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Anomalous Energy Flux in Critical L^p-Based Spaces

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Abstract. We construct a three-dimensional vector field that exhibits positive energy flux at every Littlewood–Paley shell and has the best possible regularity in L^p-based spaces, $p \leq 3$; in particular, it belongs to $H^{\left(\frac{5}{6}\right)^-}$.

1. Introduction

1.1. Anomalous Dissipation: What is Known

In his famous 1949 work [1] Onsager conjectured energy dissipation 'without final assistance of viscosity' in fluids. In today's mathematical literature such phenomenon is referred to as 'anomalous dissipation' of solutions to the Euler equation. Onsager's original conjecture was stated in terms of Hölder spaces and is by now fully resolved: Constantin, E and Titi [2] proved conservation of energy of any weak Euler solution which is $C^{1/3+\varepsilon}$-regular in space, while Isett [3] constructed non-conservative $C^{1/3-\varepsilon}$ Euler solutions, using the convex integration technique developed by De Lellis and Székelyhidi [4]. For the truly dissipative case, see Buckmaster, De Lellis, Székelyhidi, and Vicol [5].

Both results remain valid if stated in terms of L^3-based Sobolev or Besov spaces, i.e. for $W^{1/3\pm\varepsilon,3}$ or $B^{1/3\pm\varepsilon}_{3,\infty}$: the dissipative one trivially, the conservative part was shown by Duchon and Robert [6]. The exponent 3 appears naturally via the cubic term in the energy balance equation, but less is known if the integrability is below 3, in particular for $H^s = W^{s,2}$. The best known result on the conservative side is the trivial one: by Sobolev embedding $H^{5/6+\varepsilon} \hookrightarrow W^{1/3+\varepsilon,3}$. On the non-conservative side, recently $H^{1/2-\varepsilon}$ has been reached in Buckmaster, Masmoudi, Novack, and Vicol [7] (which was later extended by Novack and Vicol [8]). While the exponent $1/2$ may be critical for full (h-principle-type) flexibility of Euler solutions, it is conjectured (cf. Problem 5 in the survey [9] by Buckmaster and Vicol) that $5/6$ is the threshold for energy conservation.

Conjecture 1. *For any $s < \frac{5}{6}$ there are weak solutions to the Euler equation in $C_t H^s_x$ with strictly decreasing kinetic energy.*

In this work we do not prove the conjecture, but provide evidence in its favour.

1.2. Energy Flux and Statement of the Main Result

The first step for proving conservation of energy is usually to regularize the Euler equation and to test it with the regularised solution:

$$\frac{d}{dt}\frac{1}{2}\int (S_q u)^2\, dx = \int S_q u \cdot \partial_t S_q u\, dx = -\int S_q u \cdot S_q\,(u \cdot \nabla u)\, dx. \tag{1}$$

Here S_q denotes any linear regularizing operator such that $S_q \to Id$ as $q \to \infty$; in our case it will be the Littlewood-Paley projection onto functions with frequencies $\lesssim 2^q$. Note that the pressure term vanishes due to incompressibility and linearity of S_q.

The second step is showing that the right hand side of (1), called the energy flux (towards high frequencies), vanishes as $q \to \infty$, which holds if u is a divergence free field with enough regularity; this step may be thus completely detached from the fact that u solves the Euler equation!

By (1), vanishing of the energy flux is a sufficient condition for conservation of energy. Even though it may not be a necessary condition, non-vanishing of the flux suggests an energy cascade as described by Richardson (cf. Frisch [10]), i.e. continuous transport of energy from low to high and higher frequency structures, which is in turn seen as the mechanism of anomalous dissipation. Therefore, constructing a non-vanishing energy flux of certain regularity is a strong premise for non-conservative Euler solutions with that regularity. Such flux construction, with certain further desired properties, is the subject of this note.

Let us define for a distributionally divergence-free vector field $u : \mathbb{T}^3 \to \mathbb{R}^3$

$$\Pi_q(u) = \int_{\mathbb{T}^3} S_q(u \otimes u) : \nabla S_q(u)\, dx,$$

where S_q denotes a Littlewood-Paley projection, to be precised later. The right hand side of (1), with S_q disambiguated as a Littlewood-Paley projection, after applying incompressibility and integration by parts, is precisely our $\Pi_q(u)$. With this definition, and the standard notion of Besov spaces $B^s_{p,r}$ (defined in Sect. 2) we state

Theorem 1. *For any $c \in \mathbb{R}$ and $\delta > 0$, there exists a divergence-free vector field $U \in B^{5/6}_{2,\infty}(\mathbb{T}^3)$ such that*

$$\liminf_{q\to\infty} \Pi_q(U) > c - \delta \qquad and \qquad \limsup_{q\to\infty} \Pi_q(U) < c + \delta. \tag{2}$$

In fact, $U \in B^{\frac{3}{p}-\frac{2}{3}}_{p,\infty}(\mathbb{T}^3)$ with any $p \in (1, \frac{9}{2})$ and $U \in \dot{B}^{\frac{3}{p}-\frac{2}{3}}_{p,\infty}(\mathbb{T}^3)$ with any $p \in (1,\infty]$.

1.3. Remarks and Comparison to Previous Results

(i) The theorem is sharp concerning regularity of U. Indeed, for any $p < \infty$ the flux $\Pi_q(U)$ vanishes for $q \to \infty$ for every $U \in B^{5/6}_{2,r}$. This follows from embedding into $B^{1/3}_{3,r}$ and Theorem 3.3 of Cheskidov, Constantin, Friedlander, and Shvydkoy [11].

(ii) A proper limit seems possible with a more careful choice of parameters. We did not pursue this here, as the main objective was obtaining a positive lim inf for critical spaces in three space dimensions.

(iii) Our construction is strongly inspired by Cheskidov, Lopes Filho, Nussenzveig Lopes, and Shvydkoy [12], which in turn develops ideas of Eyink [13]. Regularity-wise, a two dimensional sharp result is sketched in [12]. The vector field $\mathcal{U}$ described there enjoys $\limsup_{q\to\infty} \Pi_q(\mathcal{U}) \neq 0$, while $\liminf_{q\to\infty} \Pi_q(\mathcal{U}) = 0$. In fact, [12] points at Cheskidov and Shvydkoy [14] for details of the construction aimed at sharp regularity for $p < 3$. A straightforward attempt to fill in those details may result in non-zero values appearing extremely sparsely in the sequence $\Pi_q(\mathcal{U})$, which means that 'typically' $\Pi_q(\mathcal{U}) = 0$. This is unsatisfactory from the perspective of energy cascade. Indeed, any $\Pi_q(\mathcal{U}) \approx 0$ disrupts the transport of energy, which is predominantly local (from lower to slightly higher frequencies; this locality remains in accordance with turbulence literature). Hence there cannot be a continuous cascade related to $\mathcal{U}$.

(iv) Another approach towards anomalous dissipation is proposed by Cheskidov and Luo [15]: the authors construct almost fully intermittent vector fields in 3 or more dimensions, which are not stationary but instead use time as another degree of freedom. More precisely, the construction features a 'time-delayed' energy cascade and the definition of energy flux involves averaging in time. This is a very interesting approach, especially towards possible blow-up scenarios, but it does not satisfy the energy cascade heuristics sketched above.

1.4. Organisation

In Sect. 2 we detail the construction of the vector field U and gather its properties. In Sect. 3 we decompose the energy flux of U, which facilitates computations of the final Sect. 4, where the proof of Theorem 1 is concluded.

2. The Construction

Take a smooth function

$$\varphi : [0, \infty) \to [0, 1], \quad \varphi(\xi) = \begin{cases} 1 & \xi \leq \frac{\sqrt{5}}{2} + 2\epsilon \\ 0 & \xi \geq 2 - 4\epsilon, \end{cases}$$

with ϵ small enough, to be fixed later. Let $\psi(\xi) = \varphi(2\xi) - \varphi(\xi)$. The Littlewood-Paley low-frequency and dyadic projections are, respectively, $\widehat{S_q u}(k) := \varphi(\lambda_q^{-1}|k|)\hat{u}(k)$ and $\widehat{\Delta_q u}(k) := \psi(\lambda_q^{-1}|k|)\hat{u}(k)$, where $\lambda_q = 2^q$.

2.1. The Planar Construction

In this section we construct the planar vector field U_0 along [12]. Fix the plane $P_0 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \, x_3 = 0\}$. Define the frequency vectors

$$F_0^{\textcircled{1}} := (0, 1, 0), \ F_0^{\textcircled{2}} := (2, 0, 0), \ F_0^{\textcircled{3}} := (2, 1, 0)$$

and amplitude vectors

$$V_0^{\textcircled{1}} := (1, 0, 0), \ V_0^{\textcircled{2}} := (0, 1, 0), \ V_0^{\textcircled{3}} := (-1, 2, 0).$$

The planar 'skeleton field' S is the sum over $q \in 3\mathbb{N}$ of $s_q = s_q^{\textcircled{1}} + s_q^{\textcircled{2}} + s_q^{\textcircled{3}}$, where

$$s_q^{\textcircled{1}} := \lambda_q^{-\frac{1}{3}} V_0^{\textcircled{1}} \sin\left(\lambda_q F_0^{\textcircled{1}} \cdot x\right),$$

$$s_q^{\textcircled{2}} := \lambda_q^{-\frac{1}{3}} V_0^{\textcircled{2}} \cos\left(\lambda_q F_0^{\textcircled{2}} \cdot x\right),$$

$$s_q^{\textcircled{3}} := \lambda_q^{-\frac{1}{3}} V_0^{\textcircled{3}} \cos\left(\lambda_q F_0^{\textcircled{3}} \cdot x\right)$$

The skeleton field S suffices to obtain the lim sup result in $B_{3,\infty}^{\frac{1}{3}}(\mathbb{T}^3)$; in particular $\operatorname{div} S = 0$ thanks to $F_0^{\textcircled{1}} \cdot V_0^{\textcircled{1}} = 0$.

Since the Bernstein inequality implies the Sobolev inequality:

$$\|\Delta_q f\|_{L^3(\mathbb{T}^3)} \leq C\lambda_q^{\frac{1}{2}} \|\Delta_q f\|_{L^2(\mathbb{T}^3)} \implies \|f\|_{B_{3,\infty}^{\frac{1}{3}}(\mathbb{T}^3)} \leq C\|f\|_{B_{2,\infty}^{\frac{5}{6}}(\mathbb{T}^3)},$$

constructing $U \in B_{2,\infty}^{\frac{5}{6}}$ with a positive flux could be reached by replicating the interactions arising in S, while saturating the Bernstein inequality. This strategy is realised by a three-dimensional blurring of the skeleton frequencies, with the new amplitudes preserving solenoidality and being appropriately rescaled. To this end define the active regions ('blurs' of frequencies $\lambda_q F_0^{\textcircled{1}}$ that are 'active' in S) as follows:

$$A_{0,q}^{\textcircled{1}} := \mathbb{Z}^3 \cap \lambda_q \left(F_0^{\textcircled{1}} + [0, \epsilon]^3\right),$$

$$A_{0,q}^{\textcircled{2}} := \mathbb{Z}^3 \cap \lambda_q \left(F_0^{\textcircled{2}} + [0, \epsilon]^3\right),$$

$$A_{0,q}^{\textcircled{3}} := \mathbb{Z}^3 \cap \lambda_q \left(F_0^{\textcircled{3}} + [0, 2\epsilon]^3\right).$$

Define π_ξ to be the orthogonal projection onto the plane $\xi^\perp$. Note that $\pi_\xi(v)e^{i\xi\cdot x}$ is a divergence-free vector field and agrees with the classical Leray-Helmholtz projection of $ve^{i\xi\cdot x}$, since $\pi_\xi = \mathbf{1} - \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}$.

We can now define the components of $u_q = u_q^{\textcircled{1}} + u_q^{\textcircled{2}} + u_q^{\textcircled{3}}$

$$u_q^{\textcircled{1}} := \epsilon^{-2}\lambda_q^{-\frac{7}{3}} \sum_{\xi \in A_{0,q}^{\textcircled{1}}} \pi_\xi\left(V_0^{\textcircled{1}}\right) \sin\left(\xi \cdot x\right),$$

$$u_q^{\textcircled{2}} := \epsilon^{-2}\lambda_q^{-\frac{7}{3}} \sum_{\xi \in A_{0,q}^{\textcircled{2}}} \pi_\xi\left(V_0^{\textcircled{2}}\right) \cos\left(\xi \cdot x\right), \tag{3}$$

$$u_q^{\textcircled{3}} := \epsilon^{-2}\lambda_q^{-\frac{7}{3}} \sum_{\xi \in A_{0,q}^{\textcircled{3}}} \pi_\xi\left(V_0^{\textcircled{3}}\right) \cos\left(\xi \cdot x\right).$$

Finally, the (almost) planar vector field is $U_0 := \sum_{q\in 3\mathbb{N}} u_q$. It is real-valued and (weakly) divergence-free.

2.2. The Rotations and the Full Field U

The flux of U_0 is controlled only for $q \in 3\mathbb{N}$, thus merely a $\limsup$ result is possible. Straightforward 'condensation' $3\mathbb{N} \to \mathbb{N}$ results however in undesirable interactions. We overcome this problem by combining almost-planar fields at three distinct planes. Each of these three types of almost-planar fields is obtained by rotating the construction from Sect. 2.1.

Fix the line $L := \{x_1 + 2x_2 = 0 = x_3\}$ within the plane P_0. Let $\mathcal{R}$ denote the rotation around L by $\frac{\pi}{3}$, which is given by the matrix

$$\mathcal{R} = \frac{1}{10}\begin{bmatrix} 9 & -2 & -\sqrt{15} \\ -2 & 6 & -2\sqrt{15} \\ \sqrt{15} & 2\sqrt{15} & 5 \end{bmatrix}$$

The rotated components will be similar to u_q from the previous section, but anchored instead at the plane $P_1 = \mathcal{R}P_0$ or $P_2 = \mathcal{R}^2 P_0$. More precisely: fix $q \in 3\mathbb{N} + 1$, for $i = 1, 2, 3$ define $V_1^{\textcircled{1}} := \mathcal{R}V_0^{\textcircled{1}}$, $F_1^{\textcircled{1}} := \mathcal{R}F_0^{\textcircled{1}}$ and the regions

$$A_{1,q}^{\textcircled{1}} := \mathbb{Z}^3 \cap \lambda_q\left(F_1^{\textcircled{1}} + [0, \epsilon]^3\right),$$

$$A_{1,q}^{\textcircled{2}} := \mathbb{Z}^3 \cap \lambda_q\left(F_1^{\textcircled{2}} + [0, \epsilon]^3\right),$$

$$A_{1,q}^{\textcircled{3}} := \mathbb{Z}^3 \cap \lambda_q\left(F_1^{\textcircled{3}} + [0, 2\epsilon]^3\right).$$

(Note that we first rotate the skeleton frequencies and blur afterwards, since rotating the blurs $A_{0,q}^{\textcircled{1}}$ results in non-integer values.) The components $u_q^{\textcircled{1}}$ and u_q are then defined as in Sect. 2.1, now with $V_1^{\textcircled{1}}$ replacing $V_0^{\textcircled{1}}$ and $A_{1,q}^{\textcircled{3}}$ replacing $A_{0,q}^{\textcircled{3}}$. We write $U_1 := \sum_{q\in 3\mathbb{N}+1} u_q$.

The $\frac{2\pi}{3}$-rotated field $U_2 := \sum_{q\in 3\mathbb{N}+2} u_q$ is defined with $\mathcal{R}$ in the definition of $V_1^{\textcircled{1}}, F_1^{\textcircled{1}}$ replaced by $\mathcal{R}^2$ (yielding $V_2^{\textcircled{1}}, F_2^{\textcircled{1}}$, and $A_{2,q}^{\textcircled{1}}$). Finally

$$U = U_0 + U_1 + U_2.$$

For brevity, we will at times suppress the lower index j in $A_{j,q}^{①}$ (denoting rotation), since it is unambiguously determined by $q \mod 3$. In other words

$$U = \sum_q u_q = \sum_q u_q^{①} + u_q^{②} + u_q^{③}, \qquad \text{with}$$

$$u_q^{①} = \epsilon^{-2} \lambda_q^{-\frac{7}{3}} \sum_{\xi \in A_q^{①}} \pi_\xi \left(V_{q \bmod 3}^{①} \right) \sin(\xi \cdot x)$$

$$u_q^{②} = \epsilon^{-2} \lambda_q^{-\frac{7}{3}} \sum_{\xi \in A_q^{②}} \pi_\xi \left(V_{q \bmod 3}^{②} \right) \cos(\xi \cdot x)$$

$$u_q^{③} = \epsilon^{-2} \lambda_q^{-\frac{7}{3}} \sum_{\xi \in A_q^{③}} \pi_\xi \left(V_{q \bmod 3}^{③} \right) \cos(\xi \cdot x).$$

2.3. Regularity of U

By $O(\epsilon)$ we will denote a real scalar, vector or tensor of magnitude $\lesssim \epsilon$, i.e. such that $|O(\epsilon)| \leq C\epsilon$, with a uniform constant C (i.e. independent from any parameters).

For objects defined in Sects. 2.1, 2.2 we have

Proposition 1. *For $j = 0, 1, 2$ (rotations) and $i = 1, 2, 3$ it holds*

$$\pi_\xi \left(V_j^{①} \right) - V_j^{①} = O(\epsilon) \qquad \forall \xi \in A_{j,q}^{①}, \tag{4}$$

$$|A_{j,q}^{①}| \in [(\epsilon\lambda_q - 1)^3, (\epsilon\lambda_q + 1)^3] \quad for \quad i = 1, 2, \tag{5a}$$

$$|A_{j,q}^{③}| \in [(2\epsilon\lambda_q - 1)^3, (2\epsilon\lambda_q + 1)^3]. \tag{5b}$$

Moreover, for any $p \in (1, \infty]$

$$\|u_q\|_{L^p} \leq C_p \epsilon^{1 - \frac{3}{p}} \lambda_q^{\frac{2}{3} - \frac{3}{p}}, \tag{6}$$

$$\|U\|_{\dot{B}_{p,\infty}^{\frac{3}{p} - \frac{2}{3}}} \leq C_p \epsilon^{1 - \frac{3}{p}}. \tag{7}$$

Proof. From the construction in Sects. 2.1, 2.2 we have

$$\xi \in A_{j,q}^{①} \implies \frac{\xi}{|\xi|} = \frac{F_j^{①}}{|F_j^{①}|} + O(\epsilon) \tag{8}$$

and $F_j^{①} \perp V_j^{①}$. The projection $\pi_\xi = \mathbf{1} - \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}$, therefore

$$\pi_\xi \left(V_j^{①} \right) = V_j^{①} - \frac{\xi}{|\xi|} \left(\frac{F_j^{①}}{|F_j^{①}|} + O(\epsilon) \right) \cdot V_j^{①} = V_j^{①} - \frac{\xi}{|\xi|} O(\epsilon)$$

from which (4) follows immediately.

In any cube $x_0 + [0, \epsilon\lambda_q]^3$, there are at most $(\lfloor \epsilon\lambda_q \rfloor + 1)^3 \leq (\epsilon\lambda_q + 1)^3$ and at least $(\lfloor \epsilon\lambda_q \rfloor)^3 \geq (\epsilon\lambda_q - 1)^3$ lattice points. This proves (5) in the case $i = 1, 2$. For $i = 3$ the only difference is that the cube has twice longer edges.

Since π_ξ agrees with the Leray-Helmholtz projector, we can use the L^p theory for the latter, obtaining

$$\|u_q^{①}\|_p \le C_p \epsilon^{-2} \lambda_q^{-\frac{7}{3}} \| \sum_{\xi \in A_q^{①}} V_{q \bmod 3}^{①} \sin(\xi \cdot) \|_p$$

$$\le C_p \epsilon^{-2} \lambda_q^{-\frac{7}{3}} \| \sum_{\xi \in A_q^{①}} \sin(\xi \cdot) \|_p$$

$$\le C_p \epsilon^{-2} \lambda_q^{-\frac{7}{3}} (\epsilon \lambda_q)^{3(1 - \frac{1}{p})},$$

where the last but one inequality holds, because $V_{q \bmod 3}^{①}$ is constant. The last inequality, since A_q is a cube, follows along the lines of the Dirichlet kernel estimate. An analogous computation for $u_q^{②}$, $u_q^{③}$ yields (6) for $p \in (1, \infty)$. The case $p = \infty$ follows from the (non-optimal) straightforward bound $\|u_q\|_{L^\infty} \le \|\hat{u}_q\|_{L^\infty} \mathrm{spt}(\hat{u}_q)$ and application of (4) and (5).

The final regularity statement (7) follows from (6) and the fact that by (12) below at most 3 components u_q interfere with a single Littlewood-Paley shell. $\qquad\square$

2.4. Fourier Side of U

For the sake of the next section, let us write u_q in terms of Fourier modes, restricting our attention to the leading order amplitudes, provided by (4):

$$\widehat{u_q^{①}} = -i \lambda_q^{-\frac{7}{3}} \epsilon^{-2} \left(V_{q \bmod 3}^{①} + O(\epsilon) \right) \left(\chi_{A_q^{①}} - \chi_{-A_q^{①}} \right),$$

$$\widehat{u_q^{②}} = \lambda_q^{-\frac{7}{3}} \epsilon^{-2} \left(V_{q \bmod 3}^{②} + O(\epsilon) \right) \left(\chi_{A_q^{②}} + \chi_{-A_q^{②}} \right), \tag{9}$$

$$\widehat{u_q^{③}} = \lambda_q^{-\frac{7}{3}} \epsilon^{-2} \left(V_{q \bmod 3}^{③} + O(\epsilon) \right) \left(\chi_{A_q^{③}} + \chi_{-A_q^{③}} \right),$$

where χ_A denotes the characteristic function of A. Differentiation yields

$$\widehat{\nabla u_q^{①}} = -\lambda_q^{-\frac{4}{3}} \epsilon^{-2} \left(V_{q \bmod 3}^{①} \otimes F_{q \bmod 3}^{①} + O(\epsilon) \right) \left(\chi_{A_q^{①}} + \chi_{-A_q^{①}} \right),$$

$$\widehat{\nabla u_q^{②}} = i \lambda_q^{-\frac{4}{3}} \epsilon^{-2} \left(V_{q \bmod 3}^{②} \otimes F_{q \bmod 3}^{②} + O(\epsilon) \right) \left(\chi_{A_q^{②}} - \chi_{-A_q^{②}} \right), \tag{10}$$

$$\widehat{\nabla u_q^{③}} = i \lambda_q^{-\frac{4}{3}} \epsilon^{-2} \left(V_{q \bmod 3}^{③} \otimes F_{q \bmod 3}^{③} + O(\epsilon) \right) \left(\chi_{A_q^{③}} - \chi_{-A_q^{③}} \right).$$

Observe that the amplitudes of $\nabla u_q^{①}$ are real while the other ones are purely imaginary.

We will also need the following precise estimates about the shells in which the Fourier support of U is contained:

$$\mathrm{spt}(\widehat{u_q^{①}}) \subset \{\lambda_q(1 - \sqrt{3}\epsilon) \le |\xi| \le \lambda_q(1 + \sqrt{3}\epsilon)\},$$

$$\mathrm{spt}(\widehat{u_q^{②}}) \subset \{\lambda_q(2 - \sqrt{3}\epsilon) \le |\xi| \le \lambda_q(2 + \sqrt{3}\epsilon)\}, \tag{11}$$

$$\mathrm{spt}(\widehat{u_q^{③}}) \subset \{\lambda_q(\sqrt{5} - 2\sqrt{3}\epsilon) \le |\xi| \le \lambda_q(\sqrt{5} + 2\sqrt{3}\epsilon)\}.$$

Hence

$$\mathrm{spt}(\widehat{u_q}) \subset \{\lambda_q(1 - 2\epsilon) < |\xi| < \lambda_q(\sqrt{5} + 4\epsilon)\}. \tag{12}$$
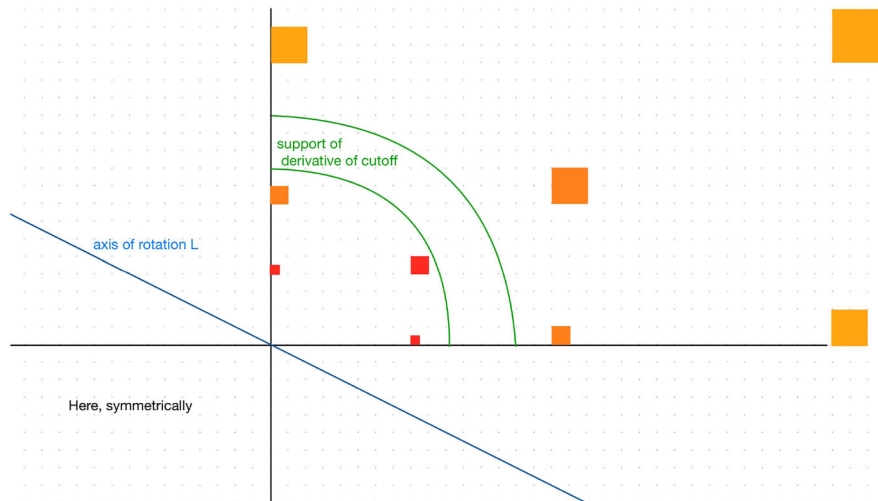
FIG. 1. Active regions from three consecutive generations $q-1$, q, $q+1$ and support of $\nabla\varphi(\lambda_q^{-1}\cdot)$. Different colours of active regions indicate that in fact they are anchored at three different planes

Bounds (11) can be seen using Fig. 1. Let us justify them more extensively. It holds $\mathrm{spt}(\widehat{u_q^{①}}) = \pm A_q^{①}$; in particular the supports are symmetric, so we can ignore the 'negative' part in our computation. For $i = 1, 2$, $A_q^{①} \subset Q^i := \lambda_q(F_{q \bmod 3}^{①} + [0, \epsilon]^3)$, while for Q^3 we replace ϵ with 2ϵ. The diameter of Q^i equals $\sqrt{3}\lambda_q\epsilon$ for $i = 1, 2$ and $2\sqrt{3}\lambda_q\epsilon$ for $i = 3$. Observe that

$$|F_{q \bmod 3}^{①}| = 1, \ |F_{q \bmod 3}^{②}| = 2, \ |F_{q \bmod 3}^{③}| = \sqrt{5}$$

irrespective of rotation. This, the fact that $\lambda_q F_{q \bmod 3}^{①} \in Q^i$ and the values of the diameters yield the generous estimate (11).

3. Flux Decomposition

Let v_1, v_2, v_3 be any vector fields from $\mathbb{T}^3$ to $\mathbb{C}^3$. Then, for $\xi_j \in \mathbb{Z}^3$, $j = 1, 2, 3$, via Parseval

$$\int (v_1 \otimes v_2) : \nabla v_3 = \sum_{\xi_1 + \xi_2 + \xi_3 = 0} \widehat{v_1}(\xi_1) \otimes \widehat{v_2}(\xi_2) : \widehat{\nabla v_3}(\xi_3). \tag{13}$$

Thus, any contribution to the integral necessitates $\xi_1 + \xi_2 + \xi_3 = 0$. This observation is an underlying principle for the following analysis of the flux. In particular, modes satisfying $\xi_1 + \xi_2 + \xi_3 = 0$ will be referred to as 'interacting' (i.e. contributing to the flux), exhibiting 'non-zero interactions' etc.

The aim of this section is to split the flux $\Pi_q(U)$ into a local part Π_q^{local}, which is restricted to interactions within a single component u_q, and a non-local part $\Pi_q^{\mathrm{non-local}}$, which involves modes further apart. We will show that Π_q^{local} behaves precisely like the flux of the skeleton field S. However, unlike in the $\Pi_q(S)$ case, $\Pi_q^{\mathrm{non-local}}$ will contribute to the flux. Indeed, a blur of a very high frequency interacts with its almost-symmetric counterpart, producing a much lower non-zero mode, which in turn may interact with a low-frequency mode.

Recall that $|O(\epsilon)| \leq C\epsilon$ with a uniform C. We will from now onwards tacitly assume that $\epsilon \leq \epsilon_0$, where $C\epsilon_0 \leq \frac{1}{C}$. Consequently, $1 - |O(\epsilon)| \geq 1 - \frac{1}{C}$. Even though $O(\epsilon)$ is unsigned and even not necessarily a real number, we will write at times $1 - O(\epsilon)$, meaning $1 - |O(\epsilon)|$. Denoting $a \odot b := a \otimes b + b \otimes a$, we state

Lemma 1 *(Flux decomposition).* *The energy flux $\Pi_q(U) = \Pi_q^{\text{local}} + \Pi_q^{\text{non-local}}$, where*

$$\Pi_q^{\text{local}} = \int u_q^{②} \odot u_q^{③} : \nabla u_q^{①} \tag{14}$$

and

$$\Pi_q^{\text{non-local}} = \sum_{i=1,2,3} \sum_{l \leq q} \sum_{k > max(q, l+N_\epsilon)} \int u_k^{①} \otimes u_k^{①} : \nabla u_l^{①} \tag{15}$$

with $N_\epsilon = \lfloor 3 - \log \epsilon \rfloor$.

The remainder of this section contains proof of Lemma 1.

3.1. Initial Splitting of $\Pi_q(U)$

Let us define for $U = \sum_q u_q$ of Sect. 2.2

$$U_{<q} := \sum_{l<q} u_l \qquad \text{and analogously:} \quad U_{>q}, \ U_{\leq q}, \ U_{\geq q}.$$

The splitting $U = U_{<q} + u_q + U_{>q}$ is usually called Bony decomposition. By (11) and the choice of the cutoff function φ, implying there are no active modes of U in the support of derivative of φ, we have (cf. Fig. 1)

$$S_q(U) = U_{<q} + u_q^{①} \quad \text{and} \quad U - S_q(U) = u_q^{②} + u_q^{③} + U_{>q}. \tag{16}$$

Therefore

$$\begin{aligned} \Pi_q(U) &= \int S_q(U \otimes U) : \nabla S_q(U) = \int S_q(u \otimes U) : \nabla (U_{<q} + u_q^{①}) \\ &= \int (U \otimes U) : \nabla (U_{<q} + u_q^{①}). \end{aligned} \tag{17}$$

U is divergence-free and integrable, while $U_{<q} + u_q^{①}$ is smooth, thus integrating by parts

$$\int (U_{<q} + u_q^{①}) \otimes U : \nabla (U_{<q} + u_q^{①}) = 0. \tag{18}$$

Subtracting (18) from (17) we obtain

$$\Pi_q(U) = \int \left(u_q^{②} + u_q^{③} + U_{>q} \right) \otimes U : \nabla \left(U_{<q} + u_q^{①} \right). \tag{19}$$

We split now $\Pi_q(U)$ of (19) into the part $\Pi_q^{\text{non-local}}$ where modes are separated and the remainder Π_q^{local} as follows:

$$\Pi_q^{\text{local}} = \int (u_q^{②} + u_q^{③} + u_{q+1}) \otimes (u_{q-1} + u_q + u_{q+1}) : \nabla (u_{q-1} + u_q^{①}), \tag{20}$$

$$\begin{aligned} \Pi_q^{\text{non-local}} &= \int U_{>q+1} \otimes U : \nabla \left(U_{<q} + u_q^{①} \right) \\ &\quad + \int (u_q^{②} + u_q^{③} + u_{q+1}) \otimes U : \nabla U_{<q-1} \\ &\quad + \int (u_q^{②} + u_q^{③} + u_{q+1}) \otimes (U_{<q-1} + U_{>q+1}) : \nabla (u_{q-1} + u_q^{①}). \end{aligned} \tag{21}$$

We need to show that (20) is in fact (14), and that (21) is in fact (15).

3.2. Exclusion of Three Different Planes Interacting

We will say that a point ξ is anchored to a set Ω if $dist(\xi, \Omega) \leq 2\epsilon|\xi|$. We claim that each frequency ξ of $\widehat{U}$ is unambiguously anchored at one of three different planes P_0, P_1 or P_2, thanks to $\epsilon \leq \epsilon_0$ small in our definition of active frequency regions $A_q^{\textcircled{1}}$.

Indeed, $\lambda_q F_{q \bmod 3}^{\textcircled{1}} \in P_{q \bmod 3}$, so $dist(\xi, P_{q \bmod 3}) \leq |\xi - \lambda_q F_{q \bmod 3}^{\textcircled{1}}|$. At the same time, if $\xi \in A_q^{\textcircled{1}}$ then by construction ξ belongs to a cube with a vertex $\lambda_q F_{q \bmod 3}^{\textcircled{1}}$ and with its side length $\epsilon \lambda_q$ or $2\epsilon \lambda_q$. This reads for $\xi \in A_q^{\textcircled{1}}$ with $i = 1, 2$

$$dist(\xi, P_{q \bmod 3}) \leq |\xi - \lambda_q F_{q \bmod 3}^{\textcircled{1}}| \leq \sqrt{3}\epsilon \lambda_q \leq \frac{\sqrt{3}\epsilon}{1 - 2\epsilon}|\xi|,$$

where the last inequality stems from (12). Hence for $i = 1, 2$, $\xi \in A_q^{\textcircled{1}}$ are anchored to $P_{q \bmod 3}$, provided $\frac{\sqrt{3}}{1 - 2\epsilon_0} \leq 2$. Similarly for $\xi \in A_q^{\textcircled{3}}$, because

$$dist(\xi, P_{q \bmod 3}) \leq |\xi - \lambda_q F_{q \bmod 3}^{\textcircled{3}}| \leq 2\sqrt{3}\epsilon \lambda_q \leq \frac{2\sqrt{3}\epsilon}{\sqrt{5} - 2\sqrt{3}\epsilon}|\xi|,$$

applying now (11) and again an uniform upper bound on ϵ_0.

Since ϵ is small, we see that the anchoring is unambiguous, as claimed.

Our next goal is to exclude any interactions between modes related to three different planes P.

Proposition 2. *If each* $\mathrm{spt}(\widehat{u_{q'}})$, $\mathrm{spt}(\widehat{u_{q''}})$, $\mathrm{spt}(\widehat{u_{q'''}})$ *is anchored at a different plane* P, *then*

$$\int u_{q'} \otimes u_{q''} : \nabla u_{q'''} = 0. \tag{22}$$

Proof. Every P stems from a rotation around axis L, whose normal plane we call $L^\perp$. Let us denote the orthogonal projection onto $L^\perp$ by $\pi_{L^\perp}$. For modes $\xi_j \in \mathrm{spt}(\widehat{u_{q_j}})$ with $j = 1, 2, 3$ to interact, it is necessary that $\pi_{L^\perp}(\xi_i)$ do interact, i.e. that

$$\pi_{L^\perp}(\xi_1) + \pi_{L^\perp}(\xi_2) + \pi_{L^\perp}(\xi_3) = 0. \tag{23}$$

For the following considerations, it may be beneficial to visualise U projected onto $L^\perp$, i.e. Fig. 1 seen along the axis L, see Fig. 2.

Observe that $\pi_{L^\perp}$ projections of modes of u_q have norms equal either $\frac{2\sqrt{5}}{5}\lambda_q(1 + O(\epsilon)) =: \tilde{\lambda}_q(1 + O(\epsilon))$ or $2\tilde{\lambda}_q(1 + O(\epsilon))$.

We assumed that each mode ξ_i is anchored at a different P plane. Consequently, each $\pi_{L^\perp}(\xi_i)$ is anchored to a different line l_i which is $\pi_{L^\perp}$ projection of a plane P. Take the mode with the largest norm, say ξ_1 of $\widehat{u_{q_0}}$, anchored at l_1. We have

$$\text{either} \quad |\pi_{L^\perp}(\xi_1)| = \tilde{\lambda}_{q_0}(1 + O(\epsilon)), \tag{24a}$$

$$\text{or} \quad |\pi_{L^\perp}(\xi_1)| = 2\tilde{\lambda}_{q_0}(1 + O(\epsilon)). \tag{24b}$$

The remaining two active $\pi_{L^\perp}(\xi_i)$'s, $i = 2, 3$ are anchored, respectively, at different lines l_2, l_3. Therefore we have

$$|\pi_{L^\perp}(\xi_2)| = 2^{-j}\tilde{\lambda}_{q_0}(1 + O(\epsilon)) \quad \text{for a } j \geq 0 \tag{25a}$$

$$|\pi_{L^\perp}(\xi_3)| = 2^{-k-1}\tilde{\lambda}_{q_0}(1 + O(\epsilon)) \quad \text{for a } k \geq 0. \tag{25b}$$

The first consequence of (25) is that (24b) cannot happen. Indeed, using (23) we have

$$|\pi_{L^\perp}(\xi_1)| = |\pi_{L^\perp}(\xi_2) + \pi_{L^\perp}(\xi_3)| \leq \tilde{\lambda}_{q_0}(1 + O(\epsilon)) + 2^{-1}\tilde{\lambda}_{q_0}(1 + O(\epsilon)) < \frac{3}{2}\tilde{\lambda}_{q_0}(1 + O(\epsilon)). \tag{26}$$

Next, knowing that (24a) is the only possibility, we argue that (24a) requires $j = 1$ in (25a). Indeed, $j < 1$ is immediately excluded analogously to (26). Take case $j = 0$ in (25a). The involved projections of modes
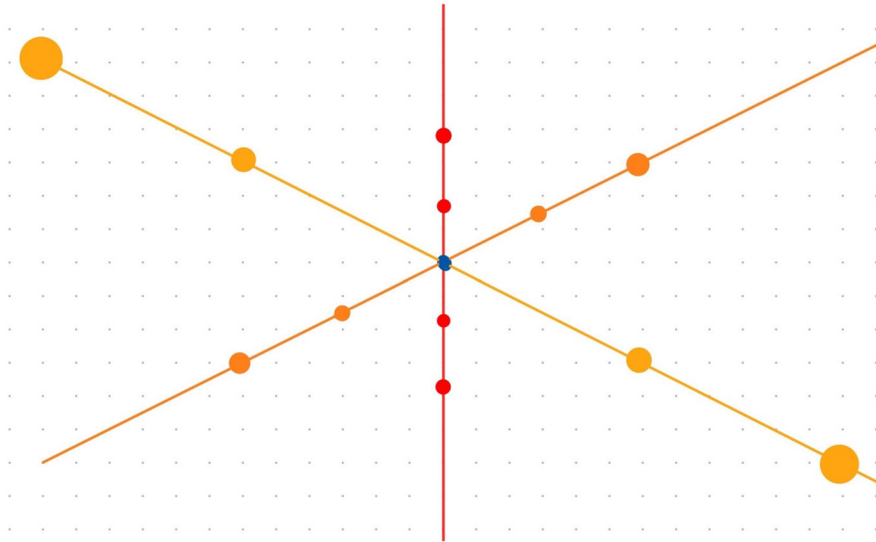
FIG. 2. Windmill: dots indicate (approximately) active regions projected on the plane of rotation $L^\perp$

$\pi_{L^\perp}(\xi_1)$, $\pi_{L^\perp}(\xi_2)$ are anchored at different lines l_1, l_2, with angle $\pi/3$ between them. This spreading means that, by adding vectors we have

$$|\pi_{L^\perp}(\xi_1) + \pi_{L^\perp}(\xi_2)| \geq \lambda_{q_0}(1 - O(\epsilon)),$$

but then (25b) makes (23) impossible. (Recall we write $1 - O(\epsilon)$ for $1 - |O(\epsilon)|$.)

Finally, having both (24a) and (25a) with $j = 1$, we see that $k > 0$ in (25b) is now excluded analogously to (26).

Altogether, (23) may occur only if

$$|\pi_{L^\perp}(\xi_1)| = \tilde{\lambda}_{q_0}(1 + O(\epsilon)), \ |\pi_{L^\perp}(\xi_2)| = \frac{\tilde{\lambda}_{q_0}}{2}(1 + O(\epsilon)), \ |\pi_{L^\perp}(\xi_3)| = \frac{\tilde{\lambda}_{q_0}}{2}(1 + O(\epsilon)),$$

but due to $\pi/3$ angle between l_2, l_3, $|\pi_{L^\perp}(\xi_2) + \pi_{L^\perp}(\xi_3)| \leq \frac{\sqrt{3}}{2}\tilde{\lambda}_{q_0}(1 + O(\epsilon)) < \tilde{\lambda}_{q_0}(1 - O(\epsilon))$, so (23) is impossible. □

3.3. Exclusion of Near-Field Interactions

It turns out that active modes which are not all from the same $\widehat{u_q}$ may interact only in the following scenario: the two higher-frequency modes are from the same $\widehat{u_q}$, while the third mode is of considerably lower frequency. More precisely, we have

Proposition 3. *Let $q_1 \leq q_2 \leq q_3$. Assume that u_{q_j}'s interact, i.e. there are ξ_1, ξ_2, ξ_3 where $\xi_j \in \mathrm{spt}\left(\widehat{u_{q_j}}\right)$ for $j = 1, 2, 3$ such that*

$$\xi_1 + \xi_2 + \xi_3 = 0.$$

Suppose $q_1 < q_3$, then $q_1 + 3 \leq q_2 = q_3$.

Proof. We distinguish three elementary cases, depending on the separation between the largest q_3 and the smallest q_1.

Case 1. $q_1 = q_3 - 1$. Since $q_2 \in \{q_1, q_3\}$ there are exactly two modes anchored at the same plane and the third mode at a different one, say ξ, η are anchored at P_0 while ζ is not. Then, on the one hand

$$dist(\xi + \eta, P_0) \leq dist(\xi, P_0) + dist(\eta, P_0) \leq 2\epsilon(|\xi| + |\eta|) \leq 2\epsilon(2\sqrt{5} + O(\epsilon))\lambda_{q_3}, \tag{27}$$

with the last inequality given by (12) and the fact that q_3 is the largest of q_j's. On the other hand, projections do not increase distances, so $dist(\zeta, P_0) \geq dist(\pi_{L^\perp}(\zeta), l_0)$, with $l_0 = \pi_{L^\perp}(P_0)$ (recall setting of proof of Proposition 2). Since the angle between the planes $P_{q \bmod 3}$ is always $\frac{\pi}{3}$ (thus between lines $l_{q \bmod 3}$ being their projections) and ζ is anchored to $l_{q \bmod 3} \neq l_0$, we have, cf Fig. 2

$$
\begin{aligned}
dist(\zeta, P_0) &\geq dist(\pi_{L^\perp}(\zeta), l_0) \geq \tilde{\lambda}_{q_1}(1 - O(\epsilon))(\sin(\frac{\pi}{3}) - O(\epsilon)) \\
&\geq (\frac{\sqrt{15}}{5} - O(\epsilon))\lambda_{q_1} = (\frac{\sqrt{15}}{10} - O(\epsilon))\lambda_{q_3}
\end{aligned}
\tag{28}
$$

Comparing (27) and (28), we see that $dist(\xi + \eta, P_0) < dist(\zeta, P_0)$ for $\epsilon \leq \epsilon_0$, so there is no interaction.

Case 2. $q_1 = q_3 - 2$. There are two possibilities:

(a) $q_2 = q_1 + 1$, i.e. q_2 is strictly inbetween q_1 and q_3. Then all ξ_j are anchored at a different plane, so by Proposition 2 there is no interaction.

(b) $q_2 \in \{q_1, q_3\}$. Then the very same argument as in Case 1 applies, with a smaller but fixed parameter ϵ_0.

Case 3. $q_1 \leq q_3 - 3$. We can estimate using (12) twice

$$(\sqrt{5} + 4\epsilon)\lambda_{q_2} > |\xi_2| = |\xi_1 + \xi_3| \geq |\xi_3| - |\xi_1| \geq (1 - \frac{\sqrt{5}}{8} - \frac{5\epsilon}{2})\lambda_{q_3} > \frac{\sqrt{5} + 4\epsilon}{4}\lambda_{q_3},$$

where the last inequality uses $\epsilon \leq \epsilon_0$. Comparing the leftmost and the rightmost quantity above, we see that necessarily $q_2 \geq q_3 - 1$. We have thus two options: either $q_2 + 1 = q_3$ or $q_2 = q_3$; the latter being the final statement. Therefore it remains to exclude $q_2 + 1 = q_3$. In such case ξ_2 and ξ_3 are anchored at different planes, say P_0 and P_1, respectively. Analogously to (28) we thus have

$$dist(\xi_3, P_0) \geq (\frac{\sqrt{15}}{5} - O(\epsilon))\lambda_{q_3}.$$

The mode ξ_2 is anchored at the plane P_0, so $-\xi_2$ is also anchored at the plane P_0. Consequently, via triangle inequality

$$|\xi_2 + \xi_3| + 2\epsilon|\xi_2| \geq dist(\xi_3, P_0).$$

Together, the above two inequalities yield

$$|\xi_2 + \xi_3| \geq dist(\xi_3, P_0) - 2\epsilon|\xi_2| \geq (\frac{\sqrt{15}}{5} - O(\epsilon))\lambda_{q_3} > (\frac{\sqrt{5}}{8} + \frac{\epsilon}{2})\lambda_{q_3} \geq |\xi_1|,$$

with the last inequality due to (12) and $q_1 \leq q_3 - 3$. Hence we excluded $q_2 + 1 = q_3$. $\qquad\square$

3.4. Proof of (14)

Observe that

$$
\begin{aligned}
\Pi_q^{\text{local}} &= \int (u_q^{②} + u_q^{③} + u_{q+1}) \otimes (u_{q-1} + u_q + u_{q+1}) : \nabla(u_{q-1} + u_q^{①}) \\
&= \int \left(u_q^{②} + u_q^{③}\right) \otimes u_q : \nabla u_q^{①}.
\end{aligned}
\tag{29}
$$

The former identity of (29) is (20); the latter holds thanks to Proposition 3. Indeed, the remaining terms involve at least two different q_j's, but they do not satisfy the separation condition $q_1 + 3 \leq q_3$ (here $q_1 = q - 1$, $q_3 = q + 1$) demanded in Proposition 3 for a non-zero interaction.

Let us now take ξ^1, ξ^2 in the Fourier support of the tensor product terms in (29). For a nonzero interaction necessarily $\xi^1 + \xi^2 \in \mathrm{spt}(\widehat{u_q^{①}})$, which requires by construction (consult Fig. 1) one of ξ^1, ξ^2 in $\mathrm{spt}(\widehat{u_q^{②}})$ and the other in $\mathrm{spt}(\widehat{u_q^{③}})$, implying (14).

3.5. Proof of (15)

First observe that none of the integrals in (21) involves only one u_q, consequently Proposition 3 applies ($q_1 < q_3$). Hence, any non-zero interaction within $\Pi_q^{\text{non-local}}$ has the form

$$\Pi_{l,k} := \int u_k \otimes u_k : \nabla u_l, \quad \text{where } l \leq q \leq k \text{ and } l+3 \leq k.$$

Let us analyse $\Pi_{l,k}$. First note that $\xi_1 + \xi_2 + \xi_3 = 0$ together with $\xi_1, \xi_2 \in \text{spt}(\widehat{u_k})$ and $\xi_3 \in \text{spt}(\widehat{u_l})$ imply that $\xi_1 \in A_k^{\textcircled{1}}$ and $\xi_2 \in -A_k^{\textcircled{1}}$ with the same i, or $\xi_1 \in -A_k^{\textcircled{1}}$ and $\xi_2 \in +A_k^{\textcircled{1}}$ with the same i. Indeed, in any other case, by the separation of active regions $\pm A_k^{\textcircled{1}}, \pm A_k^{\textcircled{1}}$ with $i \neq j$ (cf Fig. 1) one has

$$|\xi_1 + \xi_2| \geq (1 - O(\epsilon))\lambda_k = 8(1 - O(\epsilon))\lambda_l > |\xi_3|,$$

with the last inequality via (12), so there cannot be any interaction. Therefore

$$\Pi_{l,k} = \sum_{i=1,2,3} \int u_k^{\textcircled{1}} \otimes u_k^{\textcircled{1}} : \nabla u_l = \sum_{i=1,2,3} \int u_k^{\textcircled{1}} \otimes u_k^{\textcircled{1}} : \nabla u_l^{\textcircled{1}},$$

where the last identity is due to (13) and the following observation: the product of Fourier modes from opposite regions is real by (9), but by (10) modes in $\nabla u_q^{\textcircled{1}}$ are purely imaginary unless $i = 1$. The total flux is real, so any imaginary components are irrelevant.

Summarising, the non-local flux reduces to

$$\Pi_q^{\text{non-local}} = \sum_{l \leq q} \sum_{k \geq \max\{q, l+3\}} \Pi_{l,k}.$$

Using the expression for $\Pi_{l,k}$ we can strip it down even more. Since ξ_1, ξ_2 are from opposite active regions, we can estimate

$$\lambda_l(1 - 2\epsilon) \leq |\xi_3| = |\xi_1 + \xi_2| \leq 2\,\text{diam}(A_k^{\textcircled{1}}) \leq 4\sqrt{3}\epsilon\lambda_k.$$

Taking logarithms above

$$l \leq \log_2 \epsilon + k + 2 + \log_2 \frac{\sqrt{3}}{1 - 2\epsilon} \implies k > l - 3 - \log_2 \epsilon$$

for $\epsilon \leq \epsilon_0$, as long as $\epsilon_0 < \frac{2 - \sqrt{3}}{4}$. Now (15) follows immediately and the definition of $N(\epsilon)$ is justified. □

4. Proof of Theorem 1

4.1. Exact Estimates for Π_q^{local}

Recall (14); using (13) it reads

$$\Pi_q^{\text{local}} = \sum_{\xi^1 + \xi^2 + \xi^3 = 0} \widehat{u_q^{\textcircled{2}}}(\xi_2) \odot \widehat{u_q^{\textcircled{3}}}(\xi_3) : \widehat{\nabla u_q^{\textcircled{1}}}(\xi_1),$$

where $\xi^1 \in \pm A_q^{\textcircled{1}}, \xi^2 \in \pm A_q^{\textcircled{2}}, \xi^3 \in \pm A_q^{\textcircled{3}}$, since these are the active regions of respective $u_q^{\textcircled{1}}$'s, see (9). By our construction, for $\xi^1 \in A_q^{\textcircled{1}}$ and $\xi^2 \in A_q^{\textcircled{2}}$, $\xi^1 + \xi^2 \in A_q^{\textcircled{3}}$ always holds, compare Fig. 1 and recall from Sect. 2.2 that we used the blurs $A_q^{\textcircled{3}}$ with twice the side length of the blurs $A_q^{\textcircled{1}}$, $i = 1, 2$. Consequently, any $\xi^1 \in A_q^{\textcircled{1}}$ and $\xi^2 \in A_q^{\textcircled{2}}$ interacts via $\xi^1 + \xi^2 + \xi^3 = 0$ with precisely one $\xi^3 \in -A_q^{\textcircled{3}}$;

if only one of the signs of active regions of ξ^1, ξ^2 is flipped, there is no interaction; and any $\xi^1 \in -A_q^{\text{①}}$ and $\xi^2 \in -A_q^{\text{②}}$ interacts with precisely one $\xi_3 \in A_q^{\text{③}}$. We thus arrive via (9) and (10) at

$$\Pi_q^{\text{local}} =$$
$$- (\epsilon\lambda_q)^{-6} 2 \sum_{\substack{\xi^1 \in A_q^{\text{①}} \\ \xi^2 \in A_q^{\text{②}}}} \left(V_{q\,\text{mod}\,3}^{\text{②}} + O(\epsilon) \right) \odot \left(V_{q\,\text{mod}\,3}^{\text{③}} + O(\epsilon) \right) : \left(V_{q\,\text{mod}\,3}^{\text{①}} \otimes F_{q\,\text{mod}\,3}^{\text{①}} + O(\epsilon) \right)$$
$$= -2(\epsilon\lambda_q)^{-6} |A_q^{\text{①}}||A_q^{\text{②}}| \left(V_{q\,\text{mod}\,3}^{\text{②}} \odot V_{q\,\text{mod}\,3}^{\text{③}} : V_{q\,\text{mod}\,3}^{\text{①}} \otimes F_{q\,\text{mod}\,3}^{\text{①}} + O(\epsilon) \right).$$

It holds $a \odot b : c \otimes d = (b \cdot c)(a \cdot d) + (a \cdot c)(b \cdot d)$. This, the fact that rotations, in particular $\mathcal{R}$, do not alter the dot product, and the values of $V^{\text{①}}$ and $F^{\text{①}}$ provided in Sect. 2.1 yield

$$\Pi_q^{\text{local}} = 2(\epsilon\lambda_q)^{-6} |A_q^{\text{①}}||A_q^{\text{②}}| (1 + O(\epsilon)) \in 2(\epsilon\lambda_q)^{-6}[(\epsilon\lambda_q - 1)^3, (\epsilon\lambda_q + 1)^3](1 + O(\epsilon)),$$

with the latter identity using (5).

Summarizing, we obtain, for $O(\epsilon)$ independent of q

$$2\left(\frac{\epsilon\lambda_q - 1}{\epsilon\lambda_q}\right)^6 (1 - O(\epsilon)) \leq \Pi_q^{\text{local}} \leq 2\left(\frac{\epsilon\lambda_q + 1}{\epsilon\lambda_q}\right)^6 (1 + O(\epsilon)). \tag{30}$$

For any $\epsilon > 0$ fixed, the lower and upper bound in (30) becomes, respectively, $2 \pm O(\epsilon)$ for large q's.

4.2. Estimate for $\Pi_q^{\text{non-local}}$

Recall (15). It reads via (13)

$$\Pi_q^{\text{non-local}}$$
$$= \sum_{i=1,2,3} \sum_{l \leq q} \sum_{k > \max\{q, l+N_\epsilon\}} \sum_{\substack{\xi_1 + \xi_2 + \xi_3 = 0 \\ |\xi^1| \in A_k^{\text{①}},\, |\xi^3| \in A_l^{\text{①}}}} \left(\widehat{u_k^{\text{①}}}(\xi_1) \otimes \widehat{u_k^{\text{①}}}(\xi_2) : \widehat{\nabla u_l^{\text{①}}}(\xi_3) \right).$$

A brutal estimate for the innermost sum, using (9) and (10), yields

$$\sum_{|\xi^1| \in A_k^{\text{①}},\, |\xi^3| \in A_l^{\text{①}}} \left| \widehat{u_k^{\text{①}}}(\xi_1) \otimes \widehat{u_k^{\text{①}}}(\xi_2) : \widehat{\nabla u_l^{\text{①}}}(\xi_3) \right| \leq C\epsilon^{-6} \lambda_k^{-\frac{14}{3}} \lambda_l^{-\frac{4}{3}} |A_k^{\text{①}}||A_l^{\text{①}}|$$
$$\leq C\epsilon^{-3} \lambda_k^{-\frac{5}{3}} \lambda_l^{-\frac{4}{3}} (\epsilon\lambda_l + 1)^3,$$

with the latter inequality using (5) and that, since we interested in the limit $q \to \infty$, for any fixed ϵ, $\epsilon\lambda_k \geq 1$ for $k \geq q$. Therefore

$$|\Pi_q^{\text{non-local}}| \leq C \sum_{l \leq q} \sum_{k > \max\{q, l+N_\epsilon\}} \lambda_k^{-\frac{5}{3}} \max(\lambda_l^{\frac{5}{3}}, \epsilon^{-3}\lambda_l^{-\frac{4}{3}})$$
$$= C \sum_{l \leq q} \left(\max\{1, \frac{1}{\epsilon\lambda_l}\} \right)^3 \sum_{k > \max\{q, l+N_\epsilon\}} \left(\frac{\lambda_l}{\lambda_k} \right)^{\frac{5}{3}}.$$

Observe that the max in the outer sum is 1 if $l \geq l_\epsilon := - \frac{\log \epsilon}{\log 2}$. Therefore the sums split into

$$|\Pi_q^{\mathrm{non-local}}|$$

$$\leq \sum_{l<l_\epsilon} (\epsilon \lambda_l)^{-3} \sum_{k \geq q} \left(\frac{\lambda_l}{\lambda_k} \right)^{\frac{5}{3}} + \sum_{l_\epsilon \leq l \leq q - N_\epsilon} \sum_{k \geq q} \left(\frac{\lambda_l}{\lambda_k} \right)^{\frac{5}{3}} + \sum_{q - N_\epsilon < l \geq q} \sum_{k \geq l + N_\epsilon} \left(\frac{\lambda_l}{\lambda_k} \right)^{\frac{5}{3}}$$

(where the second sum might be empty). The first summand above vanishes as $q \to \infty$ (and ϵ fixed), the second term is bounded independent of q by $2^{-\frac{5}{3} N_\epsilon} = O(\epsilon)$, and the third sum is bounded by $N_\epsilon 2^{-\frac{5}{3} N_\epsilon} = O(\epsilon \log \epsilon)$, in view of $N_\epsilon = \lfloor 3 - \log(\epsilon) \rfloor$. Thus we have, for q sufficiently large with respect to ϵ,

$$|\Pi_q^{\mathrm{non-local}}| \leq O(\epsilon \log \epsilon). \tag{31}$$

4.3. Conclusion of the Proof

Recall that the energy flux of $\Pi_q(U)$ is cubic in U, so by multiplying the original construction with $(c/2)^{1/3}$ we obtain a field such that by (30) and (31)

$$c - O(\epsilon \log \epsilon) < \Pi_q(U) < c + O(\epsilon \log \epsilon)$$

holds for sufficiently large q. Choosing ϵ in the construction sufficiently small with respect to δ then proves (2).

Declarations
Conflict of interest The authors have no competing interests to declare that are relevant to this article.

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