



# A Sharp Version of the Benjamin and Lighthill Conjecture for Steady Waves with Vorticity

Evgeniy Lokharu

Communicated by A. Constantin

**Abstract.** We give a complete proof of the classical Benjamin and Lighthill conjecture for arbitrary two-dimensional steady water waves with vorticity. We show that the flow force constant of an arbitrary smooth solution is bounded by the flow force constants for the corresponding conjugate laminar flows. We prove these inequalities without any assumptions on the geometry of the surface profile and put no restrictions on the wave amplitude. Furthermore, we give a complete description of all cases when the equalities can occur. In particular, that excludes the existence of one-sided bores and multi-hump solitary waves. Our conclusions are new already for Stokes waves with a constant vorticity, while the case of equalities is new even in the classical setting of irrotational waves.

**Keywords.** Steady waves, Vorticity, Gravity waves, Flow force.

## 1. Introduction

In this paper we address the classical water wave problem for two-dimensional steady waves with vorticity on water of finite depth, formulated in terms of Euler equations with a free boundary. Every steady wave admit two non-dimensional constants of motion, which are the Bernoulli and the flow force constants. In this paper we focus on two questions, one is related to fundamental bounds for the surface profile, and the other one is about a possible range for values of the flow force constant. The last problem is known as the Benjamin and Lighthill conjecture and was introduced in [2]. The property conjectured by Benjamin and Lighthill (see also Keady and Norbury [9], Conjecture 2) can be expressed as the inequalities

$$\mathcal{S}_-(r) \leq \mathcal{S} \leq \mathcal{S}_+(r), \quad (1.1)$$

where  $\mathcal{S}$  is the non-dimensional flow force constant of a solution,  $r$  is the corresponding Bernoulli constant, while  $\mathcal{S}_-(r)$  and  $\mathcal{S}_+(r)$  are flow force constants of conjugate laminar flows (supercritical and subcritical respectively) determined by the same Bernoulli constant. According to the conjecture, inequalities in (1.1) are valid for arbitrary smooth solutions. Geometrically, it means that any steady motion realizes a point inside the cuspidal region as in Fig. 1, where the upper boundary corresponds to subcritical flows and the lower boundary represents supercritical flows respectively. We recall that subcritical flows are those with Froude numbers  $F < 1$ , while supercritical flows are determined by  $F > 1$ . Every subcritical flow allows for a local bifurcation of small-amplitude Stokes waves that correspond to the shaded region adjacent to the upper boundary in Fig. 1. The lower boundary in Fig. 1 represents supercritical flows as well as solitary waves. Note that every solitary wave satisfies  $\mathcal{S} = \mathcal{S}_+(r)$ .

The importance of the Benjamin–Lighthill conjecture was recently reemphasized in [21], where it was shown that the amplitude of any irrotational Stokes wave is bounded from above by some absolute constant times  $c^2/g$ , where  $c$  is the wave speed and  $g$  is the gravitational constant. It is also shown in [21] that there exist small-amplitude extreme waves. Both facts rely heavily on the inequality  $\mathcal{S} \leq \mathcal{S}_+$ , which is the most interesting one.

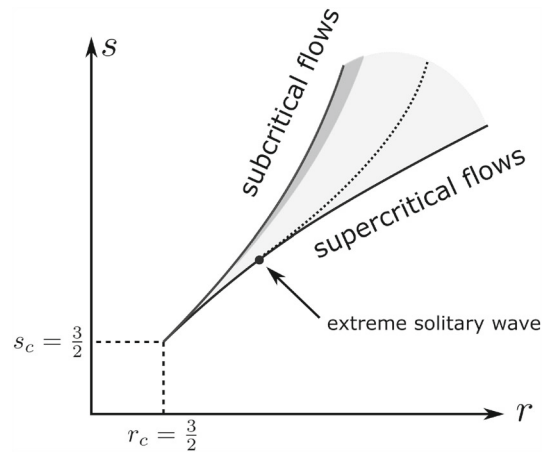


FIG. 1. The parameter region in  $(r, s)$ -variables for irrotational waves

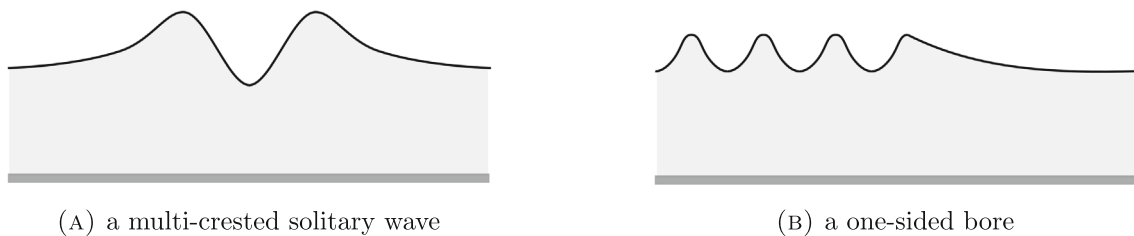


FIG. 2. Solitary-type waves with  $S = S_-(r)$

The case of equalities in (1.1) was not originally a part of the Benjamin–Lighthill conjecture, but is a significant problem itself. For instance, every subcritical solitary wave as in Fig. 2, if exists, must satisfy  $\mathcal{S} = \mathcal{S}_+(r)$ . Similarly, a one-sided bore as in Fig. 2 has to be subject to  $\mathcal{S} = \mathcal{S}_-(r)$ . The non-existence of subcritical solitary waves was recently established in [19], while the absence of supercritical bores, for instance, was still an open problem. This shows that the possibility of equalities in (1.1) is a separate question of interest itself. In this paper we completely settle both of the problems: the classical conjecture (1.1) and the case of equalities.

The Benjamin–Lighthill conjecture for irrotational Stokes waves was verified by Benjamin [4]. He also outlined a possible proof of (1.1) for sub-harmonic bifurcations of Stokes waves. The bottom bound in (1.1) was obtained earlier by Keady & Norbury [9] (also for periodic wavetrains). Kozlov and Kuznetsov [13, 15] proved (1.1) for arbitrary solutions under weak regularity assumptions, provided the Bernoulli constant  $r$  is close to its critical value  $R_c$ ; it was extended to the rotational setting in [18], again for  $r \approx R_c$ ; the latter condition guarantees that solutions are of small amplitude, that is are close to the unperturbed shear flow. The left inequality in (1.1) for periodic waves with a favorable vorticity was obtained by Keady & Norbury [11]. Whereas their result is valid under essential restrictions on the vorticity, they point out that in general the statement of the conjecture is probably false: “There is no reason to suppose that the conjectures of Benjamin and Lighthill will hold for all flows with vorticity”. Even so this comment is probably related to flows with counter-currents, it is still surprising that (1.1) turns out to be true for arbitrary vorticity distributions and arbitrary solutions that are unidirectional, which is one of the main results of the present paper.

The Benjamin and Lighthill conjecture is closely related to another problem about bounds for the surface profile. If  $y = \eta(x)$  determines the surface of the fluid in a moving frame of reference, then the

following inequalities are well known:

$$d_-(r) \leq \inf_{x \in \mathbb{R}} \eta(x) \leq d_+(r) \leq \sup_{x \in \mathbb{R}} \eta(x), \tag{1.2}$$

where  $d_-(r)$  and  $d_+(r)$  are depths of the supercritical and subcritical flows respectively. First obtained by Keady & Norbury [10] for irrotational Stokes waves, it was extended to arbitrary solutions by Kozlov and Kuznetsov [12, 14]; see also [16]. We emphasize that Kozlov and Kuznetsov [14] obtained strict inequality  $d_+(r) < \sup_{x \in \mathbb{R}} \eta(x)$  for arbitrary irrotational solutions, provided  $\eta$  is not a constant identically. While for Stokes waves it can be obtained by using the Hopf lemma, the general case is much more subtle. The argument in [14] required a careful analysis of the Fourier symbol associated with an integro-differential operator and the irrotational nature of the problem was essential. For waves with vorticity only a weak form of (1.2) is known; see [16, 17].

In this paper we consider both problems, inequalities for the flow force (1.1) and bounds (1.2). For an arbitrary wave with vorticity we prove (1.1) and (1.2) and provide a complete description of all cases when equalities can occur. The case of equalities in (1.1) is new even in the irrotational setting. In fact, for all solutions other than streams and classical solitary waves all inequalities in (1.1) and (1.2) are shown to be strict. In particular, if a given non-laminar solution satisfies  $\mathcal{S} = \mathcal{S}_-(r)$  (without any assumptions on the surface profile), then it is necessarily a classical solitary wave of elevation, whose profile decays monotonically on each side of the crest. On the other hand, the relation  $\mathcal{S} = \mathcal{S}_+(r)$  is only valid for subcritical laminar flows. This is a strong statement, because it, in particular, shows that subcritical solitary waves (with the Froude number less than one) do not exist. The latter was an open problem for a long time even in the irrotational setting; it was resolved recently in [19] by using an asymptotic analysis. In addition we prove that any steady wave is subject to strict inequalities in (1.2), provided it is not a parallel flow or a solitary wave. This generalizes a series of previous mentioned results.

## 2. Statement of the Problem

We consider the classical water wave problem for two-dimensional steady waves with vorticity on water of finite depth. We neglect effects of surface tension and consider a fluid of constant (unit) density. Thus, in an appropriate coordinate system moving along with the wave, stationary Euler equations are given by

$$(u - c)u_x + vu_y = -P_x, \tag{2.1a}$$

$$(u - c)v_x + vv_y = -P_y - g, \tag{2.1b}$$

$$u_x + v_y = 0, \tag{2.1c}$$

which holds true in a two-dimensional fluid domain  $D$ , defined by the inequality

$$0 < y < \eta(x).$$

Here  $(u, v)$  are components of the relative velocity field,  $y = \eta(x)$  is the surface profile,  $c$  is the wave speed,  $P$  is the pressure and  $g$  is the gravitational constant. The corresponding boundary conditions are

$$v = 0 \qquad \qquad \qquad \text{on } y = 0, \tag{2.1d}$$

$$v = (u - c)\eta_x \qquad \qquad \text{on } y = \eta, \tag{2.1e}$$

$$P = P_{\text{atm}} \qquad \qquad \text{on } y = \eta. \tag{2.1f}$$

It is often assumed in the literature that the flow is irrotational, that is  $v_x - u_y$  is zero everywhere in the fluid domain. Under this assumption components of the velocity field are harmonic functions, which allows to apply methods of complex analysis. Being a convenient simplification it forbids modeling of non-uniform currents, commonly occurring in nature. In the present paper we will consider rotational flows, where the vorticity function is defined by

$$\omega = -v_x + u_y. \tag{2.2}$$

Throughout the paper we assume that the flow is free from stagnation points and the horizontal component of the relative velocity field does not change sign, that is

$$u - c < 0 \tag{2.3}$$

everywhere in the fluid. We call such flows unidirectional.

In the two-dimensional case relation (2.1c) allows to reformulate the problem in terms of a stream function  $\psi$ , defined implicitly by relations

$$\psi_y = c - u, \quad \psi_x = v.$$

This determines  $\psi$  up to an additive constant, while relations (2.1d) and (2.1f) force  $\psi$  to be constant along the boundaries. Thus, by subtracting a suitable constant, we can always assume that

$$\psi = m \text{ on } y = \eta; \quad \psi = 0 \text{ on } y = 0.$$

Here  $m > 0$  is the mass flux, defined by

$$m = \int_0^\eta (c - u) dy.$$

In what follows we will use non-dimensional variables proposed by Keady & Norbury [10], where lengths and velocities are scaled by  $(m^2/g)^{1/3}$  and  $(mg)^{1/3}$  respectively; in new units  $m = 1$  and  $g = 1$ . For simplicity we keep the same notations for  $\eta$  and  $\psi$ .

Taking the curl of Euler equations (2.1a)-(2.1c) one checks that the vorticity function  $\omega$  defined by (2.2) is constant along paths tangent everywhere to the relative velocity field  $(u - c, v)$ ; see [5] for more details. Having the same property by the definition, stream function  $\psi$  is strictly monotone by (2.3) on every vertical interval inside the fluid region. These observations together show that  $\omega$  depends only on values of the stream function, that is

$$\omega = \omega(\psi).$$

This property and Bernoulli's law allow to express the pressure  $P$  as

$$P - P_{\text{atm}} + \frac{1}{2}|\nabla\psi|^2 + y + \Omega(\psi) - \Omega(1) = \text{const}, \tag{2.4}$$

where

$$\Omega(\psi) = \int_0^\psi \omega(p) dp$$

is a primitive of the vorticity function  $\omega(\psi)$ . Thus, we can eliminate the pressure from equations and obtain the following problem:

$$\Delta\psi + \omega(\psi) = 0 \tag{2.5a} \quad \text{for } 0 < y < \eta,$$

$$\frac{1}{2}|\nabla\psi|^2 + y = r \tag{2.5b} \quad \text{on } y = \eta,$$

$$\psi = 1 \tag{2.5c} \quad \text{on } y = \eta,$$

$$\psi = 0 \tag{2.5d} \quad \text{on } y = 0.$$

Here  $r > 0$  is referred to as Bernoulli's constant.

Let us define the flow force constant, another motion invariant. Following Benjamin [3], we put

$$\mathcal{S} = \int_0^\eta (P - P_{\text{atm}} + (u - c)^2) dy. \tag{2.6}$$

Taking  $x$ -derivative in (2.6) and using (2.1a) together with the formula for the pressure (2.4), one verifies that  $\mathcal{S}$  is a constant of motion independent of  $x$ . In terms of the stream function one obtains

$$\mathcal{S} = \int_0^\eta \left( \frac{1}{2}(\psi_y^2 - \psi_x^2) - y + \Omega(1) - \Omega(\psi) + r \right) dy. \tag{2.7}$$

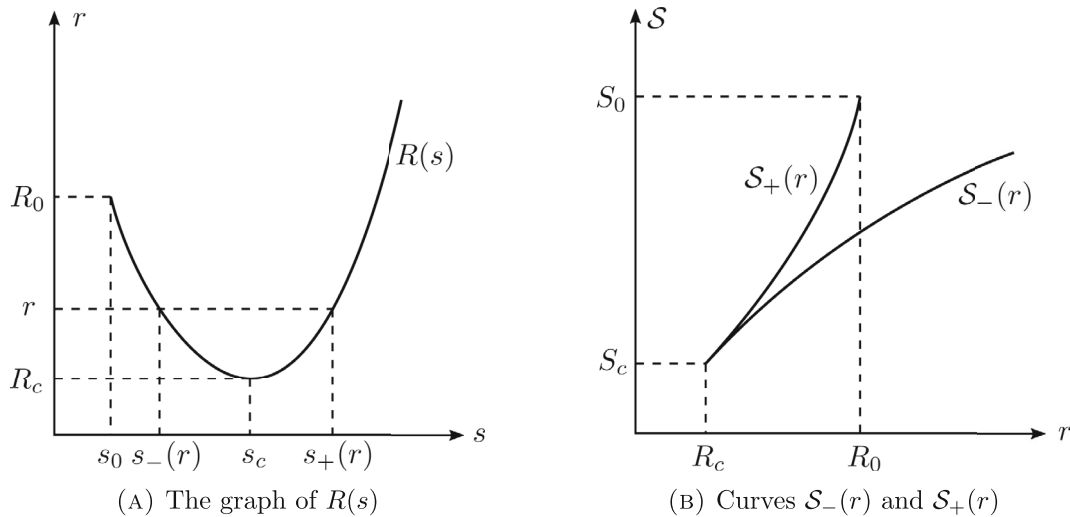


FIG. 3. Diagrams for laminar flows

This constant is important in several ways; for instance, it plays the role of the Hamiltonian in spatial dynamics; see [1].

### 2.1. Stream Solutions

Laminar flows or shear currents, for which the vertical component  $v$  of the velocity field is zero play an important role in the theory of steady waves. Let us recall some basic facts about stream solutions  $\psi = U(y)$  and  $\eta = d$  describing shear currents. It is convenient to parameterize this family by the relative speed at the bottom. Thus, we put  $U_y(0) = s$  and find that  $U = U(y; s)$  is subject to

$$U'' + \omega(U) = 0, \quad 0 < y < d; \quad U(0) = 0, \quad U(d) = 1. \tag{2.8}$$

Our assumption (2.3) implies  $U' > 0$  on  $[0; d]$ , which puts a natural constraint on  $s$ . Indeed, multiplying the first equation in (2.8) by  $U'$  and integrating over  $[0; y]$ , we find

$$U'^2 = s^2 - 2\Omega(U).$$

This shows that the expression  $s^2 - 2\Omega(p)$  is non-negative for all  $p \in [0; 1]$ , which requires

$$s > s_0 = \sqrt{\max_{p \in [0,1]} 2\Omega(p)}. \tag{2.9}$$

On the other hand, every  $s > s_0$  gives rise to a monotonically increasing function  $U(y; s)$  solving (2.8) for some unique  $d = d(s)$ , given explicitly by

$$d(s) = \int_0^1 \frac{1}{\sqrt{s^2 - 2\Omega(p)}}.$$

The formula above shows that  $d(s)$  monotonically depends on  $s$  and takes values between zero and

$$d_0 = \lim_{s \rightarrow s_0+} d(s).$$

This quantity can be finite or not depending on the vorticity. For instance, when  $\omega = 0$  we find  $s_0 = 0$  and  $d_0 = +\infty$ . On the other hand, when  $\omega = -b$  for some positive constant  $b \neq 0$ , then  $s_0 = 0$  but  $d_0 < +\infty$ .

Every stream solution  $U(y; s)$  determines the Bernoulli constant  $R(s)$ , which can be found from the relation (2.5c). This constant can be computed explicitly as

$$R(s) = \frac{1}{2}s^2 - \Omega(1) + d(s), \quad s \geq s_0.$$

Taking the derivative with respect to  $s$  we find that  $R'(s) = 0$  only for  $s = s_c$ , which is uniquely defined from the relation

$$\int_0^1 \frac{1}{(s^2 - 2\Omega(p))^{3/2}} dp = 1.$$

The corresponding value of  $R(s)$  at  $s = s_c$  is denoted by

$$R_c = R(s_c).$$

Thus, the function  $R(s)$  decreases from  $R_0$  to  $R_c$  when  $s$  varies from  $s_0$  to  $s_c$  and increases to infinity on  $(s_c, +\infty)$ . This monotonicity property of  $R(s)$  shows (see Fig. 3A) that for any  $r \in (R_c; R_0)$  the equation  $R(s) = r$  has exactly two solutions  $s = s_-(r)$  and  $s = s_+(r)$ , such that  $s_0 < s_-(r) < s_c < s_+(r)$ . The corresponding depths

$$d_-(r) = d(s_+(r)), \quad d_+(r) = d(s_-(r))$$

satisfy  $d_-(r) < d_+(r)$  and are called supercritical and subcritical depths respectively. The flow force constants corresponding to  $d_{\pm}(r)$  are given by

$$\mathcal{S}_-(r) = S(s_+(r)), \quad \mathcal{S}_+(r) = S(s_-(r)),$$

where  $S(s)$  is the flow force constant for the stream solution  $(U(y; s), d(s))$ . It is computed from (2.7) as

$$S(s) = \int_0^{d(s)} \left( \frac{1}{2}(U_y(y; s))^2 + \Omega(1) - \Omega(U(y; s)) - y + r \right) dy.$$

A direct computation gives

$$S'(s) = d(s)(s + d'(s)),$$

so that  $S(s)$  attains its global minimum  $S_c$  at  $s = s_c$ . One also checks that two curves  $(R(s), S(s))$  for  $s < s_c$  and  $(R(s), S(s))$  for  $s > s_c$  in  $(r, \mathcal{S})$  plane have the same tangent line  $\mathcal{S} = d_c r$  at  $s = s_c$ . Furthermore,  $(R(s), S(s))$  has a positive curvature for  $s < s_c$ , while  $(R(s), S(s))$  for  $s > s_c$  has a negative curvature. Taking into account that  $S > S_c = S(s_c)$  for all  $s \neq s_c$ , we conclude

$$\mathcal{S}_-(r) < \mathcal{S}_+(r), \quad r \in (R_c, R),$$

see Fig. 3B.

## 2.2. Formulations of Main Results

Following notations from the previous section, our main theorem is

**Theorem 2.1.** *Let  $\psi \in C^{2,\gamma}(\overline{D})$  and  $\eta \in C^{2,\gamma}(\mathbb{R})$  be a solution to (2.5) for some  $\gamma \in (0, 1)$  with  $\inf_D \psi_y > 0$  and  $r \in (R_c, R_0)$ . Then the flow force constant  $\mathcal{S}$  given by (2.7) enjoys the following properties:*

- (i) *inequalities  $\mathcal{S}_-(r) \leq \mathcal{S} \leq \mathcal{S}_+(r)$  are always true;*
- (ii) *the equality  $\mathcal{S} = \mathcal{S}_-(r)$  holds true only for supercritical laminar flows and symmetric solitary waves of positive elevation supported by supercritical streams;*
- (iii) *the equality  $\mathcal{S} = \mathcal{S}_+(r)$  is true only for subcritical laminar flows.*

The claim (i) of Theorem 2.1 is known as the classical Benjamin and Lighthill conjecture. We emphasise that it is new even in the irrotational case since it covers all possible smooth solutions; the original proof by Benjmain [4] deals only with Stokes waves (and small amplitude perturbations of those) and Benjmain’s argument relies heavily on that assumption.

The parts (ii) and (iii) of Theorem 2.1 are of separate interest and had never been considered in the literature before. For instance, the claim (iii), in particular, forbids the existence of subcritical solitary waves; this was an open problem for a long time and was recently proved in [19]. Both statements (ii) and (iii) are new even for irrotational waves.

The assumption  $r \in (R_c, R_0)$  appears naturally, since otherwise the quantity  $\mathcal{S}_+(r)$  is not defined. Moreover, for a broad range of solutions it is fulfilled automatically. In particular when  $R_0 = +\infty$ . When  $R_0$  is finite and the maximum of  $\Omega(p)$  on  $[0, 1]$  is strictly positive, then  $r < R_0$  is proved to be true for arbitrary non-laminar solutions; see [17]. The case  $\Omega \leq 0$  is more delicate and was considered by the author in [20]. It was shown that an arbitrary Stokes wave enjoys  $r < R_0 - \Omega(1)$ , which is not an optimal bound. Our conjecture is that  $r < R_0$  for all nontrivial solutions irrespective of the vorticity. Furthermore, when  $r$  is close to  $R_0$ , then the wave amplitude must be small. Such property was recently verified in [21] in the irrotational case, when  $R_0 = +\infty$ . It was shown that the amplitude of an arbitrary solution is of order  $r^{-2}$ . As a consequence of that we obtained the inequality  $a < Bc^2g^{-1}$  for an arbitrary Stokes wave. Here  $B$  is an absolute constant,  $a$  is the wave amplitude in original physical variables,  $c$  is the propagation speed, and  $g$  is the gravitational constant. It is worth mentioning that the argument in [21] was essentially based on the inequality  $\mathcal{S} \leq \mathcal{S}_+(r)$  for irrotational Stokes waves, which was verified by Benjamin in [4]. This emphasizes the importance of the Benjamin and Lighthill conjecture. A direct consequence of Theorem 2.1 is stated below.

**Corollary 2.2.** *Under assumptions of the theorem, the water wave profile  $\eta$  is subject to the following properties:*

- (i') for all  $x \in \mathbb{R}$  we have  $d_-(r) < \eta(x)$ ;
- (ii') denoting  $\hat{\eta} = \sup_{\mathbb{R}} \eta$  and  $\check{\eta} = \inf_{\mathbb{R}} \eta$ , we have  $\check{\eta} < d_+(r) < \hat{\eta}$ , while equalities  $\check{\eta} = d_+(r)$  or  $d_+(r) = \hat{\eta}$  are only possible if  $\check{\eta} = \hat{\eta} = d_+(r)$ ;
- (iii') the equality  $\check{\eta} = d_-(r)$  is valid only for supercritical laminar flows and supercritical solitary waves.

These statements follow from Proposition 3.2 and Theorem 2.1. The inequalities  $d_-(r) < \eta(x)$  and  $\check{\eta} < d_+(r) < \hat{\eta}$  for irrotational Stokes waves were first obtained by Keady & Norbury [9]. An extension to arbitrary irrotational solutions was done by Kozlov & Kuznetsov [12, 14]. For waves with vorticity only non-strict versions of inequalities were known; see [17] and references therein. The last claim (iii') is new even in the irrotational setting.

Such bounds as in Corollary 2.2 are useful for further analysis of large-amplitude waves since they define positive cones containing all solutions.

Our proofs are essentially based on the elliptic maximum principle, which is the main reason why the inequalities  $\mathcal{S}_-(r) \leq \mathcal{S} \leq \mathcal{S}_+(r)$  hold true. In order to use maximum principle arguments we introduce flow force flux functions, generalizing a similar construction from [19].

### 3. Preliminaries

#### 3.1. Reformulation of the Problem

Under assumption (2.3) we can apply the partial hodograph transform introduced by Dubreil-Jacotin [7]. Thus, we present new independent variables

$$q = x, \quad p = \psi(x, y),$$

while new unknown function  $h(q, p)$  (height function) is defined from the identity

$$h(q, p) = y.$$

Note that it is related to the stream function  $\psi$  through the formulas

$$\psi_x = -\frac{h_q}{h_p}, \quad \psi_y = \frac{1}{h_p}, \tag{3.1}$$

where

$$h_p > 0 \tag{3.2a}$$

throughout the fluid domain by (2.3). An advantage of using new variables is in that instead of two unknown functions  $\eta(x)$  and  $\psi(x, y)$  with an unknown domain of definition we have one function  $h(q, p)$  defined in a fixed strip  $S = \mathbb{R} \times (0, 1)$ . An equivalent problem for  $h(q, p)$  is given by

$$\left( \frac{1 + h_q^2}{2h_p^2} + \Omega \right)_p - \left( \frac{h_q}{h_p} \right)_q = 0 \tag{3.2b}$$

$$\frac{1 + h_q^2}{2h_p^2} + h = r \tag{3.2c}$$

$$h = 0 \tag{3.2d}$$

The wave profile  $\eta$  becomes the boundary value of  $h$  on  $p = 1$ :

$$h(q, 1) = \eta(q), \quad q \in \mathbb{R}.$$

Using (3.1) and Bernoulli’s law (2.4) we recalculate the flow force constant  $\mathcal{S}$  defined in (2.7) as

$$\mathcal{S} = \int_0^1 \left( \frac{1 - h_q^2}{2h_p^2} - h - \Omega + \Omega(1) + r \right) h_p dp. \tag{3.3}$$

Laminar flows defined by stream functions  $U(y; s)$  correspond to height functions  $h = H(p; s)$  that are independent of the horizontal variable  $q$ . The corresponding equations are

$$\left( \frac{1}{2H_p^2} + \Omega \right)_p = 0, \quad H(0) = 0, \quad H(1) = d(s), \quad \frac{1}{2H_p^2(1)} + H(1) = R(s).$$

Solving equations for  $H(p; s)$  explicitly, we find

$$H(p; s) = \int_0^p \frac{1}{\sqrt{s^2 - 2\Omega(\tau)}} d\tau.$$

Given a height function  $h(q, p)$  and a stream solution  $H(p; s)$ , we define

$$w^{(s)}(q, p) = h(q, p) - H(p; s). \tag{3.4}$$

This notation will be frequently used in what follows. In order to derive an equation for  $w^{(s)}$  we first write (3.2b) in a non-divergence form as

$$\frac{1 + h_q^2}{h_p^2} h_{pp} - 2 \frac{h_q}{h_p} h_{qp} + h_{qq} - \omega(p) h_p = 0.$$

Now using our ansatz (3.4), we find

$$\frac{1 + h_q^2}{h_p^2} w_{pp}^{(s)} - 2 \frac{h_q}{h_p} w_{qp}^{(s)} + w_{qq}^{(s)} - \omega(p) w_p^{(s)} + \frac{(w_q^{(s)})^2 H_{pp}}{h_p^2} - \frac{w_p^{(s)}(h_p + H_p) H_{pp}}{h_p^2 H_p^2} = 0. \tag{3.5}$$

Thus,  $w^{(s)}$  solves a homogeneous elliptic equation in  $S$  and is subject to a maximum principle; see [22] for an elliptic maximum principle in unbounded domains. The boundary conditions for  $w^{(s)}$  can be obtained directly from (3.2c) and (3.2d) by inserting (3.4) and using the corresponding equations for  $H$ . This gives

$$\frac{(w_q^{(s)})^2}{2h_p^2} - \frac{w_p^{(s)}(h_p + H_p)}{2h_p^2 H_p^2} + w^{(s)} = r - R(s) \tag{3.6a}$$

$$w^{(s)} = 0 \tag{3.6b}$$



For  $s = s_{\pm}(r)$ , we have  $r - R(s_{\pm}(r)) = 0$  and (3.6a) turns into

$$\frac{w_p^{(s_{\pm})}}{H_p^3} - w^{(s_{\pm})} = \frac{(w_q^{(s_{\pm})})^2}{2h_p^2} + \frac{(w_p^{(s_{\pm})})^2(2h_p + H_p)}{2H_p^3h_p^2}, \quad p = 1. \tag{3.7}$$

This shows that  $w_p^{(s_{\pm})}(q; 1)$  is positive whenever  $w^{(s_{\pm})}(q; 1)$  is positive. This property will be used in what follows.

In many formulas, such as (3.6a), it is often convenient to omit the dependence on  $s$  in the notation of  $H$ . The right choice of  $s$  will be always clear from the context and is the same as for  $w^{(s)}$ . Furthermore, since the Bernoulli constant  $r$  will remain unchanged, we will often omit it from notations, such as for  $s_{\pm}$  or  $\mathcal{S}_{\pm}$ .

### 3.2. Subsequence Solutions

Let  $h \in C^{2,\gamma}(\bar{S})$  be a solution to (3.2) for some  $r > 0$  and let  $\mathcal{S}$  be the corresponding flow force constant. For an arbitrary sequence  $\{q_j\}_{j=1}^{\infty} \subset \mathbb{R}$ , possibly unbounded, we consider horizontal shifts

$$h_j(q, p) = h(q + q_j, p), \quad j \geq 1.$$

Thus, every function  $h_j$  solve the same problem (3.2) with the same Bernoulli constant. Now let  $\gamma' \in (0, \gamma)$  be given. Then the embedding  $C^{2,\gamma}(K) \hookrightarrow C^{2,\gamma'}(K)$  is compact for any compact subset  $K \subset \bar{S}$ . Because the norms  $\|h_j\|_{C^{2,\gamma}(\bar{S})}$  are uniformly bounded in  $j$ , we can find a subsequence  $\{h_{j_k}\}_{k=1}^{\infty}$  and a function  $\tilde{h} \in C^{2,\gamma'}(\bar{S})$  with the following property: for any compact  $K \subset \bar{S}$  restrictions of functions  $h_{j_k}$  to  $K$  converge to  $\tilde{h}|_K$  in  $C^{2,\gamma'}(K)$ . Then it is straightforward to show that  $\tilde{h}$  has the same regularity as  $h$ , that is  $\tilde{h} \in C^{2,\gamma}(\bar{S})$ . It is also clear that  $\tilde{h}$  solves the same elliptic problem (3.2) with the same Bernoulli constant  $r$  and has the same flow force constant  $\mathcal{S}$ . Note that if the convergence takes place for some  $\gamma'$  then it is true for all  $\gamma' \in (0, \gamma)$  by the interpolation. Such function  $\tilde{h}$  will be referred to as a subsequence solution of  $h$ . This terminology will be useful in order to avoid multiple repetitions of the argument with subsequences as above. Let us give an explicit definition.

**Definition 3.1.** Given two functions  $h, \tilde{h} \in C^{2,\gamma}(\bar{S})$  we say that  $\tilde{h}$  is a subsequence solution of  $h$  if there exists a sequence  $\{q_j\}_{j=1}^{\infty} \subset \mathbb{R}$  such that functions  $h_j(q, p) = h(q + q_j, p)$  converge to  $\tilde{h}$  in  $C^{2,\gamma'}(K)$  for all compact  $K \subset \bar{S}$  and all  $\gamma' \in (0, \gamma)$ .

Note that the definition is symmetric:  $\tilde{h}$  is a subsequence solution of  $h$  if and only if  $h$  is a subsequence solution of  $\tilde{h}$ . The following property of subsequence solutions will be useful in what follows.

**Proposition 3.1.** Let  $\tilde{h} \in C^{2,\gamma}(\bar{S})$  be a subsequence solution of  $h \in C^{2,\gamma}(\bar{S})$  and  $\hat{h} \in C^{2,\gamma}(\bar{S})$  is a subsequence solution of  $\tilde{h}$ . Then  $\hat{h}$  is a subsequence solution of  $h$ .

*Proof.* Let us fix  $\epsilon > 0$  and a bounded closed interval  $I$ . Then, because  $\hat{h}$  is a subsequence solution of  $\tilde{h}$  and one finds  $\hat{q} \in \mathbb{R}$  such that the function  $\tilde{h}(q + \hat{q})$  is close to  $\hat{h}(q)$  on  $I$  in  $C^{2,\gamma'}(I \times [0, 1])$ , that is

$$\|\tilde{h}(\cdot + \hat{q}) - \hat{h}(\cdot)\|_{C^{2,\gamma'}(I \times [0,1])} < \epsilon/2.$$

Now we consider functions  $\tilde{h}(\cdot + \hat{q})$  and  $h$ . It is clear that  $h$  is a subsequence solution of  $\tilde{h}(\cdot + \hat{q})$  and a similar argument gives  $\tilde{q} \in \mathbb{R}$  such that

$$\|h(\cdot + \tilde{q}) - \tilde{h}(\cdot + \hat{q})\|_{C^{2,\gamma'}(I \times [0,1])} < \epsilon/2.$$

Combining two inequalities together, we conclude that for any  $\epsilon > 0$  and any interval  $I$  there exists  $\tilde{q} \in \mathbb{R}$  such that

$$\|h(\cdot + \tilde{q}) - \hat{h}(\cdot)\|_{C^{2,\gamma'}(I \times [0,1])} < \epsilon.$$

This shows that  $\hat{h}$  is a subsequence solutions of  $h$ . □

### 3.3. General Bounds for Solutions

Let us recall a precise statement about bounds for the surface profile, mainly borrowed from [17], that will be used in our proofs.

**Proposition 3.2.** *Let  $(\psi, \eta)$  be as in Theorem 2.1 with  $r \in (R_c, R_0)$ . Then the following is true:*

- (i) *for all  $x \in \mathbb{R}$  we have  $\eta(x) > d_-(r)$ ; furthermore, if  $\check{\eta} = d_-(r)$ , then  $\mathcal{S} = \mathcal{S}_-(r)$ ;*
- (ii)  *$\hat{\eta} \geq d_+(r)$  and if the equality holds true, then  $\mathcal{S} = \mathcal{S}_+(r)$ ;*
- (iii)  *$\check{\eta} \leq d_+(r)$  and  $\check{\eta} = d_+(r)$  implies  $\mathcal{S} = \mathcal{S}_+(r)$ .*

*Proof.* The inequalities for the profile  $\eta(x) > d_-(r)$ ,  $\hat{\eta} \geq d_+(r)$  and  $\check{\eta} \leq d_+(r)$  are well known and were obtained in [17]. Here we only need to prove the remaining claims about the flow force constant.

First we assume that  $\check{\eta} = d_-(r)$ . Let  $h$  be the height function corresponding to  $\psi$ ; see Section 3.1. Then there exists a sequence  $\{q_j\}_{j=1}^\infty$  such that  $\eta(q_j) \rightarrow d_-(r)$  (recall that  $d_-(r) = d(s_+(r))$ ) and so

$$w^{(s_+)}(q_j, 1) \rightarrow 0$$

as  $j \rightarrow +\infty$ , where  $w^{(s_+)}$  is defined by (3.4) with  $s = s_+(r)$ . Let  $\tilde{w}$  be any subsequence solution of  $w$ , corresponding to the sequence  $\{q_j\}_{j=1}^\infty$ . Then  $\tilde{w}(0, 1) = 0$  by the construction, while  $\tilde{w} \geq 0$  in  $S$ . The latter is because  $w^{(s_+)} \geq 0$  in  $S$ , which follows from the inequality  $\check{\eta} \geq d_-(r)$  and the maximum principle. Next, we claim that  $\tilde{w} = 0$  identically in  $S$ . Indeed, if it is not the case, then the Hopf lemma would give  $\tilde{w}_p(0, 1) < 0$ . But this is in a contradiction with the relation (3.7), which holds for  $\tilde{w}$  instead of  $w^{(s_\pm)}$ . The identity computed at  $q = 0$  then would give  $\tilde{w}_p(0, 1) \geq 0$ , since  $\tilde{w}(0, 1) = \tilde{w}_q(0, 1) = 0$ . Thus, we have proved that  $\tilde{w} = 0$  in  $S$  so that its flow force constant equals to  $\mathcal{S}_-(r)$ . But functions  $\tilde{w}$  and  $w^{(s_+)}$  have same flow force constants and then  $\mathcal{S} = \mathcal{S}_-(r)$ .

A similar argument with subsequence solutions works for the cases  $\hat{\eta} = d_+(r)$  and  $\check{\eta} = d_+(r)$ . □

**Proposition 3.3.** *Let  $h \in C^{2,\gamma}(\bar{S})$  be a solution to (3.2) with  $r \in (R_c, R_0)$ . Then there exists a stream solution  $H(p; s)$  with  $s \in (s_0, s_-(r))$  such that  $\sup_{q \in \mathbb{R}} h_p(q, 0) < H_p(0; s)$ .*

*Proof.* If  $s_0 = 0$ , then the statement is trivial, since then  $H_p(0; s) = 1/s \rightarrow +\infty$  as  $s \rightarrow s_0$ . If  $d_0 = +\infty$ , then we choose  $s \in (s_0, s_-(r))$  sufficiently close to  $s_0$  so that  $d(s)$  is large enough to ensure  $w^{(s)} < 0$  on  $p = 0$ . Then  $w^{(s)} < 0$  everywhere in  $S$  by the maximum principle and so  $w_p^{(s)} < 0$  on  $p = 0$  by the Hopf lemma as desired. The remaining case is when  $s_0 > 0$  and  $d_0 < +\infty$ . Then

$$\max_{[0,1]} \Omega(p) = \Omega(1) > 0.$$

Here we used the inequality  $d_0 < +\infty$ , which ensures that the maximum of  $\Omega$  can not be attained at an inner point of the interval. Indeed, otherwise we would find

$$d_0 = \int_0^1 \frac{1}{\sqrt{2\Omega(p_0) - 2\Omega(p)}} dp = +\infty.$$

Therefore, we can conclude that

$$H_p(1; s) = \frac{1}{\sqrt{s^2 - 2\Omega(1)}} \rightarrow +\infty$$

as  $s \rightarrow s_0+ = \sqrt{2\Omega(1)}$ . Then we can choose  $s \in (s_0, s_-(r))$  for which  $w_p^{(s)} < 0$  on  $p = 1$ . In particular, the maximum of  $w^{(s)}$  in  $S$  can not be attained on  $p = 1$ . Thus, it is attained at the bottom and then  $w_p^{(s)} < 0$  on  $p = 0$  as desired. This finishes the proof. □

### 3.4. Flow Force Flux Functions

Our aim is to extract some information by comparing the flow force constant  $\mathcal{S}$  of a given solution with different flow force constants  $S(s)$  of stream solutions. We start by computing the difference

$$\begin{aligned}
 \mathcal{S} - S(s) &= \int_0^1 \left( \frac{1 - h_q^2}{2h_p^2} - h - \Omega + \Omega(1) + r \right) h_p dp \\
 &\quad - \int_0^1 \left( \frac{1}{2H_p^2} - H - \Omega + \Omega(1) + R \right) H_p dp \\
 &= \int_0^1 \left( \frac{1 - h_q^2}{2h_p^2} - \frac{1}{2H_p^2} - h + H + r - R \right) H_p \\
 &\quad + \int_0^1 \left( \frac{1 - h_q^2}{2h_p^2} - h - \Omega + \Omega(1) + r \right) w_p \\
 &= \int_0^1 \left( -\frac{w_p(h_p + H_p)}{2h_p^2 H_p} - wH_p + (r - R)H_p - \frac{w_q^2 H_p}{2h_p^2} \right) dp \\
 &\quad + \int_0^1 \left( \frac{w_p - h_q^2 w_p}{2h_p^2} - hw_p - \Omega w_p + \Omega(1)w_p + rw_p \right) dp \\
 &= \int_0^1 \left( -\frac{w_p}{2h_p H_p} - \frac{w_q^2}{2h_p} \right) dp \\
 &\quad + \int_0^1 (-hw_p - \Omega w_p + \Omega(1)w_p + rw_p - wH_p + (r - R)H_p) dp \\
 &= \int_0^1 \left( \frac{w_p^2}{2h_p H_p^2} - \frac{w_q^2}{2h_p} \right) dp \\
 &\quad + \int_0^1 \left( -\frac{w_p}{2H_p^2} - hw_p - \Omega w_p + \Omega(1)w_p + rw_p - wH_p + (r - R)H_p \right) dp \\
 &= I_1 + I_2
 \end{aligned}$$

Note that

$$\frac{1}{2H_p^2} + \Omega = s^2,$$

so that we can integrate all terms in  $I_2$ . This gives

$$\begin{aligned}
 I_2 &= [-ws^2 - dw - \frac{1}{2}w^2 + \Omega(1)w + rw + (r - R)d]_{p=1} \\
 &= [-\frac{1}{2}w^2 + (r - R)w + (r - R)d]_{p=1}.
 \end{aligned}$$

Here we used the Bernoulli equation for  $H$ , which reads as

$$\frac{1}{2}s^2 - \Omega(1) + d = R.$$

Therefore, we obtained the identity

$$\mathcal{S} - S(s) = -\frac{1}{2}w^2|_{p=1} + (r - R)w|_{p=1} + (r - R)d + \int_0^1 \left( \frac{w_p^2}{2h_p H_p^2} - \frac{w_q^2}{2h_p} \right) dp,$$

which is valid for an arbitrary stream solution  $H = H(p; s)$ . Let us move the term  $(r - R)d$  to the left, so that the right-hand side consists only of terms with  $w$ . Thus, denoting

$$\sigma(s; r) = S(s) + (r - R(s))d(s),$$

we conclude

$$2(\mathcal{S} - \sigma(s; r)) = 2(r - R(s))w^{(s)}(q, 1) - (w^{(s)}(q, 1))^2 + \int_0^1 \left( \frac{(w_p^{(s)})^2}{h_p H_p^2} - \frac{(w_q^{(s)})^2}{h_p} \right) dp. \tag{3.8}$$

Now we can define the flow force function

$$\Phi^{(s)}(q, p) = \int_0^p \left( \frac{(w_p^{(s)}(q, p'))^2}{h_p(q, p')(H_p(p'; s))^2} - \frac{(w_q^{(s)}(q, p'))^2}{h_p(q, p')} \right) dp'. \tag{3.9}$$

One of the key ideas of the proof is that every function  $\Phi^{(s)}$  solves an elliptic problem and therefore is subject to the maximum principle, while its boundary values are related to the quantity  $\mathcal{S} - \sigma(s; r)$  via (3.8). Together with certain properties of  $\sigma(s; r)$  this will allow us to prove Theorem 2.1.

Let us compute the first derivatives of  $\Phi^{(s)}$ . For this purpose we find

$$\begin{aligned} \Phi_{qp}^{(s)} &= \frac{2w_p^{(s)}w_{qp}^{(s)}}{h_p H_p^2} - \frac{(w_p^{(s)})^2 w_{qp}^{(s)}}{h_p^2 H_p^2} - 2 \left( \frac{w_q^{(s)}}{h_p} \right)_q w_q^{(s)} - \frac{(w_q^{(s)})^2 w_{qp}^{(s)}}{h_p^2} \\ &= -w_{qp} \left( \frac{1 + (w_q^{(s)})^2}{h_p^2} - \frac{1}{H_p^2} \right) - 2 \left( \frac{w_q^{(s)}}{h_p} \right)_q w_q^{(s)}. \end{aligned}$$

Now we use the identity

$$2 \left( \frac{w_q^{(s)}}{h_p} \right)_q = \left( \frac{1 + (w_q^{(s)})^2}{h_p^2} - \frac{1}{H_p^2} \right),$$

which follows directly from (3.2b) if one take the difference of two equations for  $h$  and  $H$  respectively. Thus, we conclude

$$\Phi_{qp}^{(s)} = - \left[ w_q^{(s)} \left( \frac{1 + (w_q^{(s)})^2}{h_p^2} - \frac{1}{H_p^2} \right) \right]_p$$

and so

$$\Phi_q^{(s)} = -w_q^{(s)} \left( \frac{1 + (w_q^{(s)})^2}{h_p^2} - \frac{1}{H_p^2} \right), \quad \Phi_p^{(s)} = \frac{(w_p^{(s)})^2}{h_p H_p^2} - \frac{(w_q^{(s)})^2}{h_p}. \tag{3.10}$$

The same computation was made in [19] for  $s = s_-(r)$ .

Now we can state

**Proposition 3.4.** *There exist functions  $b_1, b_2 \in L^\infty(S)$  such that*

$$\frac{1 + h_q^2}{h_p^2} \Phi_{pp}^{(s)} - 2 \frac{h_q}{h_p} \Phi_{qp}^{(s)} + \Phi_{qq}^{(s)} + b_1 \Phi_q^{(s)} + b_2 \Phi_p^{(s)} = 0 \quad \text{in } S. \tag{3.11}$$

Furthermore,  $\Phi^{(s)}$  satisfies the boundary conditions

$$\Phi^{(s)} = 2(\mathcal{S} - \sigma(s; r)) - 2(r - R(s))w^{(s)}(q, 1) + (w^{(s)}(q, 1))^2 \quad \text{for } p = 1, \tag{3.12a}$$

$$\Phi^{(s)} = 0 \quad \text{for } p = 0. \tag{3.12b}$$

In the irrotational case  $b_1, b_2 = 0$  and (3.11) is equivalent to the Laplace equation.

A proof of this fact was given in [19, Proposition 3.1], where functions  $b_1$  and  $b_2$  were computed explicitly.

### 3.5. Auxiliary Functions $\sigma$ and $\kappa$

For a given  $r > R_c$  and  $s > s_0$  we already defined

$$\sigma(s; r) = \int_0^1 \left( \frac{1}{2H_p^2(p; s)} - H(p; s) - \Omega(p) + \Omega(1) + r \right) H_p(p; s) dp. \tag{3.13}$$

This expression coincides with the flow force constant for  $H(p; s)$ , but with the Bernoulli constant  $R(s)$  replaced by  $r$ . We also note that

$$\sigma(s_{\mp}(r); r) = \mathcal{S}_{\pm}(r).$$

The key property of  $\sigma(s; r)$  is stated below.

**Lemma 3.5.** *For a given  $r \in (R_c, R_0)$  the function  $s \mapsto \sigma(s; r)$  increases for  $s \in (s_0, s_-(r))$ , decreases for  $s \in (s_-(r), s_+(r))$  and increases to infinity for  $s \in (s_+(r), +\infty)$ .*

*Proof.* Because

$$H_p(p; s) = \frac{1}{\sqrt{s^2 - 2\Omega(p)}}, \quad \partial_s H_p(p; s) = -sH_p^3(p; s),$$

we can compute the derivative

$$\begin{aligned} \sigma_s(s; r) &= \int_0^1 \left( \frac{1}{2H_p^2(p; s)} - H(p; s) - \Omega(p) + \Omega(1) + r \right) \partial_s H_p(p; s) dp \\ &+ \int_0^1 \left( -\frac{\partial_s H_p(p; s)}{H_p^3(p; s)} - \partial_s H(p; s) \right) H_p(p; s) dp \\ &= \int_0^1 \left( -\frac{1}{2H_p^2(p; s)} - \Omega(p) + \Omega(1) + r \right) \partial_s H_p(p; s) dp - d(s)d'(s) \\ &= \int_0^1 \left( -\frac{1}{2}s^2 + \Omega(1) + r \right) \partial_s H_p(p; s) dp - d(s) \int_0^1 \partial_s H_p(p; s) dp \\ &= -s(r - R(s)) \int_0^1 H_p^3(p; s) dp. \end{aligned}$$

Finally, because  $R(s) < r$  for  $s_-(r) < s < s_+(r)$  and  $R(s) > r$  for  $s > s_+(r)$  or  $s < s_-(r)$  we obtain the statement of the lemma. □

Our function  $\sigma(s; r)$  and its role is similar to the function  $\sigma(h)$  introduced by Keady and Norbury in [9]. The main purpose of the latter is to be used for a comparison with the flow force constant  $\mathcal{S}$ . The following functional will be also involved in our analysis.

$$\kappa(s; r) = 2(\mathcal{S} - \sigma(s; r)) - (r - R(s))^2. \tag{3.14}$$

A direct computation gives

$$\begin{aligned} \partial_s \kappa(s; r) &= -2\partial_s \sigma(s; r) + 2(r - R(s))R'(s) \\ &= 2s(r - R(s)) \int_0^1 H_p^3(p; s) dp + 2(r - R(s))(s + d'(s)) \\ &= 2s(r - R(s)). \end{aligned}$$

Thus, we obtain

**Lemma 3.6.** *For a given  $r \in (R_c, R_0)$  the function  $s \mapsto \kappa(s; r)$  decreases for  $s \in (s_0, s_-(r))$ , increases for  $s \in (s_-(r), s_+(r))$  and decreases to minus infinity for  $s \in (s_+(r), +\infty)$ .*

The meaning of  $\kappa$  is explained below.

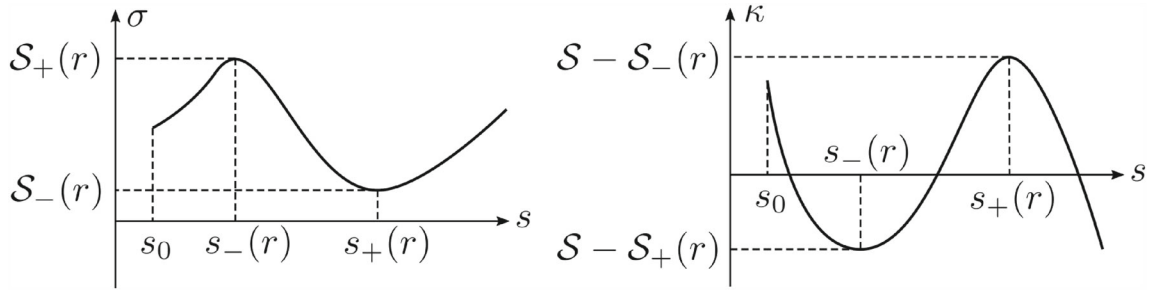


FIG. 4. Graphs of functions  $\sigma(s; r)$  and  $\kappa(s; r)$

**Proposition 3.7.** *Let  $h \in C^{2,\gamma}(\bar{S})$  be a solution to (3.2) with  $r > R_c$ . Assume that the flow force flux function  $\Phi^{(s)}$  for some  $s > s_0$  satisfies  $\inf_{q \in \mathbb{R}} \Phi^{(s)}(q; 1) \leq 0$ . Then*

$$\inf_{q \in \mathbb{R}} \Phi^{(s)}(q; 1) = \kappa(s; r) \tag{3.15}$$

and there exists a minimizing sequence  $\{q_j\}_{j=1}^\infty$  for which  $\Phi^{(s)}(q_j; 1) \rightarrow \kappa(s; r)$  as  $j \rightarrow +\infty$  and

$$\lim_{j \rightarrow +\infty} w^{(s)}(q_j, 1) = r - R(s). \tag{3.16}$$

Here  $\kappa(s; r)$  is defined by (3.14).

A similar property was proved in [20, Proposition 2.3] but without claim about the minimizing sequence. Therefore, we give a complete argument here.

*Proof.* First, we assume that the infimum is attained at some point  $(q_0; 1)$ , where  $\Phi_q^{(s)}(q_0; 1) = 0$ . Differentiating the boundary condition (3.12a), we find

$$\Phi_q^{(s)}(q_0, 1) = 2w_q^{(s)}(q_0, 1)(w^{(s)}(q_0, 1) - (r - R(s))) = 0. \tag{3.17}$$

Because  $\Phi^{(s)}$  attains its global minimum at  $(q_0, 1)$ , then the maximum principle and the Hopf lemma give  $\Phi_p^{(s)}(q_0, 1) < 0$ . In particular, we find that  $w_q^{(s)}(q_0, 1) \neq 0$  by the second formula (3.10). Thus, we necessarily obtain from (3.17) that  $w^{(s)}(q_0, 1) = (r - R(s))$ . Using this equality in (3.12a), we conclude (4.1) and (3.16) as required.

Now we assume that the infimum is attained over a sequence  $\{q_j\}_{j=1}^\infty$ . Let  $\hat{h}$  be any subsequence solution of  $h$  corresponding to  $\{q_j\}_{j=1}^\infty$ . Then we put  $\hat{w} = \hat{h} - H$  and define  $\hat{\Phi}$  by (3.9), where  $h$  and  $w$  are replaced by  $\hat{h}$  and  $\hat{w}$  respectively. Then  $\hat{\Phi}$  is a subsequence solution of  $\Phi^{(s)}$  and so

$$\inf_S \hat{\Phi} = \hat{\Phi}(0, 1) \leq 0.$$

Then the same argument as before gives

$$\hat{\Phi}(0, 1) = \kappa(s; r), \quad \hat{w}(0, 1) = (r - R(s)).$$

This immediately gives (4.1), while (3.16) follows from Definition 3.1. □

#### 4. Proof of Theorem 2.1

Throughout this section we assume that given functions  $\psi$  and  $\eta$  satisfy assumptions of the theorem. Let  $h$  be the corresponding height function defined in Section 3.1. Furthermore, let  $r$  and  $\mathcal{S}$  be the corresponding Bernoulli and flow force constants respectively. Our aim is to prove the inequalities

$$\mathcal{S}_-(r) \leq \mathcal{S} \leq \mathcal{S}_+(r)$$

and investigate when they turn into equalities. First we will prove the lower bound by applying some maximum principle arguments to properly chosen flow force flux functions. In a similar way we will argue when proving the upper bound. The case  $\mathcal{S} = \mathcal{S}_+(r)$  is simpler and will be included there. Finally we will deal with the most interesting case  $\mathcal{S} = \mathcal{S}_-(r)$ . It is more delicate and more technical. Our argument there is essentially based on an application of Lemma 4.2, which is the key point of the proof.

### 4.1. Proof of the Lower Bound

Note that the first claim of Proposition 3.2 allows us to assume  $\check{\eta} > d_-(r)$ , because otherwise  $\mathcal{S} = \mathcal{S}_-(r)$  and we have nothing to prove. In fact we are going to prove a stronger statement given below.

**Proposition 4.1.** *Under assumptions of Theorem 2.1 and assuming that  $\check{\eta} > d_-(r)$ , we have  $\mathcal{S} > \mathcal{S}_-(r)$ .*

*Proof.* Assume the contrary, that  $\mathcal{S} \leq \mathcal{S}_-(r)$ . In this case we have

$$\inf_{\mathbb{R}} \Phi^{(s_+(r))}(q, 1) > 0. \tag{4.1}$$

If the latter is not true, then the infimum above is nonpositive and Proposition 3.7 gives a sequence of points  $\{q_j\}_{j=1}^\infty$  for which

$$\lim_{j \rightarrow +\infty} w^{(s_+(r))}(q_j, 1) = r - R(s_+(r)) = 0,$$

which contradicts to the assumption  $\check{\eta} > d_-(r)$ .

Now we show that for some  $s \in (s_-(r), s_+(r))$  the function  $\Phi^{(s)}$  attains negative values somewhere along the upper boundary  $p = 1$ . Indeed, by Proposition 3.2 we have  $d_-(r) < \check{\eta} \leq d_+(r)$  and then we can find  $s_\star \in [s_-(r), s_+(r))$  such that

$$H(1; s_\star) = \inf_{\mathbb{R}} h(q, 1).$$

Note that we necessarily have  $s_\star > s_-(r)$  because otherwise  $\check{\eta} = d_+(r)$  and  $\mathcal{S} = \mathcal{S}_+(r)$  by Proposition 3.2 (iii), which is impossible, since  $\mathcal{S} \leq \mathcal{S}_-(r) < \mathcal{S}_+(r)$ .

The choice of  $s_\star$  and the boundary relation (3.12a) show that  $\inf_{\mathbb{R}} \Phi^{(s_\star)}(q, 1) \leq \mathcal{S} - \sigma(s_\star; r)$ . But according Lemma 3.5 (see Fig. 4) we have  $\sigma(s_\star; r) > \sigma(s_+(r); r) = \mathcal{S}_-(r)$  and  $\mathcal{S}_-(r) \geq \mathcal{S}$  by the assumption. Thus,  $\inf_{\mathbb{R}} \Phi^{(s_\star)}(q, 1) < 0$ . Finally, by the continuity we find  $s_\dagger \in (s_-(r), s_+(r))$  such that

$$\inf_{\mathbb{R}} \Phi^{(s_\dagger)}(q, 1) = 0.$$

Then  $\kappa(s_\dagger; r) = 0$  by Proposition 3.7 and the definition of  $\kappa$  gives  $\mathcal{S} - \sigma(s_\dagger; r) \geq 0$ . But  $\sigma(s_\dagger; r) > \mathcal{S}_-(r)$  by Lemma 3.5 and we arrive to a contradiction with the assumption  $\mathcal{S}_-(r) \geq \mathcal{S}$ . Therefore, we proved that  $\mathcal{S} > \mathcal{S}_-(r)$ . □

### 4.2. Proof of the Upper Bound

We aim to prove the strict inequality  $\mathcal{S} < \mathcal{S}_+(r)$ , provided  $h$  defines a non-laminar solution. Assume the contrary, that  $\mathcal{S} \geq \mathcal{S}_+(r)$ . Then for any  $s \in (s_0, s_+(r))$  the function  $\Phi^{(s)}$  is nonnegative in  $S$ . Indeed, if for some  $s \in (s_0, s_+(r))$  we assume that

$$\inf_{\mathbb{R}} \Phi^{(s)}(q, 1) < 0,$$

then  $\kappa(s; r) < 0$  by Proposition 3.7. On the other hand, Lemma 3.6 (see Fig. 4) shows that  $\kappa \geq 0$  for all  $s \in (s_0, s_+(r))$ , since  $\mathcal{S} \geq \mathcal{S}_+(r)$  by the assumption. Thus, all functions  $\Phi^{(s)}$  are nonnegative for all  $s \in (s_0, s_+(r))$ .

Note that since  $w^{(s_+(r))}$  is nonnegative by Proposition 3.2, then the Hopf lemma ensures  $w^{(s_+(r))}(q, 0) > 0$  for all  $q \in \mathbb{R}$ . Now by using Proposition 3.3 we can find  $s_* \in (s_0, s_+(r))$  such that

$$w_p^{(s_*)}(q_*, 0) = 0.$$

for some  $q_* \in \mathbb{R}$ . But we already know that  $\Phi^{(s_*)}$  is nonnegative and then

$$\Phi_p^{(s_*)}(q_*, 0) = \frac{[w_p^{(s_*)}(q_*, 0)]^2}{h_p H_p^2(0; s_*)} > 0$$

by the Hopf lemma, leading to a contradiction, until  $h$  is not a laminar solution. Thus, we have proved the strict inequality  $\mathcal{S} < \mathcal{S}_+(r)$  for any non-laminar solution.

### 4.3. Analysis of the Case $\mathcal{S} = \mathcal{S}_-(r)$

Here we assume that  $\mathcal{S} = \mathcal{S}_-(r)$  and so  $\tilde{\eta} = d_-(r)$  by Proposition 4.1. Our aim is to prove that

$$\lim_{q \rightarrow +\infty} h(q, 1) = d_-(r). \tag{4.2}$$

Assume this is not true. Then we show the following.

**Lemma 4.2.** *There exists  $\delta > 0$  such that for any  $0 < \epsilon < \delta$  there is an interval  $[a, b]$  such that*

(1) *the functions  $w^{(s_+)}(q, 1)$  and  $\Phi^{(s_+)}(q, 1)(q, 1)$  are strictly positive on  $[a, b]$ , while also*

$$\min_{[a, b]} \Phi^{(s_+)}(q, 1) < \epsilon;$$

(2) *we have*

$$\Phi^{(s_+)}(a, p), \quad \Phi^{(s_+)}(b, p) \geq \delta p$$

*for all  $p \in [0, 1]$ ,*

*where the constant  $\delta$  is independent of  $\epsilon$ , while the interval  $[a, b]$  depends  $\epsilon$ .*

If the statement of the lemma is true, then we can find  $s_0 < s < s_+(r)$  sufficiently close to  $s_+(r)$  so that the function  $\Phi^{(s)}$  is non-negative on  $[a, b] \times [0, 1]$ , while

$$\min_{[a, b]} \Phi^{(s)}(q, 1) = 0.$$

Let  $q_0 \in [a, b]$  be the point where the minimum is attained. If  $\epsilon$  is sufficiently smaller than  $\delta$ , then in view of (2) we must have  $q_0 \in (a, b)$ . In this case

$$\Phi_q^{(s)}(q_0, 1) = 2w_q^{(s)}(q_0, 1)(w^{(s)}(q_0, 1) - (r - R(s))) = 0.$$

By the Hopf lemma we have  $\Phi_p^{(s)}(q_0, 1) < 0$ , which in view of (3.10) guarantees that  $w_q^{(s)}(q_0, 1) \neq 0$  and so

$$w^{(s)}(q_0, 1) = (r - R(s)).$$

Thus, the boundary relation (3.8) now implies that

$$\min_{[a, b]} \Phi^{(s)}(q, 1) = \kappa(s; r) = 0.$$

But Lemma 3.6 gives

$$0 = \kappa(s; r) < \kappa(s_+(r); r) = \mathcal{S} - \mathcal{S}_-(r) = 0,$$

leading to a contradiction.

This argument shows that (4.2) is true for an arbitrary solution with  $\mathcal{S} = \mathcal{S}_-(r)$ . In particular, (4.2) is valid for  $h(-q, p)$ . Then by [8, Theorem 3.1] we have that  $h$  defines a solitary wave of positive elevation, symmetric around the crest and monotone on the sides. Thus, it is left to prove Lemma 4.2. This is a



technical result that requires some additional facts about half-solitary waves. An exact statement will be given in the next section. Then we will complete our proof of Lemma 4.2 in Section 4.5.

#### 4.4. Asymptotics for Solitary Waves

A one-sided solitary wave solution to (2.5) is defined by an asymptotic relation

$$\lim_{x \rightarrow +\infty} \eta(x) = d. \tag{4.3}$$

Note that here we do not require any monotonicity property. For unidirectional waves (4.3) guarantees that the corresponding height function  $h$  has a subsequence solution, which is a laminar flow with the depth  $d$ . In particular, this requires

$$d = d_-(r) \text{ or } d = d_+(r),$$

where  $r$  is the Bernoulli constant. It was recently proved in [19] that the assumption (4.3) requires  $d = d_-(r)$ ; that is all solitary waves are subcritical (supported by subcritical laminar flows  $H(p; s)$  with  $s > s_c$ ). Furthermore, one verifies that

$$h(q, p) \rightarrow H(p; s_+(r)), \quad h_q(q, p) \rightarrow 0, \quad q \rightarrow \pm\infty \tag{4.4}$$

in  $C^{2,\gamma'}([0, 1])$  and  $C^{1,\gamma'}([0, 1])$  respectively for all  $\gamma' \in (0, \gamma)$ , provided  $h \in C^{2,\gamma}(\bar{S})$ . Asymptotics (4.4) show that  $\mathcal{S} = \mathcal{S}_-(r)$ , which follows from (3.3) by passing to the limit  $q \rightarrow +\infty$ .

In order to obtain higher order asymptotics for  $h$ , we need to introduce the following eigenvalue problem:

$$-\left(\frac{\varphi_p}{H_p^3}\right)_p = \mu \frac{\varphi}{H_p}, \quad p \in [0, 1], \tag{4.5a}$$

where  $H = H(p; s_+(r))$  and the eigenfunction  $\varphi(p)$  is subject to the boundary conditions

$$\varphi(0) = 0, \quad \varphi_p(1) = H_p^3(1)\varphi(1). \tag{4.5b}$$

This Sturm-Liouville problem arises as a form of the dispersion relation; see [6]. The first eigenvalue  $\mu_1 = \lambda_1^2 > 0$  is always positive, provided  $H = H(p; s_+(r))$  is a supercritical stream solution. This suggests that the difference  $h - H$  must decay as  $e^{-\lambda_1 q}$  as  $q \rightarrow +\infty$ . A precise statement is given below.

**Proposition 4.3.** *Let  $h \in C^{2,\gamma}(\bar{S})$  be a non-laminar solution to (3.2) and satisfy  $\lim_{q \rightarrow +\infty} h(q, 1) = d_-(r)$ . Then*

$$h(q, p) = H(p; s_+(r)) + \xi \varphi_1(p) e^{-\lambda_1 q} + f(q, p) e^{-\lambda'_1 q}, \quad (q, p) \in S,$$

where  $\xi \neq 0$ ,  $\lambda'_1 > \lambda_1$  and  $f \in C^{2,\gamma}(\bar{S})$ .

These asymptotics were proved in [8], under an additional assumption  $\lim_{q \rightarrow -\infty} h(q, 1) = d_-(r)$ . However this is not essential and proofs from [8] are applicable with minor modifications, so we omit it here. The coefficient  $\xi$  is always positive because the function  $h(q, p) - H(p; s_+(r))$  is positive in  $S$ . Indeed, if  $\xi = 0$  then the leading order asymptotics is determined by another eigenfunction  $\phi_j$ ,  $j > 1$  of (4.5) that must change sign on  $[0, 1]$ , leading to a contradiction. Note that some exponential decay is always available due to Harnack principle and positivity of  $h(q, p) - H(p; s_+(r))$ .

#### 4.5. Proof of Lemma 4.2

Assuming (4.2) is violated we will prove that there is a non-laminar subsequence solution for which (4.2) is fulfilled. Throughout this section we use the notation

$$w(q, p) = h(q, p) - H(p; s_+(r)).$$

We first will prove the following auxiliary fact.

**Lemma 4.4.** *If (4.2) is not true there exists a non-laminar subsequence solution  $\hat{w}$  of  $w$  corresponding to a sequence  $\{\hat{q}_j\}_{j=1}^\infty$  accumulating at the positive infinity and such that  $\hat{w}(q, 1) \rightarrow 0$  as  $q \rightarrow +\infty$ .*

*Proof.* First we note that even so (4.2) is violated there exists a sequence  $q_j \rightarrow +\infty$  such that

$$w(q_j, 1) \rightarrow 0, \quad j \rightarrow +\infty. \tag{4.6}$$

This is because we have  $\check{\eta} = d_-(r)$ . On the other hand, by the assumption there is  $\rho > 0$  such that

$$\limsup_{q \rightarrow +\infty} w(q, 1) > \rho. \tag{4.7}$$

Furthermore, we can assume that  $\rho$  is small enough so that every small-amplitude subsequence solution  $\bar{h}$  with  $\|\bar{h} - d_-(r)\|_\infty \leq \rho$  is trivial. We also assume that  $w(0, 1) > \rho$  and  $w(q_j, 1) < \rho$  for all  $j \geq 1$ . Otherwise we can consider a function of the form  $w(q - q_0, p)$  for which this is true. Now let

$$I_j = (a_j, b_j), \quad a_j < q_j < b_j$$

be the largest interval containing  $q_j$  and such that  $w(q; 1) < \rho$  for all  $q \in I_j$ . Note that each  $I_j$  is bounded in view of *rho* and our assumptions. Furthermore, by the definition we find

$$w(q; 1) < \rho, \quad q \in I_j, \quad w(a_j, 1) = w(b_j, 1) = \rho$$

for all  $j \geq 1$ . On the other hand, we have

$$|I_j| = b_j - a_j \rightarrow +\infty, \quad j \rightarrow +\infty.$$

If this is not true, and the lengths  $|I_{j_k}|$  are bounded for some subsequence, then we can consider some subsequence solution  $w^\dagger$  corresponding to the sequence  $\{q_{j_k}\}_{k=1}^\infty$ . Then  $w^\dagger$  has to be a non-zero and nonnegative solution with  $w^\dagger(0, 1) = 0$  by (4.6). But this is impossible by Proposition 3.2 part (i).

Thus,  $|I_j| \rightarrow +\infty$  as  $j \rightarrow +\infty$  and we can consider a subsequence solution  $\hat{w}$  corresponding to the sequence  $\{a_j\}_{j=1}^\infty$ , where  $a_j$  is the left endpoint of  $I_j$ . Then  $\hat{w}$  must be nonnegative in  $S$  and  $\hat{w}(0, 1) = \rho$ , so it is not zero identically. Moreover, because  $|I_j| \rightarrow +\infty$  as  $j \rightarrow +\infty$  and  $w(q, 1) \leq \rho$  on each  $I_j$ , we conclude that

$$\hat{w}(q, 1) \leq \rho, \quad q > 0.$$

If (4.2) is not valid for  $\hat{w}$ , then we can find a non-zero subsequence solution  $\bar{w}$  of  $\hat{w}$  such that  $|\bar{w}(q, 1)| \leq \rho$  for all  $q \in \mathbb{R}$ . But that would be in a contradiction with the choice of  $\rho$ . This finishes the proof of Lemma 4.4.  $\square$

Now we are ready to give a proof of Lemma 4.2. Let  $\hat{w}$  be the decaying subsequence solution provided by Lemma 4.4. Then by Proposition 4.3 we have

$$\hat{w}(q, p) = \xi \varphi_1(p) e^{-\lambda_1 q} + f(q, p) e^{-\lambda'_1 q}. \tag{4.8}$$

Let  $\hat{\Phi}$  be the flow force flux function corresponding to  $\hat{w}$ . Then using (4.8) in (3.9) we obtain

$$\hat{\Phi}(q, p) = \xi^2 \frac{\varphi'_1(p) \varphi_1(p)}{H_p^3(p; s_+(r))} e^{-2\lambda_1 q} + g(q, p) e^{-(\lambda_1 + \lambda'_1) q},$$

where  $g$  is smooth and bounded. Thus, we can choose  $q_0 > 0$  such that

$$\hat{\Phi}(q_0, p) \geq A p e^{-2\lambda_1 q_0} \tag{4.9}$$

for all  $p \in [0, 1]$ . Here  $A > 0$  is a constant that depends only on  $\xi$  and  $\phi_1$  and is independent of  $q_0$ . Now we put

$$\delta = \frac{1}{2} A e^{-2\lambda_1 q_0}.$$

Let  $q_j \rightarrow +\infty$  be the sequence such that functions  $h(q + q_j, p)$  converge to  $\hat{h} = \hat{w} + H$  on every compact subset of  $S$ . In particular, functions  $\Phi(q + q_j, p)$  converge to  $\hat{\Phi}(q, p)$  to  $C^2([-2q_0, 2q_0] \times [0, 1])$  as  $j \rightarrow +\infty$ . This together with (4.9) allow us to choose some large enough  $j_1$  so that

$$\Phi(q_0 + q_{j_1}, p) \geq \delta p, \quad p \in [0, 1].$$

Now let  $\epsilon > 0$  be given. Then because  $\check{\eta} = d_-(r)$  we can find some  $q_* > q_0 + q_{j_1}$  such that  $w(q_*, 1) < \sqrt{\epsilon}$ . After that we choose another integer  $j_2$  so that  $q_0 + q_{j_2} > q_*$  and

$$\Phi(q_0 + q_{j_2}, p) \geq \delta p, \quad p \in [0, 1].$$

This is possible because  $q_j \rightarrow +\infty$ , provided (4.2) is violated. Finally, it is left to put  $a = q_0 + q_{j_1}$  and  $b = q_0 + q_{j_2}$ . Then the claim (2) of Lemma 4.2 is true by the construction. Furthermore, the second claim (2) holds true since  $\Phi = w^2$  on  $p = 1$  by (3.8). This finishes our proof of Lemma 4.2.

**Funding** Open access funding provided by Lund University.

## Declarations

**Conflict of interest** The author have no conflict of interest.

**Open Access.** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## References

- [1] Baensens, C., Mackay, R.S.: Uniformly travelling water waves from a dynamical systems viewpoint: some insights into bifurcations from Stokes' family. *J. Fluid Mech.* **241**, 333–347 (1992)
- [2] Benjamin, T. B.: *On cnoidal waves and bores*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 224, pp. 448–460 (1954)
- [3] Benjamin, T.B.: Impulse, flow force and variational principles. *IMA J. Appl. Math.* **32**, 3–68 (1984)
- [4] Benjamin, T.B.: Verification of the Benjamin-Lighthill conjecture about steady water waves. *J. Fluid Mech.* **295**, 337–356 (1995)
- [5] Constantin, A.: *Nonlinear water waves with applications to wave-current interactions and tsunamis*, vol. 81 of CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (2011)
- [6] Constantin, A., Strauss, W.: Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.* **57**, 481–527 (2004)
- [7] Dubreil-Jacotin, M.L.: Sur la détermination rigoureuse des ondes permanentes périodiques d'ampleur finite. *J. Math. Pures Appl.* **13**, 217–291 (1934)
- [8] Hur, V.M.: Symmetry of steady periodic water waves with vorticity. *Philos. Trans. R. Soc. Lond. Ser. A* **365**, 2203–2214 (2007)
- [9] Keady, G., Norbury, J.: Water waves and conjugate streams. *J. Fluid Mech.* **70**, 663–671 (1975)
- [10] Keady, G., Norbury, J.: On the existence theory for irrotational water waves. *Math. Proc. Cambridge Philos. Soc.* **83**, 137–157 (1978)
- [11] Keady, G., Norbury, J.: Waves and conjugate streams with vorticity. *Mathematika* **25**, 129–150 (1978)
- [12] Kozlov, V., Kuznetsov, N.: Bounds for arbitrary steady gravity waves on water of finite depth. *J. Math. Fluid Mech.* **11**, 325–347 (2007)
- [13] Kozlov, V., Kuznetsov, N.: The Benjamin-Lighthill conjecture for near-critical values of Bernoulli's constant. *Arch. Ration. Mech. Anal.* **197**, 433–488 (2009)
- [14] Kozlov, V., Kuznetsov, N.: Fundamental bounds for steady water waves. *Math. Ann.* **345**, 643–655 (2009)
- [15] Kozlov, V., Kuznetsov, N.: The Benjamin-Lighthill conjecture for steady water waves (revisited). *Arch. Ration. Mech. Anal.* **201**, 631–645 (2011)
- [16] Kozlov, V., Kuznetsov, N.: Bounds for steady water waves with vorticity. *J. Differential Equations* **252**, 663–691 (2012)
- [17] Kozlov, V., Kuznetsov, N., Lokharu, E.: On bounds and non-existence in the problem of steady waves with vorticity. *J. Fluid Mech.* **765**, R1 (2015)
- [18] Kozlov, V., Kuznetsov, N., Lokharu, E.: On the Benjamin-Lighthill conjecture for water waves with vorticity. *J. Fluid Mech.* **825**, 961–1001 (2017)

- [19] Kozlov, V., Lokharu, E., Wheeler, M.H.: Nonexistence of subcritical solitary waves. *Arch. Ration. Mech. Anal.* **241**, 535–552 (2021)
- [20] Lokharu, E.: On bounds for steady waves with negative vorticity. *J. Math. Fluid Mech.* **23**, 1–11 (2021)
- [21] Lokharu, E.: On the amplitude and the flow force constant of steady water waves. *J. Fluid Mech.* **921**, A2 (2021)
- [22] Vitolo, A.: A note on the maximum principle for second-order elliptic equations in general domains. *Acta Mathematica Sinica, English Series* **23**, 1955–1966 (2007)

Evgeniy Lokharu  
Centre for Mathematical Sciences  
Lund University  
223 62 Lund  
Sweden  
e-mail: evgeniy.lokharu@math.lu.se

(accepted: January 27, 2024)