J. Math. Fluid Mech. (2024) 26:26

1422-6928/24/020001-39
https://doi.org/10.1007/s00021-024-00860-9

Journal of Mathematical Fluid Mechanics

Non-Uniqueness and Energy Dissipation for 2D Euler Equations with Vorticity in Hardy Spaces

Miriam Buck and Stefano Modena

Communicated by R. Shvydkoy

Abstract. We construct by convex integration examples of energy dissipating solutions to the 2D Euler equations on $\mathbb{R}^2$ with vorticity in the Hardy space $H^p(\mathbb{R}^2)$, for any $2/3 < p < 1$.

Mathematics Subject Classification. 35Q31, 76B03.

Keywords. Convex integration, Euler equations, Nonuniqueness, Real Hardy spaces.

1. Introduction

In this paper we consider the 2-dimensional incompressible Euler equations on the full space $\mathbb{R}^2$

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(\cdot, 0) = u_0, \end{cases} \tag{1}$$

where $u : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ is the velocity field of some fluid and $p : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ is the corresponding (scalar) pressure.

It is well known that the system (1) is globally well posed in $W^{s,2}$ for $s > 2$, in the sense that for initial data $u_0 \in W^{s,2}$ there is a *unique* solution $u \in C([0,1], W^{s,2}(\mathbb{R}^2))$ defined on the whole time interval $[0,1]$ (more precisely on the whole time half-line $[0,+\infty)$).

It is however of fundamental importance, both mathematically and physically, to understand what happens in case of "rougher" initial data, and in particular if it is still possible, in case of rougher initial data, to prove existence and uniqueness of (weak) solutions to (1).

1.1. Short Literature Overview

The starting point of this analysis is the observation that (1) can be formally rewritten as a transport equation for the vorticity $\omega = \operatorname{curl} u$ via

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^{\perp} \Delta^{-1} \omega. \end{cases} \tag{2}$$

From (2) it is clear that the L^p norm of the vorticity of any smooth solution to (1) is conserved in time, for any $p \in [1,\infty]$. In the framework of weak solutions, it is thus natural to ask the following question:

The author would like to thank Jan Burczak for several interesting discussions on the topics considered in this paper. The authors would like to thank Phil Isett for a stimulating email exchange about some aspects of the present paper.

Q1: *For* $u_0 \in L^2(\mathbb{R}^2)$ *with* $\operatorname{curl} u_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ *for some* $p \in [1, \infty]$, *does there exist a unique solution* $u \in C([0,1], L^2(\mathbb{R}^2))$ *to* (1) *with* $\operatorname{curl} u \in C([0,1], L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$ *and initial datum* u_0?
or, more generally,

Q2: *For* $u_0 \in L^2(\mathbb{R}^2)$ *with* $\operatorname{curl} u_0 \in X$ *for some Banach space* X, *does there exist a unique solution* $u \in C([0,1], L^2(\mathbb{R}^2))$ *to* (1) *with* $\operatorname{curl} u \in C([0,1], X)$ *and initial datum* u_0?

The first result in this direction is due to Yudovich [30,31] for the case $p = \infty$ and it states that for any initial datum $u_0 \in L^2$ with $\omega_0 \in L^1 \cap L^\infty$, there exists a unique global solution $u \in C_t L^2_x$ with $\omega \in L^\infty_t (L^1_x \cap L^\infty_x)$ to (2). Yudovich result is based on the observation that even though a bounded vorticity ω does not imply Lipschitz bound on the velocity field u (hence the classical "smooth" theory can not be simply applied), nevertheless it is possible to deduce log-Lipschitz bounds on u, which are enough to show well posedness.

For $p < \infty$, the question turns out to be much more delicate (and still open in its generality to this date): indeed, an L^p bound on ω implies, in general, only bounds on u in some C^α space of Hölder continuous functions, and this is in general not enough to apply Yudovich techniques and show well-posedness of (2) (some partial extension of Yudovich's result appeared in [21], where functions with vorticity in $\bigcap_{p<\infty} L^p$ were considered, with strong bounds on the growth of L^p norms as $p \to \infty$).

There have been however in the last years several important results, providing partial answers to questions **Q1** and **Q2** above. We mention few of them, and in particular those concerning the problem of non-uniqueness of weak solutions.

In [28,29] Vishik gave a negative answer to **Q1**, proving nonuniqueness in the class of solutions having vorticity $\omega \in L^\infty_t (L^p_x)$, however not for the Euler system (1) (or (2)), but for the Euler system (1) with a $L^1_t (L^1_x \cap L^p_x)$ external force (thus allowing for an additional "degree of freedom"). Vishik's proof is based on a careful analysis of the linearized operator $\mathcal{L}$ associated to (1) and on the construction of an *unstable* eigenvalue for $\mathcal{L}$.

Another approach based on numerical simulations has been proposed by Bressan and Shen in [2], where an initial profile is constructed for which there is numerical evidence of non-uniqueness, but a rigorous proof of this result is still missing.

Very recently, in [22], Mengual proved that for any $2 < p < \infty$ there exists initial data $u_0 \in L^2(\mathbb{R}^2)$ with initial vorticity $\operatorname{curl} u_0 \in L^1 \cap L^p$ for which there are infinitely many admissible solutions $u \in C_t L^2$ to (1) but with the drawback that $\operatorname{curl} u(t, \cdot)$ does not belong to $L^p(\mathbb{R}^2)$ for any $t > 0$. An admissible solution is a weak solution that does not increase the kinematic energy, i.e. $\frac{1}{2}\|u(t)\|^2_{L^2} \le \frac{1}{2}\|u(0)\|^2_{L^2}$ for a.e. t.

Concerning the more general question **Q2**, Bruè and Colombo address this question in [3] for the case that X is the Lorentz space $X = L^{1,\infty}$. They construct a sequence $(u_n)_n$ of smooth "approximate" solutions to (1), converging to an "anomalous" weak solution u of (1) (in the sense that u is nonzero, but $u|_{t=0} = 0$, thus providing an example of non-uniqueness) and having the additional property that the sequence of vorticities $(\operatorname{curl} u_n)_n$ is a Cauchy sequence in $L^{1,\infty}$. An adaptation of the proof shows the same statement for $X = L^{1,q}$ for $q > 4$, see Remark 1.3 in [3].

The construction in [3] is based on an *intermittent convex integration scheme*. As we shall explain in Sect. 1.2 below, it is expected that, in general, intermittent convex integration schemes in dimension d can provide ("anomalous") weak solutions to the Euler equations having vorticity in L^p only if

$$p < \frac{2d}{d+2} \tag{3}$$

In particular, in dimension $d = 2$, it is not possible with the current techniques to construct solutions u with $\operatorname{curl} u \in L^p$, not even for $p = 1$. This motivated the authors in [3] to look for velocity fields with vorticity in $L^{1,\infty}$, a function space which is "weaker" than L^1 in terms of integrability, but which scales as L^1.

It has however to be noted that, as we mentioned before, the result in [3] shows the existence of a sequence $\{u_n\}_n$ of approximate solutions to (1) converging strongly in L^2 to an anomalous weak solution u to (1) and whose corresponding vorticities $\{\operatorname{curl} u_n\}_n$ build a Cauchy sequence in $L^{1,\infty}$ which thus has a limit ω in $L^{1,\infty}$. However, since $L^{1,\infty}$ is not a space of distributions (precisely, it does not embed into $\mathcal{D}'$; neither does $L^{1,q}$ for $1 < q < \infty$), it is not clear whether and in what sense the distributional vorticity of the solution u (or, in other words, the distributional limit of $\operatorname{curl} u_n$) coincide with the $L^{1,\infty}$ limit ω.

Indeed, in general, there is no connection between distributional limit and limit in $L^{1,q}$, $q \in (1, \infty]$. Standard examples where this absence of connection can be explicitly seen can be constructed even in one dimension, see, for instance, Sect. 1.4 below, where a sequence (f_n) of piecewise constant maps is constructed, with f_n converging to two very different "objects" in distributions and in $L^{1,q}$ respectively: a Dirac delta in $\mathcal{D}'$ and the zero function in $L^{1,q}$. Similar constructions can also be done for smooth (f_n).

1.2. Our Result

The result by Bruè and Colombo [3] motivated us to see if the methods used in [3] could be adapted to show non-uniqueness of weak solutions to (1) with vorticity in some other function space X that is "weaker" than L^1 in terms of integrability, but at the same time it does embed into $\mathcal{D}'$, avoiding the issues connected to the $L^{1,\infty}$ topology.

The real Hardy spaces H^p for $p < 1$ (thus matching with (3) in dimension $d = 2$) turns out to be a natural choice, as H^p does embed into $\mathcal{D}'$ for any $p \in (0, \infty)$ (see Definition 2.3 for the precise definition of the space H^p). Precisely, we prove the following theorem.

Theorem 1.1 (Main Theorem). *Let $\frac{2}{3} < p < 1$. For any energy profile $e \in C^\infty\left([0,1]; \left[\frac{1}{2}, 1\right]\right)$ there exists a solution $u \in C([0,1], L^2(\mathbb{R}^2))$ to (1) with*

(i) $\int_{\mathbb{R}^2} |u|^2(t)\,\mathrm{d}x = e(t)$,

(ii) $\operatorname{curl} u \in C([0,1], H^p(\mathbb{R}^2))$.

In particular, there exist energy dissipating solutions $u \in C_t L^2_x$ to (1) with $\operatorname{curl} u \in C_t H^p_x$.

Furthermore, for energy profiles e_1, e_2 such that $e_1 = e_2$ on $[0, t_0]$ for some $t_0 \in [0,1]$, there exist two distinct solutions u_1, u_2 satisfying (i), (ii) with $u_1(t) = u_2(t)$ for $t \in [0, t_0]$.

Corollary 1.2. *Let $\frac{2}{3} < p < 1$. There are two admissible (in the sense that the total kinetic energy is non-increasing in time) solutions $u_1, u_2 \in C([0,1]; L^2(\mathbb{R}^2))$ with $\operatorname{curl} u_1, \operatorname{curl} u_2 \in C([0,1]; H^p(\mathbb{R}^2))$ with the same initial datum $u_1|_{t=0} = u_2|_{t=0}$.*

Proof. The proof follows immediately from Theorem 1.1, picking two non-increasing energy profiles e_1, e_2 which coincide on $[0, 1/2]$ and are different from each other on $[1/2, 1]$. ☐

Remark 1.3. We add some remarks about the statement of Theorem 1.1.

(1) The solutions we construct are *distributional solutions* in the sense that
$$\int_0^1 \int_{\mathbb{R}^2} -u \cdot \partial_t \varphi - u \otimes u : \nabla \varphi = 0, \qquad \int_{\mathbb{R}^2} u(t) \cdot \nabla \psi = 0 \text{ for all } t \in [0,1]$$
for any divergence-free $\varphi \in C^\infty_c((0,1) \times \mathbb{R}^2; \mathbb{R}^2)$ and any $\psi \in C^\infty_c(\mathbb{R}^2)$. Observe also that our solutions belong to $C([0,1], L^2(\mathbb{R}^2))$, in particular they achieve their initial datum in a strong sense.

(2) Differently from typical results in convex integration, we work on the full space $\mathbb{R}^2$ and not on the periodic domain $\mathbb{T}^2$. This is motivated by the fact that Hardy spaces are usually defined and studied on the full space and it is quite hard to find references for Hardy spaces on $\mathbb{T}^2$ (or $\mathbb{T}^d$). This creates some technical troubles we are going to discuss in Sect. 1.3.

(3) The constraint $p > 2/3$ comes from the fact that working in real Hardy spaces requires to treat the moments of the involved functions up to a certain order. In this paper we are only keeping track of the 0th order moment of the vorticity, which is sufficient for $p \in (2/3, 1)$, compare with subsection 1.3.1 and Definition 2.4 for Hardy space atoms.

(4) Differently than in [3], condition (ii) in the statement of Theorem 1.1 means precisely that the *distributional* curl of $u(t)$ belongs to H^p, for all t (with continuous dependence on time).

We wish now to spend some words in explaining why conditions (3) plays a fundamental role (both in [3] and in our result), and therefore why we were able to show Theorem 1.1 only under the condition $p < 1$.

As in [3], we use a convex integration technique in the spirit of De Lellis and Székelyhidi works on the 3D Euler equations in the framework of Onsager's Theorem (see [5,11–13,19]). Meanwhile, Onsager's Theorem has also been proven in the 2D setting by Giri and Radu using a combination of the aforementioned convex integration technique and a Newtonian linearization of the Euler equations, see [16]. Notice that solutions constructed in [16] are Hölder continuous and no bound on their vorticity is shown in the mentioned paper.

The outline in all of these schemes is an iterative construction where, starting from an initial approximate solution, one adds fast oscillating perturbations with a higher frequency $\lambda_n \to \infty$ with respect to the typical frequencies λ_{n-1} in the previous approximation. In case of the Euler equation, given an approximate solution $(u_{n-1}, p_{n-1}, R_{n-1})$ with error term on the right hand side

$$\partial_t u_{n-1} + \operatorname{div}(u_{n-1} \otimes u_{n-1}) + \nabla p_{n-1} = -\operatorname{div} R_{n-1}, \tag{4}$$

one makes the Ansatz

$$u_n(t,x) = u_{n-1}(t,x) + w_n(t,x) + \text{lower order corrector terms}$$

with

$$\begin{aligned} w_n(t,x) &= a_{n-1}(t,x) W_{\lambda_n}, \\ W_{\lambda_n}(x) = W(\lambda_n x) &: \text{ fast oscillating building block,} \\ a_{n-1} &: \text{ slowly varying coefficient,} \quad a_{n-1} \approx |R_{n-1}|^{1/2}. \end{aligned}$$

The interaction of w_n (having frequencies λ_n) with itself from the nonlinearity of the equation produces a term having frequencies $\approx \lambda_{n-1}$ and it allows therefore for the cancellation of the previous error, provided

$$a_{n-1}^2 \int_{\mathbb{T}^2} W_{\lambda_n} \otimes W_{\lambda_n} \, \mathrm{d}x \approx R_{n-1} \int_{\mathbb{T}^2} W_{\lambda_n} \otimes W_{\lambda_n} \, \mathrm{d}x \sim R_{n-1}.$$

In particular, this forces us to choose a building block W such that

$$\int_{\mathbb{T}^2} W \otimes W \, \mathrm{d}x = \int_{\mathbb{T}^2} W_{\lambda_n} \otimes W_{\lambda_n} \, \mathrm{d}x \sim 1, \tag{5}$$

which in turn implies (taking the trace in the above relations) that

$$\|W\|_{L^2}^2 = \|W_{\lambda_n}\|_{L^2}^2 \sim 1. \tag{6}$$

Clearly, since W_{λ_n} is fast oscillating with frequency $\lambda_n \gg 1$ one expects very little control on the first derivative of W_{λ_n} (and thus also on $\operatorname{curl} u_n$). In particular, one can not expect that $\|\nabla W_{\lambda_n}\|_{L^\infty}$ or even $\|\nabla W_{\lambda_n}\|_{L^2}$ stays bounded as $n \to \infty$.

There is however some hope in controlling $\|\nabla W_{\lambda_n}\|_{L^p}$ if $p \ll 2$, or, more precisely, if (3) holds. Indeed, for those p's for which (3) does not holds, we have the embedding $W^{1,p} \hookrightarrow L^2$ and thus (6) combined with the Sobolev inequality gives

$$1 \sim \|W\|_{L^2}^2 \leq \|\nabla W\|_{L^p}$$

so that there is no hope in showing smallness of $\|\nabla W\|_{L^p}$. On the other side, if (3) holds, the Sobolev inequality fails and thus it is possible to construct a sequence of building blocks W_{λ_n} oscillating with frequencies λ_n, satisfying (6) and, at the same time, having

$$\|\nabla W_{\lambda_n}\|_{L^p} \to 0 \text{ as } n \to \infty.$$

This was the crucial observation of Buckmaster and Vicol in the groundbreaking work [7], where the authors apply a convex integration scheme to the Navier–Stokes equations and need therefore to control higher order derivatives of the perturbation, because of the presence of the dissipative term in the system.

Similar observations were used also in [4,9,10,17,23–26] for constructing counterexamples to uniqueness for the transport equations with Sobolev vector fields and other more recent works (see e.g. [6,8,14,15]).

As we observed before, in dimension $d = 2$, condition (3) corresponds to $p < 1$, hence preventing the possibility of estimating $\operatorname{curl} u$ in L^1 with the current techniques. On the other side, the key observation in [3] is that for the Lorentz space $L^{1,\infty}$, the Sobolev embedding fails,

$$\|\nabla u\|_{L^{1,\infty}} \not\gtrsim \|u\|_{L^2} \text{ in general, for } u \in C^\infty(\mathbb{T}^2),$$

and this made the construction in [3] possible.

If one were allowed to choose $p < 1$ in (3), the embedding

$$\|\nabla u\|_{L^p} \not\gtrsim \|u\|_{L^2}$$

would also fail. Even though L^p spaces are defined also for $p < 1$, they do not embed continuously into $\mathcal{D}'$, hence a construction with vorticity in L^p for $p < 1$ would suffer from the same issues as the construction in Lorentz spaces.

It turns however out that a feasible subsitute for L^p in the range $p \in (0, \infty)$ is the Hardy space H^p. Indeed, on one hand, we have $H^p(\mathbb{R}^2) \cong L^p(\mathbb{R}^2)$ for $p > 1$ and $H^p(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$ for $p = 1$. On the other hand, H^p embeds into $\mathcal{D}'$ for all $p \in (0, \infty)$ (e.g. [18], Proposition 6.4.10) and, finally, functions in H^p scale like L^p (also for $p < 1$), in the sense that

$$\|\nabla^l \varphi(\mu \cdot)\|_{H^p} = \mu^{l-\frac{2}{p}} \|\nabla^l \varphi\|_{L^p(\mathbb{R}^2)} \tag{7}$$

for $\varphi \in C_c^\infty(\mathbb{R}^2)$ and any $0 < p < \infty$, so that one can hope to have a sequence of building blocks which have L^2 norm of order 1 (as in (6)) and, at the same time, having vorticity with H^p norm arbitrarily small, if $p < 1$.

1.3. Technical Novelties

We briefly explain now the two main technical novelties of this paper compared to previous works on convex integration. They concern

(1) how elements in Hardy spaces can be estimated and, in particular, how to exploit the scaling properties (7) in Hardy spaces;
(2) how to do the construction on the full space, where also decay at ∞ has to be taken into account.

1.3.1. Concentration in Hardy Spaces. As we mentioned before, in order to control the quantity $\|\operatorname{curl} w\|_{H^p}$, we use the mechanism of *concentration* or *intermittency* that was also used in [3] for the control of the norm in $L^{1,\infty}$. The building blocks are defined via concentrated functions,

$$W := W_\mu := \varphi_\mu(x)\xi$$

where $\varphi \in C_c^\infty(\mathbb{R}^2)$, φ_μ is the periodization of the concentrated function $\mu\varphi(\mu x)$ and $\xi \in \mathbb{R}^2$ is some given direction. The scaling is such that we keep (6), i.e. $\|W_\mu\|_{L^2}$ 1. The main problem in exploiting concentration in the framework of Hardy spaces (with $p < 1$) is that there is no Hölder inequality available: in general

$$\|af\|_{H^p} \not\lesssim \|a\|_{L^\infty} \|f\|_{H^p}.$$

Hence the estimate for $\|\operatorname{curl} w\|_{H^p}$ is more subtle and we cannot use (7) directly.

To deal with this issue, one could use the definition of Hardy norm (see (10)), but this turns out to be extremely difficult. We use therefore the notion of *atoms*, which are typical functions f in Hardy space that have support in a ball B and satisfy the cancellation property $\int_B f \,\mathrm{d}x = 0$ and an L^∞ estimate, see Definition 2.4. Indeed, thanks to the intermittency, one can view the perturbation $w(x) =$

$\chi_{\kappa_0}(x)a(x)W(x)$ as a finite sum of functions, each of them supported on a very small ball of radius $\frac{1}{\mu}$, i.e.

$$w = \sum \theta_j,$$
$$\theta_j = \mathbb{1}_{B_{\frac{1}{\mu}}(x_j)} w$$

for some $x_1, \ldots, x_n$. The curl of each θ_j satisfies the cancellation property $\int_{B_{\frac{1}{\mu}}(x_j)} \operatorname{curl} \theta_j \, \mathrm{d}x = 0$ as a derivative of a compactly supported function. Therefore, $\operatorname{curl} w$ is a linear combination of atoms and thus $\operatorname{curl} w \in H^p$. One can use a standard estimate for atoms (see Lemma 2.5) on each θ_j, balancing $\|\theta_j\|_{L^\infty}$ (estimated by (7)) and the size of its support.

1.3.2. Full Spaces Versus Periodic Domain. Since we are constructing solutions in $L^2(\mathbb{R}^2)$ and not in $L^2(\mathbb{T}^d)$, we need to implement a convex integration scheme that differs from previous ones in at least two more ways:

(i) As fast oscillating perturbations are used to reduce the error, $\mathbb{T}^2$ is the natural habitat for solutions constructed by convex integration schemes. We want to keep the advantages from using fast oscillations, while also ensuring the decay at infinity.

(ii) On a more technical side, there is no bounded right inverse

$$\operatorname{div}^{-1} : L^1(\mathbb{R}^2; \mathbb{R}^2) \to L^1(\mathbb{R}^2; \mathrm{Sym}_{2\times 2}(\mathbb{R}))$$

(here $\mathrm{Sym}_{2\times 2}(\mathbb{R})$ is the space of real symmetric 2×2 matrices) for the divergence. In order to reduce R_0, it is crucial to construct an antidivergence for functions of the form fu_λ with $f \in C_c^\infty(\mathbb{R}^2), u \in C_0^\infty(\mathbb{T}^2)$ that takes advantage of the oscillation with an estimate of the form $\|\operatorname{div}^{-1}(fu_\lambda)\|_{L^1} \approx \frac{1}{\lambda}\|fu\|_{L^1}$.

Non-periodic solutions to the 3D Euler equations (with Hölder regularity) were already constructed in [20]. Compared to [20], we take here a different route, as we better explain below.

We deal with (i) by using that if $R_0 \in L^1(\mathbb{R}^2)$

$$\lim_{\kappa\to\infty} \|R_0\|_{L^1(\mathbb{R}^2 \setminus \ B_\kappa)} = 0$$

and reduce the error only on a compact set B_{κ_0} such that $\|R_0\|_{L^1(\mathbb{R}^2\setminus B_{\kappa_0})} \ll 1$, using a cutoff χ_{κ_0} in our perturbations

$$w(t,x) = \chi_{\kappa_0}(x)a(t,x)W_\lambda(x).$$

Therefore, the support of w consists of a (possibly very large) finite number (which is of order κ_0^2) of periodic boxes of the form $[0,1]^2 + k$ for some $k \in \mathbb{Z}^2$ that is fixed at the start of each iteration. This allows us to have similar estimates as for periodic functions on $\mathbb{T}^2$ with a factor depending on κ_0, while also having perturbations in $L^2(\mathbb{R}^2)$.

Concerning (ii), we gain the factor $\frac{1}{\lambda}$ by using integration by parts: On $\mathbb{T}^2$, we have the bounded (in L^1) operator $\operatorname{div}^{-1} : C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C_0^\infty(\mathbb{T}^2; \mathrm{Sym}_{2\times 2}(\mathbb{R}))$ that satisfies $\|\operatorname{div}^{-1} u_\lambda\|_{L^1(\mathbb{T}^2)} \le \frac{C}{\lambda}\|u\|_{L^1(\mathbb{T}^2)}$ (see Lemma 2.7 below or also, for instance, [8, Proposition 4]). Defining

$$R_1(f, u_\lambda) = f \operatorname{div}^{-1} u_\lambda,$$

we have $\|R_1\|_{L^1} \le \frac{C(\operatorname{supp} f)}{\lambda}\|f\|_{C(\mathbb{R}^2)}\|u\|_{L^1(\mathbb{T}^2)}$ and this matrix satisfies

$$\operatorname{div} R_1 = fu_\lambda + (\operatorname{div}^{-1} u_\lambda) \cdot \nabla f.$$

Since div^{-1} is not bounded from $L^1(\mathbb{R}^2;\mathbb{R}^2) \to L^1(\mathbb{R}^2; \mathrm{Sym}_{2\times 2}(\mathbb{R}))$, we can not write the last term as a divergence of a tensor field whose L^1 norm is bounded by the L^1 norm of $(\operatorname{div}^{-1} u_\lambda) \cdot \nabla f$. Hence we simply set

$$r_1 = -(\operatorname{div}^{-1} u_\lambda) \cdot \nabla f$$

so that

$$r_1 + \operatorname{div} R_1 = f u_\lambda \text{ and } \|R_1\|_{L^1}, \|r_1\|_{L^1} \lesssim \frac{1}{\lambda}. \tag{8}$$

We therefore work with approximate solutions that satisfy

$$\partial_t u_{n-1} + \operatorname{div}(u_{n-1} \otimes u_{n-1}) + \nabla p_{n-1} = -r_{n-1} - \operatorname{div} R_{n-1}$$

instead of (4). In order to cancel this additional error term, we include in our definition of u_n a corrector of the form

$$v(x,t) = \int_0^t r_{n-1}(x,s)\,\mathrm{d}s$$

such that $\partial_t v - r_{n-1} = 0$. Since in this way r_{n-1} enters into the definition of the perturbation through v, we have to make sure to control $\|\operatorname{curl} \int_0^t r_{n-1}(x,s)\,\mathrm{d}s\|_{H^p}$. We do this by carrying out the "integration by parts" N times, yielding (r_N, R_N) with

$$r_N + \operatorname{div} R_N = f u_\lambda \text{ and } \|\nabla r_N\|_{L^\infty} \lesssim \frac{1}{\lambda^{N-1}}$$

instead of (8). We then make sure that r_N has compact support, so that we can again use the standard estimate for atoms mentioned above (L^∞ bound together with a bound on the size of the support).

This is quite different from the approach in [20]. As in the present paper, also in [20] the authors have to deal with the absence of a bounded right inverse $\operatorname{div}^{-1} : L^1 \to L^1$ with values in the symmetric matrices. This issue is solved in [20] by constructing an antidivergence operator which is defined only on the subset of $L^1(\mathbb{R}^3;\mathbb{R}^3)$ consisting of elements which are orthogonal to translation and rotational vector fields. Therefore, the construction in [20] becomes in a sense more complicated than in the present paper (because one has to check every time that div^{-1} is applied to an vector field in the domain of definition of div^{-1}), but, on the other side, it allows the authors of [20] to construct solutions which have well defined and conserved angular momentum, a property we are not at all considering in the present work.

1.4. An Explicit Example Comparing Distributional and Lorentz Space Convergence

We conclude this introduction with an example of a sequence $(f_n)_n$ of $1D$ piecewise constant maps (but similar constructions can be done with smooth maps) converging to *different limits* in $L^{1,q}$, $q \in (1,\infty]$ and in $\mathcal{D}'$. In particular, $f_n \to \delta_0$ in distributions, whereas $f_n \to 0$ in $L^{1,q}$ for all $q \in (1,\infty]$. Set

$$f_n = \frac{1}{n} \sum_{j=0}^{n-1} 2^{n+j} \mathbb{1}_{[2^{-(n+j)},2^{-(n+j-1)}]}$$

Then $\int_{\mathbb{R}} f_n \,\mathrm{d}x = 1$ and it is not difficult to see that

$$f_n \to \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}).$$

On the other hand, it holds

$$|\{|nf_n| \geq t\}| \leq \begin{cases} 2^{-n+1}, & t \in (0, 2^n], \\ 2^{-n}, & t \in (2^n, 2^{n+1}], \\ \vdots & \\ 2^{-(2n-2)}, & t \in (2^{2n-2}, 2^{2n-1}], \\ 0, & t > 2^{2n-1}. \end{cases}$$

This yields

$$\|nf_n\|_{L^{1,q}} \leq C(q) n^{\frac{1}{q}}$$

and therefore

$$\|f_n\|_{L^{1,q}(\mathbb{R})} \leq C(q) n^{\frac{1}{q}-1} \to 0 \text{ for } n \to \infty,$$

as far as $q \in (1, \infty]$.

1.5. Notation

We fix some notation we are going to use in the paper.

- We denote by e_1, e_2 the standard basis vectors of $\mathbb{R}^2$.
- For any vector $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we will denote by $\xi^\perp$ the orthogonal vector $\xi^\perp = (\xi_2, -\xi_1)$.
- We denote by $\mathrm{Sym}_{n\times n}(\mathbb{R})$ the set of real symmetric $n \times n$ matrices.
- For a quadratic 2×2 matrix T, we denote by $\mathring{T} = T - \frac{1}{2} \operatorname{tr} T \operatorname{Id}$ its traceless part.
- For a function $f \in C^1(\mathbb{R}^2)$ we denote by $\nabla^\perp f = (\partial_2 f, -\partial_1 f)$ its orthogonal gradient.
- For $d_1, d_2 \in \mathbb{N}$ we write $f : \mathbb{T}^{d_1} \to \mathbb{R}^{d_2}$ for a function $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ defined on the full space that is periodic with period 1 in all variables, i.e. $f(x + le_k) = f(x)$ for all $k = 1, \ldots, d_1$, $l \in \mathbb{Z}$.
- For a periodic function f as above, we denote $\int_{\mathbb{T}^{d_1}} f \, \mathrm{d}x = \int_{[0,1]^{d_1}} f \, \mathrm{d}x$, i.e. the integral over just one periodic box.
- $C_0^\infty(\mathbb{T}^2; \mathbb{R}^d) = \{f : \mathbb{T}^2 \to \mathbb{R}^d \text{ smooth}, \int_{\mathbb{T}^2} f \, \mathrm{d}x = 0\}$ is the space of smooth periodic functions on $\mathbb{R}^2$ with zero mean value on one periodic box.
- For a function $g \in C^\infty(\mathbb{T}^2)$ and $\lambda \in \mathbb{N}$, we denote by $g_\lambda : \mathbb{T}^2 \to \mathbb{R}$ the $\frac{1}{\lambda}$ periodic function
$$g_\lambda(x) := g(\lambda x).$$
Notice that for every $l \in \mathbb{N}$, $s \in [1, \infty]$
$$\|D^l g_\lambda\|_{L^s(\mathbb{T}^2)} = \lambda^l \|D^l g\|_{L^s(\mathbb{T}^2)}.$$
- $\mathcal{S}(\mathbb{R}^2)$ denotes the space of Schwartz functions.
- $H^p(\mathbb{R}^2)$ is the real Hardy space, see Definition 2.3.
- $L^2_\sigma(\mathbb{R}^2) = \left\{f \in L^2(\mathbb{R}^2) : \operatorname{div} f = 0 \text{ in distributions}\right\}$ is the space of divergence-free vector fields in $L^2(\mathbb{R}^2)$.
- For a function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^d$ and $s \in [1, \infty]$, we write $\| \cdot \|_{C_t L^s_x}$ for the norm $\|f\|_{C_t L^s_x} = \max_{t\in[0,1]} \|f(t)\|_{L^s(\mathbb{R}^2)}$.
- For any function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\operatorname{supp}(\varphi) \subset (-\frac{1}{2}, \frac{1}{2})$ and $\mu > 1$ we write φ_μ for the periodic extension of the function $\mu^{\frac{1}{2}} \varphi(\mu\left(x - \frac{1}{2}\right))$, whose support is contained in intervalls of length $\frac{1}{\mu}$ centered around the points $\frac{1}{2} + \mathbb{Z}$. Note that
$$\|\varphi_\mu\|_{L^r(\mathbb{T})} = \mu^{\frac{1}{2} - \frac{1}{r}} \|\varphi\|_{L^r(\mathbb{R})} \tag{9}$$
and in particular $\|\varphi_\mu\|_{L^2(\mathbb{T})} = \|\varphi\|_{L^2(\mathbb{R})}$.
- Let $\lambda \in \mathbb{N}$, $f : \mathbb{T}^2 \to \mathbb{R}^d$. We will sometimes write f_λ for the oscillating functions $f_\lambda(x) = f(\lambda x)$. On the other hand, for $f : \mathbb{R} \to \mathbb{R}$ with compact support, we will oftentimes write f_μ for its concentrated version. To avoid confusion, we will only use the parameter λ for oscillations and μ (or μ_1, μ_2, respectively) for concentration.
- $\mathbb{P}$ denotes the Leray projector $L^2(\mathbb{R}^d; \mathbb{R}^d) \to L^2_\sigma(\mathbb{R}^d; \mathbb{R}^d)$ onto the space of divergence free (in the sense of distributions) vector fields.

2. Preliminaries

We now provide the technicals tools that are needed for the proof of the Main Theorem 1.1 and we start this section with two useful estimates for functions of the form fg_λ, where $f \in C_c^\infty(\mathbb{R}^2)$, $g \in C^\infty(\mathbb{T}^2)$.

For these estimates it is crucial that f is compactly supported. Note also that the size of $\operatorname{supp} f$ enters the estimate.

Proposition 2.1 (Improved Hölder). *Let $k, \lambda \in \mathbb{N}$, $f : \mathbb{R}^2 \to \mathbb{R}$ smooth with $\operatorname{supp} f \subset [-k,k]^2$ and $g : \mathbb{T}^2 \to \mathbb{R}$ smooth. Then it holds for all $s \in [1,\infty]$*

$$\|f g_\lambda\|_{L^s(\mathbb{R}^2)} \leq \|f\|_{L^s(\mathbb{R}^2)}\|g\|_{L^s(\mathbb{T}^2)} + \frac{C(s)(2k)^{\frac{2}{s}}}{\lambda^{\frac{1}{s}}}\|f\|_{C^1(\mathbb{R}^2)}\|g\|_{L^s(\mathbb{T}^2)}.$$

Proof. This is an adaptation of Lemma 2.1 in [24], which can be proven in the same way. □

Lemma 2.2. *Let $k, \lambda \in \mathbb{N}$, $f : \mathbb{R}^2 \to \mathbb{R}$ smooth with $\operatorname{supp} f \subset [-k,k]^2$ and $g : \mathbb{T}^2 \to \mathbb{R}$ smooth with $\int_{\mathbb{T}^2} g \,\mathrm{d}x = 0$. Then*

$$\left| \int_{[-k,k]^2} f(x) g_\lambda(x)\,\mathrm{d}x \right| \leq \frac{4\sqrt{2}k^2\|f\|_{C^1(\mathbb{R}^2)}\|g\|_{L^1(\mathbb{T}^2)}}{\lambda}.$$

Proof. This is an adaptation of Lemma 2.6 in [24] with the same proof. □

Definition 2.3 (*Hardy spaces on* $\mathbb{R}^2$). Let $\Psi \in \mathcal{S}(\mathbb{R}^2)$ be a Schwartz function with $\int_{\mathbb{R}^2} \Psi(x)\,\mathrm{d}x \neq 0$ and let $\Psi_\varepsilon(x) = \frac{1}{\varepsilon^2}\Psi(\frac{x}{\varepsilon})$. For any $f \in \mathcal{S}'(\mathbb{R}^2)$, we define the radial maximal function

$$m_\Psi f(x) = \sup_{\zeta > 0} |f * \Psi_\zeta(x)|. \tag{10}$$

Let $0 < p < \infty$. The real Hardy space $H^p(\mathbb{R}^2)$ is defined as the space of tempered distributions

$$H^p(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : m_\Psi f \in L^p(\mathbb{R}^2) \right\}$$

and we write

$$\|f\|_{H^p(\mathbb{R}^2)} = \|m_\Psi f\|_{L^p(\mathbb{R}^2)}.$$

Note that $\|\cdot\|_{H^p(\mathbb{R}^2)}$ is only a quasinorm. The definition of $H^p(\mathbb{R}^2)$ does not depend on the choice of the function Ψ and the quasinorms are equivalent. For $p > 1$, the space $H^p(\mathbb{R}^2)$ coincides with the Lebesgue space $L^p(\mathbb{R}^2)$. For $p \leq 1$, $H^p(\mathbb{R}^2)$ is a complete metric space with the metric given by $d(f,g) = \|f-g\|^p_{H^p(\mathbb{R}^2)}$ and the inclusion $H^p(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ is continuous, see [18], Proposition 6.4.10.

Definition 2.4 (*Hardy space atoms*). For $p \leq 1$, a Hardy space atom is a measurable function a with the following properties:

(i) $\operatorname{supp} a \subset B$ for some ball B,
(ii) $|a| \leq |B|^{-\frac{1}{p}}$
(iii) $\int_B x^\beta a(x)\,\mathrm{d}x = 0$ for all multiindices β with $|\beta| \leq 2(p^{-1} - 1)$.

Lemma 2.5 (Estimate for Hardy space atoms). *There is a uniform constant C such that for all atoms a it holds*

$$\|a\|_{H^p(\mathbb{R}^2)} \leq C.$$

Proof. We refer to [27], see 2.2 in Chapter III.2. □

Remark 2.6. (1) We will use that for a function f satisfying *(iii)* in Definition 2.4 with support in a ball B, we have by Lemma 2.5

$$\|f\|_{H^p(\mathbb{R}^2)} \leq C|B|^{\frac{1}{p}}\|f\|_{L^\infty(\mathbb{R}^2)}.$$

(2) Since $\frac{2}{3} < p < 1$ in our case, we only need to check the 0th moment in *(iii)*, i.e. $\int_B a(x)\,\mathrm{d}x = 0$.

Lemma 2.7 (Standard antidivergence). *There exists a linear operator*

$$\operatorname{div}^{-1} : C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C_0^\infty(\mathbb{T}^2;\operatorname{Sym}_{2\times 2}(\mathbb{R}))$$

such that $\operatorname{div}\operatorname{div}^{-1} u = u$ *and*

$$\begin{aligned}\|\nabla^l \operatorname{div}^{-1} u\|_{L^s(\mathbb{T}^2)} &\le C(s)\|\nabla^l u\|_{L^s(\mathbb{T}^2)},\\ \|\nabla^l \operatorname{div}^{-1} u_\lambda\|_{L^s(\mathbb{T}^2)} &\le \frac{C(s)}{\lambda^{1-l}}\|\nabla^l u\|_{L^s(\mathbb{T}^2)} \text{ for all } l,\lambda\in\mathbb{N}, s\in[1,\infty].\end{aligned}$$

For the proof see Proposition 4 in [8].

For $N \ge 2$ we inductively define

$$\operatorname{div}^{-N} u = \sum_{k=1,2} \operatorname{div}^{-1}\left(\operatorname{div}^{N-1} u\cdot e_k\right).$$

With that standard antidivergence operator, we will define an improved antidivergence operator for functions of the form fu_λ, $f\in C_c^\infty(\mathbb{R}^2)$, $u\in C_0^\infty(\mathbb{T}^2;\mathbb{R}^2)$, on the full space.

Lemma 2.8 (Improved antidivergence operators).

(i) For any $N\in\mathbb{N}$, there exists a bilinear operator

$$S_N : C_c^\infty(\mathbb{R}^2;\mathbb{R})\times C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)\times C_c^\infty(\mathbb{R}^2;\operatorname{Sym}_{2\times2}(\mathbb{R}))$$

such that for $S_N(f,u)=(r,R)$ it holds

$$r+\operatorname{div} R = fu$$

with

$$\begin{aligned}\|\nabla^l r\|_{L^\infty(\mathbb{R}^2)} &\le C(\operatorname{supp} f)\|\nabla^l \operatorname{div}^{-N} u\|_{L^\infty(\mathbb{T}^2)}\|f\|_{C^{N+l}(\mathbb{R}^2)} \text{ for all } l\in\mathbb{N},\\ \|R\|_{L^1(\mathbb{R}^2)} &\le C(\operatorname{supp} f)\|\operatorname{div}^{-1} u\|_{L^1(\mathbb{T}^2)}\|f\|_{C^{N-1}(\mathbb{R}^2)}.\end{aligned}$$

(ii) For any $N\in\mathbb{N}$, there exists a bilinear operator

$$\tilde S_N : C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)\times C_0^\infty(\mathbb{T}^2;\operatorname{Sym}_{2\times2}(\mathbb{R})) \to C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)\times C_c^\infty(\mathbb{R}^2;\operatorname{Sym}_{2\times2}(\mathbb{R}))$$

such that for $\tilde S_N(f,T)=(r,R)$ it holds

$$r+\operatorname{div} R = Tf$$

with

$$\begin{aligned}\|\nabla^l r\|_{L^\infty(\mathbb{R}^2)} &\le C(\operatorname{supp} f)\|\nabla^l \operatorname{div}^{-N} T\|_{L^\infty(\mathbb{T}^2)}\|f\|_{C^{N+l}(\mathbb{R}^2)} \text{ for all } l\in\mathbb{N},\\ \|R\|_{L^1(\mathbb{R}^2)} &\le C(\operatorname{supp} f)\|\operatorname{div}^{-1} T\|_{L^1(\mathbb{T}^2)}\|f\|_{C^{N-1}(\mathbb{R}^2)}.\end{aligned}$$

where, by a slight abuse of notation, we define

$$\operatorname{div}^{-N} T = \sum_{k=1,2}\operatorname{div}^{-N}(Te_k).$$

Proof. Let us inductively define

$$\begin{aligned}r_0 : C_c^\infty(\mathbb{R}^2;\mathbb{R})\times C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) &\to C_c^\infty(\mathbb{R}^2;\mathbb{R}^2),\\ r_0(f,u) &= fu,\\ R_0 : C_c^\infty(\mathbb{R}^2;\mathbb{R})\times C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) &\to C_c^\infty(\mathbb{R}^2;\operatorname{Sym}_{2\times2}(\mathbb{R})),\\ R_0(f,u) &= 0\end{aligned}$$

and for $N \geq 1$

$$\begin{aligned} r_N &: C_c^\infty(\mathbb{R}^2;\mathbb{R}) \times C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C_c^\infty(\mathbb{R}^2;\mathbb{R}^2), \\ & r_N(f,u) = -\sum_{k=1,2} r_{N-1}(\partial_k f, \operatorname{div}^{-1} u \cdot e_k), \\ R_N &: C_c^\infty(\mathbb{R}^2;\mathbb{R}) \times C_0^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C_c^\infty(\mathbb{R}^2;\mathrm{Sym}_{2\times 2}(\mathbb{R})), \\ & R_N(f,u) = f \operatorname{div}^{-1} u - \sum_{k=1,2} R_{N-1}(\partial_k f, \operatorname{div}^{-1} u \cdot e_k). \end{aligned}$$

It is clear that

$$r_0(f,u) + \operatorname{div} R_0(f,u) = fu.$$

Let us assume that

$$r_N(f,u) + \operatorname{div} R_N(f,u) = fu$$

for some $N \in \mathbb{N}$ for all $f \in C_c^\infty(\mathbb{R}^2)$, $u \in C_0^\infty(\mathbb{T}^2;\mathbb{R}^2)$. Then we also have

$$\begin{aligned} r_{N+1}(f,u) + \operatorname{div} R_{N+1}(f,u) &= \sum_{k=1,2} r_N(\partial_k f, \operatorname{div}^{-1} u \cdot e_k) \\ &\quad + \operatorname{div}\left(f \operatorname{div}^{-1} u - \sum_{k=1,2} R_N(\partial_k f, \operatorname{div}^{-1} u \cdot e_k) \right) \\ &= fu + (\operatorname{div}^{-1} u) \cdot \nabla f \\ &\quad - \sum_{k=1,2} r_N(\partial_k f, \operatorname{div}^{-1} u \cdot e_k) - \operatorname{div}\left(\sum_{k=1,2} R_N(\partial_k f, \operatorname{div}^{-1} u \cdot e_k) \right) \\ &= fu + (\operatorname{div}^{-1} u) \cdot \nabla f - \sum_{k=1,2} \partial_k f \operatorname{div}^{-1} u \cdot e_k = fu. \end{aligned}$$

Therefore, we set

$$S_N(f,u) = (r_N(f,u), R_N(f,u)).$$

For the second operator, we simply set for $f \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$, $T \in C_0^\infty(\mathbb{T}^2;\mathrm{Sym}_{2\times 2}(\mathbb{R}))$

$$\tilde{S}_N(f,T) = \sum_{k=1,2} S_N(f_k, Te_k).$$

The estimates follow by induction using the estimate for div^{-1} from Lemma 2.7 and the standard estimate for any $f \in C_c^\infty(\mathbb{R}^2), u \in C_0^\infty(\mathbb{T}^2;\mathbb{R}^2)$, $s \in [1,\infty]$

$$\|fu\|_{L^s(\mathbb{R}^2)} \leq \|u\|_{L^s(\operatorname{supp}(f))} \|f\|_{L^\infty(\mathbb{R}^2)} \leq C(\operatorname{supp} f) \|u\|_{L^s(\mathbb{T}^2)} \|f\|_{L^\infty(\mathbb{R}^2)}$$

where in the first step we consider u as a (periodic) function on $\mathbb{R}^2$. □

Remark 2.9. In particular, if $(r_N, R_N) = S_N(f, u_\lambda)$, then

$$\begin{aligned} \|\nabla^l r_N\|_{L^\infty(\mathbb{R}^2)} &\leq \frac{C(\operatorname{supp} f)}{\lambda^{N-l}} \|\nabla^l u\|_{L^\infty(\mathbb{T}^2)} \|f\|_{C^{N+l}(\mathbb{R}^2)} \text{ for all } l \in \mathbb{N}, \\ \|R_N\|_{L^1(\mathbb{R}^2)} &\leq \frac{C(\operatorname{supp} f)}{\lambda} \|u\|_{L^1(\mathbb{T}^2)} \|f\|_{C^{N-1}(\mathbb{R}^2)} \end{aligned}$$

and the same holds for $\tilde{S}_N$.

Lemma 2.10 (A helpful computation). *Let $f, g \in C^1(\mathbb{R})$. For any vector $\xi \neq 0 \in \mathbb{R}^2$ it holds*

$$\operatorname{div}\left(f(\xi\cdot x)g(\xi^\perp\cdot x)\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right) = f'(\xi\cdot x)g(\xi^\perp\cdot x)\xi,$$
$$\operatorname{div}\left(f(\xi\cdot x)g(\xi^\perp\cdot x)\frac{\xi}{|\xi|}\otimes\frac{\xi^\perp}{|\xi|}\right) = f(\xi\cdot x)g'(\xi^\perp\cdot x)\xi,$$
$$\operatorname{div}\left(f(\xi\cdot x)g(\xi^\perp\cdot x)\frac{\xi^\perp}{|\xi|}\otimes\frac{\xi}{|\xi|}\right) = f'(\xi\cdot x)g(\xi^\perp\cdot x)\xi^\perp,$$
$$\operatorname{div}\left(f(\xi\cdot x)g(\xi^\perp\cdot x)\frac{\xi^\perp}{|\xi|}\otimes\frac{\xi^\perp}{|\xi|}\right) = f(\xi\cdot x)g'(\xi^\perp\cdot x)\xi^\perp.$$

Proof. The proof is trivial. □

Definition 2.11. For ψ_1, ψ_2, $\Psi \in C^1(\mathbb{R})$ with $\Psi'' = \psi_2$ and a vector $\xi \neq 0$ we define

$$\begin{aligned} A(\psi_1,\psi_2,\xi) = \psi_1(\xi\cdot x)\Psi'(\xi^\perp\cdot x)\left(\frac{\xi}{|\xi|}\otimes\frac{\xi^\perp}{|\xi|}+\frac{\xi^\perp}{|\xi|}\otimes\frac{\xi}{|\xi|}\right) \\ -\psi_1'(\xi\cdot x)\Psi(\xi^\perp\cdot x)\frac{\xi^\perp}{|\xi|}\otimes\frac{\xi^\perp}{|\xi|}. \end{aligned}$$

and

$$B(\psi_1,\psi_2,\xi) = \psi_1(\xi\cdot x)\Psi'(\xi^\perp\cdot x)\frac{\xi^\perp}{|\xi|}\otimes\frac{\xi^\perp}{|\xi|}.$$

By Lemma 2.10, these symmetric matrices satisfy

$$\operatorname{div} A = \psi_1(\xi\cdot x)\psi_2(\xi^\perp\cdot x)\xi,$$
$$\operatorname{div} B = \psi_1(\xi\cdot x)\psi_2(\xi^\perp\cdot x)\xi^\perp.$$

Let $\mu_2 \gg \mu_1$. It is not difficult to see that for $\psi_1, \psi_2, \Psi \in C_c^\infty(\mathbb{R})$ with zero mean value and $\Psi'' = \psi_2$, supported in $(-\frac{1}{2}, \frac{1}{2})$, we have for their concentrated, fast oscillating extensions

$$A(\psi_{1,\mu_1}(\lambda\cdot),\psi_{2,\mu_2}(\lambda\cdot),\xi) \in C_0^\infty(\mathbb{T}^2,\mathrm{Sym}_{2\times 2}(\mathbb{R})),$$
$$B(\psi_{1,\mu_1}(\lambda\cdot),\psi_{2,\mu_2}(\lambda\cdot),\xi) \in C_0^\infty(\mathbb{T}^2,\mathrm{Sym}_{2\times 2}(\mathbb{R}))$$

if $\xi \in \mathbb{N}^2$ and the estimates

$$\begin{aligned} \|\nabla^l A(\psi_{1,\mu_1}(\lambda\cdot),\psi_{2,\mu_2}(\lambda\cdot),\xi)\|_{L^s(\mathbb{T}^2)} \le \lambda^{l-1}\mu_1^{\frac{1}{2}-\frac{1}{s}}\mu_2^{l-\frac{1}{2}-\frac{1}{s}}\max_{j_1,j_2=0,1}\|\psi_1^{(j_1)}\|_{L^s(\mathbb{T})}\|\Psi^{(j_2)}\|_{L^s(\mathbb{T})}, \\ \|\nabla^l B(\psi_{1,\mu_1}(\lambda\cdot),\psi_{2,\mu_2}(\lambda\cdot),\xi)\|_{L^s(\mathbb{T}^2)} \le \lambda^{l-1}\mu_1^{\frac{1}{2}-\frac{1}{s}}\mu_2^{l-\frac{1}{2}-\frac{1}{s}}\max_{j_1,j_2=0,1}\|\psi_1^{(j_1)}\|_{L^s(\mathbb{T})}\|\Psi^{(j_2)}\|_{L^s(\mathbb{T})}, \end{aligned} \quad (11)$$

where one uses $\mu_2 \gg \mu_1$.

3. Main Proposition

In this section we present the main proposition that is the key to prove Theorem 1.1. To this end, we first introduce the Reynolds defect equation:

Definition 3.1 (*Solution to the Reynolds defect equation*). A solution to the Reynolds-defect-equation is a tuple (u, p, R, r) of smooth functions

$$u \in C([0,1], L^2(\mathbb{R}^2)\cap L^3(\mathbb{R}^2)), p \in C([0,1], L^2(\mathbb{R}^2)), R \in C([0,1], L^1(\mathbb{R}^2;\mathrm{Sym}_{2\times 2}(\mathbb{R}))),$$
$$r \in C([0,1], L^\infty(\mathbb{R}^2)), \mathrm{supp}_{(t,x)} r \subseteq [0,1]\times\mathbb{R}^2 \text{ compact},$$

such that

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = -r - \operatorname{div} \mathring{R},$$
$$\operatorname{div} u = 0$$

is satisfied in the classical sense.

Proposition 3.2 (Main Proposition). *Let $e \in C^\infty\left([0,1];\left[\frac{1}{2},1\right]\right)$ be an arbitrary given energy profile. There exists a constant $M_0 > 0$ such that the following holds: Choose $\delta, \eta > 0$ with*

$$0 < \delta < 1, 0 < \eta < \frac{1}{32}\delta,$$

and assume that there exists a (smooth) solution (u_0, R_0, r_0, p_0) to the Reynolds-Defect-equation, satisfying

$$\frac{3}{4}\delta e(t) \le e(t) - \int_{\mathbb{R}^2} |u_0|(x,t)^2 \, \mathrm{d}x \le \frac{5}{4}\delta e(t), \tag{12}$$

$$40\|R_0\|_{C_t L^1_x} + \|r_0\|_{C_t L^2_x} + 2\|u_0(t)\|_{L^2(\mathbb{R}^2)} \|r_0\|_{C_t L^2_x} \le \frac{1}{32}\delta. \tag{13}$$

Then there exists another (smooth) solution (u_1, R_1, r_1, p_1) such that

(i)

$$\frac{3}{8}\delta e(t) \le e(t) - \int_{\mathbb{R}^2} |u_1|^2(x,t) \, \mathrm{d}x \le \frac{5}{8}\delta e(t),$$

(ii) r_1 satisfies

$$\|r_1\|_{C_t L^2_x} + \|u_1\|_{C_t L^2_x} \|r_1\|_{C_t L^2_x} \le \eta$$

and

(iii)

$$\| \int_0^t \operatorname{curl} r_1(s) \, \mathrm{d}s \|^p_{H^p(\mathbb{R}^2)} \le \eta,$$

(iv) $\|R_1(t)\|_{L^1(\mathbb{R}^2)} \le \eta + 4\|r_0\|_{C_t L^2_x} + 2\|r_0\|_{C_t L^2_x} \|u_0(t)\|_{L^2(\mathbb{R}^2)}$,

(v) $\|u_1(t) - u_0(t)\|_{L^2(\mathbb{R}^2)} \le M_0 \delta^{\frac{1}{2}}$,

(vi) $\| \operatorname{curl}(u_1 - u_0)(t)\|^p_{H^p(\mathbb{R}^2)} \le \eta + \| \int_0^t \operatorname{curl} r_0(s) \, \mathrm{d}s \|^p_{H^p(\mathbb{R}^n)}$.

Proof of the Main Theorem assuming Proposition 3.2. The solution to (1) is constructed iteratively. We start with the trivial solution $(u_0, p_0, R_0, r_0) \equiv 0$ and choose $\delta_0 = 1$. Then obviously (12) and (13) are satisfied. Let $\delta_n = 2^{-n}$ for $n \ge 0$ and $\eta_n = \frac{\delta_{n+1}}{11584}$ for $n \ge -1$. Assuming that the first $n+1$ solutions $(u_j, p_j, R_j, r_j)_{0 \le j \le n}$ are already constructed and that (u_n, p_n, R_n, r_n) satisfies (12), (13) with δ_n, we obtain $(u_{n+1}, p_{n+1}, R_{n+1}, r_{n+1})$ by applying Proposition 3.2 with δ_n, η_n. We show that we can proceed the iteration, i.e. that $(u_{n+1}, p_{n+1}, R_{n+1}, r_{n+1})$ satisfies (12), (13) with δ_{n+1}. First, we note that by (ii), we have

$$\|r_j\|_{C_t L^2_x} + \|u_j\|_{C_t L^2_x} \|r_j\|_{C_t L^2_x} \le \eta_{j-1} \tag{14}$$

for all $0 \le j \le n+1$. Now, by (i), the new solution satisfies

$$\frac{3}{8}\delta_n e(t) \le e(t) - \int_{\mathbb{R}^2} |u_{n+1}(t)|^2 \, \mathrm{d}x \le \frac{5}{8}\delta_n e(t)$$

and therefore

$$\frac{3}{4}\delta_{n+1} e(t) \le e(t) - \int_{\mathbb{R}^2} |u_1(t)|^2 \, \mathrm{d}x \le \frac{5}{4}\delta_{n+1} e(t),$$

i.e. (12) is satisfied. Also, by (iv) and (14) we have

$$\begin{aligned}
&40\|R_{n+1}\|_{C_tL^1_x} + \|r_{n+1}\|_{C_tL^2_x} + 2\|u_{n+1}(t)\|_{L^2(\mathbb{R}^2)}\|r_{n+1}\|_{C_tL^2_x} \\
&\quad \le 40(\eta_n + 4\|r_n\|_{C_tL^2_x} + 2\|r_n\|_{C_tL^2_x}\|u_n\|_{C_tL^2_x}) \\
&\qquad + \|r_{n+1}\|_{C_tL^2_x} + 2\|u_{n+1}(t)\|_{L^2(\mathbb{R}^2)}\|r_{n+1}\|_{C_tL^2_x} \\
&\quad \le 40\eta_n + 160\eta_{n-1} + 2\eta_n = 362\eta_n = \frac{1}{32}\delta_{n+1},
\end{aligned}$$

hence (13) holds. This shows that with our choice of $(\delta_n)_n$ and $(\eta_n)_n$ we can indeed construct a sequence $(u_n, p_n, R_n, r_n)_{n\in\mathbb{N}}$ of solutions to the Reynolds-defect-equation. By (v),

$$\sup_{t\in[0,1]} \|u_{n+1}(t) - u_n(t)\|_{L^2(\mathbb{R}^2)} \le M_0 2^{-\frac{n}{2}}$$

for all $n \in \mathbb{N}$, i.e. there exists $u \in C([0,1], L^2_\sigma(\mathbb{R}^2))$ such that $u_n \to u$ in $C([0,1], L^2_\sigma(\mathbb{R}^2))$. By (ii) and (iv),

$$\begin{aligned}
r_n &\to 0 \text{ in } C([0,1], L^1(\mathbb{R}^2)), \\
R_n, \mathring{R}_n &\to 0 \text{ in } C([0,1], L^1(\mathbb{R}^2, \mathrm{Sym}_{2\times 2}(\mathbb{R}))),
\end{aligned}$$

showing that u is a weak solution to (1). By (iii) and (vi), inductively we have

$$\|\operatorname{curl}(u_{n+1} - u_n)(t)\|^p_{H^p(\mathbb{R}^2)} \le \eta_n + \eta_{n-1},$$

which shows that there exists $v \in C([0,1], H^p(\mathbb{R}^2))$ such that

$$\operatorname{curl} u_n \to v \text{ in } C([0,1], H^p(\mathbb{R}^2)).$$

But since $H^p(\mathbb{R}^2) \hookrightarrow \mathcal{S}'(\mathbb{R}^2)$ is a continuous inclusion, this shows that $v = \operatorname{curl} u$. □

Remark 3.3. Notice that in the statement of our Main Theorem, Theorem 1.1, by *solution* we mean *distributional solution*, see Remark 1.3, Point (1). For this reason, in the proof of the Main Theorem we do not carry out any estimates on the sequence $(p_n)_n$ of (smooth) approximate pressures, nor we claim that $(p_n)_n$ is converging (in any suitable sense). For the same reason, Proposition 3.2 does not contain any estimates for the pressure.

On the other hand, the solutions to the Reynolds defect equation (u_n, p_n, R_n, r_n), introduced in Definition 3.1 and used in the iteration steps, are smooth functions and they solve the Reynolds defect equation in the classical sense. In particular they can be differentiated in space and time infinitely many times.

4. The Building Blocks

We fix the vectors

$$\xi_1 = e_1, \xi_2 = e_2, \xi_3 = e_1 + e_2, \xi_4 = e_1 - e_2$$

in $\mathbb{R}^2$. In the following, we will introduce several parameters that will be fixed in the course of this paper. They will be fixed in the order given by Table 1.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a smooth, odd function with support in $(-\frac{1}{2}, \frac{1}{2})$, and $\int \Phi \, \mathrm{d}x = 0$ such that $\varphi := \Phi'''$ satisfies $\int \varphi^2 \, \mathrm{d}x = 1$. Furthermore, we denote by φ^k_μ the translated function

$$\varphi^k_\mu(x) = \varphi_\mu\left(x - \frac{k}{16}|\xi_k|^2\right).$$

Table 1. *Occuring parameters and their meaning*

Parameter	Meaning
η, δ	Parameters in the main proposition that will ensure convergence
κ	size of the ball where the error is reduced, R_0 is small outside B_κ
ε	Smoothing of ρ (see Sect. 5)
μ_1	Concentration
μ_2	Very high concentration
ω	Phase speed
λ	Oscillation

The translation will ensure the disjointness of the supports of different building blocks, we will prove this in Lemma 4.3. Let $\mu_2 \gg \mu_1 \gg 1$ and $\lambda, \omega \gg 1$ with $\lambda \in \mathbb{N}$ to be fixed in Sect. 10. For $k = 1, 2, 3, 4$, let us introduce

$$\begin{aligned} w_k(x) &= \varphi^k_{\mu_1}(\lambda x_1)\varphi_{\mu_2}(\lambda x_2), \\ w^c_k(x) &= -\frac{\mu_1}{\mu_2}(\varphi')^k_{\mu_1}(\lambda x_1)(\Phi'')_{\mu_2}(\lambda x_2), \\ w^{cc}_k(x) &= -\frac{1}{\lambda\mu_2}\varphi^k_{\mu_1}(\lambda x_1)(\Phi'')_{\mu_2}(\lambda x_2), \\ q_k(x) &= \frac{1}{\omega}(\varphi^k_{\mu_1})^2(\lambda x_1)\varphi^2_{\mu_2}(\lambda x_2). \end{aligned}$$

Lemma 4.1. *It holds*

$$\begin{aligned} &\int_{\mathbb{T}^2} w_k^2 \,\mathrm{d}x = 1, \\ &\int_{\mathbb{T}^2} w_k \,\mathrm{d}x = \int_{\mathbb{T}^2} w^c_k \,\mathrm{d}x = \int_{\mathbb{T}^2} w^{cc}_k \,\mathrm{d}x = 0. \end{aligned}$$

For any $s \in [1, \infty]$, we have the estimates

$$\begin{aligned} \|\partial_1^{l_1}\partial_2^{l_2} w_k\|_{L^s(\mathbb{T}^2)} &\le C(s)\lambda^{l_1+l_2}\mu_1^{l_1+\frac{1}{2}-\frac{1}{s}}\mu_2^{l_2+\frac{1}{2}-\frac{1}{s}}, \\ \|\partial_1^{l_1}\partial_2^{l_2} w^c_k\|_{L^s(\mathbb{T}^2)} &\le C(s)\lambda^{l_1+l_2}\mu_1^{l_1+\frac{3}{2}-\frac{1}{s}}\mu_2^{l_2-\frac{1}{2}-\frac{1}{s}}, \\ \|\partial_1^{l_1}\partial_2^{l_2} w^{cc}_k\|_{L^r(\mathbb{T}^2)} &\le C(s)\lambda^{l_1+l_2-1}\mu_1^{l_1+\frac{1}{2}-\frac{1}{s}}\mu_2^{l_2-\frac{1}{2}-\frac{1}{s}}, \\ \|\partial_1^{l_1}\partial_2^{l_2} q_k\|_{L^s(\mathbb{T}^2)} &\le C(s)\omega^{-1}\lambda^{l_1+l_2}\mu_1^{l_1+1-\frac{1}{s}}\mu_2^{l_2+1-\frac{1}{s}}. \end{aligned}$$

Proof. We have

$$\int_{\mathbb{T}^2} w_k^2(x)\,\mathrm{d}x = \int_0^1 (\varphi^k_{\mu_1})^2(\lambda x_1)\,\mathrm{d}x_1 \cdot \int_0^1 \varphi^2_{\mu_2}(\lambda x_2)\,\mathrm{d}x_2 = 1$$

by (9) and since $\int \varphi^2 \,\mathrm{d}x = 1$. Similarly, one gets the zero mean values of w_k, w^c_k and w^{cc}_k by noting that $\int_{\mathbb{T}} \varphi \,\mathrm{d}x = 0$ since $\varphi = \Phi'''$ is a derivative. The estimates can also be proven using (9). □

For $k = 1, 2, 3, 4$ we define the linear maps

$$\begin{aligned} \Lambda_k : \mathbb{R}^2 &\to \mathbb{R}^2, \\ x &\mapsto (\xi_k \cdot x, \xi_k^\perp \cdot x). \end{aligned} \tag{15}$$

Our main building block is now defined as

$$\begin{aligned} W_k^p(x,t) &= w_k\left(\Lambda_k\left(x-\omega t\frac{\xi_k}{|\xi_k|^2}\right)\right)\frac{\xi_k}{|\xi_k|} \\ &= w_k\left(\Lambda_k x-\omega t e_1\right)\frac{\xi_k}{|\xi_k|}, \end{aligned}$$

i.e.

$$W_k^p(x,t) := W^p_{\xi_k,\mu_1,\mu_2,\lambda,\omega}(x,t) = \varphi^k_{\mu_1}(\lambda(\xi_k\cdot x-\omega t))\varphi_{\mu_2}(\lambda\xi_k^\perp\cdot x)\frac{\xi_k}{|\xi_k|},$$

which means that we first rotate w_k and move in time in the direction of ξ_k. This vector field is not divergence free. We define the corrector W_k^c by

$$\begin{aligned} W_k^c(x,t) := W^c_{\xi_k,\mu_1,\mu_2,\lambda,\omega}(x,t) &= w_k^c\left(\Lambda_k\left(x-\omega t\frac{\xi_k}{|\xi_k|^2}\right)\right)\frac{\xi_k^\perp}{|\xi_k|} \\ &= -\frac{\mu_1}{\mu_2}(\varphi')^k_{\mu_1}(\lambda(\xi_k\cdot x-\omega t))(\Phi'')_{\mu_2}(\lambda\xi_k^\perp\cdot x)\frac{\xi_k^\perp}{|\xi_k|} \end{aligned}$$

and observe that $\operatorname{div}(W_k^p+W_k^c)=0$, see Proposition 4.2. We introduce further building blocks by

$$\begin{aligned} W_k^{cc,\parallel}(x,t) := W^{cc,\parallel}_{\xi_k,\mu_1,\mu_2,\lambda,\omega}(x,t) &= w_k^{cc}\left(\Lambda_k\left(x-\omega t\frac{\xi_k}{|\xi_k|^2}\right)\right)\frac{\xi_k}{|\xi_k|} \\ &= -\frac{1}{\lambda\mu_2}\varphi^k_{\mu_1}(\lambda(\xi_k\cdot x-\omega t))(\Phi'')_{\mu_2}(\lambda\xi_k^\perp\cdot x)\frac{\xi_k}{|\xi_k|}, \\ W_k^{cc,\perp}(x,t) := W^{cc,\perp}_{\xi_k,\mu_1,\mu_2,\lambda,\omega}(x,t) &= w_k^{cc}\left(\Lambda_k\left(x-\omega t\frac{\xi_k}{|\xi_k|^2}\right)\right)\frac{\xi_k^\perp}{|\xi_k|} \\ &= -\frac{1}{\lambda\mu_2}\varphi^k_{\mu_1}(\lambda(\xi_k\cdot x-\omega t))(\Phi'')_{\mu_2}(\lambda\xi_k^\perp\cdot x)\frac{\xi_k^\perp}{|\xi_k|}. \end{aligned}$$

Finally, we introduce the building blocks for our time-corrector

$$\begin{aligned} Y_k(x,t) := Y_{\xi_k,\mu_1,\mu_2,\lambda,\omega}(x,t) &= q_k\left(\Lambda_k\left(x-\omega t\frac{\xi_k}{|\xi_k|^2}\right)\right)\xi_k \\ &= \frac{1}{\omega}(\varphi^k_{\mu_1})^2(\lambda(\xi_k\cdot x-\omega t))(\varphi_{\mu_2})^2(\lambda\xi_k^\perp\cdot x)\xi_k \end{aligned}$$

We note that our building blocks are again periodic functions on $\mathbb{R}^2$ with period 1 in both variables since $\xi_k\in\mathbb{N}^2$.

Proposition 4.2 (Building blocks). *The building blocks are λ-periodic and satisfy*

(i) $\operatorname{div}(W_k^p\otimes W_k^p)=\partial_t Y_k$,
(ii) $\int_{\mathbb{T}^2} W_k^p\otimes W_k^p(x,t)\,\mathrm{d}x = \frac{\xi_k}{|\xi_k|}\otimes\frac{\xi_k}{|\xi_k|}$,
(iii) $\|W_k^p(\cdot,t)\|_{L^s(\mathbb{T}^2)} = \|w_k\|_{L^s(\mathbb{T}^2)}$ *for all* $s\in[1,\infty]$,
(iv) $\int_{\mathbb{T}^2} W_k^p(x,t)\,\mathrm{d}x = \int_{\mathbb{T}^2} W_k^c(x,t)\,\mathrm{d}x = \int_{\mathbb{T}^2} W_k^{cc,\parallel}(x,t)\,\mathrm{d}x = \int_{\mathbb{T}^2} W_k^{cc,\perp}(x,t)\,\mathrm{d}x = 0$,
(v) $\int_{\mathbb{T}^2} Y_k\,\mathrm{d}x = \frac{1}{\omega}\xi_k$.

Furthermore, for all $k \in \mathbb{N}, l \in \mathbb{N}$ they satisfy the following estimates:

$$\|\nabla^l W_k^p\|_{L^s([-k,k]^2)} \leq C(s) k^{\frac{2}{s}} \lambda^l \mu_1^{\frac{1}{2}-\frac{1}{s}} \mu_2^{l+\frac{1}{2}-\frac{1}{s}},$$
$$\|\nabla^l W_k^c\|_{L^s([-k,k]^2)} \leq C(s) k^{\frac{2}{s}} \lambda^l \mu_1^{\frac{3}{2}-\frac{1}{s}} \mu_2^{l-\frac{1}{2}-\frac{1}{s}},$$
$$\|\nabla^l W_k^{cc,\|}\|_{L^s([-k,k]^2)} \leq C(s) k^{\frac{2}{s}} \lambda^{l-1} \mu_1^{\frac{1}{2}-\frac{1}{s}} \mu_2^{l-\frac{1}{2}-\frac{1}{s}},$$
$$\|\nabla^l W_k^{cc,\perp}\|_{L^s([-k,k]^2)} \leq C(s) k^{\frac{2}{s}} \lambda^{l-1} \mu_1^{\frac{1}{2}-\frac{1}{s}} \mu_2^{l-\frac{1}{2}-\frac{1}{s}},$$
$$\|\nabla^l Y_k\|_{L^s([-k,k]^2)} \leq C(s) k^{\frac{2}{s}} \omega^{-1} \lambda^l \mu_1^{1-\frac{1}{s}} \mu_2^{l+1-\frac{1}{s}}$$

Proof. For (i), we have by Lemma 2.10 with $f(x) = (\varphi_{\mu_1}^k)^2(\lambda(x-\omega t))$ and $g(x) = \varphi_{\mu_2}^2(\lambda x)$

$$\begin{aligned}
\operatorname{div}(W_k^p \otimes W_k^p) &= \operatorname{div}\left((\varphi_{\mu_1}^k)^2(\lambda(\xi_k \cdot x - \omega t))\varphi_{\mu_2}^2(\lambda \xi_k^\perp \cdot x)\frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right) \\
&= \lambda\left((\varphi_{\mu_1}^k)^2\right)'(\lambda(\xi_k \cdot x - \omega t))\varphi_{\mu_2}^2(\lambda \xi_k^\perp \cdot x)\xi_k \\
&= \partial_t Y_k.
\end{aligned}$$

For (ii), this is immediate for $k = 1$, since by Lemma 4.1

$$\int_{\mathbb{T}^2} W_k^p \otimes W_k^p \,\mathrm{d}x = \int_{\mathbb{T}^2} w_k^2(x - \omega t e_1)\,\mathrm{d}x \cdot \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|} = \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}.$$

The same is true for $k = 2$ by switching the roles of x_1 and x_2 in the definition of w_k. For $k = 3$, we calculate with the transformation rule by rotating the cube $[-\frac{1}{2}, \frac{1}{2}]^2$ by Λ_k

$$\begin{aligned}
\int_{\mathbb{T}^2} W_k^p \otimes W_k^p \,\mathrm{d}x &= \int_{[-\frac{1}{2},\frac{1}{2}]^2} w_k^2(\Lambda_k x - \omega t e_1)\,\mathrm{d}x \cdot \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|} \\
&= \frac{1}{|\det D\Lambda_k|}\int_Q w_k^2(x - \omega t e_1)\,\mathrm{d}x \cdot \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}
\end{aligned}$$

where $Q = \Lambda_k([-\frac{1}{2}, \frac{1}{2}]^2)$ is the by 90 degress rotated and scaled cube with vertices $\{\pm e_1, \pm e_2\}$. It is not difficult to see that, by a geometric argument, it holds $\int_Q w_k^2 \,\mathrm{d}x = 2\int_{\mathbb{T}^2} w_k^2 \,\mathrm{d}x$ because w_k is periodic. Since $|\det D\Lambda_k| = 2$ for $k = 3$, we have

$$\int_{\mathbb{T}^2} W_k^p \otimes W_k^p \,\mathrm{d}x = \int_{\mathbb{T}^2} w_k^2(x - \omega t e_1)\,\mathrm{d}x \cdot \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|} = \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|},$$

and the same reasoning holds for $k = 4$. For (iii), we do a similar calculation and obtain

$$\|W_k(\cdot,t)\|_{L^s(\mathbb{T}^2)}^s = \int_{\mathbb{T}^2} |W_k^p|^s \,\mathrm{d}x = \int_{\mathbb{T}^2} |w_k(x - \omega t e_1)|^s \,\mathrm{d}x = \|w_k\|_{L^s(\mathbb{T}^2)}^s$$

for any $s \in [1, \infty)$, and the same calculations show (iv) and (v). The estimates follow directly from Lemma 4.1 and exploiting the fact that $\mu_2 \gg \mu_1$. □

Lemma 4.3 (Disjointness of supports). *We have*

$$\operatorname{supp} W_k^p = \operatorname{supp} W_k^c = \operatorname{supp} W_k^{cc,\|} = \operatorname{supp} W_k^{cc,\perp} = \operatorname{supp} Y_k$$

and for large enough μ_1 (independent of λ, μ_2) it holds

$$\operatorname{supp} W_{k_1}^p \cap \operatorname{supp} W_{k_2}^p = \emptyset$$

for $k_1 \neq k_2$.

Proof. Looking at the definition, we see that the function w_k (and also w_k^c, w_k^{cc}, q_k) is supported in small balls of radius $\frac{1}{\lambda\mu_1}$ around the points $\frac{1}{\lambda}\left((\frac{1}{2}, \frac{1}{2}) + \frac{k}{16}|\xi_k|^2 e_1 + \mathbb{Z}^2\right)$, i.e.

$$\operatorname{supp} w_k \subset B_{\frac{1}{\lambda\mu_1}}(0) + \frac{1}{\lambda}\left(\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{k}{16}|\xi_k|^2 e_1 + \mathbb{Z}^2\right).$$

Therefore, for a fixed time t, we have since $W_k^p(x,t) = w_k\left(\Lambda_k\left(x - \omega t \frac{\xi_k}{|\xi_k|^2}\right)\right)\frac{\xi_k}{|\xi_k|}$

$$\operatorname{supp} W_k^p(\cdot,t) \subset B_{\frac{1}{\lambda\mu_1}}(0) + \frac{1}{\lambda}\Lambda_k^{-1}\left(\left(\frac{1}{2},\frac{1}{2}\right) + \frac{k}{16}|\xi_k|^2 e_1 + \mathbb{Z}^2\right) + \omega t\frac{\xi_k}{|\xi_k|^2},$$

i.e. we calculate, using $\Lambda_k^{-1} = \frac{1}{2}\Lambda_k$,

$$\begin{aligned}
\operatorname{supp} W_1^p(\cdot,t) &\subset B_{\frac{1}{\lambda\mu_1}}(0) + \frac{1}{\lambda}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{1}{\lambda}\frac{1}{16}\xi_1 + \frac{1}{\lambda}\mathbb{Z}^2 + \omega t(1,0),\\
\operatorname{supp} W_2^p(\cdot,t) &\subset B_{\frac{1}{\lambda\mu_1}}(0) + \frac{1}{\lambda}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{1}{\lambda}\frac{1}{8}\xi_2 + \frac{1}{\lambda}\mathbb{Z}^2 + \omega t(0,1),\\
\operatorname{supp} W_3^p(\cdot,t) &\subset B_{\frac{1}{\lambda\mu_1}}(0) + \frac{1}{\lambda}\left(\frac{1}{2},0\right) + \frac{1}{\lambda}\frac{3}{16}\xi_3 + \frac{1}{\lambda}\left(\frac{1}{2}\mathbb{Z}\right)^2 + \omega t\left(\frac{1}{2},\frac{1}{2}\right),\\
\operatorname{supp} W_4^p(\cdot,t) &\subset B_{\frac{1}{\lambda\mu_1}}(0) + \frac{1}{\lambda}\left(0,-\frac{1}{2}\right) + \frac{1}{\lambda}\frac{1}{4}\xi_4 + \frac{1}{\lambda}\left(\frac{1}{2}\mathbb{Z}\right)^2 + \omega t\left(\frac{1}{2},-\frac{1}{2}\right).
\end{aligned}$$

One can now check by hand that the supports are disjoint. We do this for W_2^p and W_4^p as an example. Assume there is an $x \in \operatorname{supp} W_2^p(\cdot,t) \cap W_4^p(\cdot,t)$. Then there exists $y_1, y_2 \in B_{\frac{1}{\lambda\mu_1}}(0)$ and $k \in \mathbb{Z}^2$, $l \in (\frac{1}{2}\mathbb{Z})^2$ such that

$$y_1 + \frac{1}{\lambda}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{1}{\lambda}\frac{1}{8}\xi_2 + \frac{1}{\lambda}k + \omega t(0,1) = x = y_2 + \frac{1}{\lambda}\left(0,-\frac{1}{2}\right) + \frac{1}{\lambda}\frac{1}{4}\xi_4 + \frac{1}{\lambda}l + \omega t\left(\frac{1}{2},-\frac{1}{2}\right)$$

or equivalently

$$\begin{aligned}
\underbrace{y_1 - y_2}_{\in B_{\frac{2}{\lambda\mu_1}}(0)} &= -\frac{1}{\lambda}\left(\frac{1}{2},1\right) + \lambda\left(\frac{2}{8},-\frac{3}{8}\right) + \frac{1}{\lambda}(l-k) + \omega t\left(\frac{1}{2},-\frac{3}{2}\right)\\
&= \underbrace{-\frac{1}{\lambda}\left(\frac{1}{2},1\right) + \frac{1}{\lambda}(l-k)}_{\in\frac{1}{\lambda}(\frac{1}{2}\mathbb{Z})^2} + \frac{1}{\lambda}\left(\frac{1}{8},0\right) + \underbrace{\frac{1}{\lambda}\left(\frac{1}{8},-\frac{3}{8}\right) + \omega t\left(\frac{1}{2},-\frac{3}{2}\right)}_{\in\{s(1,-3):s\in\mathbb{R}\}}.
\end{aligned}$$

But it is not difficult to see that $0 \notin \frac{1}{\lambda}(\frac{1}{2}\mathbb{Z})^2 + \frac{1}{\lambda}\left(\frac{1}{8},0\right) + \{s(1,-3) : s \in \mathbb{R}\}$. Therefore, we can choose μ_1 large enough such that $B_{\frac{2}{\lambda\mu_1}}(0) \cap \left(\frac{1}{\lambda}(\frac{1}{2}\mathbb{Z})^2 + \frac{1}{\lambda}\left(\frac{1}{8},0\right) + \{s(1,-3) : s \in \mathbb{R}\}\right) = \emptyset$. This shows $\operatorname{supp} W_2^p(\cdot,t) \cap \operatorname{supp} W_4^p(\cdot,t) = \emptyset$. □

5. The Perturbations

Before we can define the perturbations, let us decompose the error $\mathring{R}_0$ in the following way. There are smooth functions Γ_k with $|\Gamma_k| \le 1$ such that for any matrix A with $|A - I| < \frac{1}{8}$

$$A = \sum_k \Gamma_k^2(A)\frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}, \tag{16}$$

see Section 5 in [3]. Let $\kappa \in \mathbb{N}$ such that

$$\|\mathring{R}_0(t)\|_{L^1(\mathbb{R}^2\setminus B_\kappa)} \le \frac{\eta}{2} \tag{17}$$

for all $t \in [0,1]$. With condition (17), our choice of κ is set. For $\varepsilon > 0$ we further define

$$\gamma(t) = \frac{e(t)(1-\frac{\delta}{2}) - \int_{\mathbb{R}^2} |u_0|^2(x,t)\,\mathrm{d}x}{2\|\chi_\kappa\|^2_{L^2(\mathbb{R}^2)}},$$

$$\rho(x,t) = 10\sqrt{\varepsilon^2 + |\mathring{R}_0(x,t)|^2} + \gamma(t),$$

$$a_k(x,t) = \chi_\kappa(x)\rho^{\frac{1}{2}}(x,t)\Gamma_k\left(I + \frac{\mathring{R}_0(x,t)}{\rho(x,t)}\right),$$

noting that the decomposition (16) exists for $I + \frac{\mathring{R}_0}{\rho}$. The function χ_κ is a smooth cutoff with $\chi_\kappa \equiv 1$ on B_κ and $\chi_\kappa \equiv 0$ on $\mathbb{R}^2 \setminus B_{\kappa+1}$. For later use, we note that

$$\chi_\kappa^2(x)\rho(x,t)I + \chi_\kappa^2(x)\mathring{R}_0(x,t) = \sum_k a_k^2(x,t)\frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}. \tag{18}$$

We define

$$H^k(x,t) = \frac{a_k(x,t)}{|\xi_k|} w_k^{cc}\left(\Lambda_k\left(x - \omega t\frac{\xi_k}{|\xi_k|^2}\right)\right).$$

Let us define the perturbations as follows.

$$w(x,t) = \sum_{k=1}^{4} \nabla^\perp H^k(x,t),$$

$$u^t(x,t) = -\sum_{k=1}^{4} \mathbb{P}\left(a_k^2(x,t)Y_k(x,t)\right),$$

$$v(x,t) = \mathbb{P}\int_0^t r_0(x,s)\,\mathrm{d}s.$$

We note that

$$\operatorname{div} w = 0, \tag{19}$$

being an orthogonal gradient. We set

$$u_1 = u_0 + w + u^t + v.$$

By a simple calculation, we see that

$$\begin{aligned}
\nabla^\perp H^k(x,t) &= a_k(x,t)W_k^p(x,t) + a_k(x,t)W_k^c(x,t) + w_k^{cc}\left(\Lambda_k\left(x - \omega t\frac{\xi_k}{|\xi_k|^2}\right)\right)\frac{\nabla^\perp a_k(x,t)}{|\xi_k|} \\
&= a_k(x,t)W_k^p(x,t) + a_k(x,t)W_k^c(x,t) \\
&\quad + \frac{\langle \nabla^\perp a_k(x,t)\cdot \xi_k\rangle}{|\xi_k|^2} W_k^{cc,\parallel}(x,t) + \frac{\langle \nabla^\perp a_k(x,t)\cdot \xi_k^\perp\rangle}{|\xi_k|^2} W_k^{cc,\perp}(x,t)
\end{aligned}$$

and we set $w = u^p + u^c$ with

$$u^p(x,t) = \sum_{k=1}^{4} a_k(x,t)W_k^p(x,t),$$

$$u^c(x,t) = \sum_{k=1}^{4} a_k(x,t)W_k^c(x,t) + b_k^1(x,t)W_k^{cc,\parallel}(x,t) + b_k^2(x,t)W_k^{cc,\perp}(x,t) \tag{20}$$

where we denote

$$b_k^1(x,t) = \frac{\langle \nabla^\perp a_k(x,t) \cdot \xi_k \rangle}{|\xi_k|^2},$$
$$b_k^2(x,t) = \frac{\langle \nabla^\perp a_k(x,t) \cdot \xi_k^\perp \rangle}{|\xi_k|^2}.$$

Remark 5.1. We note several things:

(1) $a_k(x,t)$: Decomposition of the old error R_0, also pumping energy into the system.
(2) $\mathbb{P}$ denotes the Leray projector.
(3) Note that $\operatorname{div}(w + u^t + v) = 0$ and thus also $\operatorname{div} u_1 = 0$.
(4) We will sometimes use $w = \sum_k \nabla^\perp H^k$ as a whole and use estimates on H^k, whereas on other occasions we have to decompose $w = u^p + u^c$ and use certain properties of the individual parts.

Lemma 5.2. *The function u_1 is smooth with $u_1 \in C([0,1], L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2))$, i.e. u_1 has the desired regularity.*

Proof. The function u_0 is smooth and in $C([0,1], L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2))$ by assumption. For w this is also clear since it is smooth with compact support. For u^t, we note that $\mathbb{P} : L^s(\mathbb{R}^2) \to L^s(\mathbb{R}^2)$ is a bounded operator for all $1 < s < \infty$, see for example Lemma 1.16 in [1]. Since the function inside $\mathbb{P}$ in the definition of u^t is smooth and compactly supported and therefore in $C([0,1], L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2))$, this also holds for u^t. By assumption, r_0 is smooth and $r_0 \in C([0,1], L^\infty(\mathbb{R}^2))$ with compact support in space, in particular also $r_0 \in C([0,1], L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2))$ and therefore also $v \in C([0,1], L^2(\mathbb{R}^2) \cap L^3(\mathbb{R}^2))$ by the boundedness of $\mathbb{P}$. □

6. Estimates of the Perturbations

In this section, we provide the necessary estimates on the perturbations. We start with a preliminary estimate on the coefficients a_k and then estimate the individual parts of the perturbations separately. After that, we obtain an estimate on the energy increment and conclude the section by fixing the parameter ε.

Lemma 6.1 (Preliminary estimates I). *It holds*

$$\|a_k\|_{C^l(\mathbb{R}^2 \times [0,1])} \leq C(R_0, u_0, e, \delta, \kappa, \varepsilon, l), \tag{21}$$

and

$$\|a_k(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{10\pi}\left((\kappa+1)\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}}\right) \tag{22}$$

uniformly in t.

Proof. For the first part, we only note that by (12)

$$0 \leq e(t)\left(1 - \frac{\delta}{2}\right) - \int_{\mathbb{R}^2} |u_0|^2(x,t)\,\mathrm{d}x \leq \frac{3}{4}\delta,$$

so we have $0 \leq \gamma(t) \leq \frac{3}{4}\delta/(2|B_\kappa|)$. This together with the definition of a_k implies the L^∞-estimates. For the second part, we calculate using $|\Gamma_k| \leq 1$

$$\begin{aligned}
\int_{\mathbb{R}^2} a_k^2(x,t)\,\mathrm{d}x &= \int_{\mathbb{R}^2} \chi_\kappa^2(x)\rho(x,t)\Gamma_k^2\left(I + \frac{\mathring{R}_0(x,t)}{\rho(x,t)}\right)\mathrm{d}x \leq \int_{B_{\kappa+1}} 10\sqrt{\varepsilon^2 + |\mathring{R}_0(t)|^2(x,t)} + \gamma(t)\,\mathrm{d}x \\
&\leq 10\pi(\kappa+1)^2\varepsilon + 20\|R_0\|_{C_t L_x^1} + 10\pi(\kappa+1)^2\gamma \\
&\leq 10\pi(\kappa+1)^2\varepsilon + 20\|R_0\|_{C_t L_x^1} + 5\pi\frac{(\kappa+1)^2}{\|\chi_\kappa\|_{L^2(\mathbb{R}^2)}^2}\left(e(t)(1 - \frac{\delta}{2}) - \int_{\mathbb{R}^2} |u_0|^2(x,t)\,\mathrm{d}x\right).
\end{aligned}$$

Using again (12) and

$$5\pi\frac{(\kappa+1)^2}{\|\chi_\kappa\|^2_{L^2(\mathbb{R}^2)}}\le 5\frac{(\kappa+1)^2}{\kappa^2}\le 20,$$

and also that $20\|R_0\|_{C_tL^1_x}\le\delta$ by assumption, we obtain

$$\int_{\mathbb{R}^2} a_k^2(x,t)\,\mathrm{d}x\le 10\pi(\kappa+1)^2\varepsilon+16\delta. \tag{23}$$

From this (22) follows. □

Lemma 6.2 (Estimate of the principal perturbation). *It holds*

$$\|u^p(t)\|_{L^s(\mathbb{R}^2)}\le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\mu_1^{\frac{1}{2}-\frac{1}{s}}\mu_2^{\frac{1}{2}-\frac{1}{s}}$$

and for $p=2$ *more refined*

$$\|u^p(t)\|_{L^2(\mathbb{R}^2)}\le\sqrt{10\pi}\left((\kappa+1)\varepsilon^{\frac{1}{2}}+\delta^{\frac{1}{2}}\right)+\frac{C(R_0,u_0,e,\delta,\kappa,\varepsilon)}{\lambda^{\frac{1}{2}}} \tag{24}$$

uniformly in t.

For the first estimate, we use Proposition 4.2, (21) and the fact that u^p is supported in $B_{\kappa+1}$. For the second estimate, we use Proposition 2.1, noting again that $\operatorname{supp} a_k(\cdot,t)\subset[-\kappa-1,\kappa+1]^2$, Lemma 4.1, Proposition 4.2 and (22)

$$\begin{aligned}\|\sum_{k=1}^4 a_k(\cdot,t)W_k^p(\cdot,t)\|_{L^2(\mathbb{R}^2)}&=\sum_{k=1}^4\|a_k(\cdot,t)W_k^p(\cdot,t)\|_{L^2([-\kappa-1,\kappa+1]^2)}\\&\le\|a_k(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|W_k^p(\cdot,t)\|_{L^2(\mathbb{T}^2)}\\&\quad+C\frac{2\kappa+2}{\lambda^{\frac{1}{2}}}\|a_k(\cdot,t)\|_{C^1(\mathbb{R}^2)}\|W_k^p(\cdot,t)\|_{L^2(\mathbb{T}^2)}\\&\le\sqrt{10\pi}\left((\kappa+1)\varepsilon^{\frac{1}{2}}+\delta^{\frac{1}{2}}\right)+\frac{C(R_0,u_0,e,\kappa,\delta,\kappa,\varepsilon)}{\lambda^{\frac{1}{2}}}.\end{aligned}$$ □

.

Lemma 6.3 (Estimates of the correctors). *We have*

$$\|u^c(t)\|_{L^s(\mathbb{R}^2)}\le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\mu_1^{\frac{3}{2}-\frac{1}{s}}\mu_2^{-\frac{1}{2}-\frac{1}{s}}$$

and

$$\|u^t(t)\|_{L^2(\mathbb{R}^2)}\le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}$$

uniformly in t.

Proof. This proof follows by using Proposition 4.2 together with (21) and the fact that u^c is supported in $[-\kappa-1,\kappa+1]^2$. For u^t, we also use that $\mathbb{P}$ is bounded from L^2 to L^2 and the argument inside $\mathbb{P}$ in the definition of u^t is supported in $[-\kappa-1,\kappa+1]^2$. □

Lemma 6.4. *It holds*

$$\|\nabla^l H^k(t)\|_{L^s(\mathbb{T}^2)}\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,l)\lambda^{l-1}\mu_1^{\frac{1}{2}-\frac{1}{s}}\mu_2^{l-\frac{1}{2}-\frac{1}{s}}$$

uniformly in t.

Proof. This follows immediately from Lemma 4.1, (21) and the definition of H^k, exploiting also the fact that $\mu_2\gg\mu_1$. □

Lemma 6.5. *It holds for all $s \in (1,\infty)$*

$$\|v(t)\|_{L^s(\mathbb{R}^2)} \leq \|\mathbb{P}\|_{\mathcal{L}(L^s(\mathbb{R}^2))}\|r_0\|_{C_tL^s_x}$$

and for $s = 2$

$$\|v(t)\|_{L^2(\mathbb{R}^2)} \leq \|r_0\|_{C_tL^2_x}$$

uniformly in t.

Proof. This follows using Minkowski's inequality and the fact that $\mathbb{P} : L^s(\mathbb{R}^2) \to L^s(\mathbb{R}^2)$ for all $s \in (1,\infty)$. For $s = 2$, $\mathbb{P}$ is an orthogonal projection, therefore $\|\mathbb{P}\|_{\mathcal{L}(L^2(\mathbb{R}^2))} \leq 1$. ☐

Lemma 6.6 (Estimate of the energy increment). *We have*

$$\left|e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{R}^2} |u_1|^2(x,t)\,\mathrm{d}x\right| \leq \frac{1}{32}\delta + 20\pi(\kappa+1)^2\varepsilon + C(R_0,r_0,u_0,e,\delta,\kappa,\varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega} + \frac{\mu_1}{\mu_2} + \frac{1}{\lambda}\right). \tag{25}$$

Looking at (18), we consider

$$u^p \otimes u^p - \chi_\kappa^2 \mathring{R}_0 = \chi_\kappa^2 \rho I + \sum_{k=1}^4 a_k^2 \left(W_k^p \otimes W_k^p - \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right).$$

We take the trace and use that $\mathring{R}_0$ is traceless, hence we get

$$|u^p|^2 - 2\chi_\kappa^2\gamma(t) = 20\chi_\kappa^2\sqrt{\varepsilon^2 + |\mathring{R}_0|^2} + \sum_{k=1}^4 a_k^2\left(|W_k^p|^2 - 1\right)$$

Integrating this and using $\sqrt{\varepsilon^2+|x|^2} \leq \varepsilon + |x|$, we get

$$\left|\int_{\mathbb{R}^2} |u^p|^2(x,t)\,\mathrm{d}x - \left(e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{R}^2} |u_0|^2(x,t)\,\mathrm{d}x\right)\right| \leq 20\pi(\kappa+1)^2\varepsilon + 40\|R_0\|_{C_tL^1_x} + \sum_k \left|\int_{\mathbb{R}^2} a_k^2(x,t)\left(|W_k^p|^2(x,t) - 1\right)\,\mathrm{d}x\right|. \tag{26}$$

We can estimate each summand in the second line with Lemma 2.2, using that a_k is supported in $[-\kappa-1,\kappa+1]^2$, (21) and Proposition 4.2 by

$$\begin{aligned}\left|\int_{\mathbb{R}^2} a_k^2(x,t)\left(|W_k^p|^2(x,t)-1\right)\,\mathrm{d}x\right| &\leq \frac{4\sqrt{2}(\kappa+1)^2\|a_k^2(\cdot,t)\|_{C^1(\mathbb{R}^2)}\||W_k^p(\cdot,t)|^2 - 1\|_{L^1(\mathbb{T}^2)}}{\lambda} \\ &\leq \frac{C(R_0,u_0,e,\delta,\kappa,\varepsilon)}{\lambda}.\end{aligned} \tag{27}$$

Writing $u_1 = u_0 + u^p + u^c + u^t + v$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |u_1|^2(x,t)\,\mathrm{d}x = {} & \|u_0(t)\|^2_{L^2(\mathbb{R}^2)} + \|u^p(t)\|^2_{L^2(\mathbb{R}^2)} + \|v(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_{\mathbb{R}^2} u_0 \cdot v(x,t)\,\mathrm{d}x \\ & + 2\int_{\mathbb{R}^2} u_0 \cdot (u^p+u^c+u^t)(x,t)\,\mathrm{d}x + 2\int_{\mathbb{R}^2} u^p\cdot(u^c+u^t+v)(x,t)\,\mathrm{d}x \\ & + 2\int_{\mathbb{R}^2} v\cdot(u^c+u^t)(x,t)\,\mathrm{d}x + \int_{\mathbb{R}^2} |u^c+u^t|^2(x,t)\,\mathrm{d}x,\end{aligned}$$

where by Lemma 6.2, Lemma 6.3 and Lemma 6.5

$$
\begin{aligned}
2\left|\int_{\mathbb{R}^2} u_0\cdot(u^p+u^c+u^t)(x,t)\,\mathrm{d}x\right| &\le 2\|u_0(t)\|_{L^2(\mathbb{R}^2)}\left(\|u^c(t)\|_{L^2(\mathbb{R}^2)}+\|u^t(t)\|_{L^2(\mathbb{R}^2)}\right)\\
&\quad+2\|u_0(t)\|_{L^3(\mathbb{R}^2)}\|u^p(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}\\
&\le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\left(\frac{\mu_1}{\mu_2}+\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}}+\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}\right),\\
2\left|\int_{\mathbb{R}^2} u^p\cdot(u^c+u^t+v)(x,t)\,\mathrm{d}x\right| &\le 2\|u^p(t)\|_{L^2(\mathbb{R}^2)}\left(\|u^c(t)\|_{L^2(\mathbb{R}^2)}+\|u^t(t)\|_{L^2(\mathbb{R}^2)}\right)\\
&\quad+2\|\mathbb{P}\|_{\mathcal{L}(L^3(\mathbb{R}^2))}\|u^p(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}\|r_0\|_{C_tL^3_x},\\
&\le C(R_0,r_0,u_0,e,\delta,\kappa,\varepsilon)\left(\frac{\mu_1}{\mu_2}+\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}}+\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}\right),\\
2\left|\int_{\mathbb{R}^2} v\cdot(u^c+u^t)(x,t)\,\mathrm{d}x\right| &\le 2\|v(t)\|_{L^2(\mathbb{R}^2)}\left(\|u^c(t)\|_{L^2(\mathbb{R}^2)}+\|u^t(t)\|_{L^2(\mathbb{R}^2)}\right)\\
&\le C(R_0,r_0,u_0,e,\delta,\kappa,\varepsilon)\left(\frac{\mu_1}{\mu_2}+\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}\right),\\
\int_{\mathbb{R}^2}|u^c+u^t|^2(x,t)\,\mathrm{d}x &\le 2\left(\|u^c(t)\|^2_{L^2(\mathbb{R}^2)}+\|u^t(t)\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\left(\left(\frac{\mu_1}{\mu_2}\right)^2+\left(\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}\right)^2\right). \qquad (28)
\end{aligned}
$$

This yields

$$
\begin{aligned}
\left|e(t)\left(1-\frac{\delta}{2}\right)-\int_{\mathbb{R}^2}|u_1|^2(x,t)\,\mathrm{d}x\right| &\le \left|\int_{\mathbb{R}^2}|u^p|^2(x,t)\,\mathrm{d}x-\left(e(t)\left(1-\frac{\delta}{2}\right)-\int_{\mathbb{R}^2}|u_0|^2(x,t)\,\mathrm{d}x\right)\right|\\
&\quad+\|v(t)\|^2_{L^2(\mathbb{R}^2)}+2\|u_0(t)\|_{L^2(\mathbb{R}^2)}\|v(t)\|_{L^2(\mathbb{R}^2)}\\
&\quad+C(R_0,r_0,u_0,e,\delta,\kappa,\varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}}+\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}+\frac{\mu_1}{\mu_2}\right)\\
&\le \left|\int_{\mathbb{R}^2}|u^p|^2(x,t)\,\mathrm{d}x-\left(e(t)(1-\frac{\delta}{2})-\int_{\mathbb{R}^2}|u_0|^2(x,t)\,\mathrm{d}x\right)\right|\\
&\quad+\|r_0\|^2_{C_tL^2_x}+2\|u_0(t)\|_{L^2(\mathbb{R}^2)}\|r_0\|_{C_tL^2_x}\\
&\quad+C(R_0,r_0,e,u_0,\delta,\kappa,\varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}}+\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}+\frac{\mu_1}{\mu_2}\right).
\end{aligned}
$$

Let us combine the previous inequality with (26) and (27) and then use our assumptions (13) and $e(t)\ge\frac{1}{2}$, this yields

$$
\begin{aligned}
\left|e(t)\left(1-\frac{\delta}{2}\right)-\int_{\mathbb{R}^2}|u_1|^2(x,t)\,\mathrm{d}x\right| &\le 40\|R_0\|_{C_tL^1_x}+\|r_0\|^2_{C_tL^2_x}+2\|u_0(t)\|_{L^2(\mathbb{R}^2)}\|r_0\|_{C_tL^2_x}\\
&\quad+20\pi(\kappa+1)^2\varepsilon\\
&\quad+C(R_0,r_0,u_0,e,\delta,\kappa,\varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}}+\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega}+\frac{\mu_1}{\mu_2}+\frac{1}{\lambda}\right)
\end{aligned}
$$

$$\leq \frac{1}{32}\delta + 20\pi(\kappa+1)^2\varepsilon$$
$$+ C(R_0, r_0, u_0, e, \delta, \kappa, \varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega} + \frac{\mu_1}{\mu_2} + \frac{1}{\lambda}\right). \qquad \square$$

At this point, we fix ε and choose this parameter so small such that

$$20\pi(\kappa+1)^2\varepsilon < \frac{1}{32}\delta,$$
$$\sqrt{10}\pi((\kappa+1)\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}}) \leq 10\delta^{\frac{1}{2}},$$

therefore (24) becomes

$$\|u^p(t)\|_{L^2(\mathbb{R}^2)} \leq 10\delta^{\frac{1}{2}} + \frac{C(R_0, u_0, e, \delta, \kappa, \varepsilon)}{\lambda^{\frac{1}{2}}}$$

and (25) reduces to

$$\left|e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{R}^2} |u_1|^2(x,t)\,\mathrm{d}x\right| < \frac{1}{16}\delta + C(R_0, r_0, u_0, e, \delta, \kappa, \varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega} + \frac{\mu_1}{\mu_2} + \frac{1}{\lambda}\right)$$
$$\leq \frac{1}{8}\delta e(t) + C(R_0, r_0, u_0, e, \delta, \kappa, \varepsilon)\left(\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\omega} + \frac{\mu_1}{\mu_2} + \frac{1}{\lambda}\right). \tag{29}$$

7. Estimates of the Curl in Hardy Space

In the following Lemmas, we prove that the curls of the perturbations are in the real Hardy space $H^p(\mathbb{R}^2)$ for $\frac{2}{3} < p < 1$ and estimate their Hardy space seminorms in terms of λ, μ_1 and μ_2. We will use Remark 2.6; therefore, we decompose the perturbations into finitely many functions that are supported on disjoint, very small balls of radius $\frac{1}{\lambda\mu_1}$.

Lemma 7.1 (Curl of w). *It holds* $\operatorname{curl} w(t) \in H^p(\mathbb{R}^2)$ *and*

$$\|\operatorname{curl} w(t)\|_{H^p(\mathbb{R}^2)} \leq C(R_0, u_0, e, \delta, \kappa, \varepsilon)\lambda\mu_1^{\frac{1}{2}-\frac{2}{p}}\mu_2^{\frac{3}{2}} \text{ for all } t \in [0,1].$$

By definition of H^k, $\operatorname{supp} H^k = \operatorname{supp} W_k^p$. As seen in the proof of Lemma 4.3, for a fixed time t, the perturbations are supported in small, disjoint balls of radius $\frac{1}{\lambda\mu_1}$ around the points in the finite set

$$M_k(t) = \left\{\frac{1}{\lambda}\Lambda_k^{-1}\left(\left(\frac{1}{2},\frac{1}{2}\right) + \frac{k}{16}|\xi_k|^2 e_1 + \mathbb{Z}^2\right) + \omega t\frac{\xi_k}{|\xi_k|^2}, k = 1,2,3,4\right\} \cap B_{\kappa+1}.$$

Let us abbreviate $B_{x_0} = B_{\frac{1}{\lambda\mu_1}}(x_0)$ for $x_0 \in M(t)$, and let us decompose w as

$$w(x,t) = \sum_{x_0 \in M(t)} \theta_{x_0}(x,t)$$

where

$$\theta_{x_0}(x,t) = \mathbb{1}_{B(x_0)}(x)w(x,t).$$

Since θ_{x_0} is smooth and has compact support, $\operatorname{curl}\theta_{x_0} \in H^p(\mathbb{R}^2)$ since, as a derivative of a compactly supported function, it satisfies $\int_{\mathbb{R}^2} \operatorname{curl}\theta_{x_0}\,\mathrm{d}x = 0$. We estimate the H^p-seminorm for each $\operatorname{curl}\theta_{x_0}$. We have

$$\operatorname{curl}\theta_{x_0}(x,t) = \mathbb{1}_{B(x_0)}(x)\operatorname{curl} w(x,t) = -\mathbb{1}_{B(x_0)}(x)\sum_{k=1}^{4}\Delta H^k(x,t).$$

As already said, each θ_{x_0} is supported on one ball of measure $\frac{C}{\lambda\mu_1}$. By Lemma 6.4,

$$\|\operatorname{curl}\theta_{x_0}(t)\|_{L^\infty(\mathbb{R}^2)} \le \sum_{k=1}^{4}\|\nabla^2 H^k(t)\|_{L^\infty(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\lambda\mu_1^{\frac{1}{2}}\mu_2^{\frac{3}{2}}$$

This gives us by Remark 2.6

$$\|\operatorname{curl}\theta_{x_0}(t)\|_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\lambda^{1-\frac{2}{p}}\mu_1^{\frac{1}{2}-\frac{2}{p}}\mu_2^{\frac{3}{2}}.$$

Since $|M(t)|$ is of order $\kappa^2\lambda^2$, $\operatorname{curl} w$ is made up of $\approx \lambda^2\kappa^2$- many functions $\operatorname{curl}\theta_{x_0}$, and we obtain

$$\begin{aligned}\|\operatorname{curl} w(t)\|^p_{H^p(\mathbb{R}^2)} &\le \sum_{x_0\in M(t)}\|\operatorname{curl}\theta_{x_0}(t)\|^p_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\lambda^2\lambda^{p-2}\mu_1^{\frac{p}{2}-2}\mu_2^{\frac{3p}{2}}\\ &= C(R_0,u_0,e,\delta,\kappa,\varepsilon)\lambda^p\mu_1^{\frac{p}{2}-2}\mu_2^{\frac{3p}{2}}.\end{aligned}$$

□

Lemma 7.2 (Curl of u^t). *It holds* $\operatorname{curl} u^t(t)\in H^p(\mathbb{R}^2)$ *and*

$$\|\operatorname{curl} u^t(t)\|_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\omega^{-1}\lambda\mu_1^{1-\frac{2}{p}}\mu_2^2 \text{ for all } t\in[0,1].$$

We write again

$$u^t(x,t) = \sum_{x_0\in M(t)}\theta_{x_0}(x,t)$$

with the same decomposition as in the previous Lemma. Since $\operatorname{curl}(\mathbb{P}f)=\operatorname{curl} f$ for all smooth $f:\mathbb{R}^2\to\mathbb{R}^2$, we have

$$\begin{aligned}\operatorname{curl}\theta_{x_0}(x,t) &= \mathbb{1}_{B(x_0)}\sum_{k=1}^{4}a_k^2(x,t)\frac{\lambda}{\omega}(\varphi^k_{\mu_1})^2(\lambda(\xi_k\cdot x-\omega t))(\varphi^2_{\mu_2})'(\lambda\xi_k^\perp\cdot x)|\xi_k|^2\\ &= \mathbb{1}_{B(x_0)}\sum_{k=1}^{4}a_k^2(x,t)(\partial_2 q_k)(\Lambda_k x-\omega t e_1).\end{aligned}$$

Arguing in the same way as before, we just need to estimate with Lemma 4.1

$$\|\operatorname{curl}\theta_{x_0}(t)\|_{L^\infty(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\omega^{-1}\lambda\mu_1\mu_2^2,$$

hence

$$\|\operatorname{curl} u^t(t)\|_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon)\omega^{-1}\lambda\mu_1^{1-\frac{2}{p}}\mu_2^2. \quad \square$$

Lemma 7.3 (Curl of v). *It holds*

$$\|\operatorname{curl} v(t)\|_{H^p(\mathbb{R}^2)} = \Big\|\int_0^t \operatorname{curl} r_0(s)\,\mathrm{d}s\Big\|_{H^p(\mathbb{R}^2)}.$$

This is true since $\operatorname{curl}(\mathbb{P}f)=\operatorname{curl} f$ for all smooth $f:\mathbb{R}^2\to\mathbb{R}^2$, which gives

$$\operatorname{curl} v(t) = \int_0^t \operatorname{curl} r_0(s)\,\mathrm{d}s. \quad \square$$

8. The New Error

This section is devoted to the definition of the new error (r_1,R_1), which will be estimated in the next section.

8.1. The New Reynolds-Defect-Equation

Plugging u_1 into the new Reynolds-defect-equation and writing $u_1 = u_0 + w + u^t + v$, we need to define (r_1, R_1, p_1) such that

$$\begin{aligned} &- r_1 - \operatorname{div} \mathring{R}_1 \\ &\quad = \operatorname{div}(u_0 \otimes (u_1 - u_0) + (u_1 - u_0) \otimes u_0) \\ &\qquad + \operatorname{div}((u_1 - u_0 - u^p) \otimes u^p) + \operatorname{div}(u^p \otimes (u_1 - u_0 - u^p)) \\ &\qquad + \operatorname{div}((u_1 - u_0 - u^p) \otimes (u_1 - u_0 - u^p)) \\ &\qquad + \partial_t u^t + \operatorname{div}(u^p \otimes u^p - \mathring{R}_0) \\ &\qquad + \partial_t (u^p + u^c) \\ &\qquad + \partial_t v - r_0 \\ &\qquad + \nabla (p_1 - p_0). \end{aligned} \tag{30}$$

We will analyse each line in (30) in separate subsections.

8.2. Analysis of the First Three Lines of (30)

Let us define

$$\begin{aligned} R^{\mathrm{lin},1} &= u_0 \otimes (u_1 - u_0) + (u_1 - u_0) \otimes u_0, \\ R^{\mathrm{lin},2} &= (u_1 - u_0 - u^p) \otimes u^p + u^p \otimes (u_1 - u_0 - u^p), \\ R^{\mathrm{lin},3} &= (u_1 - u_0 - u^p) \otimes (u_1 - u_0 - u^p), \end{aligned}$$

i.e.

$$\begin{aligned} &\operatorname{div}(u_0 \otimes (u_1 - u_0) + (u_1 - u_0) \otimes u_0) \\ &\qquad + \operatorname{div}((u_1 - u_0 - u^p) \otimes u^p) + \operatorname{div}(u^p \otimes (u_1 - u_0 - u^p)) \\ &\qquad + \operatorname{div}((u_1 - u_0 - u^p) \otimes (u_1 - u_0 - u^p)) \\ &\quad = \operatorname{div}(R^{\mathrm{lin},1} + R^{\mathrm{lin},2} + R^{\mathrm{lin},3}). \end{aligned}$$

8.3. Analysis of the Fourth Line of (30)

8.3.1. Rewriting the Fourth Line of (30).

Using that

$$u^t(x,t) = -\sum_{k=1}^{4} \mathbb{P}\left(a_k^2(x,t) Y_k(x,t)\right) = -\sum_{k=1}^{4} a_k^2(x,t) Y_k(x,t) - \nabla p^t$$

for some p^t and (18), let us start by calculating

$$\begin{aligned} \partial_t u^t + \operatorname{div}(u^p \otimes u^p - \mathring{R}_0) = &- \sum_{k=1}^{4} \partial_t a_k^2 Y_k - \sum_{k=1}^{4} a_k^2 \partial_t Y_k \\ &+ \operatorname{div}\left(\sum_{k=1}^{4} a_k^2 W_k^p \otimes W_k^p\right) \\ &+ \operatorname{div}\left(-\sum_{k} a_k^2 \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right) \end{aligned}$$

$$+ \operatorname{div}(\chi_\kappa^2 \mathring{R}_0 - \mathring{R}_0) - \nabla(\partial_t p^t) + \nabla(\chi_\kappa^2 \rho)$$

We consider the second and third summand on the right hand side of the previous calculation. By Proposition 4.2, we have

$$-\sum_{k=1}^{4} a_k^2 \partial_t Y_k + \operatorname{div}\left(\sum_{k=1}^{4} a_k^2 W_k^p \otimes W_k^p\right) = \sum_{k=1}^{4} a_k^2 \underbrace{\left(\operatorname{div}\left(W_k^p \otimes W_k^p\right) - \partial_t Y_k\right)}_{=0} + \sum_{k=1}^{4} (W_k^p \otimes W_k^p) \cdot \nabla a_k^2$$

Also, we have

$$\operatorname{div}\left(-\sum_k a_k^2 \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right) = -\sum_{k=1}^{4}\left(\frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right) \cdot \nabla a_k^2.$$

Putting together the previous two calculations, the fourth line in (30) equals

$$\begin{aligned}\partial_t u^t + \operatorname{div}(u^p \otimes u^p - \mathring{R}_0) &= -\sum_{k=1}^{4} \partial_t a_k^2 Y_k + \sum_{k=1}^{4}\left(W_k^p \otimes W_k^p - \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right) \cdot \nabla a_k^2 \\ &\quad + \operatorname{div}(\chi_\kappa^2 \mathring{R}_0 - \mathring{R}_0) - \nabla(\partial_t p^t) + \nabla(\chi_\kappa^2 \rho) \\ &= r^Y + \operatorname{div} R^Y + r^{\text{quad}} + \operatorname{div} R^{\text{quad}} + \operatorname{div} R^\kappa - \nabla \pi_1,\end{aligned}$$

where we can directly define

$$\begin{aligned} R^\kappa &= \chi_\kappa^2 \mathring{R}_0 - \mathring{R}_0, \\ \pi_1 &= \partial_t p^t - \chi_\kappa^2 \rho. \end{aligned}$$

8.3.2. Definition of R^{quad} and r^{quad}. We define R^{quad}, r^{quad} as

$$(r^{\text{quad}}, R^{\text{quad}}) = \sum_{k=1}^{4} \tilde{S}_N\left(\nabla a_k^2, W_k^p \otimes W_k^p - \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right),$$

with an $N \in \mathbb{N}$ to be chosen in Sect. 10. Hence, by construction

$$r^{\text{quad}} + \operatorname{div} R^{\text{quad}} = \sum_{k=1}^{4}\left(W_k^p \otimes W_k^p - \frac{\xi_k}{|\xi_k|} \otimes \frac{\xi_k}{|\xi_k|}\right) \cdot \nabla a_k^2.$$

8.3.3. Definition of R^Y and r^Y. We add and subtract

$$-\sum_{k=1}^{4} \partial_t a_k^2(x,t) Y_k(x,t) = -\sum_{k=1}^{4} \partial_t a_k^2(x,t)\left(Y_k(x,t) - \frac{1}{\omega}\xi_k\right) - \sum_{k=1}^{4} \frac{1}{\omega} \partial_t a_k^2(x,t) \xi_k.$$

Noting that $\int_{\mathbb{T}^2} Y_k \, \mathrm{d}x = \frac{1}{\omega}\xi_k$, see Proposition 4.2, we can define

$$(r^{Y,1}, R^Y) = -\sum_{k=1}^{4} S_N(\partial_t a_k^2, Y_k - \frac{1}{\omega}\xi_k)$$

so that by definition

$$r^{Y,1} + \operatorname{div} R^Y = -\sum_{k=1}^{4} \partial_t a_k^2 \left(Y_k - \frac{1}{\omega}\xi_k\right).$$

We further define

$$r^{Y,2} = -\sum_{k=1}^{4} \frac{1}{\omega} \partial_t a_k^2(x,t)\xi_k$$

and set

$$r^Y = r^{Y,1} + r^{Y,2},$$

hence

$$r^Y + \operatorname{div} R^Y = -\sum_{k=1}^{4} \partial_t a_k^2 Y_k.$$

8.4. Analysis of the Fifth Line of (30)

We will write the third line in the form

$$\partial_t(u^p + u^c) = r^{\text{time}} + \operatorname{div} R^{\text{time}}.$$

We will use the operators from Definition 2.11. Calculating, we see that

$$\begin{aligned}
\partial_t u^p(x,t) &= \sum_{k=1}^{4} \partial_t a_k(x,t) W_k^p(x,t) + \sum_{k=1}^{4} a_k(x,t)\partial_t W_k^p(x,t) \\
&= \sum_{k=1}^{4} \partial_t a_k(x,t) W_k^p(x,t) + \omega\lambda\mu_1 \sum_{k=1}^{4} a_k(x,t)(\varphi')^k_{\mu_1}(\lambda(\xi_k \cdot x - \omega t))\varphi_{\mu_2}(\lambda\xi_k^\perp \cdot x)\frac{\xi_k}{|\xi_k|} \\
&= \left(r^{\text{time},1} + \operatorname{div} R^{\text{time},1}\right) + \operatorname{div}\left(\frac{\omega\lambda\mu_1}{|\xi_k|}\sum_{k=1}^{4} a_k A((\varphi')^k_{\mu_1}(\lambda(\cdot - \omega t)), \varphi_{\mu_2}(\lambda\cdot), \xi_k)\right) \\
&\quad - \frac{\omega\lambda\mu_1}{|\xi_k|}\sum_{k=1}^{4} A((\varphi')^k_{\mu_1}(\lambda(\cdot - \omega t)), \varphi_{\mu_2}(\lambda\cdot), \xi_k) \cdot \nabla a_k \\
&= (r^{\text{time},1} + \operatorname{div} R^{\text{time},1}) + \operatorname{div}\tilde{R}^{\text{time},2} + (r^{\text{time},2} + \operatorname{div} R^{\text{time},2}).
\end{aligned}$$

with

$$\begin{aligned}
(r^{\text{time},1}, R^{\text{time},1}) &= \sum_{k=1}^{4} S_N(\partial_t a_k, W_k^p), \\
\tilde{R}^{\text{time},2} &= \frac{\omega\lambda\mu_1}{|\xi_k|}\sum_{k=1}^{4} a_k A((\varphi')^k_{\mu_1}(\lambda(\cdot - \omega t)), \varphi_{\mu_2}(\lambda\cdot), \xi_k), \\
(r^{\text{time},2}, R^{\text{time},2}) &= -\frac{\omega\lambda\mu_1}{|\xi_k|}\sum_{k=1}^{4} \tilde{S}_N(\nabla a_k, A((\varphi')^k_{\mu_1}(\lambda(\cdot - \omega t)), \varphi_{\mu_2}(\lambda\cdot), \xi_k)).
\end{aligned}$$

Analogously, let us write, using (20)

$$\begin{aligned}
\partial_t u^c(x,t) &= \sum_{k=1}^{4} \partial_t a_k(x,t) W_k^c(x,t) + a_k(x,t)\partial_t W_k^c(x,t) \\
&\quad + \sum_{k=1}^{4} \partial_t b_k^1(x,t) W_k^{cc,\parallel}(x,t) + b_k^1(x,t)\partial_t W_k^{cc,\parallel}(x,t)) \\
&\quad + \sum_{k=1}^{4} \partial_t b_k^2(x,t) W_k^{cc,\perp}(x,t) + b_k^2(x,t)\partial_t W_k^{cc,\perp}(x,t).
\end{aligned} \tag{31}$$

The first line of (31) can be written as

$$
\begin{aligned}
&\sum_{k=1}^{4} \partial_t a_k(x,t) W_k^c(x,t) + \sum_{k=1}^{4} a_k(x,t)\partial_t W_k^c(x,t)\\
&\quad = \sum_{k=1}^{4} \partial_t a_k(x,t) W_k^c(x,t) + \frac{\omega\lambda\mu_1^2}{\mu_2} \sum_{k=1}^{4} a_k(x,t) (\varphi'')_{\mu_1}^k(\lambda(\xi_k\cdot x - \omega t))(\Phi'')_{\mu_2}(\lambda \xi_k^\perp \cdot x)\frac{\xi_k}{|\xi_k|}\\
&\quad = \left(r^{\text{time},3} + \operatorname{div} R^{\text{time},3}\right) + \operatorname{div}\left(\frac{\omega\lambda\mu_1^2}{\mu_2|\xi_k|}\sum_{k=1}^{4} a_k B((\varphi'')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\right)\\
&\qquad - \frac{\omega\lambda\mu_1^2}{\mu_2|\xi_k|}\sum_{k=1}^{4} B((\varphi'')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\cdot\nabla a_k\\
&\quad = \left(r^{\text{time},3} + \operatorname{div} R^{\text{time},3}\right) + \operatorname{div}\tilde{R}^{\text{time},4} + (r^{\text{time},4} + \operatorname{div} R^{\text{time},4})
\end{aligned}
$$

with

$$
\begin{aligned}
(r^{\text{time},3}, R^{\text{time},3}) &= \sum_{k=1}^{4} S_N(\partial_t a_k, W_k^c),\\
\tilde{R}^{\text{time},4} &= \frac{\omega\lambda\mu_1^2}{\mu_2|\xi_k|}\sum_{k=1}^{4} a_k B((\varphi'')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k),\\
(r^{\text{time},4}, R^{\text{time},4}) &= -\frac{\omega\lambda\mu_1^2}{\mu_2|\xi_k|}\sum_{k=1}^{4} \tilde{S}_N\left(\nabla a_k, B((\varphi'')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\right).
\end{aligned}
$$

For the second line of (31), we write

$$
\begin{aligned}
&\sum_{k=1}^{4} \partial_t b_k^1(x,t) W_k^{cc,\parallel}(x,t) + \sum_{k=1}^{4} b_k^1(x,t)\partial_t W_k^{cc,\parallel}(x,t)\\
&\quad = \sum_{k=1}^{4} \partial_t b_k^1(x,t) W_k^{cc,\parallel}(x,t) + \frac{\omega\mu_1}{\mu_2} \sum_{k=1}^{4} b_k^1(x,t) (\varphi')_{\mu_1}^k(\lambda(\xi_k\cdot x - \omega t))(\Phi'')_{\mu_2}(\lambda \xi_k^\perp \cdot x)\frac{\xi_k}{|\xi_k|}\\
&\quad = (r^{\text{time},5} + \operatorname{div} R^{\text{time},5}) + \operatorname{div}\left(\frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} b_k^1 A((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\right)\\
&\qquad - \frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} A((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\cdot\nabla b_k^1\\
&\quad = (r^{\text{time},5} + \operatorname{div} R^{\text{time},5}) + \operatorname{div}\tilde{R}^{\text{time},6} + (r^{\text{time},6} + \operatorname{div} R^{\text{time},6})
\end{aligned}
$$

with

$$
\begin{aligned}
(r^{\text{time},5}, \operatorname{div} R^{\text{time},5}) &= \sum_{k=1}^{4} S_N(\partial_t b_k^1, W_k^{cc,\parallel}),\\
\tilde{R}^{\text{time},6} &= \frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} b_k^1 A((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k),\\
(r^{\text{time},6}, \operatorname{div} R^{\text{time},6}) &= -\frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} \tilde{S}_N\left(\nabla b_k^1, A((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\right).
\end{aligned}
$$

Similarly, we have for the third line of (31)

$$
\begin{aligned}
&\sum_{k=1}^{4} \partial_t b_k^2(x,t) W_k^{cc,\perp}(x,t) + \sum_{k=1}^{4} b_k^2(x,t) \partial_t W_k^{cc,\perp}(x,t)\\
&\quad = \sum_{k=1}^{4} \partial_t b_k^2(x,t) W_k^{cc,\perp}(x,t) + \frac{\omega\mu_1}{\mu_2} \sum_{k=1}^{4} b_k^2(x,t) (\varphi')_{\mu_1}^k(\lambda(\xi_k\cdot x - \omega t))(\Phi'')_{\mu_2}(\lambda \xi_k^\perp \cdot x) \frac{\xi_k^\perp}{|\xi_k|}\\
&\quad = (r^{\text{time},7} + \operatorname{div} R^{\text{time},7}) + \operatorname{div}\left(\frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} b_k^2 B((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\right)\\
&\qquad - \frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} B((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\cdot \nabla b_k^2\\
&\quad = (r^{\text{time},7} + \operatorname{div} R^{\text{time},7}) + \operatorname{div} \tilde{R}^{\text{time},8} + (r^{\text{time},8} + \operatorname{div} R^{\text{time},8})
\end{aligned}
$$

with

$$
\begin{aligned}
(r^{\text{time},7}, \operatorname{div} R^{\text{time},7}) &= \sum_{k=1}^{4} S_N(\partial_t b_k^2, W_k^{cc,\perp}),\\
\tilde{R}^{\text{time},8} &= \frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} b_k^2 B((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k),\\
(r^{\text{time},8}, \operatorname{div} R^{\text{time},8}) &= -\frac{\omega\mu_1}{\mu_2|\xi_k|}\sum_{k=1}^{4} \tilde{S}_N\left(\nabla b_k^2, B((\varphi')_{\mu_1}^k(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda\cdot), \xi_k)\right).
\end{aligned}
$$

Finally, we set

$$
\begin{aligned}
R^{\text{time}} &= \sum_{i=1}^{8} R^{\text{time},i} + \sum_{i=1}^{4} \tilde{R}^{\text{time},2i}\\
r^{\text{time}} &= \sum_{i=1}^{8} r^{\text{time},i}.
\end{aligned}
$$

8.5. Analysis of the Sixth and Seventh Line of (30)

Since $\mathbb{P} r_0 = r_0 - \nabla p^r$ for some p^r, we see that

$$
\partial_t v - r_0 = -\nabla p^r,
$$

i.e. it only remains a part that can be put into the new pressure and we define

$$
\pi_2 = p^r.
$$

8.6. Definition of the New Error

Altogether, we define

$$
\begin{aligned}
R^1 &= -\left(R^{\text{lin},1} + R^{\text{lin},2} + R^{\text{lin},3} + R^{\kappa} + R^{\text{quad}} + R^{Y} + R^{\text{time}}\right),\\
r_1 &= -\left(r^{\text{quad}} + r^{Y} + r^{\text{time}}\right),\\
p_1 &= p_0 + \pi_1 + \pi_2 + \frac{1}{2}\operatorname{tr} R^1.
\end{aligned}
$$

9. Estimates of the New Error

We will now estimate the different parts of R_1 and r_1 that were defined in the previous section.

9.1. Estimates of the Symmetric Tensor R_1

Lemma 9.1 (Estimate of $R^{\mathrm{lin},1}$). *It holds*

$$\|R^{\mathrm{lin},1}(t)\|_{L^1(\mathbb{R}^2)} \leq C(R_0, u_0, e, \delta, \kappa, \varepsilon) \left(\mu_1^{-\frac{1}{6}} \mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\omega} \right) + 2\|r_0\|_{C_t L^2_x} \|u_0(t)\|_{L^2(\mathbb{R}^2)}.$$

Using Hölder's inequality and Lemma 6.2, Lemma 6.3 and Lemma 6.5, we have

$$\begin{aligned} \|R^{\mathrm{lin},1}(t)\|_{L^1(\mathbb{R}^2)} &\leq 2\|u_0(t)\|_{L^2(\mathbb{R}^2)} \left(\|u^c(t)\|_{L^2(\mathbb{R}^2)} + \|u^t(t)\|_{L^2(\mathbb{R}^2)} + \|v(t)\|_{L^2(\mathbb{R}^2)} \right) \\ &\quad + 2\|u_0(t)\|_{L^3(\mathbb{R}^2)} \|u^p(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \\ &\leq C(R_0, u_0, e, \delta, \kappa, \varepsilon) \left(\frac{\mu_1}{\mu_2} + \mu_1^{-\frac{1}{6}} \mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\omega} \right) + 2\|r_0\|_{C_t L^2_x} \|u_0(t)\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

□

Lemma 9.2 (Estimate of $R^{\mathrm{lin},2}$). *It holds*

$$\|R^{\mathrm{lin},2}(t)\|_{L^1(\mathbb{R}^2)} \leq C(R_0, r_0, e, \delta, \kappa, \varepsilon) \left(\frac{\mu_1}{\mu_2} + \mu_1^{-\frac{1}{6}} \mu_2^{-\frac{1}{6}} + \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\omega} \right).$$

Proof. We have

$$\begin{aligned} \|R^{\mathrm{lin},2}(t)\|_{L^1(\mathbb{R}^2)} &\leq 2\|u^p(t)\|_{L^2(\mathbb{R}^2)} \left(\|u^c(t)\|_{L^2(\mathbb{R}^2)} + \|u^t(t)\|_{L^2(\mathbb{R}^2)} \right) \\ &\quad + 2\|u^p(t)\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \|v(t)\|_{L^3(\mathbb{R}^2)} \end{aligned}$$

and this was already estimated in (28). □

Lemma 9.3 (Estimate of $R^{\mathrm{lin},3}$). *It holds*

$$\|R^{\mathrm{lin},3}(t)\|_{L^1(\mathbb{R}^2)} \leq C(R_0, u_0, e, \delta, \kappa, \varepsilon) \left(\left(\frac{\mu_1}{\mu_2} \right)^2 + \left(\frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\omega} \right)^2 \right) + 4\|r_0\|^2_{C_t L^2_x}.$$

Proof. Since

$$R^{\mathrm{lin},3} = (u_1 - u_0 - u^p) \otimes (u_1 - u_0 - u^p) = (u^c + u^t + v) \otimes (u^c + u^t + v),$$

we have

$$\|R^{\mathrm{lin},3}(t)\|_{L^1(\mathbb{R}^2)} \leq 4 \left(\|u^c(t)\|^2_{L^2(\mathbb{R}^2)} + \|u^t(t)\|^2_{L^2(\mathbb{R}^2)} + \|v(t)\|^2_{L^2(\mathbb{R}^2)} \right),$$

hence the assertion follows from Lemma 6.3 and Lemma 6.5. □

Lemma 9.4 (Estimate of R^{κ}). *It holds*

$$\|R^{\kappa}(t)\|_{L^1(\mathbb{R}^2)} \leq \frac{\eta}{2}.$$

Proof. This holds because of our choice of κ in (17). □

Lemma 9.5 (Estimate of R^{quad}). *It holds*

$$\|R^{\mathrm{quad}}(t)\|_{L^1(\mathbb{R}^2)} \leq \frac{C(R_0, u_0, e, \delta, \kappa, \varepsilon, N)}{\lambda}.$$

Proof. This follows directly from Remark 2.9, the scaling of W^p_k (see Proposition 4.2) and the estimates on a_k in (21). □

Lemma 9.6 (Estimate of R^Y). *It holds*

$$\|R^Y(t)\|_{L^1(\mathbb{R}^2)} \leq \frac{C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)}{\omega\lambda}.$$

Proof. As for R^{quad}, this is a direct application of Remark 2.9. □

Lemma 9.7 (Estimate of R^{time}). *It holds*

$$\|R^{\text{time}}(t)\|_{L^1(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\left(\lambda^{-1}\mu_1^{-\frac{1}{2}}\mu_2^{-\frac{1}{2}} + \omega\mu_1^{\frac{1}{2}}\mu_2^{-\frac{3}{2}}\right).$$

By Remark 2.9 and the estimates for the operators A and B in (11), Proposition 4.2 and (21) we have

$$\begin{aligned}
\|R^{\text{time},1}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\|\partial_t a_k\|_{C^{N-1}(\mathbb{R}^2)}\frac{1}{\lambda}\|W_k^p\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\lambda^{-1}\mu_1^{-\frac{1}{2}}\mu_2^{-\frac{1}{2}},\\
\|\tilde{R}^{\text{time},2}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\omega\lambda\mu_1\|a_k\|_{C(\mathbb{R}^2)}\|A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),\varphi_{\mu_2}(\lambda\cdot),\xi_k)\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\mu_1^{\frac{1}{2}}\mu_2^{-\frac{3}{2}},\\
\|R^{\text{time},2}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\omega\lambda\mu_1\|a_k\|_{C^N(\mathbb{R}^2)}\|\operatorname{div}^{-1}A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),\varphi_{\mu_2}(\lambda\cdot),\xi_k)\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\lambda^{-1}\mu_1^{\frac{1}{2}}\mu_2^{-\frac{3}{2}},\\
\|R^{\text{time},3}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\|\partial_t a_k\|_{C^{N-1}(\mathbb{R}^2)}\frac{1}{\lambda}\|W_k^c\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\lambda^{-1}\mu_1^{\frac{1}{2}}\mu_2^{-\frac{3}{2}},\\
\|\tilde{R}^{\text{time},4}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\frac{\omega\lambda\mu_1^2}{\mu_2}\|a_k\|_{C(\mathbb{R}^2)}\|B((\varphi'')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\mu_1^{\frac{3}{2}}\mu_2^{-\frac{5}{2}},\\
\|R^{\text{time},4}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\frac{\omega\lambda\mu_1^2}{\mu_2}\|a_k\|_{C^N(\mathbb{R}^2)}\left\|\operatorname{div}^{-1}B((\varphi'')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\lambda^{-1}\mu_1^{\frac{3}{2}}\mu_2^{-\frac{5}{2}},\\
\|R^{\text{time},5}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\|\partial_t b_k^1\|_{C^{N-1}(\mathbb{R}^2)}\frac{1}{\lambda}\|W_k^{cc,\parallel}\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\lambda^{-2}\mu_1^{-\frac{1}{2}}\mu_2^{-\frac{3}{2}},\\
\|\tilde{R}^{\text{time},6}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\frac{\omega\mu_1}{\mu_2}\|b_k^1\|_{C(\mathbb{R}^2)}\left\|A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\lambda^{-1}\mu_1^{\frac{1}{2}}\mu_2^{-\frac{5}{2}},\\
\|R^{\text{time},6}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\frac{\omega\mu_1}{\mu_2}\|b_k^1\|_{C^N(\mathbb{R}^2)}\left\|\operatorname{div}^{-1}A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\lambda^{-2}\mu_1^{\frac{1}{2}}\mu_2^{-\frac{5}{2}},\\
\|R^{\text{time},7}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\|\partial_t b_k^2\|_{C^{N-1}(\mathbb{R}^2)}\frac{1}{\lambda}\|W_k^{cc,\perp}\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\lambda^{-2}\mu_1^{-\frac{1}{2}}\mu_2^{-\frac{3}{2}},\\
\|\tilde{R}^{\text{time},8}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\frac{\omega\mu_1}{\mu_2}\|b_k^2\|_{C(\mathbb{R}^2)}\left\|B((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^1(\mathbb{T}^2)}\\
&\leq C(R_0,e,\kappa,\delta,\varepsilon)\omega\lambda^{-1}\mu_1^{\frac{1}{2}}\mu_2^{-\frac{5}{2}},\\
\|R^{\text{time},8}\|_{L^1(\mathbb{R}^2)} &\leq C(\kappa)\frac{\omega\mu_1}{\mu_2}\|b_k^2\|_{C^N(\mathbb{R}^2)}\left\|\operatorname{div}^{-1}B((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^1(\mathbb{T}^2)}
\end{aligned}$$

$$\leq C(R_0,u_0,e,\kappa,\delta,\varepsilon,N)\omega\lambda^{-2}\mu_1^{\frac{1}{2}}\mu_2^{-\frac{5}{2}}.$$
□

Putting those estimate together yields the claim.

9.2. Estimates of the Vector r_1

In this subsection, we estimate the new error r_1. Since r_1 also enters into the next iteration (see the definition of v in Sect. 5), we need an estimate on $\int_0^t \operatorname{curl} r_1(x,s)\,\mathrm{d}s$ in $H^p(\mathbb{R}^2)$ as well. The operators S_N and $\tilde S_N$ guarantee that all parts of r_1 have compact supports, therefore one can use Remark 2.6, and we control the quantity $\left\|\int_0^t \operatorname{curl} r_1(\cdot,s)\,\mathrm{d}s\right\|_{H^p(\mathbb{R}^2)}$ by $\|r_1\|_{L^\infty(\mathbb{R}^2)}$.

Lemma 9.8 (Estimate of r^{quad}). *The function r^{quad} has compact support and satisfies*

$$\|r^{\text{quad}}(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2}{\lambda^N}$$
$$\|\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\lambda^{N-1}}.$$

Proof. The compact support follows from the properties of $\tilde S_N$. By Remark 2.9, Proposition 4.2 and (21), we can estimate

$$\begin{aligned}\|r^{\text{quad}}(s)\|_{L^\infty(\mathbb{R}^2)} &\leq C(\kappa)\frac{1}{\lambda^N}\|a_k^2\|_{C^{N+1}(\mathbb{R}^2)}\left\|W_k^p\otimes W_k^p-\frac{\xi_k}{|\xi_k|}\otimes\frac{\xi_k}{|\xi_k|}\right\|_{L^\infty(\mathbb{T}^2)}\\ &\leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2}{\lambda^N}.\end{aligned}$$

For the curl, we use again Remark 2.9 and obtain

$$\begin{aligned}\|\operatorname{curl} r^{\text{quad}}(s)\|_{L^\infty(\mathbb{R}^2)} &\leq C(\kappa)\|a^2\|_{C^{N+2}(\mathbb{R}^2)}\left\|\nabla \operatorname{div}^{-N}(W_k^p\otimes W_k^p)\right\|_{L^\infty(\mathbb{T}^2)}\\ &\leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\lambda^{N-1}} \text{ for all } s\in[0,1]\end{aligned}$$

and therefore also

$$\|\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s\|_{L^\infty(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\lambda^{N-1}}.$$

Using that $\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s$ is supported in $B_{\kappa+1}$, we can apply Remark 2.6 and obtain

$$\|\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\lambda^{N-1}}.$$

□

Lemma 9.9 (Estimate of r^Y). *The function r^Y has compact support and satisfies*

$$\|r^Y(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\left(\frac{\mu_1\mu_2}{\omega\lambda^N}+\frac{1}{\omega}\right),$$
$$\|\int_0^t \operatorname{curl} r^Y(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} \leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\left(\frac{\mu_1\mu_2^2}{\omega\lambda^{N-1}}+\frac{1}{\omega}\right).$$

The compact support follows from the properties of S_N for $r^{Y,1}$ and the compact support of a_k for $r^{Y,2}$, respectively. By Remark 2.9, Proposition 4.2 and (21), we can estimate

$$\begin{aligned}\|r^{Y,1}(s)\|_{L^\infty(\mathbb{R}^2)} &\leq C(\kappa)\frac{1}{\lambda^N}\|\partial_t a_k^2\|_{C^{N+1}(\mathbb{R}^2)}\left\|Y_k-\frac{1}{\omega}\xi_k\right\|_{L^\infty(\mathbb{T}^2)}\\ &\leq C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2}{\omega\lambda^N}.\end{aligned}$$

For the curl, we use again Remark 2.9 and obtain

$$\begin{aligned}\|\operatorname{curl} r^{Y,1}(s)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\|\partial_t a^2\|_{C^{N+2}(\mathbb{R}^2)}\|\nabla \operatorname{div}^{-N} Y_k\|_{L^\infty(\mathbb{T}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\omega\lambda^{N-1}} \text{ for all } s\in[0,1]\end{aligned}$$

and therefore also

$$\|\int_0^t \operatorname{curl} r^{Y,1}(s)\,\mathrm{d}s\|_{L^\infty(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\omega\lambda^{N-1}}.$$

Using that $\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s$ is supported in $B_{\kappa+1}$, we can apply Remark 2.6 and obtain

$$\|\int_0^t \operatorname{curl} r^{Y,1}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1\mu_2^2}{\omega\lambda^{N-1}}.$$

For $r^{Y,2}$ we immediately get

$$\|r^{Y,2}(s)\|_{L^\infty(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{1}{\omega}$$

and also

$$\|\int_0^t \operatorname{curl} r^{Y,2}(s)\,\mathrm{d}s\|_{L^\infty(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{1}{\omega}$$

so that since $\operatorname{supp}\left(\int_0^t \operatorname{curl} r^{Y,2}(s)\,\mathrm{d}s\right)\subset B_{\kappa+1}$, we have by Remark 2.6

$$\|\int_0^t \operatorname{curl} r^{Y,2}(s)\,\mathrm{d}s\|_{H^p\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{1}{\omega}. \quad \square$$

Lemma 9.10 (Estimate of r^{time}). *The function r^{time} has compact support and satisfies*

$$\|r^{\text{time}}(t)\|_{L^\infty(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\left(\frac{\mu_1^{\frac12}\mu_2^{\frac12}}{\lambda^N}+\frac{\omega\mu_1^{\frac32}\mu_2^{-\frac12}}{\lambda^N}\right),$$

$$\|\int_0^t \operatorname{curl} r^{\text{time}}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\left(\frac{\mu_1^{\frac12}\mu_2^{\frac32}}{\lambda^{N-1}}+\frac{\omega\mu_1^{\frac32}\mu_2^{\frac12}}{\lambda^{N-1}}\right).$$

We estimate the different parts of r^{time} separately. Again, by Remark 2.9, Proposition 4.2 and (21) we have

$$\|r^{\text{time},1}(t)\|_{L^\infty(\mathbb{R}^2)} \le C(\kappa)\frac{1}{\lambda^N}\|\partial_t a_k\|_{C^N(\mathbb{R}^2)}\|W_k^p\|_{L^\infty(\mathbb{T}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac12}\mu_2^{\frac12}}{\lambda^N}$$

and

$$\begin{aligned}\|\operatorname{curl} r^{\text{time},1}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\|\partial_t a\|_{C^{N+1}(\mathbb{R}^2)}\left\|\nabla \operatorname{div}^{-N} W_k^p\right\|_{L^\infty(\mathbb{T}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac12}\mu_2^{\frac32}}{\lambda^{N-1}} \text{ for all } t\in[0,1]\end{aligned}$$

and therefore also

$$\left\|\int_0^t \operatorname{curl} r^{\text{time},1}(s)\,\mathrm{d}s\right\|_{H^p(\mathbb{R}^2)} \le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac12}\mu_2^{\frac32}}{\lambda^{N-1}}$$

by Remark 2.6. For $r^{\text{time},2}$, we have, using (11)

$$\begin{aligned}\|r^{\text{time},2}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\omega\lambda\mu_1\|a_k\|_{C^{N+1}}\left\|\operatorname{div}^{-N} A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),\varphi_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\omega\lambda\mu_1\frac{\mu_1^{\frac12}\mu_2^{-\frac12}}{\lambda^{N+1}} = C(R_0,e,\delta,\kappa,\varepsilon)\frac{\omega\mu_1^{\frac32}\mu_2^{-\frac12}}{\lambda^N},\end{aligned}$$

$$\begin{aligned}\|\operatorname{curl} r^{\mathrm{time},2}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\omega\lambda\mu_1\|a_k\|_{C^{N+2}}\left\|\nabla\operatorname{div}^{-N}A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),\varphi_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\omega\lambda\mu_1\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\lambda^N}=C(R_0,e,\delta,\kappa,\varepsilon)\frac{\omega\mu_1^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}{\lambda^{N-1}}\end{aligned}$$

and therefore also

$$\|\int_0^t\operatorname{curl} r^{\mathrm{time},2}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)}\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\omega\mu_1^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}{\lambda^{N-1}}.$$

In the same manner, we estimate

$$\begin{aligned}\|r^{\mathrm{time},3}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\frac{1}{\lambda^N}\|\partial_t a_k\|_{C^N(\mathbb{R}^2)}\|W^c_k\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac{3}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^N},\\ \|r^{\mathrm{time},4}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\frac{\omega\lambda\mu_1^2}{\mu_2}\|a_k\|_{C^{N+1}(\mathbb{R}^2)}\left\|\operatorname{div}^{-N}B((\varphi'')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^\infty(\mathbb{T}^2)},\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\omega\lambda\mu_1^2}{\mu_2}\frac{\mu_1^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^{N+1}}=C(R_0,e,\delta,\kappa,\varepsilon)\frac{\omega\mu_1^{\frac{5}{2}}\mu_2^{-\frac{3}{2}}}{\lambda^N},\\ \|r^{\mathrm{time},5}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\frac{1}{\lambda^N}\|\partial_t b^1_k\|_{C^N(\mathbb{R}^2)}\|W^{cc,\parallel}_k\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^{N+1}},\\ \|r^{\mathrm{time},6}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\frac{\omega\mu_1}{\mu_2}\|b^1_k\|_{C^{N+1}(\mathbb{R}^2)}\left\|\operatorname{div}^{-N}A((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\omega\mu_1}{\mu_2}\frac{\mu_1^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^{N+1}}=C(R_0,e,\delta,\kappa,\varepsilon)\frac{\omega\mu_1^{\frac{3}{2}}\mu_2^{-\frac{3}{2}}}{\lambda^{N+1}},\\ \|r^{\mathrm{time},7}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\frac{1}{\lambda^N}\|\partial_t b^2_k\|_{C^N(\mathbb{R}^2)}\|W^{cc,\perp}_k\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^{N+1}},\\ \|r^{\mathrm{time},8}(t)\|_{L^\infty(\mathbb{R}^2)} &\le C(\kappa)\frac{\omega\mu_1}{\mu_2}\|b^2_k\|_{C^{N+1}(\mathbb{R}^2)}\left\|\operatorname{div}^{-N}B((\varphi')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^\infty(\mathbb{R}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\omega\mu_1}{\mu_2}\frac{\mu_1^{\frac{1}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^{N+1}}=C(R_0,e,\delta,\kappa,\varepsilon)\frac{\omega\mu_1^{\frac{3}{2}}\mu_2^{-\frac{3}{2}}}{\lambda^{N+1}},\end{aligned}$$

and

$$\begin{aligned}\|\int_0^t\operatorname{curl} r^{\mathrm{time},3}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} &\le C(\kappa)\|\partial_t a_k\|_{C^{N+1}(\mathbb{R}^2)}\left\|\nabla\operatorname{div}^{-N}W^c_k\right\|_{L^\infty(\mathbb{T}^2)}\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\mu_1^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}{\lambda^{N-1}},\\ \|\int_0^t\operatorname{curl} r^{\mathrm{time},4}(s)\,\mathrm{d}s\|_{H^p(\mathbb{R}^2)} &\le C(\kappa)\frac{\omega\lambda\mu_1^2}{\mu_2}\|a_k\|_{C^{N+2}(\mathbb{R}^2)}\\ &\quad\cdot\left\|\nabla\operatorname{div}^{-N}B((\varphi'')^k_{\mu_1}(\lambda(\cdot-\omega t)),(\Phi'')_{\mu_2}(\lambda\cdot),\xi_k)\right\|_{L^\infty(\mathbb{T}^2)},\\ &\le C(R_0,u_0,e,\delta,\kappa,\varepsilon,N)\frac{\omega\lambda\mu_1^2}{\mu_2}\frac{\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\lambda^N}=C(R_0,e,\delta,\kappa,\varepsilon)\frac{\omega\mu_1^{\frac{5}{2}}\mu_2^{-\frac{1}{2}}}{\lambda^{N-1}},\end{aligned}$$

$$\begin{aligned}
\| \int_0^t \operatorname{curl} r^{\mathrm{time},5}(s)\,\mathrm{d}s \|_{H^p(\mathbb{R}^2)} &\le C(\kappa) \|\partial_t b_k^1\|_{C^{N+1}(\mathbb{R}^2)} \|\nabla \operatorname{div}^{-N} W_k^{cc,\|}\|_{L^\infty(\mathbb{R}^2)} \\
&\le C(R_0, u_0, e, \delta, \kappa, \varepsilon, N) \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\lambda^N}, \\
\| \int_0^t \operatorname{curl} r^{\mathrm{time},6}(s)\,\mathrm{d}s \|_{H^p(\mathbb{R}^2)} &\le C(\kappa) \frac{\omega \mu_1}{\mu_2} \|b_k^1\|_{C^{N+2}(\mathbb{R}^2)} \\
&\quad \cdot \left\| \nabla \operatorname{div}^{-N} A((\varphi')^k_{\mu_1}(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda \cdot), \xi_k) \right\|_{L^\infty(\mathbb{R}^2)} \\
&\le C(R_0, u_0, e, \delta, \kappa, \varepsilon, N) \frac{\omega \mu_1}{\mu_2} \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\lambda^N} = C(R_0, e, \delta, \kappa, \varepsilon) \frac{\omega \mu_1^{\frac{3}{2}} \mu_2^{-\frac{1}{2}}}{\lambda^N}, \\
\| \int_0^t \operatorname{curl} r^{\mathrm{time},7}(s)\,\mathrm{d}s \|_{H^p(\mathbb{R}^2)} &\le C(\kappa) \|\partial_t b_k^2\|_{C^{N+1}(\mathbb{R}^2)} \|\nabla \operatorname{div}^{-N} W_k^{cc,\perp}\|_{L^\infty(\mathbb{R}^2)} \\
&\le C(R_0, u_0, e, \delta, \kappa, \varepsilon, N) \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\lambda^N}, \\
\| \int_0^t \operatorname{curl} r^{\mathrm{time},8}(s)\,\mathrm{d}s \|_{H^p(\mathbb{R}^2)} &\le C(\kappa) \frac{\omega \mu_1}{\mu_2} \|b_k^2\|_{C^{N+2}(\mathbb{R}^2)} \\
&\quad \cdot \left\| \nabla \operatorname{div}^{-N} B((\varphi')^k_{\mu_1}(\lambda(\cdot - \omega t)), (\Phi'')_{\mu_2}(\lambda \cdot), \xi_k) \right\|_{L^\infty(\mathbb{R}^2)} \\
&\le C(R_0, u_0, e, \delta, \kappa, \varepsilon, N) \frac{\omega \mu_1}{\mu_2} \frac{\mu_1^{\frac{1}{2}} \mu_2^{\frac{1}{2}}}{\lambda^N} = C(R_0, u_0, e, \delta, \kappa, \varepsilon, N) \frac{\omega \mu_1^{\frac{3}{2}} \mu_2^{-\frac{1}{2}}}{\lambda^N}. \qquad \square
\end{aligned}$$

10. Proof of the Main Proposition

Proposition 3.2 is proved by choosing all the parameters appropriately, which we do in this section. Let us set

- $\mu_1 = \lambda^\alpha$,
- $\mu_2 = \lambda \mu_1 = \lambda^{1+\alpha}$,
- $\omega = \lambda^\beta$

for some $\alpha, \beta > 0$ to be chosen below. We collect the estimates from Sect. 6 and 7 where the parameters μ_1, μ_2 and ω need to be balanced in Table 2.

We choose α, β and N such that all the exponents in the fourth column of the previous table are negative. This is clear for the first and the third row. Since $2 - \frac{2}{p} < 0$, we can choose $\alpha \gg 1$ so large such that

$$\frac{5}{2} + \alpha(2 - \frac{2}{p}) < 0$$

i.e. we have negative exponents in Line 4. Furthermore, since $3 - \frac{2}{p} < 1$, let us choose α large enough such that

$$3 + \alpha(3 - \frac{2}{p}) < \alpha + \frac{1}{2}.$$

With this choice of α, we only need β to satisfy

$$3 + \alpha(3 - \frac{2}{p}) < \alpha + \frac{1}{2} < \beta < \alpha + \frac{3}{2}.$$

With such a β, Line 1–6 in Table 2 have negative exponents of λ. Having α and β fixed, it only remains to choose N. Since N enters all the remaining exponents with a negative sign, we can simply pick $N \in \mathbb{N}$

Table 2. *All quantities that have to be controlled in terms of oscillation and concentration parameters and the phase speed*

Lemma	Term	Order	=
6.3	u^c in $L^2(\mathbb{R}^2)$	$\mu_1\mu_2^{-1}$	λ^{-1}
6.3	u^t in $L^2(\mathbb{R}^2)$	$\omega^{-1}\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}$	$\lambda^{-\beta+\alpha+\frac{1}{2}}$
6.6	Energy increment	$\mu_1^{-\frac{1}{6}}\mu_2^{-\frac{1}{6}}$	$\lambda^{-\frac{1}{3}\alpha-\frac{1}{6}}$
7.1	$\operatorname{curl} w$ in $H^p(\mathbb{R}^2)$	$\lambda\mu_1^{\frac{1}{2}-\frac{2}{p}}\mu_2^{\frac{3}{2}}$	$\lambda^{\frac{5}{2}+\alpha(2-\frac{2}{p})}$
7.2	$\operatorname{curl} u^t$ in $H^p(\mathbb{R}^2)$	$\omega^{-1}\lambda\mu_1^{1-\frac{2}{p}}\mu_2^2$	$\lambda^{3-\beta+\alpha(3-\frac{2}{p})}$
9.7	R^{time} in $L^1(\mathbb{R}^2)$	$\omega\mu_1^{\frac{1}{2}}\mu_2^{-\frac{3}{2}}$	$\lambda^{\beta-\alpha-\frac{3}{2}}$
9.8	r^{quad} in $L^\infty(\mathbb{R}^2)$	$\lambda^{-N}\mu_1\mu_2$	$\lambda^{2\alpha+1-N}$
9.9	r^Y in $L^\infty(\mathbb{R}^2)$	$\omega^{-1}\lambda^{-N}\mu_1\mu_2$	$\lambda^{-\beta+2\alpha+1-N}$
9.10	r^{time} in $L^\infty(\mathbb{R}^2)$	$\lambda^{-N}\mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{2}}+\omega\lambda^{-N}\mu_1^{\frac{3}{2}}\mu_2^{-\frac{1}{2}}$	$\lambda^{\alpha+\frac{1}{2}-N}+\lambda^{\beta+\alpha-\frac{1}{2}-N}$
9.8	$\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s$ in $H^p(\mathbb{R}^2)$	$\lambda^{1-N}\mu_1\mu_2^2$	$\lambda^{3\alpha+3-N}$
9.9	$\int_0^t \operatorname{curl} r^Y(s)\,\mathrm{d}s$ in $H^p(\mathbb{R}^2)$	$\omega^{-1}\lambda^{1-N}\mu_1\mu_2^2$	$\lambda^{-\beta+3\alpha+3-N}$
9.10	$\int_0^t \operatorname{curl} r^{\text{time}}(s)\,\mathrm{d}s$ in $H^p(\mathbb{R}^2)$	$\lambda^{1-N}\mu_1^{\frac{1}{2}}\mu_2^{\frac{3}{2}}+\omega\lambda^{1-N}\mu_1^{\frac{3}{2}}\mu_2^{\frac{1}{2}}$	$\lambda^{2\alpha+\frac{5}{2}-N}$ $+\lambda^{\beta+2\alpha+\frac{3}{2}-N}$

large enough such that all exponents are negative. Let

$$\gamma_0 = \text{ exponent in the table with the smallest magnitude}$$

which satisfies $\gamma_0 < 0$ by our choice of α, β, N. We can now verify the claims of Proposition 3.2. For (i), we have by (29)

$$\left| e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{R}^2} |u_1|^2 \,\mathrm{d}x \right| < \frac{1}{8}\delta e(t) + C(R_0, r_0, u_0, e, \delta, \kappa, \varepsilon)\lambda^{\gamma_0}$$

and we can choose λ large enough such that (i) is satisfied. For (v), we use Lemma 6.2, Lemma 6.3 and Lemma 6.5

$$\begin{aligned}\|(u_1-u_0)(t)\|_{L^2(\mathbb{R}^2)} &\le \|u^p(t)\|_{L^2(\mathbb{R}^2)} + \|u^c(t)\|_{L^2(\mathbb{R}^2)} + \|u^t(t)\|_{L^2(\mathbb{R}^2)} + \|v(t)\|_{L^2(\mathbb{R}^2)} \\ &\le 10\delta^{\frac{1}{2}} + \frac{C(\kappa,\varepsilon)}{\lambda^{\frac{1}{2}}} + C(R_0, u_0, e, \delta, \kappa, \varepsilon)\lambda^{\gamma_0} + \|r_0\|_{C_tL^2_x}\end{aligned}$$

Using that $\|r_0\|_{C_tL^2_x} \le \frac{1}{32}\delta$ by assumption, we can choose λ large enough such that

$$\|u_1 - u_0(t)\|_{L^2(\mathbb{R}^2)} \le 11\delta^{\frac{1}{2}},$$

i.e. (v) is satisfied with $M_0 = 11$. For (vi), we use Lemmas 7.1, 7.2 and 7.3

$$\begin{aligned}\|\operatorname{curl}(u_1-u_0)(t)\|^p_{H^p(\mathbb{R}^2)} &\le \|\operatorname{curl} w(t)\|^p_{H^p(\mathbb{R}^2)} + \|\operatorname{curl} u^t(t)\|^p_{H^p(\mathbb{R}^2)} + \|\operatorname{curl} v(t)\|^p_{H^p(\mathbb{R}^2)} \\ &\le C(R_0, u_0, e, \delta, \kappa, \varepsilon)\lambda^{p\gamma_0} + \|\int_0^t \operatorname{curl} r_0(s)\,\mathrm{d}s\|^p_{H^p(\mathbb{R}^2)}\end{aligned}$$

and we can choose λ large enough such that (vi) is satisfied. For (iv), we have by Lemma 9.1, 9.2, 9.3, 9.4, 9.5, 9.6 and 9.7

$$\begin{aligned}\|R_1(t)\|_{L^1(\mathbb{R}^2)} &\le \|R^{\text{lin},1}(t)\|_{L^1(\mathbb{R}^2)} + \|R^{\text{lin},2}(t)\|_{L^1(\mathbb{R}^2)} + \|R^{\text{lin},3}(t)\|_{L^1(\mathbb{R}^2)} \\ &\quad + \|R^\kappa(t)\|_{L^1(\mathbb{R}^2)} + \|R^{\text{quad}}(t)\|_{L^1(\mathbb{R}^2)} + \|R^Y(t)\|_{L^1(\mathbb{R}^2)} + \|R^{\text{time}}(t)\|_{L^1(\mathbb{R}^2)} \\ &\le \frac{\eta}{2} + 4\|r_0\|^2_{C_tL^2_x} + 2\|r_0\|_{C_tL^2_x}\|u_0(t)\|_{L^2(\mathbb{R}^2)} + C(R_0, u_0, e, \delta, \kappa, \varepsilon, N)\lambda^{\gamma_0}.\end{aligned}$$

Noting that $\|r_0(t)\|^2_{L^2(\mathbb{R}^2)} \le \|r_0(t)\|_{L^2(\mathbb{R}^2)}$ since $\|r_0(t)\|_{L^2(\mathbb{R}^2)} \le 1$ by assumption, we can choose λ large enough to obtain (iv). For (ii), we have because of the compact support of r_1

$$\|r_1(t)\|_{L^2(\mathbb{R}^2)} \le C(\kappa)\|r_1(t)\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq C(\kappa)\left(\|r^{\text{quad}}(t)\|_{L^\infty(\mathbb{R}^2)} + \|r^Y(t)\|_{L^\infty(\mathbb{R}^2)} + \|r^{\text{time}}(t)\|_{L^\infty(\mathbb{R}^2)}\right)$$
$$\leq C(R_0, u_0, e, \delta, \kappa, \varepsilon, N)\lambda^{\gamma_0},$$

and by the previous estimate on $\|u_1(t) - u_0(t)\|_{L^2(\mathbb{R}^2)}$

$$\|u_1\|_{C_t L^2_x}\|r_1(t)\|_{L^2(\mathbb{R}^2)} \leq \|u_0\|_{C_t L^2_x}\|r_1(t)\|_{L^2(\mathbb{R}^2)} + \|u_1 - u_0\|_{C_t L^2_x}\|r_1(t)\|_{L^2(\mathbb{R}^2)}$$
$$\leq C(R_0, r_0, u_0, \delta, \kappa, \varepsilon, N)\lambda^{\gamma_0}.$$

Finally, we also have

$$\|\int_0^t \operatorname{curl} r_1(s)\,\mathrm{d}s\|^p_{H^p(\mathbb{R}^2)} \leq \|\int_0^t \operatorname{curl} r^{\text{quad}}(s)\,\mathrm{d}s\|^p_{H^p(\mathbb{R}^2)} + \|\int_0^t \operatorname{curl} r^Y(s)\,\mathrm{d}s\|^p_{H^p(\mathbb{R}^2)}$$
$$+ \|\int_0^t \operatorname{curl} r^{\text{time}}(s)\,\mathrm{d}s\|^p_{H^p(\mathbb{R}^2)}$$
$$\leq C(R_0, u_0, e, \delta, \kappa, \varepsilon, N)\lambda^{p\gamma_0}.$$

Again, λ can be chosen large enough such that (iii) is satisfied. Assume we have given two energy profiles e_1, e_2 with $e_1 = e_2$ on $[0, t_0]$ for some $t_0 \in [0, 1]$. The values that we add with $w(t), u_c(t), u^t(t)$ depend only on pointwise (in time) values of the previous steps, while $v(t)$ depends only on values of the previous steps on $[0, t]$. Therefore, one can do the construction for e_1 and e_2 simultaneously, choosing the same values for all the parameters in each iteration step, thereby producing two solutions u_1, u_2 to (1) that satisfy $u_1 = u_2$ on $[0, t_0]$.

Funding Open access funding provided by Gran Sasso Science Institute - GSSI within the CRUI-CARE Agreement.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest and that the manuscript has no associated data.

References

[1] Bedrossian, J., Vicol, V.: The mathematical analysis of the incompressible Euler and Navier–Stokes equations: an introduction, volume 225. American Mathematical Society, (2022)
[2] Bressan, A., Shen, W.: A posteriori error estimates for self-similar solutions to the euler equations. Discrete Contin. Dynam. Syst. (2020). https://doi.org/10.3934/dcds.2020168
[3] Bruè, E., Colombo, M.: Nonuniqueness of solutions to the euler equations with vorticity in a lorentz space. Commun. Math. Phys. (2023). https://doi.org/10.1007/s00220-023-04816-4
[4] Bruè, E., Colombo, M., De Lellis, C.: Positive solutions of transport equations and classical nonuniqueness of characteristic curves. Arch. Ration. Mech. Anal. **240**(2), 1055–1090 (2021). https://doi.org/10.1007/s00205-021-01628-5
[5] Buckmaster, T., Lellis, C., Székelyhidi, J., Vicol, V.: Onsager's conjecture for admissible weak solutions. Comm. Pure Appl. Mat **72**(2), 229–274 (2018). https://doi.org/10.1002/cpa.21781
[6] Buckmaster, T., Masmoudi, N., Novack, M., Vicol, V.: Intermittent convex integration for the 3d euler equations. Ann. of Math. Studies, 217, (2023)

[7] Buckmaster, T., Vicol, V.: Nonuniqueness of weak solutions to the Navier–Stokes equation. Annals Math. **189**(1), 101–144 (2019). https://doi.org/10.4007/annals.2019.189.1.3

[8] Burczak, J., Modena, S., Székelyhidi, L.: Non uniqueness of power-law flows. Commun. Math. Phys. **388**(1), 199–243 (2021). https://doi.org/10.1007/s00220-021-04231-7

[9] Cheskidov, A., Luo, X.: Nonuniqueness of weak solutions for the transport equation at critical space regularity. Annals PDE **7**(2), 1–45 (2021). https://doi.org/10.1007/s40818-020-00091-x

[10] Cheskidov, A., Luo, X.: Extreme temporal intermittency in the linear sobolev transport: almost smooth nonunique solutions. https://doi.org/10.48550/arXiv.2204.08950. *arXiv preprint*arXiv:2204.08950, (2022)

[11] De Lellis, C., Székelyhidi, L.: Dissipative continuous Euler flows. Invent. Math. **193**, 377–407 (2013). https://doi.org/10.1007/s00222-012-0429-9

[12] De Lellis, C., Székelyhidi, L., Jr.: The Euler equations as a differential inclusion. Ann. Math. **2**(3), 1417–1436 (2009). https://doi.org/10.4007/annals.2009.170.1417

[13] De Lellis, C., Székelyhidi, L., Jr.: Dissipative Euler flows and Onsager's conjecture. J. Eur. Math. Soc. **16**(7), 1467–1505 (2014). https://doi.org/10.4171/JEMS/466

[14] Giri, V., Kwon, H., Novack, M.: The L^3-based strong onsager theorem. https://doi.org/10.48550/arXiv.2305.18509. *arXiv preprint*arXiv:2305.18509, (2023)

[15] Giri, V., Kwon, H., Novack, M.: A wavelet-inspired L^3-based convex integration framework for the euler equations. https://doi.org/10.48550/arXiv.2305.18142. *arXiv preprint*arXiv:2305.18142, (2023)

[16] Giri, V., Radu, R.-O.: The 2d onsager conjecture: a newton-nash iteration. https://doi.org/10.48550/arXiv.2305.18105. arXiv:2305.18105, (2023)

[17] Giri, V., Sorella, M.: Non-uniqueness of integral curves for autonomous hamiltonian vector fields. Differ. Integral Equ, (2021). https://doi.org/10.57262/die035-0708-411 https://doi.org/10.57262/die035-0708-411 https://doi.org/10.57262/die035-0708-411

[18] Grafakos, L.: Modern Fourier Analysis, vol. 250. Springer, New York (2009)

[19] Isett, P.: A proof of Onsager's conjecture. Ann. Math. **188**(3), 871–963 (2018). https://doi.org/10.4007/annals.2018.188.3.4

[20] Isett, P., Oh, S.-J.: On nonperiodic Euler flows with hölder regularity. Arch. Ration. Mech. Anal. **221**(2), 725–804 (2016). https://doi.org/10.1007/s00205-016-0973-3

[21] Loeper, G.: Uniqueness of the solution to the Vlasov–Poisson system with bounded density. J. de mathématiques pures et appliquées **86**(1), 68–79 (2006). https://doi.org/10.1016/j.matpur.2006.01.005

[22] Mengual, F.: Non-uniqueness of admissible solutions for the 2d euler equation with L^p vortex data. https://doi.org/10.48550/arXiv.2304.09578. *arXiv preprint*arXiv:2304.09578, (2023)

[23] Modena, S., Sattig, G.: Convex integration solutions to the transport equation with full dimensional concentration. Annales de l'Institut Henri Poincaré C, Analyse non linéaire **37**, 1075–1108 (2020). https://doi.org/10.1016/J.ANIHPC.2020.03.002

[24] Modena, S., Székelyhidi, L.: Non-uniqueness for the transport equation with Sobolev vector fields. Annals PDE **4**(2), 18 (2018). https://doi.org/10.1007/s40818-018-0056-x

[25] Modena, S., Székelyhidi, L.: Non-renormalized solutions to the continuity equation. Calc. Var. Partial. Differ. Equ. **58**, 208 (2019). https://doi.org/10.1007/s00526-019-1651-8

[26] Pitcho, J., Sorella, M.: Almost everywhere non-uniqueness of integral curves for divergence-free sobolev vector fields. https://doi.org/10.48550/arXiv.2108.03194. *arXiv preprint*arXiv:2108.03194, (2022)

[27] Stein, E. M., Murphy, T. S.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, vol. 3. Princeton University Press (1993). https://doi.org/10.1515/9781400883929

[28] Vishik, M.: Instability and non-uniqueness in the cauchy problem for the euler equations of an ideal incompressible fluid. Part i. https://doi.org/10.48550/arXiv.1805.09426. *arXiv preprint*arXiv:1805.09426, (2018)

[29] Vishik, M.: Instability and non-uniqueness in the cauchy problem for the Euler equations of an ideal incompressible fluid. Part ii. https://doi.org/10.48550/arXiv.1805.09440. *arXiv preprint*arXiv:1805.09440, (2018)

[30] Yudovich, V.I.: Some bounds for solutions of elliptic equations. Matematicheskii Sbornik **101**, 229–244 (1962)

[31] Yudovich, V.I.: Non-stationary flows of an ideal incompressible fluid. Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki **3**(6), 1032–1066 (1963)

Miriam Buck
Department of Mathematics
Technische Universität Darmstadt
64285 Darmstadt
Germany
e-mail: mbuck@mathematik.tu-darmstadt.de

Stefano Modena
Gran Sasso Science Institute
67100 L'Aquila
Italy
e-mail: stefano.modena@gssi.it

(accepted: January 27, 2024; published online: March 28, 2024)