nabla v)^T) - \mu R \nabla \operatorname{div} v + R\nabla(R^{-1}) = f \text{ in } [0,T]\times\Omega_{\mathrm{f}}, \tag{5.1}$$

$$R_t - R \operatorname{div} v = 0 \text{ in } [0,T]\times\Omega_{\mathrm{f}}, \tag{5.2}$$

$$w_{tt} - \Delta w = 0 \text{ in } [0,T]\times\Omega_{\mathrm{e}}, \tag{5.3}$$

with the boundary conditions

$$v = w_t \text{ on } [0,T]\times\Gamma_{\mathrm{c}}, \tag{5.4}$$

$$\lambda(\partial_k v_j + \partial_j v_k)\nu^k + \mu\partial_k v_k \nu^j = \partial_k w_j \nu^k + R^{-1}\nu^j + h_j \text{ on } [0,T]\times\Gamma_{\mathrm{c}}, \tag{5.5}$$

$$v, w \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{5.6}$$

$$v = 0 \text{ on } [0,T]\times\Gamma_{\mathrm{f}}, \tag{5.7}$$

for $j = 1,2,3$, and the initial data

$$\begin{aligned} &(v,R,w,w_t)(0) = (v_0,R_0,w_0,w_1) \text{ in } \Omega_{\mathrm{f}} \times \Omega_{\mathrm{f}} \times \Omega_{\mathrm{e}} \times \Omega_{\mathrm{e}}, \\ &(v_0,R_0,w_0,w_1) \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \\ &w_0 = 0. \end{aligned} \tag{5.8}$$

In order to avoid issues of dependence of constants for small time, we introduce a cutoff function in time and work on the unit time interval $(0,1)$. Let $\tilde{T}\in(0,1/4)$, and let $\phi_{\tilde{T}}(t)$ be a smooth cutoff function valued in $[0,1]$ such that

$$\phi_{\tilde{T}}(t) = \begin{cases} 1 \text{ on } [0,\tilde{T}], \\ 0 \text{ on } [2\tilde{T},1], \end{cases} \tag{5.9}$$

and $\|\phi'_{\tilde{T}}\|_{L^\infty(0,1)} \lesssim 1/\tilde{T}$. The following lemma provides a necessary estimate for the cutoff function.

Lemma 5.1. *We have* $\|\phi_{\tilde{T}}\|_{H_t^{(s-2)/2}} \lesssim 1$.

Proof of Lemma 5.1. By the Sobolev interpolation inequality, we have

$$\|\phi_{\tilde{T}}\|_{H_t^{(s-2)/2}} \lesssim \|\phi_{\tilde{T}}\|_{H_t^1}^{(s-2)/2} \|\phi_{\tilde{T}}\|_{L_t^2}^{(4-s)/2} \lesssim (1+\tilde{T}^{-1/2})^{(s-2)/2}\tilde{T}^{(4-s)/4} \lesssim \tilde{T}^{(3-s)/2} \lesssim 1,$$

since $s<3$.

To obtain the existence of solutions and avoid issues with the dependence of constants for small time, we replace (5.1)–(5.3) and (5.4)–(5.7) with

$$v_t - \lambda R \operatorname{div} (\nabla v + (\nabla v)^T) - \mu R \nabla \operatorname{div} v + R\nabla(R^{-1}) = f \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \tag{5.10}$$

$$R_t - \phi_{\tilde{T}} R \operatorname{div} v = 0 \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \tag{5.11}$$

$$w_{tt} - \Delta w = 0 \text{ in } [0,1]\times\Omega_{\mathrm{e}}, \tag{5.12}$$

with the boundary conditions

$$w(t,x) = \int_0^t \phi_{\tilde{T}}(\tau)v(\tau,x)\,d\tau + \left(t - \int_0^t \phi_{\tilde{T}}(\tau)\,d\tau\right) v_0(x) \text{ on } [0,1]\times\Gamma_{\mathrm{c}}, \tag{5.13}$$

$$\lambda(\partial_k v_j + \partial_j v_k)\nu^k + \mu\partial_k v_k \nu^j = \partial_k w_j \nu^k + R^{-1}\nu^j + h_j \text{ on } [0,1]\times\Gamma_{\mathrm{c}}, \tag{5.14}$$

$$v, w \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{5.15}$$

$$v = 0 \text{ on } [0,1]\times\Gamma_{\mathrm{f}}, \tag{5.16}$$

for $j=1,2,3$, where $\phi_{\tilde{T}}(t)$ is as in (5.9). Note that from (5.13) it follows that

$$w_t(t,x) = \phi_{\tilde{T}}(t)(v(t,x) - v_0(x)) + v_0(x) \text{ on } [0,1]\times\Gamma_{\mathrm{c}}, \tag{5.17}$$

and thus the boundary condition (5.17) agrees with (5.4) on the time interval $[0,\tilde{T}]$, and the solutions of (5.10)–(5.16) agree with the solution of (5.1)–(5.7) on the time interval $[0,\tilde{T}]$, with the same initial and boundary conditions (5.8).

To provide the maximal regularity for the system (5.10)–(5.16), we state the following necessary a priori density estimates.

Lemma 5.2. *Let $s \in (2, 2+\epsilon_0]$, where $\epsilon_0 \in (0, 1/2)$ is arbitrary. Consider the ODE system*

$$R_t - R\phi_{\tilde{T}} \text{ div } v = 0 \text{ in } [0,1] \times \Omega_f, \tag{5.18}$$

$$R(0) = R_0 \text{ on } \Omega_f. \tag{5.19}$$

Assume that $(R_0, R_0^{-1}, v_0) \in H^s(\Omega_f) \times H^s(\Omega_f) \times H^s(\Omega_f)$ and $\|v\|_{K^{s+1}((0,1)\times\Omega_f)} \le M$, where $M \ge 1$. Let $\delta \in (0, 1/2)$. Then for a sufficiently small constant $\tilde{T} > 0$, depending on M and δ, we have

(i) $\|R\|_{L_t^\infty L_x^\infty} + \|R^{-1}\|_{L_t^\infty L_x^\infty} + \|R\|_{L_t^\infty H_x^s} + \|R^{-1}\|_{L_t^\infty H_x^s} \lesssim 1$,
(ii) $\|R^{-1}\|_{H_t^1 H_x^{3/2+\delta}} + \|R\|_{H_t^1 H_x^{3/2+\delta}} \lesssim 1$,
(iii) $\|R\|_{H_t^1 H_x^s} \lesssim M$,

where the norm of dependence is $(0,1) \times \Omega_f$.

We emphasize that the implicit constants in the above inequalities (i)–(iii) are independent of M and δ.

Proof of Lemma 5.2. (i) The solution of the ODE system (5.18)–(5.19) reads

$$R(t,x) = R_0(x) e^{\int_0^t \phi_{\tilde{T}}(\tau) \text{ div } v(\tau)\, d\tau} \text{ in } [0,1] \times \Omega_{\rm f}. \tag{5.20}$$

Let $\tilde{T} \in (0,T]$ be a small time to be determined below. From Hölder's and Sobolev inequalities it follows that

$$\|R\|_{L_t^\infty L_x^\infty} \lesssim \|R_0\|_{H^s} e^{\int_0^{2\tilde{T}} \|\phi_{\tilde{T}}(\tau) \text{ div } v(\tau)\|_{L^\infty}\, d\tau} \lesssim C^{\tilde{T}^{1/2} M} \lesssim 1$$

and

$$\|R^{-1}\|_{L_t^\infty L_x^\infty} \lesssim \|R_0^{-1}\|_{H^s} e^{\int_0^{2\tilde{T}} \|\phi_{\tilde{T}}(\tau) \text{ div } v(\tau)\|_{L^\infty}\, d\tau} \lesssim C^{\tilde{T}^{1/2} M} \lesssim 1,$$

for some sufficiently small $\tilde{T} > 0$. Similarly, we have

$$\|R\|_{L_t^\infty H_x^s} \lesssim \|R_0\|_{H^s} \|e^{\int_0^{2\tilde{T}} \phi_{\tilde{T}}(\tau) \text{ div } v(\tau)\, d\tau}\|_{L_t^\infty H_x^s} \lesssim 1$$

and

$$\|R^{-1}\|_{L_t^\infty H_x^s} \lesssim \|R_0^{-1}\|_{H^s} \|e^{\int_0^{2\tilde{T}} \phi_{\tilde{T}}(\tau) \text{ div } v(\tau)\, d\tau}\|_{L_t^\infty H_x^s} \lesssim 1.$$

(ii) From (5.18), we use Hölder's and the Sobolev inequalities to get

$$\|(R^{-1})_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim \|R^{-2} R_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim \|\text{ div } v\|_{L^2((0,2\tilde{T}), H^{3/2+\delta}(\Omega_{\rm f}))} \lesssim \|v\|_{L^2((0,2\tilde{T}), H^{5/2+\delta}(\Omega_{\rm f}))}. \tag{5.21}$$

Recall that for any $0 < r < r'$ and $f \in H^{r'}$, we have the Sobolev interpolation inequality

$$\|f\|_{H^r} \lesssim \epsilon \|f\|_{H^{r'}} + \epsilon^{r/(r-r')} \|f\|_{L^2}, \tag{5.22}$$

for any $\epsilon \in (0,1]$. From (5.21)–(5.22) it follows that

$$\begin{aligned} \|(R^{-1})_t\|_{L_t^2 H_x^{3/2+\delta}} &\lesssim \epsilon \|v\|_{L^2((0,2\tilde{T}), H^{s+1}(\Omega_{\rm f}))} + C_\epsilon \|v\|_{L^2((0,2\tilde{T}), L^2(\Omega_{\rm f}))} \lesssim \epsilon M + C_\epsilon \tilde{T}^{1/2} \|v\|_{L_t^\infty L_x^2} \\ &\lesssim \epsilon M + C_\epsilon \tilde{T}^{1/2} \|v\|_{H_t^{(s+1)/2} L_x^2} \lesssim (\epsilon + C_\epsilon \tilde{T}^{1/2}) M, \end{aligned} \tag{5.23}$$

since $s > 2$. Taking $\epsilon = 1/M$ in (5.23), we arrive at

$$\|(R^{-1})_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim 1,$$

for some sufficiently small $\tilde{T} > 0$. Similarly, we have

$$\|R_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim 1.$$

Thus, we conclude the proof of (ii) by combining (i).

(iii) From (5.18) and Hölder's inequality it follows that

$$\|R_t\|_{L_t^2 H_x^s} \lesssim \|R\phi_{\tilde{T}}\|_{L_t^\infty H_x^s}\| \operatorname{div} v\|_{L_t^2 H_x^s} \lesssim \|v\|_{L_t^2 H_x^{s+1}} \lesssim M.$$

Therefore, we conclude the proof of (iii).

The following lemma provides necessary estimates for the structure displacement and velocity on the boundary.

Lemma 5.3. *Let $s \in (2, 2+\epsilon_0]$, where $\epsilon_0 \in (0, 1/2)$ is arbitrary. Assume that $\|v\|_{K^{s+1}((0,1)\times\Omega_f)} \leq M$ for some $M \geq 1$. Suppose that v and w satisfy (5.13) and (5.15) with the initial data satisfying $(v_0, w_0, w_1) \in H^s(\Omega_f) \times H^{s+1/2}(\Omega_e) \times H^{s-1/2}(\Omega_e)$ and $v_0|_{\Gamma_c} \in H^{s+1/2}(\Gamma_c)$. Then we have*

(i) $\|w\|_{L_t^2 H_x^{s+1/2}(\Gamma_c)} \lesssim \tilde{T}^{1/2} M + 1,$

(ii) $\|w_t\|_{H_t^{s/2-3/4} H_x^{s/2+1/4}(\Gamma_c)} + \|w\|_{H_t^{s/2-3/4} H_x^{s/2+1/4}(\Gamma_c)} \lesssim (\epsilon + \tilde{\epsilon} C_\epsilon + C_{\tilde{\epsilon},\epsilon} \tilde{T}^{1/2}) M + C_\epsilon,$

(iii) $\|w\|_{H_t^{s/2+3/4} L_x^2(\Gamma_c)} \lesssim (\epsilon + \tilde{\epsilon} C_\epsilon + C_{\epsilon,\tilde{\epsilon}} \tilde{T}^{1/2}) M + C_\epsilon,$

for any $\epsilon, \tilde{\epsilon} \in (0, 1]$, where the implicit constants depend on the initial data.

Here and below, when not indicated, the time and space domains are understood to be $(0, 1)$ and Ω_f, respectively.

Proof of Lemma 5.3. (i) Using (5.13) we get

$$\begin{aligned} \|w\|_{L_t^2 H_x^{s+1/2}(\Gamma_c)} &\lesssim \left(\int_0^1 \left\| \int_0^t \phi_{\tilde{T}} v \, d\tau \right\|_{H^{s+1/2}(\Gamma_c)}^2 dt \right)^{1/2} + \left(\int_0^1 \left\| \left(t - \int_0^t \phi_{\tilde{T}} \right) v_0 \right\|_{H^{s+1/2}(\Gamma_c)}^2 dt \right)^{1/2} \\ &\lesssim \tilde{T}^{1/2} \|v\|_{L_t^2 H_x^{s+1}} + 1 \lesssim \tilde{T}^{1/2} M + 1, \end{aligned} \tag{5.24}$$

since $v_0|_{\Gamma_c} \in H^{s+1/2}(\Gamma_c)$, where we also used that for every $t \in [0, 1]$ we have

$$\left(\int_0^t \phi_{\tilde{T}} \|v\|_{H^{s+1/2}(\Gamma_c)} \, d\tau \right)^2 = \left(\int_0^{2\tilde{T}} \phi_{\tilde{T}} \|v\|_{H^{s+1/2}(\Gamma_c)} \, d\tau \right)^2 \lesssim \tilde{T} \int_0^{2\tilde{T}} \|v\|_{H^{s+1/2}(\Gamma_c)}^2 \, d\tau.$$

(ii) We use the Sobolev interpolation and Young inequalities to write

$$\begin{aligned} \|w_t\|_{H_t^{s/2-3/4} H_x^{s/2+1/4}(\Gamma_c)} &\lesssim \|w_t\|_{H_t^1 H_x^{s-3/2}(\Gamma_c)}^{s/2-3/4} \|w_t\|_{L_t^2 H_x^{(-4s^2+16s-7)/2(7-2s)}(\Gamma_c)}^{7/4-s/2} \\ &\lesssim \epsilon \|w_t\|_{H_t^1 H_x^{s-3/2}(\Gamma_c)} + C_\epsilon \|w_t\|_{L_t^2 H_x^{(-4s^2+16s-7)/2(7-2s)}(\Gamma_c)} := \mathcal{I}_1 + \mathcal{I}_2, \end{aligned} \tag{5.25}$$

for any $\epsilon \in (0, 1]$.

Note that the implicit constant in the first inequality is independent of $\tilde{T}$ since the interpolation is applied on a fixed domain $(0, 1) \times \Gamma_c$. For the term $\mathcal{I}_1$, we use (5.17), the trace inequality, and the Leibniz rule, to obtain

$$\mathcal{I}_1 \lesssim \epsilon \|\phi'_{\tilde{T}}(v - v_0)\|_{L_t^2 H_x^{s-1}} + \epsilon \|v'\|_{L_t^2 H_x^{s-1}} + \epsilon \|\phi_{\tilde{T}}(v - v_0) + v_0\|_{L_t^2 H_x^{s-1}} =: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}. \tag{5.26}$$

The term $\mathcal{I}_{11}$ is estimated using the Sobolev and Hölder inequalities as

$$\mathcal{I}_{11} \lesssim \epsilon \tilde{T}^{-1} \|v - v_0\|_{L^2((0,2\tilde{T}), H^{s-1}(\Omega_f))} \lesssim \epsilon \tilde{T}^{-1} \left\| \int_0^t v_t \right\|_{L^2((0,2\tilde{T}), H^{s-1}(\Omega_f))} \lesssim \epsilon \|v'\|_{L_t^2 H_x^{s-1}} \lesssim \epsilon M, \tag{5.27}$$

where we used

$$\begin{aligned} \left\| \int_0^t v_t \right\|^2_{L^2((0,2\tilde T),H^{s-1}(\Omega_{\rm f}))} &\le \int_0^{2\tilde T} \left(\int_0^{2\tilde T} \|v_t(s)\|_{H^{s-1}}\, ds \right)^2 dt \\ &= 2\tilde T \left(\int_0^{2\tilde T} \|v_t(s)\|_{H^{s-1}}\, ds \right)^2 \lesssim \tilde T^2 \int_0^{2\tilde T} \|v_t(s)\|^2_{H^{s-1}}\, ds \end{aligned} \tag{5.28}$$

in the second inequality and Corollary 3.4 in the last. Next, the terms $\mathcal{I}_{12}$ and $\mathcal{I}_{13}$ are estimated as

$$\mathcal{I}_{12} \lesssim \epsilon M \tag{5.29}$$

and

$$\mathcal{I}_{13} = \epsilon \|\phi_{\tilde T} v + (1-\phi_{\tilde T}) v_0\|_{L^2_t H^{s-1}_x} \lesssim \epsilon \|v\|_{L^2_t H^{s-1}_x} + \|(1-\phi_{\tilde T}) v_0\|_{L^2_t H^{s-1}_x} \lesssim \epsilon M + 1. \tag{5.30}$$

For the term $\mathcal{I}_2$, we use (5.17), (5.22), and the trace inequality to get

$$\begin{aligned} \mathcal{I}_2 &\lesssim C_\epsilon \|v\|_{L^2((0,2\tilde T),H^{(-4s^2+16s-7)/(14-4s)+1/2}(\Omega_{\rm f}))} + C_\epsilon \|v_0(1-\phi_{\tilde T})\|_{L^2((0,2\tilde T),H^{(-4s^2+16s-7)/(14-4s)+1/2}(\Omega_{\rm f}))} \\ &\lesssim C_\epsilon \|v\|_{L^2((0,2\tilde T),H^{s}(\Omega_{\rm f}))} + C_\epsilon \|v_0\|_{L^2((0,2\tilde T),H^{s}(\Omega_{\rm f}))}, \end{aligned}$$

where the last inequality follows from the identity $(-4s^2+16s-7)/(14-4s)+1/2 = s$. Using the Sobolev interpolation inequality, we get

$$\mathcal{I}_2 \lesssim \tilde\epsilon C_\epsilon \|v\|_{L^2((0,2\tilde T),H^{s+1}(\Omega_{\rm f}))} + C_{\tilde\epsilon,\epsilon} \|v\|_{L^2((0,2\tilde T),L^2(\Omega_{\rm f}))} + C_\epsilon \lesssim \tilde\epsilon C_\epsilon M + C_{\tilde\epsilon,\epsilon} \tilde T^{1/2} M + C_\epsilon, \tag{5.31}$$

for any $\tilde\epsilon \in (0,1]$. Combining (5.25)–(5.31), we arrive at

$$\|w_t\|_{H_t^{s/2-3/4} H_x^{s/2+3/4}(\Gamma_{\rm c})} \lesssim \epsilon M + \tilde\epsilon C_\epsilon M + C_{\tilde\epsilon,\epsilon} \tilde T^{1/2} M + C_\epsilon. \tag{5.32}$$

For the second term on the left side of (ii), we proceed as in (5.25), obtaining

$$\begin{aligned} \|w\|_{H_t^{s/2-3/4} H_x^{s/2+1/4}(\Gamma_{\rm c})} &\lesssim \|w\|^{s/2-3/4}_{H^1_t H^{1/2}_x(\Gamma_{\rm c})} \|w\|^{7/4-s/2}_{L^2_t H_x^{(2s+5)/2(7-2s)}(\Gamma_{\rm c})} \\ &\lesssim \|w\|_{H^1_t H^{1/2}_x(\Gamma_{\rm c})} + \|w\|_{L^2_t H_x^{(2s+5)/2(7-2s)}(\Gamma_{\rm c})} \\ &\lesssim \|\phi_{\tilde T} v\|_{L^2_t H^1_x} + \|(1-\phi_{\tilde T}) v_0\|_{L^2_t H^1_x} + \|w\|_{L^2_t H_x^{(2s+5)/(14-4s)}(\Gamma_{\rm c})}, \end{aligned}$$

since $1/2 \le (2s+5)/(14-4s)$. Note that $(2s+5)/(14-4s) < s+1/2$ for $2 < s < 5/2$. Thus, using (5.22) and (5.24), we obtain

$$\|w\|_{H_t^{s/2-3/4} H_x^{s/2+1/4}(\Gamma_{\rm c})} \lesssim (\epsilon + C_\epsilon \tilde T^{1/2}) M + 1, \tag{5.33}$$

for any $\epsilon \in (0,1]$.

(iii) First, we write

$$\begin{aligned} \|w\|_{H_t^{s/2+3/4} L^2_x(\Gamma_{\rm c})} &\lesssim \|w_t\|_{H_t^{s/2-1/4} L^2_x(\Gamma_{\rm c})} + \|w\|_{L^2_t L^2_x(\Gamma_{\rm c})} \\ &\lesssim \|w_t\|^{s/2-1/4}_{H^1_t L^2_x(\Gamma_{\rm c})} \|w_t\|^{5/4-s/2}_{L^2_t L^2_x(\Gamma_{\rm c})} + \|w\|_{L^2_t H_x^{s+1/2}(\Gamma_{\rm c})} \\ &\lesssim \epsilon \|w_{tt}\|_{L^2_t L^2_x(\Gamma_{\rm c})} + C_\epsilon \|w_t\|_{L^2_t L^2_x(\Gamma_{\rm c})} + \tilde T^{1/2} M + 1, \end{aligned} \tag{5.34}$$

for any $\epsilon \in (0,1]$, where the last inequality follows from (5.24). Note that the implicit constant in the second inequality is independent of $\tilde T$ since the interpolation is performed on $(0,1) \times \Gamma_{\rm c}$. From (5.17) it follows that

$$w_{tt}(t) = \phi'_{\tilde T}(t)(v(t) - v_0) + \phi_{\tilde T}(t) v_t(t,x) \text{ on } [0,1] \times \Gamma_{\rm c}. \tag{5.35}$$

For the first term on the far right side of (5.34), we use (5.35) and obtain

$$\begin{aligned} \epsilon \|w_{tt}\|_{L^2_t L^2_x(\Gamma_{\rm c})} &\lesssim \epsilon \|\phi'_{\tilde T}(v - v_0)\|_{L^2_t L^2_x(\Gamma_{\rm c})} + \epsilon \|\phi_{\tilde T} v_t\|_{L^2_t L^2_x(\Gamma_{\rm c})} \\ &\lesssim \epsilon \tilde T^{-1} \|v - v_0\|_{L^2((0,2\tilde T),H^1(\Omega_{\rm f}))} + \epsilon \|v_t\|_{L^2_t H^1_x} \lesssim \epsilon M, \end{aligned} \tag{5.36}$$

where the last inequality follows from (5.27) and Corollary 3.4. For the second term on the far right side of (5.34), we use (5.17) to arrive at

$$C_\epsilon \|w_t\|_{L_t^2 L_x^2(\Gamma_c)} \lesssim C_\epsilon \|\phi_{\tilde T} v\|_{L_t^2 L_x^2(\Gamma_c)} + C_\epsilon \|v_0(1-\phi_{\tilde T})\|_{L_t^2 L_x^2(\Gamma_c)} \lesssim (\tilde\epsilon C_\epsilon + C_{\epsilon,\tilde\epsilon}\tilde T^{1/2})M + C_\epsilon, \tag{5.37}$$

for any $\tilde\epsilon \in (0,1]$, where we used the trace inequality and (5.22). The proof of (iii) is concluded by combining (5.34) and (5.36)–(5.37).

The following theorem provides the local existence for the parabolic-wave system (5.10)–(5.16).

Theorem 5.4. *Let $s \in (2, 2+\epsilon_0]$, where $\epsilon_0 \in (0,1/2)$. Assume the compatibility conditions*

$$\begin{aligned}
&w_{1j} = v_{0j} \ \text{on } \Gamma_c,\\
&v_{0j} = 0 \ \text{on } \Gamma_f,\\
&\lambda(\partial_k v_{0j} + \partial_j v_{0k})\nu^k + \mu \partial_i v_{0i}\nu^j - R_0^{-1}\nu^j - \partial_k w_{0j}\nu^k = h_j(0) \ \text{on } \Gamma_c,\\
&\lambda R_0 \partial_k(\partial_k v_{0j} + \partial_j v_{0k}) + \mu R_0 \partial_j \partial_k v_{0k} - R_0 \partial_j(R_0^{-1}) = -f_j(0) \ \text{on } \Gamma_f,
\end{aligned} \tag{5.38}$$

for $j = 1,2,3$. Suppose that the initial data satisfy

$$(v_0, w_0, w_1, R_0^{-1}, R_0, f(0)) \in H^s(\Omega_f) \times H^{s+1/2}(\Omega_e) \times H^{s-1/2}(\Omega_e) \times H^s(\Omega_f) \times H^s(\Omega_f) \times H^{s-2}(\Omega_f)$$

and

$$(v_0|_{\Gamma_c}, \partial_3 v_0|_{\Gamma_f}) \in H^{s+1/2}(\Gamma_c) \times H^{s-1/2}(\Gamma_f)$$

with the nonhomogeneous terms satisfying

$$(f,h) \in K^{s-1}((0,1)\times\Omega_f) \times K^{s-1/2}((0,1)\times\Gamma_c).$$

Then there exists a unique solution

$$\begin{aligned}
(v,R,w,w_t) \in\ & K^{s+1}((0,\tilde T)\times\Omega_f) \times H^1((0,\tilde T), H^s(\Omega_f)) \times C([0,\tilde T], H^{s+1/4-\epsilon_0}(\Omega_e))\\
&\times C([0,\tilde T], H^{s-3/4-\epsilon_0}(\Omega_e)),
\end{aligned}$$

to the system (5.10)–(5.16), where $\tilde T > 0$ is a constant and the corresponding norms are bounded by a function of the initial data and the nonhomogeneous terms.

Let

$$\begin{aligned}
\mathcal{Z} = \{ v \in K^{s+1}((0,1)\times\Omega_{\mathrm f}) : v(0) = v_0 \ \text{in} \ \Omega_{\mathrm f}, v = 0 \text{ on } [0,1]\times\Gamma_{\mathrm f},\\
v \ \text{periodic in the } y_1 \text{ and } y_2 \text{ directions, and } \|v\|_{K^{s+1}((0,1)\times\Omega_{\mathrm f})} \le M\},
\end{aligned} \tag{5.39}$$

where $M \ge 1$ is a constant to be determined below. For $v \in \mathcal{Z}$, define R by (5.20). Next, we solve the wave equation (5.12) for w with the boundary condition (5.13) and the initial data $(w,w_t)(0) = (w_0,w_1)$ in $\Omega_{\mathrm e}$. With (R,w) constructed this way, we define a mapping

$$\Lambda\colon v(\in \mathcal{Z}) \mapsto \bar v,$$

where $\bar v$ is the solution of the nonhomogeneous parabolic problem

$$\bar v_t - \lambda R \operatorname{div}(\nabla \bar v + (\nabla \bar v)^T) - \mu R \nabla \operatorname{div} \bar v = f - R\nabla R^{-1} \text{ in } [0,1]\times\Omega_{\mathrm f}, \tag{5.40}$$

with the boundary conditions and the initial data

$$\begin{aligned}
&\lambda(\partial_k \bar v_j + \partial_j \bar v_k)\nu^k + \mu \partial_k \bar v_k \nu^j = \partial_k w_j \nu^k + R^{-1}\nu^j + h_j \text{ on } [0,1]\times\Gamma_{\mathrm c},\\
&\bar v = 0 \text{ on } [0,1]\times\Gamma_{\mathrm f},\\
&\bar v \ \text{periodic in the } y_1 \text{ and } y_2 \text{ directions},\\
&\bar v(0) = v_0 \text{ in } \Omega_{\mathrm f},
\end{aligned} \tag{5.41}$$

for $j = 1,2,3$. We shall prove below that Λ is a contraction mapping and then use the Banach fixed-point theorem.

5.1. Uniform Boundedness of the Iterative Sequence

In this section, we show that the mapping Λ is well-defined from $\mathcal{Z}$ to $\mathcal{Z}$, for some sufficiently large constant $M \geq 1$. Let $\tilde{T} \in (0, 1/4)$ be a constant. We emphasize that the implicit constants in this section below depend on the initial data but are independent of M and $\tilde{T}$. Denote the right side of $(5.41)_1$ by $\tilde{h}_j$. One may easily verify that

$$\tilde{h}_j(0) = \lambda(\partial_k v_{0j} + \partial_j v_{0k})\nu^k + \mu \partial_i v_{0i} \nu^j \text{ on } \Gamma_{\mathrm{c}} \tag{5.42}$$

by $(5.38)_3$. From $(5.38)_2$, (5.42), Lemma 4.2, and Lemma 5.2, it follows that

$$\begin{aligned}\|\bar{v}\|_{K^{s+1}} \lesssim \|\tilde{h}\|_{K^{s-1/2}_{\Gamma_{\mathrm{c}}}} + \|v_0|_{\Gamma_{\mathrm{c}}}\|_{H^{s+1/2}(\Gamma_{\mathrm{c}})} + \|\partial_3 v_0|_{\Gamma_{\mathrm{f}}}\|_{H^{s-1/2}(\Gamma_{\mathrm{f}})} + \|v_0\|_{H^s} + \|f\|_{K^{s-1}} \\ + \|R^{-1}\nabla R\|_{K^{s-1}} + \|f(0)\|_{H^{s-2}} + \|R_0 \nabla R_0^{-1}\|_{H^{s-2}}\end{aligned}$$

from where

$$\begin{aligned}\|\bar{v}\|_{K^{s+1}} \lesssim \left\|\frac{\partial w}{\partial \nu}\right\|_{K^{s-1/2}_{\Gamma_{\mathrm{c}}}} + \|R^{-1}\nabla R\|_{K^{s-1}} + \|R^{-1}\|_{K^{s-1/2}_{\Gamma_{\mathrm{c}}}} + \|h\|_{K^{s-1/2}_{\Gamma_{\mathrm{c}}}} + \|f\|_{K^{s-1}} + \|v_0\|_{H^s} \\ + \|v_0|_{\Gamma_{\mathrm{c}}}\|_{H^{s+1/2}(\Gamma_{\mathrm{c}})} + \|\partial_3 v_0|_{\Gamma_{\mathrm{f}}}\|_{H^{s-1/2}(\Gamma_{\mathrm{f}})} + \|f(0)\|_{H^{s-2}} + \|R_0^{-1}\nabla R_0\|_{H^{s-2}}.\end{aligned} \tag{5.43}$$

Here and below, when not indicated, the time and space domains are understood to be $(0, 1)$ and Ω_{f}, respectively.

For the space component of the first term on the right side of (5.43), we appeal to Lemma 3.6 to obtain

$$\begin{aligned}\left\|\frac{\partial w}{\partial \nu}\right\|_{L^2_t H^{s-1/2}_x(\Gamma_{\mathrm{c}})} &\lesssim \|w_0\|_{H^{s+1/2}(\Omega_{\mathrm{e}})} + \|w_1\|_{H^{s-1/2}(\Omega_{\mathrm{e}})} + \|w\|_{L^2_t H^{s+1/2}_x(\Gamma_{\mathrm{c}})} + \|w\|_{H^{s/2+1/4}_t H^{s/2+1/4}_x(\Gamma_{\mathrm{c}})} \\ &\lesssim \|w\|_{L^2_t H^{s+1/2}_x(\Gamma_{\mathrm{c}})} + \|w_t\|_{H^{s/2-3/4}_t H^{s/2+1/4}_x(\Gamma_{\mathrm{c}})} + \|w\|_{H^{s/2-3/4}_t H^{s/2+1/4}_x(\Gamma_{\mathrm{c}})} + 1.\end{aligned} \tag{5.44}$$

From (5.44) and Lemma 5.3 it follows that

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{L^2_t H^{s-1/2}_x(\Gamma_{\mathrm{c}})} \lesssim (\epsilon + \tilde{\epsilon} C_\epsilon + C_{\tilde{\epsilon},\epsilon}\tilde{T}^{1/2})M + C_\epsilon, \tag{5.45}$$

for any $\epsilon, \tilde{\epsilon} \in (0, 1]$. For the time component of the first term on the right side of (5.43), we use Lemma 3.5 to get

$$\begin{aligned}\left\|\frac{\partial w}{\partial \nu}\right\|_{H^{s/2-1/4}_t L^2_x(\Gamma_{\mathrm{c}})} \lesssim \|w_0\|_{H^{s/2+3/4}(\Omega_{\mathrm{e}})} + \|w_1\|_{H^{s/2-1/4}(\Omega_{\mathrm{e}})} + \|w\|_{L^2_t H^{s/2+3/4}_x(\Gamma_{\mathrm{c}})} \\ + \|w\|_{H^{s/2+3/4}_t L^2_x(\Gamma_{\mathrm{c}})}.\end{aligned} \tag{5.46}$$

For the third term on the right side of (5.46), we appeal to (5.24) to get

$$\|w\|_{L^2_t H^{s/2+3/4}_x(\Gamma_{\mathrm{c}})} \lesssim \tilde{T}^{1/2}M + 1, \tag{5.47}$$

since $s/2 + 3/4 \leq s + 1/2$. Applying Lemma 5.3 and (5.47) in (5.46), we get

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{H^{s/2-1/4}_t L^2_x(\Gamma_{\mathrm{c}})} \lesssim (\epsilon + \tilde{\epsilon} C_\epsilon + C_{\epsilon,\tilde{\epsilon}}\tilde{T}^{1/2})M + C_\epsilon, \tag{5.48}$$

for any $\epsilon, \tilde{\epsilon} \in (0, 1]$.

For the space component of the second term on the right side of (5.43), we use the Hölder's and the Sobolev inequalities to obtain

$$\|R^{-1}\nabla R\|_{L^2_t H^{s-1}_x} \lesssim \|R^{-1}\|_{L^\infty_t H^s_x} \|\nabla R\|_{L^2_t H^{s-1}_x} \lesssim 1, \tag{5.49}$$

where we appealed to Lemma 5.2. For the time component, we use Hölder's and the Sobolev inequalities with Lemma 5.2 to obtain

$$\begin{aligned}\|R^{-1}\nabla R\|_{H_t^{(s-1)/2}L_x^2} &\lesssim \|R^{-1}\nabla R\|_{H_t^1L_x^2} \lesssim \|R_t\nabla R\|_{L_t^2L_x^2} + \|\nabla R_t\|_{L_t^2L_x^2} + \|\nabla R\|_{L_t^2L_x^2} \\ &\lesssim \|R_t\|_{L_t^2L_x^6}\|\nabla R\|_{L_t^\infty L_x^3} + \|R_t\|_{L_t^2H_x^1} + \|R\|_{L_t^2H_x^s} \lesssim 1.\end{aligned} \tag{5.50}$$

For the space component of the third term on the right side of (5.43), we use the trace inequality to obtain

$$\|R^{-1}\|_{L_t^2H_x^{s-1/2}(\Gamma_c)} \lesssim \|R^{-1}\|_{L_t^2H_x^s} \lesssim 1, \tag{5.51}$$

where the last inequality follows from Lemma 5.2. For the time component, we proceed analogously to (5.50), obtaining

$$\|R^{-1}\|_{H_t^{s/2-1/4}L^2(\Gamma_c)} \lesssim \|R^{-1}\|_{H_t^1H_x^1} \lesssim 1, \tag{5.52}$$

since $s \leq 5/2$.

For the last term on the right side of (5.43), we proceed analogously as in (5.49), obtaining

$$\|R_0^{-1}\nabla R_0\|_{H^{s-2}} \lesssim \|R_0^{-1}\|_{H^s}\|\nabla R_0\|_{H^{s-2}} \lesssim 1. \tag{5.53}$$

Combining (5.43), (5.45), and (5.48)–(5.53), we arrive at

$$\|\bar v\|_{K^{s+1}} \lesssim (\epsilon + \tilde\epsilon C_\epsilon + C_{\epsilon,\tilde\epsilon}\tilde T^{1/2})M + C_\epsilon,$$

for any $\epsilon, \tilde\epsilon \in (0,1]$. Taking appropriate ϵ, $\tilde\epsilon$, and $\tilde T > 0$ (first ϵ sufficiently small, then $\tilde\epsilon$ sufficiently small depending on ϵ, and then $\tilde T$ sufficiently small, depending on ϵ and $\tilde\epsilon$), we get

$$\|\bar v\|_{K^{s+1}} \leq M, \tag{5.54}$$

by allowing $M \geq 1$ sufficiently large.

Thus, we have shown that the mapping $\Lambda\colon v \mapsto \bar v$ is well-defined from $\mathcal{Z}$ to $\mathcal{Z}$ and satisfies (5.54) for some $M \geq 1$, which depends on the size of the initial data and nonhomogeneous terms.

5.2. Contracting Property

In this section, we prove

$$\|\Lambda(v_1) - \Lambda(v_2)\|_{K^{s+1}} \leq \frac{1}{2}\|v_1 - v_2\|_{K^{s+1}}, \qquad v_1, v_2 \in \mathcal{Z}, \tag{5.55}$$

where $M \geq 1$ is fixed as in (5.54) and $\tilde T > 0$ is a sufficiently small constant as in the previous section, which is further restricted below. We emphasize that the implicit constants below are allowed to depend on M.

Proof of Theorem 5.4. Let $v_1, v_2 \in \mathcal{Z}$. Let $(R_1, \xi_1, \xi_{1t}, \bar v_1)$ and $(R_2, \xi_2, \xi_{2t}, \bar v_2)$ be the corresponding solutions of (5.18)–(5.19), (5.12)–(5.13), and (5.40)–(5.41) with the same initial data (R_0, w_0, w_1, v_0) and the same nonhomogeneous terms (f, h). We denote $\tilde V = \bar v_1 - \bar v_2$, $\tilde v = v_1 - v_2$, $\tilde R = R_1 - R_2$, and $\tilde\xi = \xi_1 - \xi_2$. The difference $\tilde V$ satisfies

$$\tilde V_t - \lambda R_1 \operatorname{div}(\nabla\tilde V + (\nabla\tilde V)^T) - \mu R_1\nabla \operatorname{div}\tilde V = g \text{ in } [0,1]\times\Omega_{\rm f},$$

with the boundary conditions and the initial data

$$\begin{aligned}&\lambda(\partial_k\tilde V_j + \partial_j\tilde V_k)\nu^k + \mu\partial_k\tilde V_k\nu^j = \partial_k\tilde\xi_j\nu^k - R_1^{-1}R_2^{-1}\tilde R\nu^j \text{ on } [0,1]\times\Gamma_{\rm c},\\ &\tilde V = 0 \text{ on } [0,1]\times\Gamma_{\rm f},\\ &\tilde V \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions},\\ &\tilde V(0) = 0 \text{ in } \Omega_{\rm f},\end{aligned}$$

for $j=1,2,3$, where

$$g=-R_1\nabla R_1^{-1}+R_2\nabla R_2^{-1}+\lambda\tilde{R}\ \mathrm{div}\ (\nabla\bar{v}_2+(\nabla\bar{v}_2)^T)+\mu\tilde{R}\nabla\ \mathrm{div}\ \bar{v}_2. \tag{5.56}$$

Note that $g(0)=0$. We proceed as in (5.43) to obtain

$$\begin{aligned}\|\tilde{V}\|_{K^{s+1}}\lesssim \left\|\frac{\partial\tilde{\xi}}{\partial\nu}\right\|_{K^{s-1/2}_{\Gamma_c}}&+\|\tilde{R}R_1^{-1}R_2^{-1}\nabla R_2\|_{K^{s-1}}+\|R_1^{-1}\nabla\tilde{R}\|_{K^{s-1}}+\|\tilde{R}D_x^2\bar{v}_2\|_{K^{s-1}}\\&+\|R_1^{-1}R_2^{-1}\tilde{R}\|_{K^{s-1/2}_{\Gamma_c}},\end{aligned} \tag{5.57}$$

where the last inequality follows from (5.56) and $-R_1\nabla R_1^{-1}+R_2\nabla R_2^{-1}=R_1^{-1}\nabla\tilde{R}-R_1^{-1}R_2^{-1}\tilde{R}\nabla R_2$. The difference $\tilde{\xi}$ satisfies the wave equation

$$\tilde{\xi}_{tt}-\Delta\tilde{\xi}=0 \text{ in } [0,1]\times\Omega_{\mathrm{e}},$$

with the boundary condition and the initial data

$$\begin{aligned}&\tilde{\xi}(t,x)=\int_0^t\phi_{\tilde{T}}\tilde{v}(\tau,x)\,d\tau \text{ on } [0,1]\times\Gamma_{\mathrm{c}},\\&(\tilde{\xi},\tilde{\xi}_t)(0,x)=(0,0) \text{ in } \Omega_{\mathrm{e}}.\end{aligned}$$

For the first term on the right side of (5.57), we proceed as in (5.44)–(5.48) to obtain

$$\left\|\frac{\partial\tilde{\xi}}{\partial\nu}\right\|_{K^{s-1/2}(\Gamma_c)}\lesssim(\epsilon+\tilde{\epsilon}C_\epsilon+C_{\epsilon,\tilde{\epsilon}}\tilde{T}^{1/2})\|\tilde{v}\|_{K^{s+1}}, \tag{5.58}$$

for any $\epsilon,\tilde{\epsilon}\in(0,1]$.

Since the difference $\tilde{R}$ satisfies the ODE system

$$\tilde{R}_t-\phi_{\tilde{T}}\tilde{R}\ \mathrm{div}\ v_2=R_1\phi_{\tilde{T}}\ \mathrm{div}\ \tilde{v} \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \tag{5.59}$$

$$\tilde{R}(0)=0 \text{ in } \Omega_{\mathrm{f}}, \tag{5.60}$$

the solution is given by

$$\tilde{R}(t,x)=\int_0^t e^{\int_\tau^t\phi_{\tilde{T}}\ \mathrm{div}\ v_2}\phi_{\tilde{T}}(\tau)R_1(\tau)\ \mathrm{div}\ \tilde{v}(\tau)\,d\tau \text{ in } [0,1]\times\Omega_{\mathrm{f}}. \tag{5.61}$$

For the second term on the right side of (5.57), we obtain

$$\|\tilde{R}R_1^{-1}R_2^{-1}\nabla R_2\|_{L_t^2H_x^{s-1}}\lesssim\|\tilde{R}\|_{L_t^2H_x^s}\|R_1^{-1}\|_{L_t^\infty H_x^s}\|R_2^{-1}\|_{L_t^\infty H_x^s}\|R_2\|_{L_t^\infty H_x^s}\lesssim\|\tilde{R}\|_{L_t^\infty H_x^s}, \tag{5.62}$$

where we used Hölder's inequality and Lemma 5.2, and then from (5.61) it follows that

$$\|\tilde{R}R_1^{-1}R_2^{-1}\nabla R_2\|_{L_t^2H_x^{s-1}}\lesssim\|\tilde{R}\|_{L_t^\infty H_x^s}\lesssim\tilde{T}^{1/2}\|\tilde{v}\|_{L_t^2H_x^{s+1}}, \tag{5.63}$$

where we used the Cauchy-Schwarz inequality. For the time component (note that $s/2-1/4\le 1$), we have

$$\begin{aligned}&\|(\tilde{R}R_1^{-1}R_2^{-1}\nabla R_2)_t\|_{L_t^2L_x^2}\\&\quad\lesssim\|\tilde{R}_t\nabla R_2\|_{L_t^2L_x^2}+\|\tilde{R}R_{1t}\nabla R_2\|_{L_t^2L_x^2}+\|\tilde{R}R_{2t}\nabla R_2\|_{L_t^2L_x^2}+\|\tilde{R}\nabla R_{2t}\|_{L_t^2L_x^2}\\&\quad\lesssim\|\tilde{R}\|_{L_t^\infty L_x^\infty}\|\ \mathrm{div}\ v_2\|_{L_t^2L_x^4}\|\nabla R_2\|_{L_t^\infty L_x^4}+\|R_1\|_{L_t^\infty L_x^\infty}\|\ \mathrm{div}\ \tilde{v}\|_{L^2((0,2\tilde{T}),L^\infty(\Omega_f))}\|\nabla R_2\|_{L_t^\infty L_x^2}\\&\qquad+\|\tilde{R}\|_{L_t^\infty L_x^\infty}\|\ \mathrm{div}\ v_1\|_{L_t^2L_x^4}\|\nabla R_2\|_{L_t^\infty L_x^4}+\|\tilde{R}\|_{L_t^\infty L_x^\infty}\|\ \mathrm{div}\ v_2\|_{L_t^2L_x^4}\|\nabla R_2\|_{L_t^\infty L_x^4}\\&\qquad+\|\tilde{R}\|_{L_t^\infty L_x^\infty}\|\nabla R_{2t}\|_{L_t^2L_x^2}\\&\quad\lesssim(\epsilon+C_\epsilon\tilde{T}^{1/2})\|\tilde{v}\|_{K^{s+1}},\end{aligned} \tag{5.64}$$

for any $\epsilon \in (0,1]$, where we used Hölder's inequality, Lemma 5.2, (5.59), and (5.61), as well as $\|\tilde{R}\|_{L_t^\infty H_x^s} \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{L_t^2 H_x^{s+1}}$. Note that $\|\tilde{R}R_1^{-1}R_2^{-1}\nabla R_2\|_{L_t^2 L_x^2}$ does not need to be estimated since it is dominated by (5.63). Similarly, the third term on the right side of (5.57) is estimated as

$$\|R_1^{-1}\nabla\tilde{R}\|_{L_t^2 H_x^{s-1}} \lesssim \|R_1^{-1}\|_{L_t^\infty H_x^s}\|\tilde{R}\|_{L_t^\infty H_x^s} \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}}, \tag{5.65}$$

and for the time component

$$\begin{aligned}\|(R_1\nabla\tilde{R})_t\|_{L_t^2 L_x^2} &\lesssim \|R_{1t}\nabla\tilde{R}\|_{L_t^2 L_x^2} + \|R_1\nabla\tilde{R}_t\|_{L_t^2 L_x^2}\\ &\lesssim \|R_{1t}\|_{L_t^2 L_x^\infty}\|\nabla\tilde{R}\|_{L_t^\infty L_x^2} + \|R_1\|_{L_t^\infty L_x^\infty}\|\nabla\tilde{R}_t\|_{L_t^2 L_x^2} \lesssim (\epsilon + C_\epsilon\tilde{T}^{1/2})\|\tilde{v}\|_{K^{s+1}}.\end{aligned} \tag{5.66}$$

Again, the term $\|R_1\nabla\tilde{R}\|_{L_t^2 L_x^2}$ is dominated by (5.65). Regarding the fourth term on the right side of (5.57), we use Corollary 3.4 to obtain

$$\|\tilde{R}D_x^2\bar{v}_2\|_{L_t^2 H_x^{s-1}} \lesssim \|\tilde{R}\|_{L_t^\infty H_x^s}\|\bar{v}_2\|_{L_t^2 H_x^{s+1}} \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}}. \tag{5.67}$$

To treat $\|\tilde{R}D_x^2\bar{v}_2\|_{H_t^{(s-1)/2}L_x^2}$, we claim that for any $\alpha > 1/2$ and $\delta > 0$, we have

$$\|AB\|_{H_t^\alpha L_x^2} \lesssim \|A\|_{H_t^\alpha H_x^{3/2+\delta}}\|B\|_{H_t^\alpha L_x^2} \tag{5.68}$$

on the domain $(0,1)\times\Omega_{\mathrm{f}}$. Using extensions, we may assume that the domain is actually $\mathbb{R}\times\mathbb{R}^3$. Then

$$\begin{aligned}\|AB\|_{H_t^\alpha L_x^2} &= \|AB\|_{L_x^2 H_t^\alpha} \lesssim \big\|\|A\|_{H_t^\alpha}\|B\|_{H_t^\alpha}\big\|_{L_x^2}\\ &\lesssim \|A\|_{L_x^\infty H_t^\alpha}\|B\|_{L_x^2 H_t^\alpha} \lesssim \|A\|_{H_x^{3/2+\delta}H_t^\alpha}\|B\|_{L_x^2 H_t^\alpha}\\ &= \|A\|_{H_t^\alpha H_x^{3/2+\delta}}\|B\|_{H_t^\alpha L_x^2},\end{aligned} \tag{5.69}$$

since $\alpha > 1/2$, and (5.68) follows. Using (5.68), we then write

$$\|\tilde{R}D_x^2\bar{v}_2\|_{H_t^{(s-1)/2}L_x^2} \lesssim \|\tilde{R}\|_{H_t^1 H_x^2}\|D_x^2\bar{v}_2\|_{H_t^{(s-1)/2}L_x^2} \lesssim \|\tilde{R}_t\|_{L_t^2 H_x^2} + \|\tilde{R}\|_{L_t^2 H_x^2}, \tag{5.70}$$

where we used Corollary 3.4 in the last inequality. From (5.22) and (5.59), it follows that

$$\begin{aligned}\|\tilde{R}_t\|_{L_t^2 H_x^2} &\lesssim \|\tilde{R}\|_{L_t^\infty H_x^2}\|v_2\|_{L_t^2 H_x^{s+1}} + \|R_1\|_{L_t^\infty H_x^2}\|\phi_{\tilde{T}}\tilde{v}\|_{L_t^2 H_x^3}\\ &\lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{L_t^2 H_x^{s+1}} + \epsilon\|\tilde{v}\|_{L_t^2 H_x^{s+1}} + C_\epsilon\tilde{T}^{1/2}\|\tilde{v}\|_{H_t^{(s+1)/2}L_x^2} \lesssim (\epsilon + C_\epsilon\tilde{T}^{1/2})\|\tilde{v}\|_{K^{s+1}},\end{aligned} \tag{5.71}$$

for any $\epsilon \in (0,1]$, since $s > 2$. Combining (5.70) and (5.71), we arrive at

$$\|\tilde{R}D_x^2\bar{v}_2\|_{H_t^{(s-1)/2}L_x^2} \lesssim (\epsilon + C_\epsilon\tilde{T}^{1/2})\|\tilde{v}\|_{K^{s+1}}. \tag{5.72}$$

For the last term on the right side of (5.57), we use the trace inequality and arrive at

$$\|R_1^{-1}R_2^{-1}\tilde{R}\|_{L_t^2 H_x^{s-1/2}(\Gamma_{\mathrm{c}})} \lesssim \|\tilde{R}\|_{L_t^2 H_x^s} \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}} \tag{5.73}$$

and

$$\|R_1^{-1}R_2^{-1}\tilde{R}\|_{H_t^{s/2-1/4}L_x^2(\Gamma_{\mathrm{c}})} \lesssim \|R_1^{-1}R_2^{-1}\tilde{R}\|_{H_t^1 H_x^1} \lesssim \|(R_1^{-1}R_2^{-1}\tilde{R})_t\|_{L_t^2 H_x^1} + \|R_1^{-1}R_2^{-1}\tilde{R}\|_{L_t^2 H_x^1}, \tag{5.74}$$

since $s \le 5/2$. For the first term on the right side of (5.74), we proceed as in (5.64) to obtain

$$\|(R_1^{-1}R_2^{-1}\tilde{R})_t\|_{L_t^2 H_x^1} \lesssim \|R_{1t}\tilde{R}\|_{L_t^2 H_x^1} + \|R_{2t}\tilde{R}\|_{L_t^2 H_x^1} + \|\tilde{R}_t\|_{L_t^2 H_x^1} \lesssim (\epsilon + C_\epsilon\tilde{T}^{1/2})\|\tilde{v}\|_{K^{s+1}}. \tag{5.75}$$

The second term on the right side of (5.74) is estimated analogously to (5.73), and we get

$$\|R_1^{-1}R_2^{-1}\tilde{R}\|_{L_t^2 H_x^1} \lesssim \tilde{T}\|\tilde{v}\|_{K^{s+1}}. \tag{5.76}$$

Applying the above estimates in (5.57), we obtain

$$\|\tilde{V}\|_{K^{s+1}} \lesssim (\epsilon + \tilde{\epsilon} + \tilde{T}^{1/2}C_{\tilde{\epsilon},\epsilon})\|\tilde{v}\|_{K^{s+1}},$$

for any $\epsilon, \tilde{\epsilon} \in (0,1]$. Taking appropriate ϵ, $\tilde{\epsilon}$, and $\tilde{T} > 0$ (first ϵ sufficiently small, then $\tilde{\epsilon}$ sufficiently small depending on ϵ, and then $\tilde{T}$ sufficiently small, depending on ϵ and $\tilde{\epsilon}$), we conclude the proof of (5.55).

Thus, the mapping Λ is a contraction from $\mathcal{Z}$ to $\mathcal{Z}$. Using the Banach fixed point theorem, there exists a unique solution $v \in \mathcal{Z}$ such that $\Lambda(v) = v$ and which also satisfies (5.54) for some $M \geq 1$.

Now we fix the constant $\tilde{T} > 0$ as above. Using Lemma 3.5, we have the interior regularity estimate

$$\begin{aligned} &\|w\|_{C([0,1],H^{s+1/4-\epsilon_0}(\Omega_e))} + \|w_t\|_{C([0,1],H^{s-3/4-\epsilon_0}(\Omega_e))} \\ &\quad \lesssim \|w_0\|_{H^{s+1/4-\epsilon_0}(\Omega_e)} + \|w_1\|_{H^{s-3/4-\epsilon_0}(\Omega_e)} + \|w\|_{H^{s+1/4-\epsilon_0,s+1/4-\epsilon_0}(\Gamma_c)}. \end{aligned} \tag{5.77}$$

For the last term on the right side, we appeal to (5.17), yielding

$$\begin{aligned} \|w\|_{H^{s+1/4-\epsilon_0,s+1/4-\epsilon_0}(\Gamma_c)} &\lesssim \|w_t\|_{H_t^{s-3/4-\epsilon_0}L_x^2(\Gamma_c)} + \|w\|_{L_t^2H_x^{s+1/4-\epsilon_0}(\Gamma_c)} \\ &\lesssim \|\phi_{\tilde{T}}v\|_{H_t^{s-3/4-\epsilon_0}L_x^2(\Gamma_c)} + \|(1-\phi_{\tilde{T}})v_0\|_{H_t^{s-3/4-\epsilon_0}L_x^2(\Gamma_c)} + \|w\|_{L_t^2H_x^{s+1/2}(\Gamma_c)}. \end{aligned} \tag{5.78}$$

For the first term on the far right side of (5.78), we appeal to Corollary 3.2 and Sobolev inequality to get

$$\|\phi_{\tilde{T}}v\|_{H_t^{s-3/4-\epsilon_0}L_x^2(\Gamma_c)} \lesssim \|v\|_{H_t^{s-3/4-\epsilon_0}L_x^2(\Gamma_c)} \lesssim \|v\|_{H_t^{(s+1)/2}L_x^2} + \|v\|_{L_t^2H_x^{s+1}}, \tag{5.79}$$

since $s \leq 2 + 2\epsilon_0$. From (5.24) and (5.77)–(5.79), it follows that

$$\|w\|_{C([0,1],H^{s+1/4-\epsilon_0}(\Omega_e))} + \|w_t\|_{C([0,1],H^{s-3/4-\epsilon_0}(\Omega_e))} \leq C, \tag{5.80}$$

where $C > 0$ is a constant. By (5.80) and Lemma 5.2, there exists a unique solution

$$\begin{aligned} (v, R, w, w_t) \in\ & K^{s+1}((0,\tilde{T}) \times \Omega_f) \times H^1((0,\tilde{T}), H^s(\Omega_f)) \\ & \times C([0,\tilde{T}], H^{s+1/4-\epsilon_0}(\Omega_e)) \times C([0,\tilde{T}], H^{s-3/4-\epsilon_0}(\Omega_e)) \end{aligned}$$

to the system (5.10)–(5.16), with the corresponding norms bounded by a function of the initial data and the nonhomogeneous terms. □

Remark 5.5. As pointed out at the end of Sect. 2, the approach extends to more general pressure laws. For general equation of state $q(r)$, we assume that $q(r)$ is smooth such that $q(0) = 0$ and $q(r_1) - q(r_2) = (r_1 - r_2)\tilde{q}(r_1, r_2)$ for any r_1 and r_2, where $\tilde{q}$ is a smooth function. We shall briefly outline the modifications needed for this general pressure law. In Sect. 5.1, we have $\|R\nabla(q(R^{-1}))\|_{K^{s-1}}$ instead of the second term on the right side of (5.43). For the space component, we use the Hölder and Sobolev inequalities to get

$$\|R\nabla(q(R^{-1}))\|_{L_t^2H_x^{s-1}} \lesssim \|q'(R^{-1})R^{-1}\nabla R\|_{L_t^2H_x^{s-1}} \lesssim \|q'(R^{-1})\|_{L_t^\infty H_x^s}\|R^{-1}\nabla R\|_{L_t^2H_x^{s-1}} \lesssim 1,$$

where the last inequality follows from (5.49). For the time component, we appeal to (5.69), yielding

$$\|R\nabla(q(R^{-1}))\|_{H_t^{(s-1)/2}L_x^2} \lesssim \|q'(R^{-1})\|_{H_t^1H_x^{3/2+\delta}}\|R^{-1}\nabla R\|_{H_t^{(s-1)/2}L_x^2} \lesssim 1$$

where we used Lemma 5.2 and (5.50) in the last inequality. The third term on the right side of (5.43) is replaced by $\|q(R^{-1})\|_{K_{\Gamma_c}^{s-1/2}}$, which can be estimated in a similar fashion. In Sect. 5.2, the first two terms on the right side of (5.56) are replaced by $-R_1\nabla(q(R_1^{-1})) + R_2\nabla(q(R_2^{-1}))$ and the K^{s-1} norm can be estimated using the structural assumption on $q(r)$. □

6. Solution to the Navier–Stokes-Wave System

In this section, we provide the local existence for the coupled Navier-Stokes-wave system (2.3)–(2.5) with the boundary conditions (2.6)–(2.10) and the initial data (2.14). Let $v \in \mathcal{Z}$ where $\mathcal{Z}$ is as in (5.39), with constant $M \geq 1$ to be determined below. Let $\phi_{\tilde{T}}(t)$ be a smooth cutoff function as defined in Sect. 5; here, $\tilde{T} \in (0, 1/4)$ is a constant to be determined below; it is assumed to be smaller than the constant $\tilde{T}$ from the previous section, which we from here on denote by $\tilde{T}_0$. We allow all constants to depend on $\tilde{T}_0$ (but not on $\tilde{T}$).

We again modify the system to be able to construct a solution on a unit time interval. Let

$$\eta(t,x) = x + \int_0^t \phi_{\tilde{T}}(\tau)v(\tau,x)\,d\tau \text{ in } [0,1] \times \Omega_f \tag{6.1}$$

be a modified Lagrangian flow map and $a(t,x) = (\nabla\eta(t,x))^{-1}$ its inverse matrix, while we denote by $J(t,x) = \det(\nabla\eta(t,x))$ the corresponding Jacobian. The density equations we consider is

$$R_t - R\phi_{\tilde{T}} a_{kj}\partial_k v_j = 0 \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \tag{6.2}$$

$$R(0) = R_0 \text{ on } \Omega_{\mathrm{f}}, \tag{6.3}$$

with the solution given by

$$R(t,x) = R_0(x) e^{\int_0^t \phi_{\tilde{T}}(\tau) a_{kj}(\tau,x)\partial_k v_j(\tau,x)\,d\tau} \text{ in } [0,1]\times\Omega_{\mathrm{f}}.$$

Next, we consider the solution w to the wave equation (5.12) with the boundary condition (5.13)–(5.16) and the initial data $(w,w_t)(0) = (w_0,w_1)$ in Ω_{e}. With (η,a,J,R,w) constructed, we define

$$\Pi\colon v\in\mathcal{Z}\mapsto \bar{v},$$

where $\bar{v}$ is the solution of the nonhomogeneous parabolic problem

$$\partial_t\bar{v}_j - \lambda R\partial_k(\partial_j\bar{v}_k + \partial_k\bar{v}_j) - \mu R\partial_j\partial_k\bar{v}_k = f_j \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \tag{6.4}$$

$$\lambda(\partial_k\bar{v}_j + \partial_j\bar{v}_k)\nu^k + \mu\partial_k\bar{v}_k\nu^j = \partial_k w_j\nu^k + h_j \text{ in } [0,1]\times\Gamma_{\mathrm{c}}, \tag{6.5}$$

$$\bar{v} \text{ periodic in the } y_1 \text{ and } y_2 \text{ directions}, \tag{6.6}$$

$$\bar{v}(0) = v_0 \text{ in } \Omega_{\mathrm{f}}; \tag{6.7}$$

in (6.4)–(6.5), we set

$$\begin{aligned} f_j &= \lambda R\partial_k(b_{mk}\partial_m\bar{v}_j + b_{mj}\partial_m\bar{v}_k) + \lambda R b_{kl}\partial_k(b_{ml}\partial_m\bar{v}_j + b_{mj}\partial_m\bar{v}_l) + \lambda R b_{kl}\partial_k(\partial_l\bar{v}_j + \partial_j\bar{v}_l) \\ &\quad + \mu R\partial_j(b_{mi}\partial_m\bar{v}_i) + \mu R b_{kj}\partial_k(b_{mi}\partial_m\bar{v}_i) + \mu R b_{kj}\partial_k\partial_i\bar{v}_i - R b_{kj}\partial_k R^{-1} - R\partial_j R^{-1} \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} h_j &= \lambda(1-J)(\partial_k\bar{v}_j + \partial_j\bar{v}_k)\nu^k + \mu(1-J)\partial_k\bar{v}_k\nu^j - \lambda J b_{kl}(b_{ml}\partial_m\bar{v}_j + b_{mj}\partial_m\bar{v}_l)\nu^k \\ &\quad + J b_{kj}R^{-1}\nu^k + (J-1)R^{-1}\nu^j - \lambda J(b_{mk}\partial_m\bar{v}_j + b_{mj}\partial_m\bar{v}_k)\nu^k - \lambda J b_{kl}(\partial_l\bar{v}_j + \partial_j\bar{v}_l)\nu^k \\ &\quad - \mu J b_{kj} b_{mi}\partial_m\bar{v}_i\nu^k - \mu J b_{mi}\partial_m\bar{v}_i\nu^j - \mu J b_{kj}\partial_i\bar{v}_i\nu^k + R^{-1}\nu^j \\ &=: K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10} + K_{11}, \end{aligned} \tag{6.9}$$

for $j=1,2,3$, where

$$b = a - \mathbb{I}_3,$$

and $\mathbb{I}_3$ is the three-dimensional identity matrix.

Before we bound the terms in (6.8)–(6.9) and prove the contracting property, as in Sect. 5, we provide some necessary estimates on the variable coefficients.

6.1. The Lagrangian Flow Map, Jacobian Matrix, and Density Estimates

We start with estimates on the Jacobian and the inverse matrix of the flow map.

Lemma 6.1. *Suppose that $\|v\|_{K^{s+1}((0,1)\times\Omega_f)} \le M$, where $M\ge 1$, and let $\delta\in(0,1/5)$. Then for $\tilde{T}>0$ sufficiently small depending on M and δ, the following statements hold:*

(i) $\|b\|_{L_t^\infty H_x^s} + \|b\|_{H_t^1 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20}$,
(ii) $\|b\|_{H_t^1 H_x^s} \lesssim M$,
(iii) $\|1-J\|_{L_t^\infty H_x^s} \lesssim \tilde{T}^{1/20}$,
(iv) $\|J\|_{L_t^\infty L_x^\infty} + \|J^{-1}\|_{L_t^\infty L_x^\infty} + \|J\|_{L_t^\infty H_x^s} + \|J^{-1}\|_{L_t^\infty H_x^s} \lesssim 1$,
(v) $\|J^{-1}\|_{H_t^1 H_x^{3/2+\delta}} + \|J\|_{H_t^1 H_x^{3/2+\delta}} \lesssim 1$,
(vi) $\|J\|_{H_t^1 H_x^s} \lesssim M$,

where the region of dependence is understood to be $(0,1)\times\Omega_f$.

We emphasize that the implicit constants in the above inequalities (i)–(vi) are independent of M and δ.

Proof of Lemma 6.1. (i) From (2.11) and (6.1) it follows that

$$b_t = -\phi_{\tilde{T}}(b\nabla v b + b\nabla v + \nabla v b + \nabla v) \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \tag{6.10}$$

while $b(0)=0$. By the Fundamental Theorem of Calculus, it follows that for $t\in(0,2\tilde{T})$ we have

$$\begin{aligned} \|b(t)\|_{H^s} &\lesssim \int_0^t \|b\|_{H^s}^2 \|\nabla v\|_{H^s}\,d\tau + \int_0^t \|b\|_{H^s}\|\nabla v\|_{H^s}\,d\tau + \int_0^t \|\nabla v\|_{H^s}\,d\tau \\ &\lesssim \int_0^t \|v\|_{H^{s+1}}(\|b\|_{H^s}^2 + \|b\|_{H^s})\,d\tau + \tilde{T}^{1/2}M, \end{aligned}$$

where we appealed to the Cauchy-Schwarz inequality in the last step. Using Gronwall's inequality, we obtain

$$\|b\|_{L^\infty((0,2\tilde{T}),H^s(\Omega_{\mathrm{f}}))} \lesssim \tilde{T}^{1/2}M \lesssim \tilde{T}^{1/20},$$

for $\tilde{T}>0$ sufficiently small; the choice of the power $1/20$ is apparent in (6.12) below. Since also $b_t=0$ on $(2\tilde{T},1)$, we then infer that

$$\|b\|_{L_t^\infty H_x^s} \lesssim \tilde{T}^{1/20}. \tag{6.11}$$

Applying (6.11) in (6.10) and using (5.22), we obtain

$$\begin{aligned} \|b_t\|_{L_t^2 H_x^{3/2+\delta}} &\lesssim \|v\|_{L^2((0,2\tilde{T}),H^{5/2+\delta}(\Omega_{\mathrm{f}}))} \lesssim \epsilon\|v\|_{L_t^2 H_x^{s+1}} + \epsilon^{(5+2\delta)/(3+2\delta-2s)}\|v\|_{L^2((0,2\tilde{T}),L^2(\Omega_{\mathrm{f}}))} \\ &\lesssim \epsilon M + \epsilon^{(5+2\delta)/(3+2\delta-2s)}\tilde{T}^{1/2}M, \end{aligned}$$

for any $\epsilon\in(0,1]$. Letting $\epsilon=\tilde{T}^{1/20}M^{-1}$, we get

$$\|b_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20} + \tilde{T}^{1/2+(5+2\delta)/20(3+2\delta-2s)}M^{1+(5+2\delta)/(2s-3-2\delta)} \lesssim \tilde{T}^{1/20}, \tag{6.12}$$

for $\tilde{T}>0$ sufficiently small. Combining (6.11)–(6.12), we conclude the proof of (i).

(ii) From (6.10) and Hölder's inequality it follows that

$$\|b_t\|_{L_t^2 H_x^s} \lesssim \|\nabla v\|_{L_t^2 H_x^s}\|b\|_{L_t^\infty H_x^s}^2 + \|\nabla v\|_{L_t^2 H_x^s}\|b\|_{L_t^\infty H_x^s} + \|\nabla v\|_{L_t^2 H_x^s} \lesssim M,$$

which gives (ii).

(iii) From (2.13) and (6.1) we infer that J satisfies the ODE system

$$\begin{aligned} &J_t = \phi_{\tilde{T}} J a_{kj}\partial_k v_j \text{ in } [0,1]\times\Omega_{\mathrm{f}}, \\ &J(0) = 1 \text{ in } \Omega_{\mathrm{f}}. \end{aligned} \tag{6.13}$$

The solution is given by

$$J(t,x) = e^{\int_0^t \phi_{\tilde{T}}(\tau)a_{kj}(\tau,x)\partial_k v_j(\tau,x)\,d\tau} \text{ in } [0,1]\times\Omega_{\mathrm{f}}.$$

Using the nonlinear Sobolev estimate, we have

$$\|J-1\|_{L_t^\infty H_x^s} \lesssim C^{\tilde{T}^{1/2}M} - 1 \lesssim \tilde{T}^{1/20},$$

for $\tilde{T}>0$ sufficiently small.

The proofs of (iv), (v), and (vi) are analogous to the proof of Lemma 5.2, and thus we omit the details.

The following lemma provides the necessary a priori density estimates.

Lemma 6.2. *Assume that*

$$(R_0, R_0^{-1}, b) \in H^s(\Omega_f) \times H^s(\Omega_f) \times L^\infty((0,T), H^s(\Omega_f))$$

and $\|v\|_{K^{s+1}((0,1)\times\Omega_f)} \le M$*, where* $M \ge 1$*. Let* $\delta \in (0,1/2)$*. Then for* $\tilde T > 0$ *sufficiently small depending on* M *and* δ*, the solution to the ODE system* (6.2)–(6.3) *satisfies*

(i) $\|R\|_{L_t^\infty L_x^\infty} + \|R^{-1}\|_{L_t^\infty L_x^\infty} + \|R\|_{L_t^\infty H_x^s} + \|R^{-1}\|_{L_t^\infty H_x^s} \lesssim 1$,
(ii) $\|R^{-1}\|_{H_t^1 H_x^{3/2+\delta}} + \|R\|_{H_t^1 H_x^{3/2+\delta}} \lesssim 1$,
(iii) $\|R\|_{H_t^1 H_x^s} \lesssim M$,

where the norm of dependence is $(0,1)\times\Omega_f$*.*

We emphasize that the implicit constants in the above inequalities (i)–(iii) are independent of M and δ. The proof of Lemma 6.2 is analogous to the proof of Lemma 5.2. Thus we omit the details.

6.2. Uniform Boundedness of the Iterative Sequence

In this section we shall prove that the mapping Π is well-defined from $\mathcal{Z}$ to $\mathcal{Z}$, for a sufficiently large constant $M \ge 1$ and a sufficiently small constant $\tilde T > 0$. From Lemmas 4.2 and 6.2, it follows that

$$\begin{aligned}\|\bar v\|_{K^{s+1}} \lesssim \left\|\frac{\partial w}{\partial \nu}\right\|_{K_{\Gamma_c}^{s-1/2}} + \|f\|_{K^{s-1}} + \|h\|_{K_{\Gamma_c}^{s-1/2}} + \|f(0)\|_{H^{s-2}} + \|v_0\|_{H^s} \\ + \|v_0|_{\Gamma_c}\|_{H^{s+1/2}(\Gamma_c)} + \|\partial_3 v_0|_{\Gamma_f}\|_{H^{s-1/2}(\Gamma_f)},\end{aligned} \tag{6.14}$$

where f and h are as in (6.8)–(6.9). Here and below, the time and space domains in the norms are understood to be $(0,1)$ and Ω_f, respectively, unless indicated otherwise. We emphasize that the implicit constants in this section are independent of M.

For the first term on the right side of (6.14), we proceed as in (5.44)–(5.48) to obtain

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{K^{s-1/2}(\Gamma_c)} \lesssim (\epsilon + \tilde\epsilon C_\epsilon + \tilde T^{1/2} C_{\tilde\epsilon,\epsilon})M + C_\epsilon, \tag{6.15}$$

for any $\epsilon, \tilde\epsilon \in (0,1]$. Next, we estimate the K^{s-1} norm of the terms on the right side of (6.8) for $j = 1,2,3$. For the space component of the term I_1 in (6.8), we use Hölder's inequality and Lemmas 6.1–6.2 to get

$$\|I_1\|_{L_t^2 H_x^{s-1}} \lesssim \|R\nabla b \nabla \bar v\|_{L_t^2 H_x^{s-1}} + \|R b D_x^2 \bar v\|_{L_t^2 H_x^{s-1}} \lesssim \|b\|_{L_t^\infty H_x^s} \|\bar v\|_{L_t^2 H_x^{s+1}} \lesssim \tilde T^{1/20} \|\bar v\|_{K^{s+1}}. \tag{6.16}$$

For the time component, we have

$$\|I_1\|_{H_t^{(s-1)/2} L_x^2} \lesssim \|R\nabla b\nabla \bar v\|_{H_t^{(s-1)/2} L_x^2} + \|R b D_x^2 \bar v\|_{H_t^{(s-1)/2} L_x^2}. \tag{6.17}$$

To treat the first term on the right side, we claim that for any $\alpha > 1/2$ we have

$$\|AB\|_{H_t^\alpha L_x^2} \lesssim \|A\|_{H_t^\alpha H_x^1} \|B\|_{H_t^\alpha H_x^{1/2}} \tag{6.18}$$

on the domain $(0,1)\times\Omega_f$. Using extensions, we may assume that the domain is actually $\mathbb{R}\times\mathbb{R}^3$. We proceed as in (5.69) and estimate

$$\begin{aligned}\|AB\|_{H_t^\alpha L_x^2} &= \|AB\|_{L_x^2 H_t^\alpha} \lesssim \big\| \|A\|_{H_t^\alpha} \|B\|_{H_t^\alpha} \big\|_{L_x^2} \\ &\lesssim \|A\|_{L_x^6 H_t^\alpha} \|B\|_{L_x^3 H_t^\alpha} \lesssim \|A\|_{H_x^1 H_t^\alpha} \|B\|_{H_x^{1/2} H_t^\alpha} \\ &= \|A\|_{H_t^\alpha H_x^1} \|B\|_{H_t^\alpha H_x^{1/2}},\end{aligned}$$

since $\alpha > 1/2$, and (6.18) follows. For the first term on the right side of (6.17), we now use (6.18) and write

$$\begin{aligned}\|R\nabla b\nabla\bar v\|_{H_t^{(s-1)/2}L_x^2} &\lesssim \|\nabla\bar v\|_{H_t^{(s-1)/2}H_x^1}\|R\nabla b\|_{H_t^1H_x^{1/2}}\\ &\lesssim \|\bar v\|_{K^{s+1}}(\|R_t\nabla b\|_{L_t^2H_x^{1/2}}+\|R\nabla b_t\|_{L_t^2H_x^{1/2}}+\|R\nabla b\|_{L_t^2H_x^{1/2}})\\ &\lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}},\end{aligned} \tag{6.19}$$

for any $\epsilon \in (0,1]$, where we used Corollary 3.4 and Lemmas 6.1–6.2. For the second term on the right side of (6.17), we appeal to (5.68) to get

$$\begin{aligned}\|RbD_x^2\bar v\|_{H_t^{(s-1)/2}L_x^2} &\lesssim \|Rb\|_{H_t^{(s-1)/2}H_x^{3/2+\delta}}\|D_x^2\bar v\|_{H_t^{(s-1)/2}L_x^2} \lesssim \|Rb\|_{H_t^1H_x^{3/2+\delta}}\|\bar v\|_{K^{s+1}}\\ &\lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}},\end{aligned} \tag{6.20}$$

for $\delta \in (0,1/2)$, where we used Corollary 3.4 and Lemmas 6.1–6.2. Combining (6.16)–(6.17) and (6.19)–(6.20), we obtain

$$\|I_1\|_{K^{s-1}} \lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}}.$$

The terms I_2, I_3, I_4, I_5, and I_6 are estimated analogously to I_1, and we get

$$\|I_2\|_{K^{s-1}}+\|I_3\|_{K^{s-1}}+\|I_4\|_{K^{s-1}}+\|I_5\|_{K^{s-1}}+\|I_6\|_{K^{s-1}} \lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}}.$$

For the term I_7, we use Hölder's inequality and obtain

$$\|I_7\|_{L_t^2H_x^{s-1}} \lesssim \|R^{-1}b\nabla R\|_{L_t^2H_x^{s-1}} \lesssim \|R^{-1}\|_{L_t^\infty H_x^s}\|b\|_{L_t^\infty H_x^s}\|R\|_{L_t^\infty H_x^s} \lesssim 1$$

and

$$\begin{aligned}\|I_7\|_{H_t^{(s-1)/2}L_x^2} &\lesssim \|R^{-1}b\nabla R\|_{H_t^{(s-1)/2}L_x^2} \lesssim \|R^{-1}b\nabla R\|_{H_t^1L_x^2}\\ &\lesssim \|R^{-1}\|_{H_t^1L_x^\infty}\|b\|_{L_t^\infty L_x^\infty}\|\nabla R\|_{L_t^\infty L_x^2}+\|R^{-1}\|_{L_t^\infty L_x^\infty}\|b\|_{L_t^\infty L_x^\infty}\|\nabla R\|_{H_t^1L_x^2}\\ &\quad+\|R^{-1}\|_{L_t^\infty L_x^\infty}\|b\|_{H_t^1L_x^\infty}\|\nabla R\|_{L_t^\infty L_x^2} \lesssim 1,\end{aligned}$$

where we appealed to Lemmas 6.1–6.2. The term I_8 is estimated analogously to I_7, leading to

$$\|I_8\|_{K^{s-1}} \lesssim 1.$$

Using the estimates on I_1–I_8 in (6.8), we conclude that

$$\|f\|_{K^{s-1}} \lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}}+1. \tag{6.21}$$

Next, we bound the $K^{s-1/2}(\Gamma_c)$ norm of the terms on the right side of (6.9), for every fixed $j=1,2,3$. For K_1, we use Hölder's and trace inequalities along with Lemma 6.1 to obtain

$$\|K_1\|_{L_t^2H_x^{s-1/2}(\Gamma_c)} \lesssim \|(1-J)\nabla\bar v\|_{L_t^2H_x^s} \lesssim \|1-J\|_{L_t^\infty H_x^s}\|\bar v\|_{L_t^2H_x^{s+1}} \lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}}. \tag{6.22}$$

For the time component, we appeal to Corollary 3.2, obtaining

$$\begin{aligned}\|K_1\|_{H_t^{s/2-1/4}L_x^2(\Gamma_c)} &\lesssim \epsilon_1\|(1-J)\nabla\bar v\|_{H_t^{s/2}L_x^2}+\epsilon_1^{1-2s}\|(1-J)\nabla\bar v\|_{L_t^2H_x^s}\\ &\lesssim \epsilon_1\|\nabla\bar v\|_{H_t^{s/2}L_x^2}\|1-J\|_{H_t^1H_x^{3/2+\delta}}+\epsilon_1\|\nabla\bar v\|_{H_t^{(s-1)/2}H_x^1}\|1-J\|_{H_t^{s/2}H_x^{1/2}}\\ &\quad+\epsilon_1^{1-2s}\|1-J\|_{L_t^\infty H_x^s}\|\nabla\bar v\|_{L_t^2H_x^s} =: K_{11}+K_{12}+K_{13},\end{aligned} \tag{6.23}$$

for any $\epsilon_1 \in (0,1]$, where $\delta \in (0,1/5)$. For the term K_{11}, we use Corollary 3.4 and Lemma 6.1 and obtain

$$K_{11} \lesssim \epsilon_1\|\bar v\|_{K^{s+1}}. \tag{6.24}$$

Similarly, the term K_{12} is estimated as

$$K_{12} \lesssim \epsilon_1\|\bar v\|_{K^{s+1}}\|1-J\|_{H_t^{s/2}H_x^{1/2}}. \tag{6.25}$$

From (6.13), Corollary 3.4, and Lemma 6.1, it follows that

$$
\begin{aligned}
\|J_t\|_{H_t^{(s-2)/2}H_x^{1/2}} &\lesssim \|\phi_{\tilde T}\|_{H_t^{(s-2)/2}}\|Ja\nabla v\|_{H_t^{(s-1)/2}H_x^{1/2}} \\
&\lesssim \|\phi_{\tilde T}\|_{H_t^{(s-2)/2}}\|J\|_{H_t^{(s-1)/2}H_x^{3/2+\delta}}\|a\|_{H_t^{(s-1)/2}H_x^{3/2+\delta}}\|\nabla v\|_{H_t^{(s-1)/2}H_x^{1}} \\
&\lesssim \|\phi_{\tilde T}\|_{H_t^{(s-2)/2}}M,
\end{aligned}
\tag{6.26}
$$

since $1/2 < (s-1)/2 < 1$ and $\delta \in (0,1/5)$. Using Lemma 5.1 in (6.26) and applying the resulting inequality in (6.25), we get

$$
K_{12} \lesssim \epsilon_1\|\bar v\|_{K^{s+1}}(\|J_t\|_{H_t^{(s-2)/2}H_x^{1/2}} + \|J\|_{H_t^{(s-2)/2}H_x^{1/2}}) \lesssim \epsilon_1\|\bar v\|_{K^{s+1}}(M+1) \lesssim \epsilon\|\bar v\|_{K^{s+1}}, \tag{6.27}
$$

where $\epsilon \in (0,1]$, by taking $\epsilon_1 = \epsilon M^{-1}$. For the term K_{13}, we have

$$
K_{13} \lesssim C_\epsilon M^{2s-1}\tilde T^{1/20}\|\bar v\|_{K^{s+1}}, \tag{6.28}
$$

for $\tilde T > 0$ sufficiently small. Combining (6.22)–(6.24) and (6.27)–(6.28), we arrive at

$$
\|K_1\|_{K_{\Gamma_c}^{s-1/2}} \lesssim \epsilon\|\bar v\|_{K^{s+1}}, \tag{6.29}
$$

for any $\epsilon \in (0,1]$, by taking $\tilde T > 0$ sufficiently small. The term K_2 is estimated analogously to K_1, and we obtain

$$
\|K_2\|_{K_{\Gamma_c}^{s-1/2}} \lesssim \epsilon\|\bar v\|_{K^{s+1}}. \tag{6.30}
$$

For the space component of the term K_3, we use Hölder's and trace inequalities to obtain

$$
\|K_3\|_{L_t^2H_x^{s-1/2}} \lesssim \|J\|_{L_t^\infty H_x^s}\|b\|_{L_t^\infty H_x^s}^2\|\bar v\|_{L_t^2H_x^{s+1}} \lesssim \tilde T^{1/20}\|\bar v\|_{K^{s+1}}, \tag{6.31}
$$

where we appealed to Lemma 6.1. For the time component, using Corollary 3.2, we have

$$
\begin{aligned}
\|K_3\|_{H_t^{s/2-1/4}L_x^2(\Gamma_c)} &\lesssim \epsilon_1\|Jbb\nabla\bar v\|_{H_t^{s/2}L_x^2} + \epsilon_1^{1-2s}\|Jbb\nabla\bar v\|_{L_t^2H_x^s} \\
&\lesssim \epsilon_1\|J\|_{H_t^{s/2}H_x^{1/2}}\|\nabla\bar v\|_{H_t^{(s-1)/2}H_x^1} + \epsilon_1\|b\|_{H_t^{s/2}H_x^{1/2}}\|\nabla\bar v\|_{H_t^{(s-1)/2}H_x^1} \\
&\quad + \epsilon_1\|\nabla\bar v\|_{H_t^{s/2}L_x^2} + \epsilon_1^{1-2s}\|b\|_{L_t^\infty H_x^s}\|\bar v\|_{L_t^2H_x^{s+1}} =: K_{31}+K_{32}+K_{33}+K_{34},
\end{aligned}
\tag{6.32}
$$

for any $\epsilon_1 \in (0,1]$. The term K_{31} is estimated analogously to (6.25)–(6.27), and we obtain

$$
K_{31} \lesssim \epsilon\|\bar v\|_{K^{s+1}},
$$

by taking $\epsilon_1 = \epsilon M^{-1}$ in (6.32), where $\epsilon \in (0,1]$ is a constant. The term $\|b\|_{H_t^{s/2}H_x^{1/2}}$ is estimated analogously to (6.25)–(6.27), and we get

$$
\|b\|_{H_t^{s/2}L_x^3} \lesssim M+1.
$$

Therefore, we infer that

$$
K_{32} \lesssim \epsilon\|\bar v\|_{K^{s+1}}.
$$

The term K_{33} is estimated using Corollary 3.4 as

$$
K_{33} \lesssim \epsilon\|\bar v\|_{K^{s+1}},
$$

while the term K_{34} is estimated analogously to (6.28) as

$$
K_{34} \lesssim \epsilon\|\bar v\|_{K^{s+1}},
$$

by taking $\tilde T > 0$ sufficiently small. Combining (6.31)–(6.32) and the estimates on K_{31}–K_{34}, we conclude that

$$
\|K_3\|_{K_{\Gamma_c}^{s-1/2}} \lesssim \epsilon\|\bar v\|_{K^{s+1}}, \tag{6.33}
$$

for any $\epsilon \in (0,1]$. Regarding the term K_4, we proceed as in (5.51)–(5.52) to obtain

$$
\|K_4\|_{K_{\Gamma_c}^{s-1/2}} \lesssim \|JbR^{-1}\|_{L_t^2H_x^s} + \|JbR^{-1}\|_{H_t^1H_x^1} \lesssim 1 + \|J_t\|_{L_t^2H_x^1} + \|b_t\|_{L_t^2H_x^1} + \|R_t\|_{L_t^2H_x^1} \lesssim 1, \tag{6.34}
$$

where we used Lemmas 6.1–6.2. The term K_5 is estimated in a similar fashion as K_4, and we arrive at

$$\|K_5\|_{K^{s-1/2}_{\Gamma_c}} \lesssim 1. \tag{6.35}$$

The terms K_6, K_7, K_8, K_9, and K_{10} are estimated analogously to K_3, and we have

$$\|K_6\|_{K^{s-1/2}_{\Gamma_c}} + \|K_7\|_{K^{s-1/2}_{\Gamma_c}} + \|K_7\|_{K^{s-1/2}_{\Gamma_c}} + \|K_8\|_{K^{s-1/2}_{\Gamma_c}} + \|K_9\|_{K^{s-1/2}_{\Gamma_c}} + \|K_{10}\|_{K^{s-1/2}_{\Gamma_c}} \lesssim \epsilon\|\bar{v}\|_{K^{s+1}}, \tag{6.36}$$

for any $\epsilon \in (0,1]$. For the term K_{11}, we proceed as in (5.51)–(5.52) using Lemma 6.2 to obtain

$$K_{11} \lesssim 1. \tag{6.37}$$

Collecting the estimates (6.29)–(6.30) and (6.33)–(6.37), we conclude

$$\|h\|_{K^{s-1/2}_{\Gamma_c}} \lesssim \epsilon\|\bar{v}\|_{K^{s+1}} + 1, \tag{6.38}$$

for any $\epsilon \in (0,1]$. For the fourth term on the right side of (6.14), we have

$$\|f(0)\|_{H^{s-2}} \lesssim \|R_0^{-1}\nabla R_0\|_{H^{s-2}} \lesssim 1. \tag{6.39}$$

From (6.14)–(6.15), (6.21), and (6.38)–(6.39) it follows that

$$\|\bar{v}\|_{K^{s+1}} \lesssim (\epsilon + \tilde{T}^{1/20})\|\bar{v}\|_{K^{s+1}} + (\epsilon + \tilde{\epsilon}C_\epsilon + \tilde{T}^{1/2}C_{\tilde{\epsilon},\epsilon})M + C_\epsilon,$$

for any $\epsilon, \tilde{\epsilon} \in (0,1]$. We first take ϵ sufficiency small, and $\tilde{\epsilon}$ sufficiently small depending on ϵ, and then $\tilde{T}$ sufficiently small depending on $\epsilon, \tilde{\epsilon}$, yielding

$$\|\bar{v}\|_{K^{s+1}} \leq M, \tag{6.40}$$

by allowing $M \geq 1$ sufficiently large. Thus, the mapping $\Pi\colon v \mapsto \bar{v}$ is well-defined from $\mathcal{Z}$ to $\mathcal{Z}$, for some constant $M \geq 1$, which depends on the size of the initial data.

6.3. Contracting Property

In this section, we prove

$$\|\Pi(v_1) - \Pi(v_2)\|_{K^{s+1}} \leq \frac{1}{2}\|v_1 - v_2\|_{K^{s+1}}, \qquad v_1, v_2 \in \mathcal{Z}, \tag{6.41}$$

where $M \geq 1$ is fixed as in (6.40) and $\tilde{T}$ sufficiently small. Note that the implicit constants below are allowed to depend on M. Let $\tilde{T} > 0$ be a sufficiently small constant such that Lemmas 6.1–6.2 hold.

Let $v_1, v_2 \in \mathcal{Z}$ and (η_1, η_2) be the corresponding Lagrangian flow maps as in (6.1). Denote by (J_1, a_1) and (J_2, a_2) the Jacobians and the inverse matrices of the corresponding flow map. First we solve for (R_1, R_2) from (6.2)–(6.3) with the same initial data R_0. Then we solve for (ξ_1, ξ_{1t}) and (ξ_2, ξ_{2t}) from (5.12) with the boundary conditions (5.13)–(5.16) and the same initial data (w_0, w_1). To obtain the next iterate $(\bar{v}_1, \bar{v}_2)$, we solve (5.40) with the boundary conditions and the initial data (5.41). Denote $b_1 = a_1 - \mathbb{I}_3$, $b_2 = a_2 - \mathbb{I}_3$, $\tilde{b} = b_1 - b_2$, $\tilde{V} = \bar{v}_1 - \bar{v}_2$, $\tilde{v} = v_1 - v_2$, $\tilde{R} = R_1 - R_2$, $\tilde{\xi} = \xi_1 - \xi_2$, $\tilde{\eta} = \eta_1 - \eta_2$, and $\tilde{J} = J_1 - J_2$. The difference $\tilde{V}$ satisfies

$$\begin{aligned}
&\partial_t \tilde{V}_j - \lambda\tilde{R}\partial_k(\partial_j\tilde{V}_k + \partial_k\tilde{V}_j) - \mu\tilde{R}\partial_j\partial_k\tilde{V}_k = \tilde{f}_j \text{ in } [0,1]\times\Omega_{\text{f}}\,,\\
&\lambda(\partial_k\tilde{V}_j + \partial_j\tilde{V}_k)\nu^k + \mu\partial_k\tilde{V}_k\nu^j = \partial_k\tilde{\xi}_j\nu^k + \tilde{h}_j \text{ in } [0,1]\times\Gamma_{\text{c}}\,,\\
&\tilde{V} \;\text{ periodic in the } y_1 \text{ and } y_2 \text{ directions},\\
&\tilde{V}(0) = 0 \text{ in } \Omega_{\text{f}}\,,
\end{aligned} \tag{6.42}$$

where

$$\begin{aligned}\tilde{f}_j &= \lambda\tilde{R}\partial_k(\partial_j\bar{v}_{2k}+\partial_k\bar{v}_{2j}) + \mu\tilde{R}\partial_j\partial_k\bar{v}_{2k} + \lambda\tilde{R}\partial_k(b_{1mk}\partial_m\bar{v}_{1j}+b_{1mj}\partial_m\bar{v}_{1k})\\
&\quad + \lambda R_2\partial_k(b_{1mk}\partial_m\tilde{V}_j + b_{1mj}\partial_m\tilde{V}_k) + \lambda R_2\partial_k(\tilde{b}_{mk}\partial_m\bar{v}_{2j}+\tilde{b}_{mj}\partial_m\bar{v}_{2k})\\
&\quad + \lambda\tilde{R}b_{1kl}\partial_k(b_{1ml}\partial_m\bar{v}_{1j}+b_{1mj}\partial_m\bar{v}_{1l}) + \lambda R_2\tilde{b}_{kl}\partial_k(b_{1ml}\partial_m\bar{v}_{1j}+b_{1mj}\partial_m\bar{v}_{1l})\\
&\quad + \lambda R_2 b_{2kl}\partial_k(\tilde{b}_{ml}\partial_m\bar{v}_{1j}+\tilde{b}_{mj}\partial_m\bar{v}_{1l}) + \lambda R_2 b_{2kl}\partial_k(b_{2ml}\partial_m\tilde{V}_j + b_{2mj}\partial_m\tilde{V}_l)\\
&\quad + \lambda\tilde{R}b_{1kl}\partial_k(\partial_l\bar{v}_{1j}+\partial_j\bar{v}_{1l}) + \lambda R_2\tilde{b}_{kl}\partial_k(\partial_l\bar{v}_{1j}+\partial_j\bar{v}_{1l}) + \lambda R_2 b_{2kl}\partial_k(\partial_l\tilde{V}_j+\partial_j\tilde{V}_l)\\
&\quad + \mu R\partial_j(b_{1mi}\partial_m\bar{v}_{1i}) + \mu R_2\partial_j(\tilde{b}_{mi}\partial_m\bar{v}_{1i}) + \mu R_2\partial_j(b_{2mi}\partial_m\tilde{V})\\
&\quad + \mu\tilde{R}b_{1kj}\partial_k(b_{1mi}\partial_m\bar{v}_{1i}) + \mu R_2\tilde{b}_{kj}\partial_k(b_{1mi}\partial_m\bar{v}_{1i}) + \mu R_2 b_{2kj}\partial_k(\tilde{b}_{mi}\partial_m\bar{v}_{1i})\\
&\quad + \mu R_2 b_{2kj}\partial_k(b_{2mi}\partial_m\tilde{V}_i) + \mu\tilde{R}b_{1kj}\partial_k\partial_i\bar{v}_{1i} + \mu R_2\tilde{b}_{kj}\partial_k\partial_i\bar{v}_{1i} + \mu R_2 b_{2kj}\partial_k\partial_i\tilde{V}_i\\
&\quad - R_1^{-1}R_2^{-1}\tilde{R}b_{1kj}\partial_k R_1 + R_2^{-1}\tilde{b}_{kj}\partial_k R_1 + R_2^{-1}b_{2kj}\partial_k\tilde{R} - R_1^{-1}R_2^{-1}\tilde{R}\partial_j R_1\\
&\quad + R_2^{-1}\partial_j\tilde{R}\end{aligned} \tag{6.43}$$

and

$$\begin{aligned}\tilde{h}_j &= -\lambda\tilde{J}(\partial_k\bar{v}_{1j}+\partial_j\bar{v}_{1k})\nu^k + \lambda(1-J_2)(\partial_k\tilde{V}_j+\partial_j\tilde{V}_k)\nu^k - \mu\tilde{J}\partial_k\bar{v}_{1k}\nu^j + \mu(1-J_2)\partial_k\tilde{V}_k\nu^j\\
&\quad + \lambda\tilde{J}b_{1kl}(b_{1ml}\partial_m\bar{v}_{1j}+b_{1mj}\partial_m\bar{v}_{1l})\nu^k + \lambda J_2\tilde{b}_{kl}(b_{1ml}\partial_m\bar{v}_{1j}+b_{1mj}\partial_m\bar{v}_{1l})\nu^k\\
&\quad + \lambda J_2 b_{2kl}(\tilde{b}_{ml}\partial_m\bar{v}_{1j}+\tilde{b}_{mj}\partial_m\bar{v}_{1l})\nu^k + \lambda J_2 b_{2kl}(b_{2ml}\partial_m\tilde{V}_j+b_{2mj}\partial_m\tilde{V}_l)\nu^k\\
&\quad + \tilde{J}b_{1kj}R_1^{-1}\nu^k + J_2\tilde{b}_{kj}R_1^{-1}\nu^k - J_2 b_{2kj}R_1^{-1}R_2^{-1}\tilde{R}\nu^k - \tilde{J}R_1^{-1}\nu^j - (J_2-1)R_1^{-1}R_2^{-1}\tilde{R}\nu^j\\
&\quad - \lambda\tilde{J}(b_{1mk}\partial_m\bar{v}_{1j}+b_{1mj}\partial_m\bar{v}_{1k})\nu^k - \lambda J_2(\tilde{b}_{mk}\partial_m\bar{v}_{1j}+\tilde{b}_{mj}\partial_m\bar{v}_{1k})\nu^k\\
&\quad - \lambda J_2(b_{2mk}\partial_m\tilde{V}_j+b_{2mj}\partial_m\tilde{V}_{1k})\nu^k - \lambda\tilde{J}b_{1kl}(\partial_l\bar{v}_{1j}+\partial_j\bar{v}_{1l})\nu^k - \lambda J_2\tilde{b}_{kl}(\partial_l\bar{v}_{1j}+\partial_j\bar{v}_{1l})\nu^k\\
&\quad - \lambda J_2 b_{2kl}(\partial_l\tilde{V}_j+\partial_j\tilde{V}_l)\nu^k - \mu\tilde{J}b_{1kj}b_{1mi}\partial_m\bar{v}_{1i}\nu^k - \mu J_2\tilde{b}_{kj}b_{1mi}\partial_m\bar{v}_{1i}\nu^k\\
&\quad - \mu J_2 b_{2kj}\tilde{b}_{mi}\partial_m\bar{v}_{1i}\nu^k - \mu J_2 b_{2kj}b_{2mi}\partial_m\tilde{V}_i\nu^k - \mu\tilde{J}b_{1mi}\partial_m\bar{v}_{1i}\nu^j - \mu J_2\tilde{b}_{mi}\partial_m\bar{v}_{1i}\nu^j\\
&\quad - \mu J_2 b_{2mi}\partial_m\tilde{V}_i\nu^j - \mu\tilde{J}b_{1kj}\partial_i\bar{v}_{1i}\nu^k - \mu J_2\tilde{b}_{kj}\partial_i\bar{v}_{1i}\nu^k - \mu J_2 b_{2kj}\partial_i\tilde{V}_i\nu^k - R_1^{-1}R_2^{-1}\tilde{R}\nu^j,\end{aligned} \tag{6.44}$$

for $j=1,2,3$.

Before we bound the terms on the right sides of (6.43) and (6.44), we provide necessary a priori estimates for the differences of densities, Jacobians, and inverse matrices of the flow map.

Lemma 6.3. *Let $v_1, v_2 \in \mathcal{Z}$. Suppose $\|v_1\|_{K^{s+1}((0,1)\times\Omega_f)} \le M$ and $\|v_2\|_{K^{s+1}((0,1)\times\Omega_f)} \le M$, where $M \ge 1$ is fixed as in (5.54). Let $\delta \in (0,1/5)$. Then, for $\tilde{T} > 0$ sufficiently small depending on δ, we have*

(i) $\|\tilde{b}\|_{L_t^\infty H_x^s} + \|\tilde{b}\|_{H_t^1 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20}\|\tilde{v}\|_{K^{s+1}}$,
(ii) $\|\tilde{R}\|_{L_t^\infty H_x^s} + \|\tilde{R}\|_{H_t^1 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20}\|\tilde{v}\|_{K^{s+1}}$,
(iii) $\|\tilde{J}\|_{L_t^\infty H_x^s} + \|\tilde{J}\|_{H_t^1 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20}\|\tilde{v}\|_{K^{s+1}}$,
(iv) $\|\tilde{R}\|_{H_t^1 H_x^s} + \|\tilde{b}\|_{H_t^1 H_x^s} + \|\tilde{J}\|_{H_t^1 H_x^s} \lesssim \|\tilde{v}\|_{K^{s+1}}$,

for any $\delta \in (0,1)$, where the norm of dependence is $(0,1)\times\Omega_f$.

Proof of Lemma 6.3. (i) From (6.10) it follows that the difference $\tilde{b}$ satisfies

$$\begin{aligned}-\tilde{b}_t = \phi_{\tilde{T}}\Big(&\tilde{b}(\nabla v_1)b_1 + b_2(\nabla\tilde{v})b_1 + b_2(\nabla v_2)\tilde{b} + (\nabla\tilde{v})b_1 + (\nabla v_2)\tilde{b} + \tilde{b}(\nabla v_1)\\
&+ b_2(\nabla\tilde{v}) + \nabla\tilde{v}\Big) \text{ in } [0,1]\times\Omega_{\mathrm{f}},\end{aligned} \tag{6.45}$$

with the initial data $\tilde{b}(0) = 0$. Using the fundamental theorem of calculus, we obtain that for $t \in (0, 2\tilde{T})$

$$\begin{aligned}\|\tilde{b}(t)\|_{H^s} &\lesssim \int_0^t \|\tilde{b}\|_{H^s}\|\nabla v_1\|_{H^s}\|b_1\|_{H^s} + \int_0^t \|b_2\|_{H^s}\|\nabla\tilde{v}\|_{H^s}\|b_1\|_{H^s} \\ &\quad + \int_0^t \|b_2\|_{H^s}\|\nabla v_2\|_{H^s}\|\tilde{b}\|_{H^s} + \int_0^t \|\nabla\tilde{v}\|_{H^s}\|b_1\|_{H^s} + \int_0^t \|\nabla\tilde{v}\|_{H^s} \\ &\quad + \int_0^t \|\nabla v_2\|_{H^s}\|\tilde{b}\|_{H^s} + \int_0^t \|\tilde{b}\|_{H^s}\|\nabla v_1\|_{H^s} + \int_0^t \|b_2\|_{H^s}\|\nabla\tilde{v}\|_{H^s} \\ &\lesssim \int_0^t \|\tilde{v}\|_{H^{s+1}} + \int_0^t \|\tilde{b}\|_{H^s}(\|v_1\|_{H^{s+1}} + \|v_2\|_{H^{s+1}}),\end{aligned}$$

where the last inequality follows from Lemma 6.1. Using Gronwall's inequality, we arrive at

$$\|\tilde{b}\|_{L^\infty((0,2\tilde{T}),H^s(\Omega_f))} \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}}. \tag{6.46}$$

Therefore, we have

$$\|\tilde{b}\|_{L_t^\infty H_x^s} \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}}, \tag{6.47}$$

since $\tilde{b}_t = 0$ on $(2\tilde{T}, 1)$. From (6.45) and Hölder's and the Sobolev inequalities it follows that

$$\begin{aligned}\|\tilde{b}_t\|_{L_t^2 H_x^{3/2+\delta}} &\lesssim \|\tilde{b}\|_{L_t^\infty H_x^{3/2+\delta}} + \|\nabla\tilde{v}\|_{L^2((0,2\tilde{T}),H^{3/2+\delta}(\Omega_f))} \\ &\lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}} + \epsilon_1\|\tilde{v}\|_{L_t^2 H_x^{s+1}} + \epsilon_1^{(3+2\delta)/(3+2\delta-2s)}\|\tilde{v}\|_{L^2((0,2\tilde{T}),L^2(\Omega_f))},\end{aligned} \tag{6.48}$$

for any $\epsilon_1 \in (0, 1]$, where we used Corollary 3.4 and Lemma 6.1. Letting $\epsilon_1 = \tilde{T}^{1/20}$, we obtain

$$\|\tilde{b}_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20}\|\tilde{v}\|_{K^{s+1}} + \tilde{T}^{1/2+(3+2\delta)/20(3+2\delta-2s)}\|\tilde{v}\|_{K^{s+1}} \lesssim \tilde{T}^{1/20}\|\tilde{v}\|_{K^{s+1}}. \tag{6.49}$$

Combining (6.47) and (6.49), we conclude the proof of (i).

(ii) Since the difference $\tilde{R}$ satisfies the ODE system

$$\tilde{R}_t - \phi_{\tilde{T}}\tilde{R}(\operatorname{div} v_2 + b_{1kj}\partial_k v_{1j}) = \phi_{\tilde{T}}(R_1 \operatorname{div} \tilde{v} + R_2\tilde{b}_{kj}\partial_k v_{1j} + R_2 b_{2kj}\partial_k\tilde{v}_j) \text{ in } [0, 1] \times \Omega_f, \tag{6.50}$$

$$\tilde{R}(0) = 0 \text{ in } \Omega_f, \tag{6.51}$$

the solution is given by

$$\begin{aligned}\tilde{R}(t, x) &= e^{\int_0^t \phi_{\tilde{T}}(\operatorname{div} v_2 + b_{1kj}\partial_k v_{1j})\,d\tau} \int_0^t e^{-\int_0^\tau \phi_{\tilde{T}}(\operatorname{div} v_2 + b_{1kj}\partial_k v_{1j})} \\ &\qquad \times \phi_{\tilde{T}}(R_1 \operatorname{div} \tilde{v} + R_2\tilde{b}_{kj}\partial_k v_{1j} + R_2 b_{2kj}\partial_k\tilde{v}_j)\,d\tau \text{ in } [0, 1] \times \Omega_f.\end{aligned}$$

From Hölder's inequality, it follows that

$$\|\tilde{R}\|_{L_t^\infty H_x^s} \lesssim \int_0^{2\tilde{T}} (\|\tilde{v}\|_{H^{s+1}} + \|\tilde{b}\nabla v_1\|_{H^s})\,d\tau \lesssim \tilde{T}^{1/2}\|\tilde{v}\|_{K^{s+1}}, \tag{6.52}$$

where we used (6.47) in the last inequality. Using (6.50) and Hölder's and Sobolev inequalities, we obtain

$$\begin{aligned}\|\tilde{R}_t\|_{L_t^2 H_x^{3/2+\delta}} &\lesssim \|\tilde{R}\|_{L_t^\infty H_x^{3/2+\delta}}\|\nabla v_2\|_{L_t^2 H_x^{3/2+\delta}} + \|\tilde{R}\|_{L_t^\infty H_x^{3/2+\delta}}\|b_1\|_{L_t^\infty H_x^{3/2+\delta}}\|\nabla v_1\|_{L_t^2 H_x^{3/2+\delta}} \\ &\quad + \|\tilde{v}\|_{L^2((0,2\tilde{T}),H^{5/2+\delta}(\Omega_f))} + \|\tilde{b}\|_{L_t^\infty H_x^{3/2+\delta}}\|\nabla v_1\|_{L_t^2 H_x^{3/2+\delta}} + \|\tilde{v}\|_{L^2((0,2\tilde{T}),H^{5/2+\delta}(\Omega_f))}.\end{aligned} \tag{6.53}$$

We proceed analogously to Lemma 6.2 to get

$$\|\tilde{R}_t\|_{L_t^2 H_x^{3/2+\delta}} \lesssim \tilde{T}^{1/20}\|\tilde{v}\|_{K^{s+1}}.$$

By combining (6.52)–(6.53), we conclude the proof of (ii)

The proofs of (iii) and (iv) are analogous to the proofs of (i)–(iii), and thus we omit the details.

Proof of Theorem 2.1. From Lemmas 4.2 and 6.2, it follows that the solution $\tilde V$ of (6.42) satisfies

$$\|\tilde V\|_{K^{s+1}((0,1)\times\Omega_{\rm f})}\lesssim\left\|\frac{\partial\tilde\xi}{\partial\nu}\right\|_{K^{s-1/2}((0,1)\times\Gamma_{\rm c})}+\|\tilde h\|_{K^{s-1/2}((0,1)\times\Gamma_{\rm c})}+\|\tilde f\|_{K^{s-1}((0,1)\times\Omega_{\rm f})}, \tag{6.54}$$

where $\tilde f_j$ and $\tilde h_j$ are as in (6.43)–(6.44), for $j=1,2,3$.

For the first term on the right side of (6.54), we proceed as in (5.44)–(5.48) to obtain

$$\left\|\frac{\partial\tilde\xi}{\partial\nu}\right\|_{K^{s-1/2}(\Gamma_{\rm c})}\lesssim(\epsilon+\tilde\epsilon C_\epsilon+\tilde T^{1/2}C_{\tilde\epsilon,\epsilon})\|\tilde v\|_{K^{s+1}}, \tag{6.55}$$

for any $\epsilon,\tilde\epsilon\in(0,1]$.

Next we estimate the K^{s-1} norm of terms on the right side of (6.43) for $j=1,2,3$. The space component of the term $\tilde R b_{1kj}\partial_k(b_{1mi}\partial_m\bar v_{1i})$ is bounded as

$$\|\tilde R b_{1kj}\partial_k(b_{1mi}\partial_m\bar v_{1i})\|_{L_t^2H_x^{s-1}}\lesssim\|\tilde R\|_{L_t^\infty H_x^s}\|b_1\|^2_{L_t^\infty H_x^s}\|\bar v_1\|_{L_t^2H_x^{s+1}}\lesssim\tilde T^{1/20}\|\tilde v\|_{K^{s+1}},$$

while for the time component, we have

$$\begin{aligned}\|\tilde R b_{1kj}\partial_k(b_{1mi}\partial_m\bar v_{1i})\|_{H_t^{(s-1)/2}L_x^2}&\lesssim\|\tilde R b_1\partial_k b_1\nabla\bar v_1\|_{H_t^{(s-1)/2}L_x^2}+\|\tilde R b_1 b_1 D_x^2\bar v_1\|_{H_t^{(s-1)/2}L_x^2}\\&\lesssim\|\tilde R\|_{H_t^{(s-1)/2}H_x^{3/2+\delta}}\|\nabla\bar v_1\|_{H_t^{(s-1)/2}H_x^1}+\|\tilde R\|_{H_t^{(s-1)/2}H_x^{3/2+\delta}}\|D_x^2\bar v_1\|_{H_t^{(s-1)/2}L_x^2}\\&\lesssim\tilde T^{1/20}\|\tilde v\|_{K^{s+1}},\end{aligned}$$

where we used Corollary 3.4 and Lemmas 6.1–6.3. Similarly, the term $\mu R_2 b_{2kj}\partial_k\partial_i\tilde V_i$ is estimated as

$$\|\mu R_2 b_{2kj}\partial_k\partial_i\tilde V_i\|_{L_t^2H_x^{s-1}}\lesssim\|R_2\|_{L_t^\infty H_x^s}\|b_2\|_{L_t^\infty H_x^s}\|D_x^2\tilde V\|_{L_t^2H_x^{s-1}}\lesssim\tilde T^{1/20}\|\tilde V\|_{K^{s+1}}$$

and

$$\|\mu R_2 b_{2kj}\partial_k\partial_i\tilde V_i\|_{H_t^{(s-1)/2}L_x^2}\lesssim\|b_2\|_{H_t^{(s-1)/2}H_x^{3/2+\delta}}\|D_x^2\tilde V\|_{H_t^{(s-1)/2}L_x^2}\lesssim\tilde T^{1/20}\|\tilde V\|_{K^{s+1}}.$$

Other terms on the right side of (6.43) are treated analogously as in the proof of Theorem 5.4 using Lemmas 6.1–6.3, and we arrive at

$$\|\tilde f\|_{K^{s-1}}\lesssim\tilde T^{1/20}\|\tilde v\|_{K^{s+1}}+\tilde T^{1/20}\|\tilde V\|_{K^{s+1}}. \tag{6.56}$$

Next we estimate the $K_{\Gamma_{\rm c}}^{s-1/2}$ norm of the terms on the right side of (6.44), for $j=1,2,3$. The term $\lambda(1-J_2)\partial_k\tilde V_j\nu^k$ is estimated using the trace inequality and Lemma 6.1 as

$$\begin{aligned}&\|\lambda(1-J_2)(\partial_k\tilde V_j+\partial_j\tilde V_k)\nu^k\|_{L_t^2H_x^{s-1/2}(\Gamma_{\rm c})}\\&\quad\lesssim\|(1-J_2)\nabla\tilde V\|_{L_t^2H_x^s}\lesssim\|1-J_2\|_{L_t^\infty H_x^s}\|\tilde V\|_{L_t^2H_x^{s+1}}\lesssim\tilde T^{1/20}\|\tilde V\|_{K^{s+1}}.\end{aligned}$$

For the time component, we proceed analogously to (6.23)–(6.28), obtaining

$$\|\lambda(1-J_2)(\partial_k\tilde V_j+\partial_j\tilde V_k)\nu^k\|_{H_t^{s/2-1/4}L_x^2(\Gamma_{\rm c})}\lesssim(\epsilon_1+C_{\epsilon_1}\tilde T^{1/30})\|\tilde V\|_{K^{s+1}},$$

for any $\epsilon_1\in(0,1]$. Other terms on the right side of (6.44) are treated analogously to Theorem 5.4 using Lemmas 6.1–6.3, and we arrive at

$$\|\tilde h\|_{K_{\Gamma_{\rm c}}^{s-1/2}}\lesssim\tilde T^{1/20}\|\tilde v\|_{K^{s+1}}+(\epsilon_1+C_{\epsilon_1}\tilde T^{1/30})\|\tilde V\|_{K^{s+1}}, \tag{6.57}$$

for any $\epsilon_1\in(0,1]$.

Since the terms involving $\|\tilde V\|_{K^{s+1}}$ on the right side of (6.56)–(6.57) are absorbed to the left side (6.54) by taking ϵ_1 and $\tilde T>0$ sufficiently small, we obtain from (6.54)–(6.57) that

$$\|\tilde V\|_{K^{s+1}}\le\frac{1}{2}\|\tilde v\|_{K^{s+1}},$$

by taking suitable ϵ, $\tilde{\epsilon}$, and $\tilde{T} > 0$. Thus the mapping Π is contracting from $\mathcal{Z}$ to $\mathcal{Z}$. Using the Banach fix point theorem, there exists a unique solution $v \in \mathcal{Z}$ such that $\Pi(v) = v$.

Fix the constant $\tilde{T} > 0$. We proceed as in (5.77)–(5.80) to obtain the interior regularity estimate

$$\|w\|_{C([0,1],H^{s+1/4-\epsilon_0}(\Omega_e))} + \|w_t\|_{C([0,1],H^{s-3/4-\epsilon_0}(\Omega_e))} \le C, \tag{6.58}$$

where $C > 0$ is a constant. From (6.58) and Lemma 6.2 it follows that the system (2.3)–(2.10) admits a unique solution

$$(v, R, w, w_t) \in K^{s+1}((0,\tilde{T}) \times \Omega_f) \times H^1((0,\tilde{T}), H^s(\Omega_f)) \\ \times C([0,\tilde{T}], H^{s+1/4-\epsilon_0}(\Omega_e)) \times C([0,\tilde{T}], H^{s-3/4-\epsilon_0}(\Omega_e)),$$

for some constant $\tilde{T} > 0$, with the corresponding norms bounded by a function of the initial data. □

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