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On Some Singular Limits Arising in Fluid Dynamic Modelling

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Abstract. Fluid dynamic equations are used to model various phenomena arising from physics, engineering, astrophysics, geophysics. One feature is that they take place at different time and length scales and it is important to understand which phenomena occur according to the use of single scales or to the interactions of them. From a mathematical point of view, these various physical behaviours give rise to different singular limits and, consequently to a different analysis of the asymptotic state of the governing equations. In this paper we will analyse a very simplified model given by a linearised continuity equation and by the classical momentum equation which include terms that take into account of rotation and we will show, according to the values of different scales, that the asymptotic behaviour of the model will be those of an incompressible fluid or of a geostrophic flow. Finally we point out, that the set of equations analysed in the paper may also fit in the artificial compressibility approximation methods.

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1. Introduction

In this paper we analyse a simplified fluid dynamical model given by a linearised continuity equation and by the classical momentum equation which includes a terms that takes into account of rotation and we show, according to the values of different scales, that the asymptotic behaviour of the model will be those of an incompressible fluid or of a geostrophic flow. The model under consideration is given by

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \dfrac{1}{\varepsilon}(\mathbf{g} \times \mathbf{u}^\varepsilon) + \dfrac{1}{\varepsilon^{2\beta}} \nabla p^\varepsilon = \mu \Delta \mathbf{u}^\varepsilon - (\mathbf{u}^\varepsilon \cdot \nabla)\, \mathbf{u}^\varepsilon - \dfrac{1}{2}(\operatorname{div} \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon \\ \varepsilon^{2\beta} \partial_t p^\varepsilon + \operatorname{div} \mathbf{u}^\varepsilon = 0, \end{cases} \tag{1}$$

where $x \in \Omega \subset \mathbb{R}^3$, $t \geq 0$, $\mathbf{u}^\varepsilon$ is the fluid velocity and p is the pressure, $\mathbf{g} = (0, 0, 1)$ is the rotation axis parallel to the vertical variable x_3 and $\beta > 1$ or $\beta = 1/2$. The aim of this paper is to perform a rigorous analysis of the singular limit as $\varepsilon \to 0$.

1.1. Motivation

The interest in studying the system (1) lies in the fact that, according to the physical model we are considering, it can be regarded in various ways and it can be considered as a toy model to understand the asymptotic equations resulting from different time and space scales.

This article is part of the topical collection "In memory of Antonin Novotny" edited by Eduard Feireisl, Paolo Galdi, and Milan Pokorny.

The present manuscript reflects part of the talk I gave at the workshop held in Prague in February 2019 in honour of Antonín Novotný and his sixtieth birthday. Not only was he a distinguished scholar with a deep sense of mathematics, but Antonín Novotný was also a man with an innate sense of humanity and empathy. This paper is dedicated to him.

As is well known, a very simple model for a rotating fluid is given by the incompressible Navier Stokes equations where to the momentum equation a term that considers the Coriolis effects is added. To be more precise a rotating fluid is characterised by the Rossby number Ro that takes into account the Coriolis effect and, for example, it can be used to model the oceans or atmosphere dynamics and meteorological phenomena. If we denote by U, L, T the characteristic fluid velocity, length and time scale respectively and by g_{ref} the local vertical component of the earth's rotation and we scale the equation in order to work in non dimensional setting, we have that the set of equations is given by:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \dfrac{1}{Ro}(\mathbf{g}\times\mathbf{u}) + \dfrac{1}{Ma^2}\nabla p = \Delta\mathbf{u} \\ \operatorname{div}\mathbf{u} = 0, \end{cases}$$

where we set equal to one the Strouhal and Reynolds number, Ma stands for the Mach number and the Rossby number Ro is given by the formula

$$Ro = \frac{U}{\sqrt{g_{ref}L}}.$$

Keeping Ma fixed, performing the limit as $Ro \to 0$ means that the scale motion of the fluid is much smaller than that of the earth. Despite the fact that there is a large literature concerning theoretical results for the above model (see [1,15,16]), the incompressibility constraint $\operatorname{div}\mathbf{u} = 0$ from a computational point of view is very expensive. In fact discretisation errors accumulate at each iteration and after a significant amount of error accumulation, the approximating algorithm breaks down ([23]). Moreover, the incompressibility assumption may be not very realistic in modelling many physical phenomena. A way to overcome these computational difficulties and, at the same time, to have a simple model, is to construct a family of perturbed equations. Indeed one takes the compressible Navier Stokes system with density ϱ and pressure $p = p(\varrho)$ and linearises it around a constant density state that for simplicity we can take as $\varrho = 1$. Then, the linearised continuity equation assumes the form

$$Ma\partial_t p + \operatorname{div}\mathbf{u} = 0, \tag{2}$$

where Ma is the Mach Number given by

$$Ma = \frac{U}{\sqrt{p'(1)}}.$$

The Eq. (2) is a "linearised" compressibility condition, namely we added to the incompressible constraint $\operatorname{div}\mathbf{u} = 0$ an "artificial compressibility" term $Ma\partial_t p$ that vanishes as $Ma \to 0$. This approximation method goes under the name of *artificial compressibility method* and was introduced by Chorin [2,3], Temam [21,22] and Oskolkov [18]. If in (1) we set the Rossby number equal to $Ro = \varepsilon$ and the Mach number as $Ma = \varepsilon^{2\beta}$ then, the system (1) is nothing else than the artificial compressibility approximation for a rotating fluid. Notice that the first equation of the system (1) compared to the momentum equation in the incompressible Navier Stokes equations has the extra term $-1/2(\operatorname{div} u^\varepsilon)u^\varepsilon$ which in the artificial compressible method is added as a correction to avoid the paradox of an increasing kinetic energy along the motion. Now, one of the main issue is to investigate in a rigorous way the limit as $\varepsilon \to 0$.

In the case we do not consider rotating terms, hence the singular limit is due only to the Mach number, the convergence of the artificial compressibility system to the incompressible Navier Stokes equation has been proved in several papers. In [21–23] the convergence is studied on bounded domains, while in [10–12] in the case of the whole space $\mathbb{R}^3$ and of the exterior domain. There are also some other results, where the artificial compressibility approximating system has been modified in a suitable way, for the Navier Stokes Fourier system in $\mathbb{R}^3$ in [6], for the MHD system in [7] and Navier–Stokes–Maxwell–Stefan system in [8].

In this paper compared to the previous mentioned paper we have an additional difficulty in the limit analysis because of the rotating term and the vanishing Rossby number. We will clarify this issue in the next Sect. 1.2.

However, we can look at the system (1) from a different point of view. As is well known, fluid dynamic equations are used to model various phenomena arising from physics, engineering, astrophysics. One of the models for atmosphere flows is given by the classical compressible fluid equation which include terms that take into account gravitation and rotation (see [17]). One feature of the atmospheric flows is that they take place at different time and length scales and it is important to understand which phenomena occur according to the use of single scales or to the interactions of them (i.e. internal gravity waves, Rossby waves, cloud formation). From a mathematical point of view, these various physical behaviour give rise to different singular limits and to different asymptotic behaviours of the governing equations.

A simplified set of equations (in the sense that we don't take into account temperature effects) that describes the atmosphere flow is given by

$$\frac{\epsilon^{\alpha_t}}{\epsilon^{\alpha_x}}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\mathbf{g}\times\mathbf{u}}{\epsilon^{\alpha_x-1}} + \frac{\epsilon^{\alpha_\pi}}{\epsilon^3}\nabla\tilde{\pi} = \mathbf{Q}_{\mathbf{u}}$$
$$\frac{\epsilon^{\alpha_t}}{\epsilon^{\alpha_x}}\partial_t \tilde{\pi} + \mathbf{u}\cdot\nabla\tilde{\pi} + \frac{\gamma\pi}{\epsilon^{\alpha_\pi}}(\operatorname{div}\mathbf{u}) = \mathbf{Q}_\pi,$$

with

$$\pi(t,x) = \bar{\pi}(x) + \epsilon^{\alpha_\pi}\Gamma\tilde{\pi}(t,x), \qquad \alpha_\pi > 0.$$

and $\mathbf{T} = t_{ref}/\epsilon^{\alpha_t}]$ the characteristic time, with $\alpha_t > 0$, $\mathbf{L} = l_{ref}/\epsilon^{\alpha_x}]$ the characteristic length $\alpha_x > 0$, where ϵ is the ratio between the thermal wind velocity u_{ref} and the internal wave speed c_{int}, $l_{ref} = h_{sc}$, $t_{ref} = h_{sc}/u_{ref}$ and h_{sc} is the density scale height. Moreover $\mathbf{Q}_{\mathbf{u}}$ and $\mathbf{Q}_\pi$ are source terms and we assume that they have an appropriate high order in ϵ to not affect the leading order asymptotic analysis, see [17].

From the previous equations it is clear that according to the different values of α_x, α_t, α_π we obtain a hierarchy of models that corresponds to the undergoing physical process of multiple-scale regime. In particular, if we focus on the so called advection timescales, that is $\alpha_x = \alpha_t$, we can observe that for $0 \le \alpha_\pi \le 3$, $\alpha_\pi \ne 2$ we get as the leading order equations the classical incompressible Navier Stokes equations. On the other hand if, in the advection time scale regimes we take $\alpha_x = \alpha_t = 2$ and we balance the Coriolis and the pressure gradient terms by taking $\alpha_\pi = 2$ we find out that the leading order dynamics is given by the geostrophic balance which corresponds to the quasi-geostrophic model in meteorology. The rigorous proof of those asymptotic behaviours cannot be performed with standard compactness methods since there are many competing processes that take place at the same time. One way to investigate and to go mathematically inside those behaviours is to work with a simplified model which includes the main feature of those singular limits. As we will better see in the next Sect. 1.2, the systems (1) serves as simplified model for the previous cases, in particular the case $\beta > 1$ corresponds to the first regime while $\beta = 1/2$ is the geostrophic balance regime.

1.2. Formal Limit Analysis and Main Mathematical Difficulties

As already mentioned in the Sect. 1.1 in the system (1) we can consider $\varepsilon^{2\beta}$ as the Mach Number Ma and ε as the Rossby number Ro and our purpose is to investigate the limit as $\varepsilon \to 0$. Clearly, in this scenario, we have the competition of two effects that act simultaneously. If we consider the low Mach number limit which corresponds to the physical state in which the fluid speed is much smaller than the sound speed, the fluid density becomes constant, the velocity is soleinoidal and the fluid is incompressible. Low Rossby number corresponds to fast rotation and, from experimental data, a high rotating fluid becomes planar. Therefore we have to distinguish two cases according to the different values of β:

[$\boldsymbol{\beta} > 1$] As $\varepsilon \to 0$ first, the low Mach number regime dominates and the fluid becomes incompressible then, it stabilises to a planar flow and so we and up with an incompressible planar fluid which, if we denote by $\mathbf{u}$ the limiting velocity, is described formally by the set of equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} = \nabla\pi, \quad \operatorname{div}\mathbf{u} = 0. \tag{3}$$

[$\boldsymbol{\beta} = \frac{1}{2}$] As $\varepsilon \to 0$ the speed of rotation and the incompressibility act on the same scale, so the fluid becomes solenoidal and planar at the same time and we end up, at least formally, with a single linear equation given by

$$\mathbf{g} \times \mathbf{u} + \nabla \pi = 0, \quad \operatorname{div} \mathbf{u} = 0. \tag{4}$$

The Eq. (4) describe the geostrophic balance, but from them it is not possible to determine the time evolution of the flow, so we have to find a more complete description of the limiting behaviour. Formally, this can be done by setting up an asymptotic expansion and, by looking at high order of ε, one obtains the following flow evolution (for more details on this formal derivation see [17]),

$$\partial_t(\nabla_h \pi - \pi) + \nabla_h^{\perp} \pi \cdot \nabla_h(\Delta_h \pi) = \Delta_h^2 \pi.$$

When we try to make rigorous the previous analysis, one of the main problems is that, as $\varepsilon \to 0$, the velocity field develops very fast oscillating waves in time (the so called *acoustic waves*). These waves are supported by the gradient part of the velocity field, they propagate along the motion and give rise to the loss of compactness for the nonlinear terms. It is quite obvious that if we take the initial data constructed in such a way that they are supported in the kernel of the acoustic wave operator, namely "well-prepared initial data" the limiting process is not affected by these waves.

In this paper we set the domain to be an infinite slab and we take very general initial data, we only require the boundedness of the initial energy and therefore we have to deal with the presence of the acoustic waves. Given the domain geometry, we expect that at a certain time these waves will loose and disperse their energy in the space domain. So, in order to control the oscillations we will develop a rigorous and detailed analysis of the local decay and dispersive behaviour of these waves.

This analysis will be different according to the different values of β. In fact we have to consider the dispersive behaviour of the acoustic waves and at the same time the fact that the fluid is under the effects of the centrifugal force that becomes large as $\varepsilon \to 0$. For these reasons we cannot use classical dispersive estimate of Strichartz type as in [10]. To be precise, as $\beta > 1$ the decay of the acoustic wave is strong enough to eliminate the centrifugal force and we will perform some rigorous decay estimate of the acoustic waves in the spirit of D'Ancona and Racke [5] and Sogge [20], see also [9]. If $\beta = 1/2$ then, the incompressible regime and the high rotation occur at the same scale, we cannot exploit the local decay of the oscillating waves but we have to analyse the spectral properties of the rotating operator. We have to show that the fast oscillating parts of the gradient live in the space orthogonal to the kernel of the rotating operator and so they don't affect our limiting process. The basic tool in this case will be the RAGE theorem which relates the solutions long time behaviour and the spectral properties of the corresponding operator see for example [9].

1.3. Plan of the Paper

In Sect. 2 we set the problem, we define the notion of weak solutions we are going to use and we state the main results of this paper, Theorems 2.2 and 2.3. In Sect. 3 we perform the uniform a priori estimate (with respect to ε) satisfied by the solutions of the system (1) for any $\beta > 1$ and $\beta = 1/2$. In Sect. 4 we perform the limit analysis for the case $\beta > 1$ and we prove the Theorem 2.2. Finally, in Sect. 5, we deal with the case $\beta = 1/2$ and we prove the Theorem 2.3.

2. Setting of the Problem and Main Results

2.1. Notation

We fix here the main notations we are going to use through the paper.

- $C_0^\infty([0,T) \times \Omega)$ is the space of C^∞ functions with compact support.

- $W^{k,p}(\Omega)$ is the usual Sobolev space on Ω and $H^k(\Omega) = W^{k,2}(\Omega)$.
- The notations $L^p_t L^q_x$ and $L^p_t W^{k,q}_x$ will abbreviate respectively the spaces $L^p([0,T); L^q(\Omega))$, and $L^p([0,T); W^{k,q}(\Omega))$.
- Q and P are the Leray's projectors on the space of gradients vector fields and on the space of divergence - free vector fields respectively, namely

$$P = I - Q.$$

- $\nabla_h^\perp f$ denotes the vector $(\partial_{x_2} f, -\partial_{x_1} f)$. The differential operators ∇_h, div_h, Δ_h denote the usual ∇, div, Δ applied on the horizontal variables $x_h = (x_1, x_2)$.
- For a function f the vertical average on the one dimensional torus $\mathbb{T}^1$ is defined by

$$\langle f(x_h) \rangle = \frac{1}{|\mathbb{T}^1|} \int_{\mathbb{T}^1} f(x_h, x_3) dx_3.$$

2.2. Setting of the Problem

We consider the following system

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \frac{1}{\varepsilon}(\mathbf{g} \times \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^{2\beta}} \nabla p^\varepsilon = \mu \Delta \mathbf{u}^\varepsilon - (\mathbf{u}^\varepsilon \cdot \nabla)\, \mathbf{u}^\varepsilon - \frac{1}{2}(\mathrm{div}\, \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon \\ \\ \varepsilon^{2\beta} \partial_t p^\varepsilon + \mathrm{div}\, \mathbf{u}^\varepsilon = 0, \end{cases} \tag{5}$$

where $x \in \Omega \subset \mathbb{R}^3$, $t \geq 0$, $\mu \in \mathbb{R}$ and $\mathbf{g} = (0,0,1)$ is the rotation axis parallel to the the vertical variable x_3. The geometry of the physical space Ω is given by an infinite slab,

$$\Omega = \mathbb{R}^2 \times (0,1).$$

and we will denote the horizontal variable as $x_h = (x_1, x_2)$.

For the velocity field $\mathbf{u}^\varepsilon$ we assume the complete slip boundary conditions

$$\mathbf{u}^\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0, \tag{6}$$

where $\mathbf{n}$ denotes the outer normal vector to the boundary and $\mathbb{S}$ is the viscous stress tensor given by

$$\mathbb{S}(\nabla \mathbf{u}^\varepsilon) = \mu \left(\nabla \mathbf{u}^\varepsilon + \nabla^t \mathbf{u}^\varepsilon - \frac{2}{3}\, \mathrm{div}\, \mathbb{I} \right), \quad \mu > 0.$$

From now on, without loss of generality, for simplicity we set $\mu = 1$.

In order do deal with the boundary conditions (6) it is more convenient to reformulate the state variable in a periodic setting for the variable x_3. In fact we will take

$$\Omega = \mathbb{R}^2 \times \mathbb{T}^1,$$

where $\mathbb{T}^1$ is the one dimensional torus and where the pressure is extended even in the third variable,

$$p^\varepsilon(x_1, x_2, -x_3) = p^\varepsilon(x_1, x_2, x_3),$$

as well as the horizontal component of the velocity $\mathbf{u}_h^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$,

$$u_j^\varepsilon(x_1, x_2, -x_3) = u_j^\varepsilon(x_1, x_2, x_3), \quad j = 1, 2,$$

while the vertical component u_3^ε is taken odd,

$$u_3^\varepsilon(x_1, x_2, -x_3) = -u_3^\varepsilon(x_1, x_2, x_3).$$

Furthermore we assign to the system (5) the following initial conditions

$$\mathbf{u}^\varepsilon(x, 0) = \mathbf{u}_0^\varepsilon(x), \;\; p^\varepsilon(x, 0) = p_0^\varepsilon(x). \tag{7}$$

The regularity and the limiting behaviour as $\varepsilon \to 0$ of the initial data (7) deserve a little discussion. Indeed the system (5) requires the initial conditions (7) while the target Eqs. (3) and (4) require only the initial condition for the velocity $\mathbf{u}$. Hence, our approximation will be consistent if the initial datum on the pressure p^ε will be eliminated by an "initial layer" phenomenon. Since we deal with weak solutions in the sense of the Definition 2.1 it is reasonable to require the finite energy constraint to be satisfied by the approximating sequences $(\mathbf{u}^\varepsilon, p^\varepsilon)$. So we can deduce a natural behaviour to be imposed on the initial data $(\mathbf{u}_0^\varepsilon, p_0^\varepsilon)$, namely

$$\begin{aligned} &\mathbf{u}_0^\varepsilon, \quad p_0^\varepsilon \in L^2(\Omega) \\ &\mathbf{u}_0^\varepsilon \rightharpoonup \mathbf{u}_0, \quad p_0^\varepsilon \rightharpoonup p_0 \quad \text{weakly in } L^2(\Omega). \end{aligned} \tag{8}$$

For completeness we recall the notion of weak solutions for the system (5) we are going to use.

Definition 2.1. We say that a pair $\mathbf{u}^\varepsilon, p^\varepsilon$ is a weak solution to the system (5) in $(0,T)\times\Omega$ if $u^\varepsilon \in L^\infty([0,T];L^2(\mathbb{R}^3)) \cap L^2([0,T];\dot{H}^1(\mathbb{R}^3))$. $p^\varepsilon \in L^\infty([0,T];L^2(\mathbb{R}^3))$ and they satisfy (5) in the sense of distributions, namely

$$\begin{aligned} &\int_0^T\int_\Omega \Big(\mathbf{u}^\varepsilon\partial_t\varphi - ((\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon + \frac{1}{2}(\operatorname{div}\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon)\cdot\varphi \\ &\qquad - \frac{1}{\varepsilon}(\mathbf{g}\times\mathbf{u}^\varepsilon)\cdot\varphi + \frac{1}{\varepsilon^{2\beta}}p^\varepsilon \operatorname{div}\varphi\Big)dxdt \\ &\quad = \int_0^T\int_\Omega \nabla\mathbf{u}^\varepsilon : \nabla\varphi dxdt - \int_\Omega \mathbf{u}_0^\varepsilon\cdot\varphi(0,\cdot)dx, \end{aligned} \tag{9}$$

for any $\varphi \in C_0^\infty([0,T)\times\Omega;\mathbb{R}^3)$ and

$$\int_0^T\int_\Omega \left(\varepsilon^{2\beta}p^\varepsilon\partial_t\varphi + \mathbf{u}^\varepsilon\cdot\nabla\varphi\right)dxdt = -\int_\Omega \varepsilon^{2\beta}p_0^\varepsilon\varphi(0,\cdot)dx,$$

for any $\varphi \in C_0^\infty([0,T)\times\Omega)$. Moreover the following energy inequality holds

$$\begin{aligned} &\frac{1}{2}\int_\Omega (|\mathbf{u}^\varepsilon(x,t)|^2 + |p^\varepsilon(x,t)|^2)dx + \int_0^t\int_\Omega |\nabla\mathbf{u}(x,s)|^2dxds \\ &\le \frac{1}{2}\int_\Omega (|\mathbf{u}_0^\varepsilon(x)|^2 + |p_0^\varepsilon(x)|^2)dx, \qquad \text{for all } t\ge 0. \end{aligned}$$

The proof of the existence of global in time weak solutions for (5) is omitted since, in the spirit of Temam (see Chapter III, Theorem 8.1 in [23]), it follows by standard finite dimensional Galerkin type approximations with the necessary modification due to the domain Ω.

2.3. Main Results

Now we are ready to state the main results of this paper.

Theorem 2.2 [(**Case $\beta \ge 1$**)]. Assume that $\mathbf{u}^\varepsilon, p^\varepsilon$ are weak solutions of the system (5) with initial data (7) satisfying (8), then there exists $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega))$, such that

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{10}$$

$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \quad \text{strongly in } L^2_{loc}((0,T)\times\Omega), \tag{11}$$

where $\mathbf{u} = [\mathbf{u}_h(t,x_h),0]$ is the unique weak solution of the 2D incompressible Navier Stokes equation

$$\operatorname{div}_h\mathbf{u} = 0, \tag{12}$$

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_h)\mathbf{u} + \nabla_h\pi = \Delta_h\mathbf{u}. \tag{13}$$

Theorem 2.3 [(**Case** $\boldsymbol{\beta} = 1/2$)]. Assume that $\mathbf{u}^\varepsilon, p^\varepsilon$ are weak solutions of the system (5) with initial data (7) satisfying (8), then there exists $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega))$, $\pi \in L^\infty(0,T;L^2(\Omega))$such that

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{14}$$

$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \quad \text{strongly in } L^2_{loc}((0,T)\times\Omega), \tag{15}$$

$$p^\varepsilon \rightharpoonup \pi \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)), \tag{16}$$

where $\mathbf{u}$ and π satisfy

$$\operatorname{div}_h \mathbf{u} = 0, \tag{17}$$

$$\mathbf{g}\times\mathbf{u} + \nabla\pi = 0 \tag{18}$$

and π is a solution in the sense of distribution of the equation

$$\partial_t(\nabla_h\pi - \pi) + \nabla_h^\perp\pi\cdot\nabla_h(\Delta_h\pi) = \Delta_h^2\pi. \tag{19}$$

Concerning the limiting Eqs. (18) and (19) it is important to remark that in the geostrophic balance regime the acoustic waves are asymptotically filtered out. This type of phenomenon corresponds to quasi-geostrophic regime with the advection timescale.

3. Energy Estimate and Uniform Bounds

We define the energy functional associated to the system (5) as

$$E(t) = \frac{1}{2}\int_\Omega \left(|\mathbf{u}^\varepsilon(x,t)|^2 + |p^\varepsilon(x,t)|^2\right)dx.$$

By standard computations it is straightforward to prove that the weak solutions of the system (5), satisfy the energy inequality

$$E(t) + \int_0^t\int_{\mathbb{R}^3}|\nabla\mathbf{u}^\varepsilon(x,s)|^2 dxds \le E(0). \tag{20}$$

As a consequence of (20) we obtain the following uniform bounds

$$\mathbf{u}^\varepsilon,\ p^\varepsilon \quad \text{are bounded in } L^\infty([0,T];L^2(\Omega)), \tag{21}$$

$$\nabla\mathbf{u}^\varepsilon \quad \text{is bounded in } L^2([0,T]\times\Omega). \tag{22}$$

By combining (20) with standard Sobolev embeddings we deduce that

$$\mathbf{u}^\varepsilon \quad \text{is bounded in } L^\infty([0,T];L^2(\Omega))\cap L^2([0,T];L^6(\Omega)). \tag{23}$$

Using together (22) and (23) we have that

$$\begin{aligned}&(\mathbf{u}^\varepsilon\cdot\nabla)\mathbf{u}^\varepsilon \quad \mathbf{u}^\varepsilon \operatorname{div}\mathbf{u}^\varepsilon\\ &\text{are bounded in } L^2([0,T];L^1(\Omega))\cap L^1([0,T];L^{3/2}(\Omega)).\end{aligned} \tag{24}$$

We point out that the previous estimates are uniform in ε and hold for any value of $\beta > 0$.

4. Case $\beta > 1$

This section is devoted to the analysis of the limiting behaviour as $\varepsilon \to 0$ in the case $\beta > 1$. As a first step we recover the convergence results that follows from the bounds of the previous Sect. 3. Then, as already mentioned in the introduction we have to deal with the high oscillations of the acoustic waves. Since these waves are supported by the gradient part of the velocity, we decompose the velocity in its gradient and soleinoidal part and, since the domain Ω is infinite, we will be able to get decay estimates for the acoustic potential (gradient part of the velocity). As a last step we deal with the convergence of the velocity divergence free part and, in order to get the strong convergence, we estimate the vertical average and the oscillations of the solenoidal component of $\mathbf{u}^\varepsilon$.

4.1. First Convergence Results

From (22) and (23) we have that

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2([0,T]; H^1(\Omega)). \tag{25}$$

Hence, by taking into account (21) and (25), and letting $\varepsilon \to 0$ in (5_2) we obtain

$$\operatorname{div} \mathbf{u} = 0 \quad \text{a.e. in } (0,T) \times \Omega. \tag{26}$$

Moreover, if we apply the Leray projector P on (5_1), as $\varepsilon \to 0$ we have

$$P(\mathbf{g} \times \mathbf{u}) = 0,$$

from which it follows that $\mathbf{g} \times \mathbf{u} = \nabla G$, for a certain potential G. As a consequence, the limiting velocity horizontal component $\mathbf{u}_h = (u_1, u_2)$ does not depend on the vertical variable x_3, then by using (26), we have that $\partial_{x_3} u_3 = 0$. If we take into account the boundary condition (6) and the fact that $\mathbf{u} \in L^2_t L^2_x$ we can conclude that

$$\mathbf{u} = (\mathbf{u}_h(t, x_h), 0), \quad u_3 = 0. \tag{27}$$

Now, if we choose a test function of the form $\varphi(x,t) = (\varphi_h(t,x_h), 0)$, $\operatorname{div} \varphi = 0$, it is possible to pass into the limit in the weak formulation of (5_1), provided we know how to handle the nonlinear terms $(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon$, $\mathbf{u}^\varepsilon \operatorname{div} \mathbf{u}^\varepsilon$. In the next section, by studying separately the gradient and solenoidal part of $\mathbf{u}^\varepsilon$, we will get stronger convergence results.

4.2. Acoustic Equation and Estimates

In this section we will analyse the behaviour of the acoustic waves in order to control their fast oscillation in time. To this end we rewrite the system (5) in the following form

$$\begin{cases} \varepsilon^{2\beta} \partial_t p^\varepsilon + \operatorname{div} \mathbf{u}^\varepsilon = 0 \\ \\ \varepsilon^{2\beta} \partial_t \mathbf{u}^\varepsilon + \nabla p^\varepsilon = -\varepsilon^{2\beta-1}(\mathbf{g} \times \mathbf{u}^\varepsilon) + \varepsilon^{2\beta}(\mu \Delta \mathbf{u}^\varepsilon - (\mathbf{u}^\varepsilon \cdot \nabla)\, \mathbf{u}^\varepsilon - \frac{1}{2}(\operatorname{div} \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon). \end{cases} \tag{28}$$

We can observe that the underlying structure of the system (28) is that of a wave equation in fact it goes under the name of acoustic wave system. In order to simplify the notations we rewrite (28) as

$$\begin{cases} \varepsilon^{2\beta} \partial_t p^\varepsilon + \operatorname{div} \mathbf{u}^\varepsilon = 0 \\ \\ \varepsilon^{2\beta} \partial_t \mathbf{u}^\varepsilon + \nabla p^\varepsilon = -\varepsilon^{2\beta-1} \mathbb{F}_1^\varepsilon + \varepsilon^{2\beta} \operatorname{div} \mathbb{F}_2^\varepsilon + \varepsilon^{2\beta} \mathbb{F}_3^\varepsilon, \end{cases} \tag{29}$$

where by taking into account (22)–(24) we have $\mathbb{F}_1^\varepsilon \in L^\infty_t L^2_x$, $\mathbb{F}_2^\varepsilon \in L^2_t L^2_x$, $\mathbb{F}_3^\varepsilon \in L^2_t L^1_x$, uniformly in ε. Since the equations in (29) are satisfied only in a weak sense it is more convenient to regularise them in order to deal with smooth solutions. To this purpose, given a function v, and j_δ a standard Friedrich's mollifier we denote by

$$v^\delta = j_\delta * v$$

the regularised function. Having in mind these notations we regularise the system (29) and we get

$$\begin{cases} \varepsilon^{2\beta}\partial_t p^{\varepsilon,\delta} + \operatorname{div} \mathbf{u}^{\varepsilon,\delta} = 0 \\ \varepsilon^{2\beta}\partial_t \mathbf{u}^{\varepsilon,\delta} + \nabla p^{\varepsilon,\delta} = \varepsilon^{2\beta-1}\mathbb{F}_1^{\varepsilon,\delta} + \varepsilon^{2\beta} \operatorname{div} \mathbb{F}_2^{\varepsilon,\delta} + \varepsilon^{2\beta}\mathbb{F}_3^{\varepsilon,\delta}, \end{cases} \tag{30}$$

where, for the right hand side of (30_2) the following uniform bounds in ε hold

$$\|\mathbb{F}_1^{\varepsilon,\delta}\|_{L^2_t H^k} + \|\mathbb{F}_2^{\varepsilon,\delta}\|_{L^2_t H^k} + \|\mathbb{F}_3^{\varepsilon,\delta}\|_{L^2_t H^k} \leq c(k,\delta), \qquad \text{for any } k = 0, 1, \dots$$

Since the acoustic waves are supported by the gradient part of the velocity fields, we decompose $\mathbf{u}^{\varepsilon,\delta}$ in the following way

$$\mathbf{u}^{\varepsilon,\delta} = \mathbf{Z}^{\varepsilon,\delta} + \nabla\Psi^{\varepsilon,\delta}, \tag{31}$$

where $\mathbf{Z}^{\varepsilon,\delta} = P\mathbf{u}^{\varepsilon,\delta}$, $\nabla\Psi^{\varepsilon,\delta} = Q\mathbf{u}^{\varepsilon,\delta}$ and we rewrite (30) in terms of the acoustic potential $\Psi^{\varepsilon,\delta}$,

$$\varepsilon^{2\beta}\partial_t p^{\varepsilon,\delta} + \Delta\Psi^{\varepsilon,\delta} = 0, \tag{32}$$

$$\varepsilon^{2\beta}\partial_t \Psi^{\varepsilon,\delta} + p^{\varepsilon,\delta} = \varepsilon^{2\beta-1}\Delta^{-1}\operatorname{div} \mathbb{F}_1^{\varepsilon,\delta} + \varepsilon^{2\beta}\Delta^{-1}\operatorname{div}(\operatorname{div} \mathbb{F}_2^{\varepsilon,\delta} + \mathbb{F}_3^{\varepsilon,\delta}). \tag{33}$$

Now it is evident that to estimate $\Psi^{\varepsilon,\delta}$ we have to exploit the dispersive behaviour of the system (32), (33) from which we will deduce the local decay of the acoustic potential. So we recall here the following lemma for the proof of which see Feireisl et al. [14] or D'Ancona and Racke [5].

Lemma 4.1 *Consider $\varphi \in C_0^\infty(\mathbb{R}^2)$. Then we have*

$$\int_{-\infty}^{\infty}\int_\Omega \left|\varphi(x_h)\exp\left(\mathrm{i}\sqrt{-\Delta}t\right)[v]\right|^2 dxdt \leq c(\varphi)\|v\|^2_{L^2(\Omega)}. \tag{34}$$

Moreover, on any compact set $K \subset \Omega$, $m > 0$, we have

$$\int_0^T\int_K \left|\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon^m}\right)[v]\right|^2 dxdt \leq \varepsilon^m \int_0^\infty \int_\Omega \left|\exp\left(\mathrm{i}\sqrt{-\Delta}t\right)[v]\right|^2 dxdt \leq \varepsilon^m c\|v\|^2_{L^2(\Omega)}, \tag{35}$$

and

$$\int_0^T\int_K \left|\int_0^t \exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^m}\right)[g(s)]\right|^2 dxdt \leq cT\varepsilon^m \int_0^T \left\|\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{s}{\varepsilon^m}\right)[g(s)]\right\|^2_{L^2(\Omega)} = \varepsilon^m\|g\|_{L^2((0,T)\times\Omega)}. \tag{36}$$

By means of Duhamel's formula the gradient of the solution $\Psi^{\varepsilon,\delta}$ of (33) is given by

$$\begin{aligned}
&\nabla\Psi^{\varepsilon,\delta}(t)\\
&= \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon^{2\beta}}\right)\left[\nabla\Psi^{\varepsilon,\delta}(0)+\frac{\mathrm{i}}{\sqrt{-\Delta}}\nabla p^{\varepsilon,\delta}(0)\right]\\
&\quad+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon^{2\beta}}\right)\left[\nabla\Psi^{\varepsilon,\delta}(0)-\frac{\mathrm{i}}{\sqrt{-\Delta}}\nabla p^{\varepsilon,\delta}(0)\right]\\
&\quad+\frac{\varepsilon^{-1}}{2}\int_0^t\left(\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^{2\beta}}\right)+\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^{2\beta}}\right)\right)[\nabla\Delta^{-1}\mathrm{div}\,\mathbb{F}_1^{\varepsilon,\delta}]ds\\
&\quad+\frac{1}{2}\int_0^t\left(\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^{2\beta}}\right)+\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^{2\beta}}\right)\right)[\nabla\Delta^{-1}\mathrm{divdiv}\,\mathbb{F}_2^{\varepsilon,\delta}]ds\\
&\quad+\frac{1}{2}\int_0^t\left(\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^{2\beta}}\right)+\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t-s}{\varepsilon^{2\beta}}\right)\right)[\nabla\Delta^{-1}\mathrm{div}\,\mathbb{F}_3^{\varepsilon,\delta}]ds.
\end{aligned}$$

Now, by applying (35) and (36) with $m=2\beta$ we have the following uniform in ε decay estimate for the acoustic potential,

$$\int_0^T\|\nabla\Psi^{\varepsilon,\delta}\|^2_{L^2(K)}dt\le(\varepsilon^{2\beta-1}+\varepsilon^{2\beta})c(\delta,K,T), \tag{37}$$

for any compact set $K\subset\Omega$ and $\beta\ge 1$. Hence, we can conclude that the effects of the acoustic potential vanishes as soon as $\varepsilon\to 0$, so the limiting behaviour of the system (5) depends only on the soleinoidal component of the velocity field. Finally, we point out that, at this stage, in order to have a negligible effect of the acoustic potential, it is enough to require $\beta>1/2$.

4.3. Convergence of the Soleinoidal Part of the Velocity

From the previous section we understood that the asymptotic behaviour of the system (5) depends on the solenoidal part of $\mathbf{u}^{\varepsilon,\delta}$, hence in this section we will focus on $\mathbf{Z}^{\varepsilon,\delta}=P\mathbf{u}^{\varepsilon,\delta}$. Since in the Sect. 4.1 we have proved that $\mathbf{u}^{\varepsilon,\delta}$ converges to a function $\mathbf{u}$ which depends only on the horizontal variables, in order get the strong convergence of $P\mathbf{u}^{\varepsilon,\delta}$ a first step is to establish the compactness of the vertical average of $\mathbf{u}^{\varepsilon,\delta}$. Therefore we rewrite the equation for $\mathbf{u}^{\varepsilon,\delta}$ as follows

$$\varepsilon\partial_t\mathbf{u}^{\varepsilon,\delta}+\mathbf{g}\times\mathbf{u}^{\varepsilon,\delta}=\varepsilon\mathbf{S}^{\varepsilon,\delta}-\varepsilon^{1-2\beta}\nabla p^{\varepsilon,\delta}, \tag{38}$$

where

$$\mathbf{S}^{\varepsilon,\delta}=\mu\Delta\mathbf{u}^{\varepsilon,\delta}-(\mathbf{u}^{\varepsilon,\delta}\cdot\nabla)\mathbf{u}^{\varepsilon,\delta}-\frac{1}{2}(\mathrm{div}\,\mathbf{u}^{\varepsilon,\delta})\mathbf{u}^{\varepsilon,\delta}$$

and $\mathbf{S}^{\varepsilon,\delta}$ is bounded in $L^2_tH^k_x$, for any fixed k and δ. We take now the vertical average of (38),

$$\varepsilon\partial_t\langle\mathbf{u}^{\varepsilon,\delta}\rangle+(\mathbf{g}\times\langle\mathbf{u}^{\varepsilon,\delta}\rangle)=\varepsilon\langle\mathbf{S}^{\varepsilon,\delta}\rangle-\varepsilon^{1-2\beta}\nabla\langle p^{\varepsilon,\delta}\rangle. \tag{39}$$

Since $\mathbf{Z}^{\varepsilon,\delta}$ is soleinoidal one can easily check that $P(\mathbf{g}\times\langle\mathbf{Z}^{\varepsilon,\delta}\rangle)=0$. Hence, by taking a test function $\varphi\in C_0^\infty(\Omega;\mathbb{R}^3)$, $\mathrm{div}\,\varphi=0$ in the weak formulation of (39) we obtain

$$\partial_t\int_\Omega\langle\mathbf{u}^{\varepsilon,\delta}\rangle\cdot\varphi dx=\int_\Omega\langle\mathbf{S}^{\varepsilon,\delta}\rangle\cdot\varphi dx-\frac{1}{\varepsilon}\int_\Omega(\mathbf{g}\times\langle\nabla\Psi^{\varepsilon,\delta}\rangle)\cdot\varphi dx. \tag{40}$$

If we use (37) and taking into account that $\beta>1$, by applying Lions Aubin Lemma arguments (see [19]) we can conclude that

$$\langle\mathbf{Z}^{\varepsilon,\delta}\rangle\longrightarrow\mathbf{u}^\delta,\qquad\text{strongly in } L^2((0,T)\times K), \tag{41}$$

for any compact set $K\subset\Omega$ and any fixed δ. It is worthful to remark that at this step it is crucial (37) and here to estimate the last term in the right hand side of (40) we need the restriction $\beta>1$ for the constant β . Finally, in order to get the limiting behaviour of $\mathbf{Z}^{\varepsilon,\delta}$ it is fundamental to control some possible oscillations.

Since we proved that the horizontal component of $\mathbf{u}^{\varepsilon,\delta}$ is compact we can infer that the oscillations are due to the vector fields that depends on x_3. In the remaining part of this section we will study and estimate these oscillations and we will show that they don't interfere in the convergence of the nonlinear terms. For any function f we denote the oscillation as

$$\{f\}(x) = f(x) - \langle f\rangle(x_h).$$

Notice that $\{f\}(x)$ has zero vertical mean and so it can be written for some function I as

$$\{f\}(x) = \partial_{x_3} I(x), \quad \text{with} \quad \int_{\mathbb{T}^1} I(x)dx_3 = 0.$$

Then we define for any $i, j = 1, 2, 3$

$$\omega_{i,j}^{\varepsilon,\delta} = \partial_{x_i} Z_j^{\varepsilon,\delta} - \partial_{x_j} Z_i^{\varepsilon,\delta} = \partial_{x_i} u_j^{\varepsilon,\delta} - \partial_{x_j} u_i^{\varepsilon,\delta}.$$

From (39) we have that $\omega_{i,j}^{\varepsilon,\delta}$ satisfy the following equations

$$\varepsilon\partial_t\omega_{1,2}^{\varepsilon,\delta} + \mathrm{div}_h[\mathbf{Z}^{\varepsilon,\delta}]_h = \varepsilon\left(\partial_{x_1} S_2^{\varepsilon,\delta} - \partial_{x_2} S_1^{\varepsilon,\delta}\right) - \Delta_h \Psi^{\varepsilon,\delta}, \tag{42}$$

$$\varepsilon\partial_t\omega_{1,3}^{\varepsilon,\delta} + \partial_{x_3} Z_2^{\varepsilon,\delta} = \varepsilon\left(\partial_{x_1} S_3^{\varepsilon,\delta} - \partial_{x_3} S_1^{\varepsilon,\delta}\right) - \partial^2_{x_3 x_2}\Psi^{\varepsilon,\delta}, \tag{43}$$

$$\varepsilon\partial_t\omega_{2,3}^{\varepsilon,\delta} - \partial_{x_3} Z_1^{\varepsilon,\delta} = \varepsilon\left(\partial_{x_2} S_3^{\varepsilon,\delta} - \partial_{x_3} S_2^{\varepsilon,\delta}\right) - \partial^2_{x_3 x_1}\Psi^{\varepsilon,\delta}. \tag{44}$$

Now by using the decomposition (31) we rewrite the nonlinear terms of the system (5) as,

$$\begin{aligned}(\mathbf{u}^{\varepsilon,\delta}\cdot\nabla)\mathbf{u}^{\varepsilon,\delta} + \frac{1}{2}\mathbf{u}^{\varepsilon,\delta}\,\mathrm{div}\,\mathbf{u}^{\varepsilon,\delta} &= \mathrm{div}(\mathbf{u}^{\varepsilon,\delta}\otimes\mathbf{u}^{\varepsilon,\delta}) - \frac{1}{2}\mathbf{u}^{\varepsilon,\delta}\,\mathrm{div}\,\mathbf{u}^{\varepsilon,\delta}\\ &= \mathrm{div}(\mathbf{Z}^{\varepsilon,\delta}\otimes\mathbf{Z}^{\varepsilon,\delta}) + \mathrm{div}(\nabla\Psi^{\varepsilon,\delta}\otimes\nabla\Psi^{\varepsilon,\delta})\\ &\quad + \mathrm{div}(\mathbf{Z}^{\varepsilon,\delta}\otimes\nabla\Psi^{\varepsilon,\delta}) + \mathrm{div}(\nabla\Psi^{\varepsilon,\delta}\otimes\mathbf{Z}^{\varepsilon,\delta}) - \frac{1}{2}\mathbf{u}^{\varepsilon,\delta}\Delta\Psi^{\varepsilon,\delta}. \end{aligned}\tag{45}$$

By taking into account the local decay (37) of the acoustic potential we see that the only term of (45) who requires a detailed analysis is

$$\mathrm{div}(\mathbf{Z}^{\varepsilon,\delta}\otimes\mathbf{Z}^{\varepsilon,\delta}) = \frac{1}{2}\nabla|\mathbf{Z}^{\varepsilon,\delta}|^2 - \mathbf{Z}^{\varepsilon,\delta}\times\mathrm{curl}[\mathbf{Z}^{\varepsilon,\delta}]. \tag{46}$$

Since the first term in the right hand side of (46) is a gradient we focus on the second term,

$$\begin{aligned}\mathbf{Z}^{\varepsilon,\delta}\times\mathrm{curl}\,\mathbf{Z}^{\varepsilon,\delta} = \langle\mathbf{Z}^{\varepsilon,\delta}\rangle\times\mathrm{curl}\langle\mathbf{Z}^{\varepsilon,\delta}\rangle &\\ + \partial_{x_3}\left(\langle\mathbf{Z}^{\varepsilon,\delta}\rangle\times\mathrm{curl}\, I[\mathbf{Z}^{\varepsilon,\delta}] + I[\mathbf{Z}^{\varepsilon,\delta}]\times\mathrm{curl}\langle\mathbf{Z}^{\varepsilon,\delta}\rangle\right) &\\ + \partial_{x_3} I[\mathbf{Z}^{\varepsilon,\delta}]\times\partial_{x_3}\mathrm{curl}\langle\mathbf{Z}^{\varepsilon,\delta}\rangle.&\end{aligned}\tag{47}$$

The first term of the right hand side of (47) is compact because of (41), the second term has zero vertical mean, hence we have to study carefully only the last one. For any $j = 1, 2, 3$ we have

$$\begin{aligned}&[\partial_{x_3} I[\mathbf{Z}^{\varepsilon,\delta}]\times\partial_{x_3}\mathrm{curl}\langle\mathbf{Z}^{\varepsilon,\delta}\rangle]_j\\ &= \partial_{x_3} I[Z_i^{\varepsilon,\delta}]\partial_{x_3}(\partial_{x_i} I[Z_j^{\varepsilon,\delta}] - \partial_{x_j} I[Z_i^{\varepsilon,\delta}]) = \partial_{x_3} I[Z_i^{\varepsilon,\delta}]\partial_{x_3} I[\omega_{i,j}^{\varepsilon,\delta}].\end{aligned}\tag{48}$$

From the relations (42)–(44) it is easy to obtain,

$$\varepsilon\partial_t(\partial_{x_3} I[\omega_{1,3}^{\varepsilon,\delta}]) + \partial^2_{x_3} I[Z_2^{\varepsilon,\delta}] = \varepsilon\left(\partial_{x_1}(S_3^{\varepsilon,\delta} - \langle S_3^{\varepsilon,\delta}\rangle) - \partial_{x_3} S_1^{\varepsilon,\delta}\right) - \partial^2_{x_3 x_2}\Psi^{\varepsilon,\delta}, \tag{49}$$

$$\varepsilon\partial_t(\partial_{x_3} I[\omega_{2,3}^{\varepsilon,\delta}]) - \partial^2_{x_3} I[Z_1^{\varepsilon,\delta}] = \varepsilon\left(\partial_{x_2}(S_3^{\varepsilon,\delta} - \langle S_3^{\varepsilon,\delta}\rangle) - \partial_{x_3} S_2^{\varepsilon,\delta}\right) - \partial^2_{x_3 x_1}\Psi^{\varepsilon,\delta}. \tag{50}$$

Now we compute the three functions in (48) in terms of the functions $\omega^{\varepsilon,\delta}_{i,j}$, we start with $j = 1$,

$$\begin{aligned}&[\partial_{x_3} I[\mathbf{Z}^{\varepsilon,\delta}] \times \partial_{x_3} \operatorname{curl}\langle \mathbf{Z}^{\varepsilon,\delta}\rangle]_1\\ &\quad= \partial_{x_3} I[Z^{\varepsilon,\delta}_2]\partial_{x_3} I[\omega^{\varepsilon,\delta}_{2,1}] + \partial_{x_3} I[Z^{\varepsilon,\delta}_3]\partial_{x_3} I[\omega^{\varepsilon,\delta}_{3,1}]\\ &\quad= \partial_{x_3}\left(\partial_{x_3} I[Z^{\varepsilon,\delta}_2] I[\omega^{\varepsilon,\delta}_{2,1}]\right) - \partial^2_{x_3} I[Z^{\varepsilon,\delta}_2] I[\omega^{\varepsilon,\delta}_{2,1}] - I[\operatorname{div}_h \mathbf{Z}^{\varepsilon,\delta}_h]\partial_{x_3} I[\omega^{\varepsilon,\delta}_{3,1}].\end{aligned} \tag{51}$$

Now, if in the relation (51) we use (49) and (50) and we take into account (37) and that $\mathbf{S}^{\varepsilon,\delta}$ is bounded in $L^2_t H^k$ for any fixed k and δ, by performing the same type of computation for any $j = 1, 2, 3$ we have, as $\varepsilon \to 0$ (for more details see [16] or [14]),

$$\frac{1}{\mathbb{T}^1}\int_0^T\int_\Omega \operatorname{div}(\mathbf{Z}^{\varepsilon,\delta} \otimes \mathbf{Z}^{\varepsilon,\delta})\varphi dxdt \longrightarrow \frac{1}{\mathbb{T}^1}\int_0^T\int_{\mathbb{R}^2} \operatorname{div}(\mathbf{u}^\delta \otimes \mathbf{u}^\delta)\varphi_h dx_h dt,$$

for any fixed $\delta > 0$ and for any $\varphi = (\varphi_h(t, x_h), 0)$, $\varphi_h \in C^\infty_0([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$, $\operatorname{div}_h\varphi_h = 0$.

4.4. Proof of the Theorem 2.2

In oder to conclude the proof of the Theorem 2.2 we just need to decompose the nonlinear term $\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon = \operatorname{div}(\mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) - \mathbf{u}^\varepsilon \operatorname{div}\mathbf{u}^\varepsilon$ as follows

$$\begin{aligned}\mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon + \frac12\mathbf{u}^\varepsilon \operatorname{div}\mathbf{u}^\varepsilon &= \operatorname{div}(\mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) - \frac12\mathbf{u}^\varepsilon \operatorname{div}\mathbf{u}^\varepsilon\\ &= \operatorname{div}(\mathbf{u}^{\varepsilon,\delta} \otimes \mathbf{u}^{\varepsilon,\delta}) - \frac12\mathbf{u}^{\varepsilon,\delta} \operatorname{div}\mathbf{u}^{\varepsilon,\delta}\\ &\quad+ \operatorname{div}((\mathbf{u}^\varepsilon - \mathbf{u}^{\varepsilon,\delta}) \otimes \mathbf{u}^\varepsilon) + \operatorname{div}(\mathbf{u}^{\varepsilon,\delta} \otimes (\mathbf{u}^\varepsilon - \mathbf{u}^{\varepsilon,\delta}))\\ &\quad-\frac12(\mathbf{u}^\varepsilon - \mathbf{u}^{\varepsilon,\delta}) \operatorname{div}\mathbf{u}^\varepsilon - \frac12\mathbf{u}^{\varepsilon,\delta} \operatorname{div}(\mathbf{u}^\varepsilon - \mathbf{u}^{\varepsilon,\delta})\end{aligned} \tag{52}$$

and to combine the convergence analysis of the previous section with the estimate

$$\|\mathbf{u}^{\varepsilon,\delta} - \mathbf{u}^\varepsilon\|_{L^2(K)} \le c\delta\|\nabla\mathbf{u}^\varepsilon\|_{H^1(K)} \quad \text{uniformly for } \varepsilon > 0.$$

The final step in the proof of the Theorem 2.2 is to use in the weak formulation (9) a solenoidal test function $\varphi \in C^\infty_0([0, T] \times \mathbb{R}^2; \mathbb{R}^3)$, of the form $\varphi = (\varphi_h(t, x_h), 0)$ and send $\varepsilon \to 0$. The only term that deserves some attention is the rotating one, that we handle in the following way,

$$\begin{aligned}\frac1\varepsilon\int_0^T\int_\Omega(\mathbf{g}\times\mathbf{u}^\varepsilon)\cdot\varphi dxdt &= \frac1\varepsilon\int_0^T\int_\Omega(\mathbf{g}\times(\mathbf{Z}^\varepsilon + \nabla\Psi^\varepsilon))\cdot\varphi dxdt\\ &= \frac1\varepsilon\int_0^T\int_\Omega \mathbf{g}\times(\langle\mathbf{Z}^\varepsilon\rangle + \{\mathbf{Z}^\varepsilon\})\cdot\varphi dxdt + \frac1\varepsilon\int_0^T\int_\Omega(\mathbf{g}\times\nabla\Psi^\varepsilon)\cdot\varphi dxdt.\end{aligned}$$

By using together that $P(\mathbf{g} \times \langle\mathbf{Z}^\varepsilon\rangle) = 0$, $\{\mathbf{Z}^\varepsilon\} = \partial_{x_3} I(x)$, with $\int_{\mathbb{T}^1} I(x)dx_3 = 0$ while φ depends only on x_h and the decay (37) for $\nabla\Psi^\varepsilon$ we obtain the convergence to zero of the rotating term as $\varepsilon \to 0$.

5. Case $\beta = 1/2$

In this section we investigate the case $\beta = 1/2$, where the system reads as follows

$$\begin{cases}\partial_t\mathbf{u}^\varepsilon + \frac1\varepsilon(\mathbf{g}\times\mathbf{u}^\varepsilon) + \frac1\varepsilon\nabla p^\varepsilon = \mu\Delta\mathbf{u}^\varepsilon - (\mathbf{u}^\varepsilon\cdot\nabla)\,\mathbf{u}^\varepsilon - \frac12(\operatorname{div}\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon\\ \varepsilon\partial_t p^\varepsilon + \operatorname{div}\mathbf{u}^\varepsilon = 0.\end{cases} \tag{53}$$

As we will see in the next sections, in this case, the fast rotation of order $1/\varepsilon$ due to the Coriolis force prevails on the low Mach number regime.

5.1. Preliminary Convergence Results

As before, from the uniform energy bounds (21)–(24), we have

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2([0,T];H^1(\Omega)), \tag{54}$$

$$p^\varepsilon \rightharpoonup \pi \quad *\text{-weakly in } L^\infty([0,T];L^2(\Omega)). \tag{55}$$

Letting $\varepsilon \to 0$ in (53_2) we obtain

$$\operatorname{div}\mathbf{u} = 0 \quad \text{a.e. in } (0,T)\times\Omega. \tag{56}$$

Now, if we send ε to zero in (53_1) we have

$$\mathbf{g}\times\mathbf{u} + \nabla\pi = 0, \tag{57}$$

and similarly as in Sect. 4.1 we may infer that π is independent from the third variable x_3 and also $\mathbf{u}_h = (u_1, u_2)$ doesn't depend on the vertical variable x_3, moreover $\operatorname{div}\mathbf{u}_h = 0$. This fact together with the boundary conditions (6) and the L^2 bound for $\mathbf{u}$ yields the conclusion

$$\mathbf{u} = (\mathbf{u}_h(t,x_h),0), \quad u_3 = 0. \tag{58}$$

Since, also for the case $\beta = 1/2$, we are in the framework of a low Mach number limit with ill prepared initial data, in order to establish the convergence of the nonlinear terms in (53_1) we have to investigate the acoustic waves behaviour. This will be done in the next section.

5.2. Analysis of the Acoustic Propagator

We start by writing the system (53) as follows,

$$\begin{cases} \varepsilon\partial_t p^\varepsilon + \operatorname{div}\mathbf{u}^\varepsilon = 0 \\ \varepsilon\partial_t\mathbf{u}^\varepsilon + (\mathbf{g}\times\mathbf{u}^\varepsilon + \nabla p^\varepsilon) = \varepsilon(\mu\Delta\mathbf{u}^\varepsilon - (\mathbf{u}^\varepsilon\cdot\nabla)\,\mathbf{u}^\varepsilon - \frac{1}{2}(\operatorname{div}\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon). \end{cases}$$

Then, the acoustic system is given by

$$\begin{cases} \varepsilon\partial_t p^\varepsilon + \operatorname{div}\mathbf{u}^\varepsilon = 0 \\ \varepsilon\partial_t\mathbf{u}^\varepsilon + (\mathbf{g}\times\mathbf{u}^\varepsilon + \nabla p^\varepsilon) = \varepsilon\operatorname{div}\mathbb{G}_1^\varepsilon + \varepsilon\mathbb{G}_2^\varepsilon, \end{cases} \tag{59}$$

where by taking into account (22)–(24) we have $\mathbb{G}_1^\varepsilon \in L_t^2L_x^2$, $\mathbb{G}_2^\varepsilon \in L_t^2L_x^1$. Obviously the system (59) has to be read in its weak formulation, namely for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$ it holds

$$\int_0^T\int_\Omega (\varepsilon p^\varepsilon\partial_t\varphi + \mathbf{u}^\varepsilon\cdot\nabla\varphi)\,dxdt = -\varepsilon\int_\Omega p_0^\varepsilon\varphi(0,\cdot)dx. \tag{60}$$

and for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^3)$ we get

$$\int_0^T\int_\Omega \big(\varepsilon\mathbf{u}^\varepsilon\partial_t\varphi - (\mathbf{g}\times\mathbf{u}^\varepsilon)\cdot\varphi + p^\varepsilon\operatorname{div}\varphi\big)dxdt$$

$$= -\varepsilon\int_0^T\langle\mathbb{G}^\varepsilon,\varphi\rangle - \varepsilon\int_\Omega \mathbf{u}_0^\varepsilon\varphi(0,\cdot)dx, \tag{61}$$

where

$$\langle \mathbb{G}^\varepsilon, \varphi \rangle = \int_\Omega (\mathbb{G}_1^\varepsilon : \nabla\varphi + \mathbb{G}_2^\varepsilon \cdot \varphi)\, dx.$$

To study the behaviour of the acoustic system we formally define on $L^2(\Omega) \times L^2(\Omega; \mathbb{R}^3)$ the operator $\mathcal{W}$ as

$$\mathcal{W}\begin{pmatrix} p \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \operatorname{div} \mathbf{u} \\ \mathbf{g} \times \mathbf{u} + \nabla p \end{pmatrix}. \tag{62}$$

The operator $\mathcal{W}$ is the so called the *acoustic propagator* and the goal of this section is to prove that the component of the vector $(p^\varepsilon, \mathbf{u}^\varepsilon)$ orthogonal to the null space of $\mathcal{W}$ decays to zero as $\varepsilon \to 0$. Indeed, if this is the case, the velocity field is not affected by the fast oscillations of the acoustic waves since they are killed by the fast decay to zero. In order to achieve this goal we perform a spectral analysis of $\mathcal{W}$.

It is a straightforward computation to deduce that the null space of $\mathcal{W}$ is given by the set

$$\begin{aligned} *20l Ker(\mathcal{W}) = \Big\{ &(p, \mathbf{u}) \mid p = p(x_h),\ \mathbf{u} = \mathbf{u}(x_h), \\ &\operatorname{div}_h \mathbf{u}_h = 0,\ \nabla_h p = (u_2, -u_1) \Big\}. \end{aligned} \tag{63}$$

To study the point spectrum of the operator $\mathcal{W}$ it is more convenient to work in the frequency space. For a function w, the Fourier transform $\hat{w}$ with respect to the space variables is defined as,

$$\hat{w} = \hat{w}(\xi_h, k), \quad \xi_h = (\xi_1, \xi_2) \in \mathbb{R}^2,\ k \in \mathbb{Z},$$

where

$$\hat{w}(\xi_h, k) = \int_0^1 \int_{\mathbb{R}^2} e^{-\mathrm{i}(\xi_h \cdot x_h + k \cdot x_3)} w(x_h, x_3) dx_h dx_3.$$

The eigenvalues problem for $\mathcal{W}$ is set as follows,

$$\operatorname{div} \mathbf{u} = \lambda p, \qquad \mathbf{g} \times \mathbf{u} + \nabla p = \lambda \mathbf{u},$$

which in Fourier variables has the form

$$\mathrm{i}\left(\sum_{j=1}^{2} \xi_j \hat{\mathbf{u}}_j + k \hat{\mathbf{u}}_3 \right) - \lambda \hat{p} = 0, \qquad \mathrm{i}(\xi_1, \xi_2, k)\hat{p} - (\hat{\mathbf{u}}_2, -\hat{\mathbf{u}}_1, 0) - \lambda \hat{\mathbf{u}} = 0.$$

After some standard computations we obtain

$$\lambda^2 = -\frac{1 + |\xi|^2 + k^2 \pm \sqrt{(1 + |\xi|^2 + k^2)^2 - 4k^2}}{2}. \tag{64}$$

From (64) we deduce that the only real eigenvalue is $\lambda = 0$ that we obtain for $k = 0$ and, as a consequence, we have that the space of eigenvectors of $\mathcal{W}$ coincides with the $Ker(\mathcal{W})$ defined in (63). To prove that the components of $(p^\varepsilon, \mathbf{u}^\varepsilon)$ orthogonal to $Ker(\mathcal{W})$ decay to zero we use the RAGE theorem that we state in the following form (see Cycon et al. [4, Theorem 5.8] or [13]):

Theorem 5.1 *Let H be a Hilbert space, $A : \mathcal{D}(A) \subset H \to H$ a self-adjoint operator, $C : H \to H$ a compact operator, and P_c the orthogonal projection onto H_c, where,*

$$H = H_c \oplus \mathrm{cl}_H \Big\{ \mathrm{span}\{ w \in H \mid w \text{ an eigenvector of } A \} \Big\}.$$

Then

$$\left\| \frac{1}{\tau} \int_0^\tau \exp(-\mathrm{i}tA) C P_c \exp(\mathrm{i}tA)\, dt \right\|_{\mathcal{L}(H)} \to 0 \text{ as } \tau \to \infty.$$

Remark 5.2 If the operator C is non-negative and self-adjoint in H, then we have

$$\frac{1}{T}\int_0^T \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon}A\right) C \exp\left(\mathrm{i}\frac{t}{\varepsilon}A\right) P_c X, Y \right\rangle_H dt \le \omega(\varepsilon)\|X\|_H\|Y\|_H, \tag{65}$$

where $\omega(\varepsilon) \to 0$, as $\varepsilon \to 0$. If we take $Y = P_c X$ we get

$$\frac{1}{T}\int_0^T \left\| \sqrt{C}\exp\left(\mathrm{i}\frac{t}{\varepsilon}A\right) P_c X, \right\|_H^2 dt \le \omega(\varepsilon)\|X\|_H^2, \tag{66}$$

and for any $X \in L^2(0,T;H)$ we have

$$\frac{1}{T^2}\left\| \sqrt{C}\int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}A\right) X(s)ds \right\|_{L^2(0,T;H)}^2 dt \le \omega(\varepsilon)\int_0^T \|X(s)\|_H^2 ds. \tag{67}$$

For more details see [9] or [13].

We apply the RAGE Theorem 5.1 in the case where the Hilbert space H is

$$H = H_M = \{(p,\mathbf{u}) \mid \ \hat{p}(\xi_h,k) = 0,\ \hat{\mathbf{u}}(\xi_h,k) = 0 \text{ if } |\xi_h| + |k| > M\}$$

and the operators A, C, considered on the space H_M are given by

$$A = \mathrm{i}\mathcal{W},\ C[v] = P_M[\chi v], \quad \chi \in C_c^\infty(\Omega),\ 0 \le \chi \le 1,$$

where

$$P_M : L^2(\Omega) \times L^2(\Omega;\mathbb{R}^3) \longrightarrow H_M$$

denotes the orthogonal projection into H_M. If we denote by p_M^ε, $\mathbf{u}_M^\varepsilon$ the orthogonal projection of p^ε and $\mathbf{u}^\varepsilon$ into H_M respectively and we apply the projector P_M to the acoustic system (59) we get

$$\varepsilon\frac{d}{dt}\begin{pmatrix} p_M^\varepsilon \\ \mathbf{u}_M^\varepsilon \end{pmatrix} + \mathcal{W}\begin{pmatrix} p_M^\varepsilon \\ \mathbf{u}_M^\varepsilon \end{pmatrix} = \varepsilon\begin{pmatrix} 0 \\ \mathbb{G}_{\varepsilon,M} \end{pmatrix}, \tag{68}$$

and $\mathbb{G}_{\varepsilon,M} \in H_M$. Moreover, taking into account (22) and (24), we have the following uniform bound in ε

$$\left\| \begin{pmatrix} 0 \\ \mathbb{G}_{\varepsilon,M} \end{pmatrix} \right\|_{L^2(0,T;H_M)} \le c(M).$$

By using Duhamel's formula, the solutions of (68) are given by,

$$\begin{pmatrix} p_M^\varepsilon \\ \mathbf{u}_M^\varepsilon \end{pmatrix} = \exp\left(\mathrm{i}A\frac{t}{\varepsilon}\right)\begin{pmatrix} p_M^\varepsilon(0) \\ \mathbf{u}_M^\varepsilon(0) \end{pmatrix} + \int_0^t \exp\left(\mathrm{i}A\frac{t-s}{\varepsilon}\right)\begin{pmatrix} 0 \\ \mathbb{G}_{\varepsilon,M} \end{pmatrix} ds \tag{69}$$

Now we denote by Q the orthogonal projection into $Ker(\mathcal{W})$,

$$Q : L^2(\Omega) \times L^2(\Omega;\mathbb{R}^3) \longrightarrow Ker(\mathcal{W})$$

and we recall that the point spectrum of $\mathcal{W}$, hence of the operator A is reduced to 0. By applying (66) and (67) we get

$$Q^\perp\begin{pmatrix} p_M^\varepsilon \\ \mathbf{u}_M^\varepsilon \end{pmatrix} \to 0 \quad \text{in } L^2((0,T)\times K;\mathbb{R}^4),\ \text{as } \varepsilon \to 0, \tag{70}$$

for any compact set $K \subset \overline{\Omega}$ and fixed M.

From (69) we obtain

$$Q\begin{pmatrix} p_M^\varepsilon \\ \mathbf{u}_M^\varepsilon \end{pmatrix} \to \begin{pmatrix} \pi_M \\ \mathbf{u}_M \end{pmatrix} \quad \text{in } L^2((0,T)\times K;\mathbb{R}^4),\ \text{as } \varepsilon \to 0, \tag{71}$$

where π and $\mathbf{u}$ are the limits defined in (55) and (54).

5.3. Proof of the Theorem 2.3

Now we are ready to prove the Theorem 2.3. By combing together (70) and (71) we have

$$P_M \mathbf{u}^\varepsilon \to P_M \mathbf{u} \quad \text{in } L^2((0,T)\times K;\mathbb{R}^3), \tag{72}$$

for any compact set $K \subset \Omega$ and fixed M. Finally (72) with (54) and the compact embedding of $W^{1,2}(K)$ in $L^2(K)$ gives

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{in } L^2((0,T)\times K;\mathbb{R}^3), \text{ for any compact set } K \subset \Omega. \tag{73}$$

Having established the convergences (73) and (55) we can pass into the limit in the weak formulation of (53). For the Eq. $(53)_1$ we use a test function $\varphi \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$ of the form $\varphi = (\nabla_h^\perp \psi, 0) = (\partial_{x_2}\psi, -\partial_{x_1}\psi, 0)$, $\psi \in C_c^\infty([0,T)\times\Omega)$, hence we have

$$\begin{aligned}\int_0^T\int_\Omega \Big(\mathbf{u}^\varepsilon \partial_t\varphi + \mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon : \nabla\varphi - \frac{1}{2}\mathbf{u}^\varepsilon \operatorname{div}\mathbf{u}^\varepsilon\cdot\varphi + \frac{1}{\varepsilon}(\mathbf{g}\times\mathbf{u}^\varepsilon)\cdot\varphi\Big)dxdt \\ = \int_0^T\int_\Omega \nabla\mathbf{u}^\varepsilon\cdot\nabla\varphi dxdt - \int_\Omega \mathbf{u}_0^\varepsilon\cdot\varphi(0,\cdot)dx.\end{aligned} \tag{74}$$

By combining together (74) with

$$\int_0^T\int_\Omega \left(p^\varepsilon\partial_t\psi + \frac{1}{\varepsilon}\mathbf{u}_h^\varepsilon\cdot\nabla\psi\right)dxdt = \int_\Omega p_0^\varepsilon\psi(0,\cdot)dx, \tag{75}$$

and by performing the limit as $\varepsilon \to 0$ we obtain,

$$\begin{aligned}\int_0^T\int_\Omega \Big(\mathbf{u}\partial_t\nabla_h^\perp\psi + \mathbf{u}\otimes\mathbf{u} : \nabla\nabla_h^\perp\psi + \pi\partial_t\psi\Big)dxdt \\ = \int_0^T\int_\Omega \nabla\mathbf{u}\cdot\nabla\nabla_h^\perp dxdt - \int_\Omega \mathbf{u}_0\cdot\nabla_h^\perp(0,\cdot) + p_0\psi(0,\cdot)dx.\end{aligned} \tag{76}$$

Finally, since from (57) we have $\mathbf{u} = \nabla_h^\perp\pi$ and, recalling that $\mathbf{u}$ and π are independent on the variable x_3, from (76) we get

$$\begin{aligned}\int_0^T\int_\Omega \Big(\nabla_h^\perp\pi\partial_t\nabla_h^\perp\psi + \nabla_h^\perp\pi\otimes\nabla_h^\perp\pi : \nabla\nabla_h^\perp\psi + \pi\partial_t\psi\Big)dx_h dt \\ = \int_0^T\int_\Omega \nabla\nabla_h^\perp\pi\cdot\nabla\nabla_h^\perp\psi dx_h dt - \int_\Omega (\mathbf{u}_0\cdot\nabla_h^\perp\psi(0,\cdot) + p_0\psi(0,\cdot))dx,\end{aligned}$$

which is the Eq. (19) in the sense of distribution.

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Conflict of interest The author declares that there is no conflict of interest concerning this work.

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