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Long-Time Behavior of Global Weak Solutions for a Beris-Edwards Type Model of Nematic Liquid Crystals

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Abstract. We consider a Beris-Edwards system modeling incompressible liquid crystal flows of nematic type. This system couples a Navier–Stokes system for the fluid velocity with a time-dependent system for the Q-tensor variable, whose spectral decomposition is related to the directors of liquid crystal molecules. The long-time behavior for global weak solutions is studied, proving that each whole trajectory converges to a single equilibrium whenever a regularity hypothesis is satisfied by the energy of the weak solution.

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1. Introduction

We deal with a system, which contains the Navier–Stokes equations with additional forcing terms for the unknowns velocity $\boldsymbol{u}$ and pressure p, and a parabolic system for and tensor parameter order Q (following the Landau–De Gennes theory), such that $(\boldsymbol{u}, p, Q) : (0,T) \times \Omega \to \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^{3\times 3}$ satisfies,

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \nabla\cdot\tau(Q) + \nabla\cdot\sigma(H,Q), \\ \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t Q + (\boldsymbol{u}\cdot\nabla)Q - S(W,Q) = -\gamma\, H(Q), \\ Q = Q^t, \quad tr(Q) = 0, \end{cases} \tag{1}$$

in the time-space cylinder $(0,T)\times\Omega$, subject to the initial and boundary conditions,

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \qquad Q|_{t=0} = Q_0 \quad \text{in}\Omega, \tag{2}$$

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \qquad \partial_{\boldsymbol{n}} Q|_{\partial\Omega} = 0 \quad \text{in } (0,T), \tag{3}$$

The vector $\boldsymbol{n}$ denotes the normal outwards vector on the boundary $\partial\Omega$. The set $\Omega\subset\mathbb{R}^3$ is a smooth and bounded domain, the constant $\nu > 0$ is the viscosity coefficient and $\gamma > 0$ is a material-dependent elastic constant.

The tensors $\tau = \tau(Q) \in \mathbb{R}^{3\times3}$ and $\sigma = \sigma(H,Q) \in \mathbb{R}^{3\times3}$ given in (1) are defined by

$$\begin{cases} \tau_{ij}(Q) := -\varepsilon\,\partial_j Q : \partial_i Q = -\varepsilon\,\partial_j Q_{kl}\,\partial_i Q_{kl}, \\ \sigma(H,Q) := H\,Q - Q\,H, \end{cases}$$

where $\varepsilon > 0$. They are the symmetric and antisymmetric part of the stress tensor, respectively. The tensor $H = H(Q)$ is related to the variational derivative in $L^2(\Omega)$ of a free energy functional in the vectorial

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subspace of symmetric and traceless tensors, in fact

$$\mathcal{E}(Q) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla Q|^2 + F(Q)\Big)\, dx, \qquad H(Q) := \frac{\delta \mathcal{E}(Q)}{\delta Q}. \tag{4}$$

Here, $A : B = A_{ij}\, B_{ij}$ denote the scalar product of matrices (using the Einstein summation convention over repeated indices) and the potential function $F(Q)$ is defined by

$$F(Q) := \frac{a}{2}\, |Q|^2 - \frac{b}{3}\, (Q^2 : Q) + \frac{c}{4}\, |Q|^4, \tag{5}$$

where $a,\, b,\, c \in \mathbb{R}$ with $c > 0$. We denote by $|Q| = (Q : Q)^{1/2}$ the matrix Euclidean norm. Then, from (4) and (5), one possible form to write the variational derivative $\dfrac{\delta \mathcal{E}(Q)}{\delta Q}$ in the subspace of symmetric and traceless tensors of $\mathbb{R}^{3\times 3}$ is the following:

$$H(Q) = -\varepsilon\, \Delta Q + f(Q) \tag{6}$$

where

$$f(Q) = F'(Q) + b\, \frac{tr(Q^2)}{3} I = a\, Q - b\, \left(Q^2 - \frac{tr(Q^2)}{3} I\right) + c\, |Q|^2\, Q. \tag{7}$$

Observe that, since $Q = Q^t$ and $tr(Q) = 0$, then $f(Q) = f(Q)^t$ and $tr(f(Q)) = 0$.

Note that H defined in (6) uses the one-constant approximation for the Oseen–Frank energy of liquid crystals together with a Landau–De Gennes expression for the bulk energy given by $f(Q)$.

Finally, $W = W(\boldsymbol{u}) = (\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^t)/2$ is the antisymmetric part of $\nabla \boldsymbol{u}$ and

$$S(W, Q) = W\, Q - Q\, W \tag{8}$$

is the so-called stretching term.

The configurations of liquid crystals can be described by a director field as minimizers of an energy functional following the Oseen–Frank theory. The dynamic of the problem is considered by Ericksen–Leslie models, where the evolution of the director field is coupled with a Navier–Stokes-type equation for the underlying flow field. In the Landau–De Gennes theory, the director vector is replaced by a symmetric and traceless matrix Q, which measures the deviation of the second moment tensor from its isotropic value. Different expressions of the Q-tensor order parameter allows to represent a uniaxial, biaxial or isotropic behavior of the molecules of the nematic crystal.

The corresponding dynamic model (1)–(8) is called Beris-Edwards model [2] and was studied by Paicu & Zarnescu in [16] and Abels et al. in [1]. In these two papers, symmetry and traceless of Q are assumed but it is not proved. Other version of this model appears in [13] and [14] where it becomes from a generic model which is modified to deduce symmetry and traceless of Q for any weak solution. Then, using symmetry and traceless of Q, the model obtained in [13] can be rewritten as (1)–(8). Similar models and an extensive bibliography can be found at [4] and [5].

The large-time behavior of some models for Nematic liquid crystals with unknown vector director (following the Oseen–Frank theory) are studied in [12,19] (without stretching terms), in [11,15,18] (with stretching terms) and in [17] (where different results are deduced depending on considering or not the stretching terms).

On the other hand, the large-time behavior is also analyzed for others related models. For example, for a Cahn–Hilliard–Navier–Stokes system in $2D$ domains in [10], for a chemotaxis model in [9], for a Cahn–Hilliard–Navier–Stokes vesicle model in [7] and for a smectic-A liquid crystals model in [6]. The liquid crystal model studied in [6] also follows the Oseen–Frank theory.

In this paper we study the large-time behavior of the weak solutions of (1)–(8). Firstly, we prove that the ω-limit set for weak solutions is composed by critical points of the free-energy $\mathcal{E}(Q)$. After that, by using a Lojasiewicz-Simon's result we demonstrate the convergence of the whole trajectory of any weak solution to a single equilibrium. The lack of regularity and uniqueness force to use technical and non-standard arguments. The keys in the proof of our results are two, firstly to have a dissipative energy that leads, in particular, to prove the existence of global weak solutions in time and secondly the application of the Lojasiewicz-Simon inequality that allows to obtain the convergence of the whole trajectory of the Q-tensor. More specifically, the main novelties in this paper are:

- To choose a special regularized energy of any global weak solution, satisfying an energy's law inequality in two forms; an integral version satisfied for all time interval and a differential version satisfied a.e. in time (see (23) and (24) below).
- The proof of convergence of the whole trajectory of any weak solution to a unique equilibrium steady solution, under the hypothesis that the regularized energy coincides a.e. in time with the energy evaluated in the weak solution.

Sects. 2 gives the weak solution concept describing the main steps to prove the existence of global in time weak solutions (more details can be seen in [13]). In Sect. 3 two suitable energy inequalities are proved, a time-integral version for all time t and a time-differential version for almost every time. These inequalities as far as we know, have not been proved before (they are cited in [3,17] but do not proved) and they will be essential in the following arguments when only weak solutions have considered. In fact, the standard argument of obtaining more regularity for large enough viscosity is not clear in this case. In Sect. 4 the convergence at infinite time for global weak solutions is studied. Firstly, we prove that the ω-limit set defined only for weak solutions (strong solution is necessary in standard methods) consists of critical points of the free-energy. Finally, the convergence of the whole trajectory to a single equilibrium as time goes to infinity is proved via a Lojasiewicz-Simon's lemma.

A preliminary version of this paper appears in [8]. Now, we prove the existence of a special regularized energy satisfying the energy's law inequality for all interval which is essential to prove the convergence of the trajectory to a unique point.

Notations

The notation can be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H^1 = H^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(0,T;X)$ the Banach space $L^p(0,T;X(\Omega))$. In particular, $L^p_{loc}([0,+\infty);X)$ are the functions of $L^p(0,T;X)$ for all $T>0$ finite.

Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = \mathbf{L}^2(\Omega) = L^2(\Omega)^3$.

In order to put the symmetric and traceless of the tensors inside of the space, we introduce the Hilbert space

$$\mathbb{L}^2 = \mathbb{L}^2(\Omega) = \{Q \in L^2(\Omega)^{3\times 3},\ Q = Q^t,\ tr(Q) = 0\}$$

and likewise for $\mathbb{H}^1$, etc.

We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{u} \in C_0^\infty(\Omega)^3$ satisfying $\nabla\cdot\boldsymbol{u} = 0$. We denote $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla\cdot\boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla\cdot\boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}.$$

If B is a Banach space, $C_w([0,+\infty);B)$ is the space of weakly continuous functions f, that is $f(s)$ converges weakly to $f(t)$ in B as s converges to t.

From now on, $C>0$ will denote different constants, depending only on data of the problem.

2. Weak Solutions

We start arguing in a formal manner, assuming a sufficiently regular solution $(\boldsymbol{u}, p, Q)$ of (1)–(8). For more detailed calculations of this section, see [13].

Variational Formulation

Using that $tr(Q)=0$ and definition of $f(Q)$ given in (7), one has

$$\partial_i F(Q) = F'(Q):\partial_i Q = (F'(Q) + b\,\frac{tr(Q^2)}{3} I):\partial_i Q = f(Q):\partial_i Q.$$

Therefore, the tensor τ can be rewritten as:

$$\begin{aligned}(\nabla\cdot\tau(Q))_i &= -\varepsilon\,\partial_j(\partial_j Q:\partial_i Q) = -\varepsilon\,\Delta Q:\partial_i Q - \varepsilon\,\partial_j Q:\partial^2_{ij} Q\\ &= H(Q):\partial_i Q - \partial_i\left(F(Q) + \tfrac{\varepsilon}{2}|\nabla Q|^2\right)\end{aligned}$$

By testing the first equation of (1) by any $\widetilde{\boldsymbol{u}}:\Omega\to\mathbb{R}^3$ with $\widetilde{\boldsymbol{u}}|_{\partial\Omega}=\boldsymbol{0}$ and $\nabla\cdot\widetilde{\boldsymbol{u}}=0$ in Ω, using that $\sigma(H,Q)$ is antisymmetric (hence $(\sigma(H,Q),\nabla\boldsymbol{u}) = (\sigma(H,Q),W(\boldsymbol{u}))$), we arrive at the following variational formulation of (1):

$$\langle\partial_t\boldsymbol{u},\widetilde{\boldsymbol{u}}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\widetilde{\boldsymbol{u}}) + \nu(\nabla\boldsymbol{u},\nabla\widetilde{\boldsymbol{u}}) - ((\widetilde{\boldsymbol{u}}\cdot\nabla)Q,H) + (\sigma(H,Q),W(\widetilde{\boldsymbol{u}})) = 0, \tag{9}$$

where $\langle\cdot,\cdot\rangle$ is the duality product between V' and V and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. On the other hand, testing the Q-system of (1) by any symmetric traceless $\widetilde{H}$ and the system $-\varepsilon\,\Delta Q + f(Q) = H$ by any symmetric traceless $\widetilde{Q}$, we get the following variational formulation:

$$\begin{cases}(\partial_t Q,\widetilde{H}) + ((\boldsymbol{u}\cdot\nabla)Q,\widetilde{H}) - (S(W,Q),\widetilde{H}) + \gamma\,(H(Q),\widetilde{H}) = 0,\\ \varepsilon\,(\nabla Q,\nabla\widetilde{Q}) + (f(Q),\widetilde{Q}) - (H(Q),\widetilde{Q}) = 0.\end{cases} \tag{10}$$

From (10), we obtain, in particular that:

$$(\partial_t Q,\widetilde{Q}) + ((\boldsymbol{u}\cdot\nabla)Q,\widetilde{Q}) - (S(W,Q),\widetilde{Q}) - \varepsilon\,\gamma\,(\Delta Q,\widetilde{Q}) + \gamma\,(f(Q),\widetilde{Q}) = 0. \tag{11}$$

Dissipative Energy Law and Global in Time A Priori Estimates

By taking $\widetilde{\boldsymbol{u}}=\boldsymbol{u}$ in (9) and $(\widetilde{H},\widetilde{Q}) = (H(Q),\partial_t Q)$ in (10) the following "energy equality" holds:

$$\frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{u}\|^2_{L^2} + \mathcal{E}(Q)\right) + \nu\|\nabla\boldsymbol{u}\|^2_{L^2} + \gamma\|H(Q)\|^2_{L^2} = 0. \tag{12}$$

Observe that $\mathcal{E}(Q) = \displaystyle\int_\Omega\left(\frac{1}{2}|\nabla Q|^2 + F(Q)\right)d\boldsymbol{x}$ is not a positive term due to $F(Q)$. However, it is possible to find a large enough constant $\mu>0$ depending on parameters a, b and c given in the definition of $F(Q)$ in (5), such that

$$F_\mu(Q) := F(Q) + \mu \geq \frac{c}{8}\,|Q|^4. \tag{13}$$

By replacing $\mathcal{E}(Q)$ in (12) by $\mathcal{E}_\mu(Q) := \displaystyle\int_\Omega\left(\frac{1}{2}|\nabla Q|^2 + F_\mu(Q)\right) \geq 0$, and denoting the kinetic energy as

$$\mathcal{E}_k(\boldsymbol{u}(t)) := \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2}$$

and the total energy as $\mathcal{E}(\boldsymbol{u},Q) := \mathcal{E}_k(\boldsymbol{u}) + \mathcal{E}_\mu(Q)$, then (12) implies

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{u}(t),Q(t)) + \nu\|\nabla\boldsymbol{u}\|^2_{L^2} + \gamma\|H(Q)\|^2_{L^2} = 0. \tag{14}$$

This energy equality shows the dissipative character of the model with respect to the total free-energy $\mathcal{E}(\boldsymbol{u}(t),Q(t))$. In fact, assuming finite total energy of initial data, i.e. $\mathcal{E}(\boldsymbol{u}_0,Q_0) < +\infty$, then the following regularity hold:

$$\begin{gathered}\boldsymbol{u}\in L^\infty(0,+\infty;\boldsymbol{H})\cap L^2(0,+\infty;\boldsymbol{V}),\\ \nabla Q\in L^\infty(0,+\infty;L^2(\Omega)^{3\times3\times3}),\quad F_\mu(Q)\in L^\infty(0,+\infty;L^1(\Omega)),\\ H(Q)\in L^2(0,+\infty;\mathbb{L}^2).\end{gathered} \tag{15}$$

From (13) and (15), we deduce that $Q \in L^\infty(0,+\infty;\mathbb{L}^4)$, $Q \in L^\infty(0,+\infty;\mathbb{H}^1)$ and, in particular

$$Q \in L^\infty(0,+\infty;\mathbb{L}^6). \tag{16}$$

Since $f(Q)$ is a third order polynomial function, $|f(Q)| \le C(a,b,c)\,\big(|Q| + |Q|^2 + |Q|^3\big)$ which, together with (16), gives $f(Q) \in L^\infty(0,+\infty;\mathbb{L}^2)$.

From $H(Q) = -\varepsilon\,\Delta Q + f(Q)$, by using the H^2-regularity of the Poisson problem:

$$\begin{cases} -\varepsilon\,\Delta Q + Q = H(Q) - f(Q) + Q & \text{in } \Omega, \\ \quad\partial_{\boldsymbol{n}} Q|_\Gamma = 0 \end{cases}$$

we deduce that:

$$Q \in L^2_{loc}([0+\infty);\mathbb{H}^2).$$

In the following definition of weak solution, we will relax the energy law (14) to an energy inequality in integral form (see (18) below).

Definition 1 (*Weak solution*). It will be said that $(\boldsymbol{u},Q)$ is a weak solution in $(0,+\infty)$ of problem (1)–(3) if

$$\begin{cases} \boldsymbol{u} \in L^\infty(0,+\infty;\mathbf{H}) \cap L^2(0,+\infty;\mathbf{V}), \\ Q \in L^\infty(0,+\infty;\mathbb{H}^1) \cap L^2_{loc}([0+\infty);\mathbb{H}^2), \\ H(Q) \in L^2(0,+\infty;\mathbb{L}^2), \end{cases} \tag{17}$$

satisfies the variational formulation (9) and (10), the initial conditions (2), the boundary conditions (3) and the following energy inequality a.e. t_1,t_0 with $t_1 \ge t_0 \ge 0$:

$$\mathcal{E}(\boldsymbol{u}(t_1),Q(t_1)) - \mathcal{E}(\boldsymbol{u}(t_0),Q(t_0)) + \int_{t_0}^{t_1} (\nu\|\nabla\boldsymbol{u}(s)\|^2_{L^2} + \gamma\|H(Q(s))\|^2_{L^2})\,ds \le 0. \tag{18}$$

Note that the regularity imposed in (17) is satisfied up to infinite time excepting the $H^2(\Omega)$-regularity for Q.

By applying the regularity (17) to the systems (9) and (11), we have

$$\partial_t\boldsymbol{u} \in L^{4/3}_{loc}([0,+\infty);\mathbf{V}') \quad \text{and} \quad \partial_t Q \in L^{4/3}_{loc}([0,+\infty);\mathbb{L}^2).$$

Hence, the following time-continuity can be deduced:

$$\boldsymbol{u} \in C([0,+\infty);\mathbf{V}') \cap C_w([0,+\infty);\mathbf{H}) \quad \text{and} \quad Q \in C([0,+\infty);\mathbb{L}^2(\Omega)) \cap C_w([0,+\infty);\mathbb{H}^1).$$

In particular, the initial conditions (2) make sense because $(\boldsymbol{u}(t),Q(t)) \in \mathbf{H}\times\mathbb{H}^1$ for all $t \ge 0$.

Theorem 2 (Existence of weak solutions). *If $(\boldsymbol{u}_0,Q_0) \in \boldsymbol{H}\times\mathbb{H}^1$, there exists a weak solution $(\boldsymbol{u},Q)$ of system* (1)–(3) *in* $(0,+\infty)$.

Proof. The first part of this theorem is proved in [13] by means of a Galerkin approximation related to the variational formulation (10), which preserves the divergence-free of the velocity and the symmetry and traceless of the tensor. Therefore, it suffices to prove (18). We start from the following energy equality satisfied by the Galerkin approximate solutions (see [13]) for all t,t_0 with $t \ge t_0 \ge 0$:

$$\mathcal{E}(\boldsymbol{u}_m(t),Q_m(t)) - \mathcal{E}(\boldsymbol{u}_m(t_0),Q_m(t_0)) + \int_{t_0}^{t} (\nu\|\nabla\boldsymbol{u}_m(s)\|^2_{L^2} + \gamma\|H(Q_m(s))\|^2_{L^2})\,ds \le 0. \tag{19}$$

Moreover, $\boldsymbol{u}_m(t)$ and $Q_m(t)$ have sufficient estimates to obtain

$$\mathcal{E}(\boldsymbol{u}_m(t),Q_m(t)) \to \mathcal{E}(\boldsymbol{u}(t),Q(t)) \quad \text{in } L^1(0,T), \text{ and in particular a.e. } t \ge 0. \tag{20}$$

Since $\boldsymbol{u}_m \to \boldsymbol{u}$ weakly in $L^2(0,\infty;\boldsymbol{V})$ and $H(Q_m) \to H(Q)$ weakly in $L^2(0,\infty;\mathbb{L}^2)$, then

$$\liminf_{m\to+\infty} \int_{t_0}^{t_1} (\nu\|\nabla\boldsymbol{u}_m\|^2_{L^2} + \gamma\|H(Q_m)\|^2_{L^2}) \ \ge \int_{t_0}^{t_1} (\nu\|\nabla\boldsymbol{u}\|^2_{L^2} + \gamma\|H(Q)\|^2_{L^2}) \tag{21}$$

for all $t_1,t_0 : t_1 \ge t_0 \ge 0$.

By taking $\liminf_{m\to+\infty}$ in (19), we obtain that for all $t_1 \geq t_0 \geq 0$,

$$\begin{aligned}\liminf_{m\to+\infty} \mathcal{E}(\boldsymbol{u}_m(t_1), Q_m(t_1)) + \liminf_{m\to+\infty} \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}_m\|_{L^2}^2 + \gamma\|H(Q_m)\|_{L^2}^2) \\ \leq \limsup_{m\to+\infty} \mathcal{E}(\boldsymbol{u}_m(t_0), Q_m(t_0)).\end{aligned} \tag{22}$$

By using (20) and (21) in (22), we obtain (18). □

3. An Improved Energy Inequality

In this section, we obtain an improved time-integral energy inequality for all time, in a rigorous manner, for the weak solutions furnished by the Galerkin approximations. From this integral version we also obtain a time-differential version for almost every time. The new integral inequality (23) differs from (18) in that while (18) holds almost everywhere t_0, t_1, (23) holds for all t_0, t_1. This fact allows to obtain the differential inequality (24) a.e. $t \geq 0$, which is essential to prove the Theorem 6 below.

Lemma 3. *Let $(\boldsymbol{u}, Q)$ be a weak solution in $(0, +\infty)$ of problem (1)–(3) then, there exists an appropriate function $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(t) \in \mathbb{R}$ defined for all $t \geq 0$, which satisfies the following integral inequality:*

$$\widetilde{\mathcal{E}}(t_1) - \widetilde{\mathcal{E}}(t_0) + \int_{t_0}^{t_1} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|H(Q(s))\|_{L^2}^2)\, ds \leq 0, \quad \forall\, t_1, t_0 : t_1 \geq t_0 \geq 0 \tag{23}$$

and the following differential version:

$$\frac{d}{dt}\widetilde{\mathcal{E}}(t) + \nu\|\nabla \boldsymbol{u}(t)\|_{L^2}^2 + \gamma\|H(Q(t))\|_{L^2}^2 \leq 0, \quad a.e.\ t \geq 0. \tag{24}$$

Proof. Since the inequality (18) is satisfied for all $t_0, t_1 \in [0, +\infty)\backslash N$, where N is a set of null Lebesgue measure, then the map $t \in [0, +\infty)\backslash N \to \mathcal{E}(\boldsymbol{u}(t), Q(t)) \in \mathbb{R}$ is a real decreasing (and bounded) function. Then, we can define a special function $\widetilde{\mathcal{E}}(t)$ for all $t \in [0, +\infty)$ as:

$$\widetilde{\mathcal{E}}(0) := \mathcal{E}(u_0, Q_0), \qquad \widetilde{\mathcal{E}}(t) := \lim_{\substack{s\to t^- \\ s\in[0,+\infty)\backslash N}} \mathcal{E}(\boldsymbol{u}(s), Q(s)), \quad \forall\, t > 0.$$

The function $\widetilde{\mathcal{E}}$, thus defined, is "continuous from the left" and decreasing for all $t \geq 0$. Indeed, for any $t_1, t_2 \in [0, +\infty)$, for instance $t_1 < t_2$, we can choose sequences $\{s_n^1\}, \{s_n^2\} \subset [0, +\infty)\backslash N$ such that $s_n^1 \to t_1^-$, $s_n^2 \to t_2^-$ and, $s_n^1 \leq s_n^2$ for all $n \geq n_0$. Since s_n^1 and s_n^2 are not in N, we know that $\mathcal{E}(\boldsymbol{u}(s_n^1), Q(s_n^1)) \geq \mathcal{E}(\boldsymbol{u}(s_n^2), Q(s_n^2))$. By taking limit as $s_n^1 \to t_1^-$ and $s_n^2 \to t_2^-$, we obtain that $\widetilde{\mathcal{E}}(t_1) \geq \widetilde{\mathcal{E}}(t_2)$.

Since $\widetilde{\mathcal{E}}(t)$ is decreasing for all $t \in [0, +\infty)$, it is differentiable almost everywhere $t \in (0, +\infty)$.

Since the inequality (18) is satisfied for all $t_0, t_1 \in [0, +\infty)\backslash N$ where the measure of N is zero, given any $t_0 < t_1$, we can take $\delta_n > 0$ and $\eta_n > 0$ such that $t_0 - \delta_n$, $t_1 - \eta_n \notin N$ and $\delta_n, \eta_n \to 0$, hence

$$\widetilde{\mathcal{E}}(t_1 - \eta_n) - \widetilde{\mathcal{E}}(t_0 - \delta_n) + \int_{t_0-\delta_n}^{t_1-\eta_n} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|H(s)\|_{L^2}^2)\, ds \leq 0.$$

By taking $\delta_n \to 0$ and $\eta_n \to 0$, we obtain (23).

In particular, by choosing $t_0 = t$ and $t_1 = t + h$ in (23), we obtain

$$\frac{\widetilde{\mathcal{E}}(t+h) - \widetilde{\mathcal{E}}(t)}{h} + \frac{1}{h}\int_t^{t+h} (\nu\|\boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|H(s)\|_{L^2}^2)\, ds \leq 0, \quad \forall\, t, h \geq 0. \tag{25}$$

Observe that

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} (\nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|H(s)\|_{L^2}^2)\, ds = \nu\|\nabla \boldsymbol{u}(t)\|_{L^2}^2 + \gamma\|H(t)\|_{L^2}^2,$$

a.e. $t \geq 0$ because the map, $s \in [0, +\infty) \to \nu\|\nabla \boldsymbol{u}(s)\|_{L^2}^2 + \gamma\|H(s)\|_{L^2}^2 \in \mathbb{R}$, belongs to $L^1(0, +\infty)$. Accordingly, by taking $h \to 0$ in (25), we obtain (24) a.e. $t \geq 0$. □

4. Convergence at Infinite Time

4.1. Convergence Towards the Rest State

Let $(\boldsymbol{u}, Q)$ be a weak solution of (1)–(3) in $(0, +\infty)$ associated to an initial data $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1$ (see Definition 1). From the energy inequality (18) and the weak-continuity of $(\boldsymbol{u}, Q)$ in $\boldsymbol{H} \times \mathbb{H}^1$, one can prove that the energy $\mathcal{E}(\boldsymbol{u}(t), Q(t))$ is well-defined for all $t > 0$ and is decreasing in time (using the weakly lower semicontinuity of $\|\boldsymbol{u}(t)\|^2_{L^2}$ and $\|\nabla Q(t)\|^2_{L^2}$ and the continuity of $\int_\Omega F(Q(t))$). Therefore, there exists a real number $\mathcal{E}_\infty \geq 0$ such that the total energy evaluated in the trajectory $(\boldsymbol{u}(t), Q(t))$ for all $t \in [0, +\infty)$ satisfies

$$\mathcal{E}(\boldsymbol{u}(t), Q(t)) \searrow \mathcal{E}_\infty \text{ in } \mathbb{R} \quad \text{ as } t \uparrow +\infty. \tag{26}$$

Let us define the ω-limit set of this global weak solution $(\boldsymbol{u}, Q)$ as follows:

$$\omega(\boldsymbol{u}, Q) = \{(\boldsymbol{u}_\infty, Q_\infty) \in \boldsymbol{H} \times \mathbb{H}^1 : \exists \{t_n\} \uparrow +\infty \text{ s.t. } (\boldsymbol{u}(t_n), Q(t_n)) \to (\boldsymbol{u}_\infty, Q_\infty) \text{ weakly in } \boldsymbol{L}^2 \times \mathbb{H}^1\}.$$

Observe that this ω-limit set is defined with respect to weak convergences.

Let $\mathcal{S}$ be the set of critical points of the energy $\mathcal{E}(Q)$ defined in (4), that is

$$\mathcal{S} = \{Q \in \mathbb{H}^2 : -\varepsilon \Delta Q + f(Q) = 0 \text{ in } \Omega, \ \partial_n Q|_\Gamma = 0\}.$$

Note that the elements of $\mathcal{S}$ are symmetric and traceless tensors.

Theorem 4. *Assume that $(\boldsymbol{u}_0, Q_0) \in \boldsymbol{H} \times \mathbb{H}^1$. Fixed $(\boldsymbol{u}, Q)$ a weak solution of (1)–(3) in $(0, +\infty)$, then $\omega(\boldsymbol{u}, Q)$ is nonempty and $\omega(\boldsymbol{u}, Q) \subset \{0\} \times \mathcal{S}$. Moreover, for any $Q_\infty \in \mathcal{S}$ such that $(0, Q_\infty) \in \omega(\boldsymbol{u}, Q)$, it holds*

$$\mathcal{E}(0, Q_\infty) = \mathcal{E}_\infty.$$

In particular,

$$\boldsymbol{u}(t) \to 0 \ \text{ weakly in } \ \boldsymbol{L}^2 \ \text{ and } \ \mathcal{E}_\mu(Q(t)) \to \mathcal{E}_\mu(Q_\infty) \ \text{ in } \ \mathbb{R}$$

as $t \uparrow +\infty$.

Proof. Observe that since

$$(\boldsymbol{u}, Q) \in L^\infty(0, +\infty; \boldsymbol{H} \times \mathbb{H}^1),$$

for any sequence $\{t_n\} \uparrow +\infty$, there exists a subsequence (equally denoted) and suitable limit functions $(\boldsymbol{u}_\infty, Q_\infty) \in \boldsymbol{H} \times \mathbb{H}^1$, such that

$$\boldsymbol{u}(t_n) \to \boldsymbol{u}_\infty \text{ weakly in } \boldsymbol{H}, \quad Q(t_n) \to Q_\infty \text{ weakly in } \mathbb{H}^1. \tag{27}$$

We consider the initial and boundary-value problem associated to (1)–(3) restricted on the time interval $[t_n, t_n + 1]$ with initial values $\boldsymbol{u}(t_n)$ and $Q(t_n)$. If we define

$$\boldsymbol{u}_n(s) := \boldsymbol{u}(s + t_n), \quad Q_n(s) := Q(s + t_n), \quad H_n(s) := H(s + t_n), \quad \text{a.e. } s \in [0, 1],$$

Observe that the meaning of this notation for H_n is

$$H_n(s) = H(Q_n(s)) := H(Q(s + t_n)).$$

Then $(\boldsymbol{u}_n, Q_n)$ is a weak solution to the problem (1)–(3) in the time interval $[0, 1]$. From the energy inequality (18) and the convergence of the energy (26), we have that

$$\int_0^1 (\nu \|\nabla \boldsymbol{u}_n(s)\|^2_{L^2} + \gamma \|H(Q_n(s))\|^2_{L^2}) \, ds = \int_{t_n}^{t_n+1} (\nu |\|\nabla \boldsymbol{u}(t)\|^2_{L^2} + \gamma \|H(Q(t))\|^2_{L^2}) \, dt$$
$$\leq \mathcal{E}(\boldsymbol{u}(t_n), Q(t_n)) - \mathcal{E}(\boldsymbol{u}(t_n + 1), Q(t_n + 1)) \longrightarrow 0 \quad \text{ as } n \to \infty,$$

hence,

$$\nabla \boldsymbol{u}_n \to 0 \text{ strongly in } L^2(0, 1; \boldsymbol{L}^2)$$

and

$$H(Q_n) \to 0 \text{ strongly in } L^2(0,1;\mathbb{L}^2). \tag{28}$$

In particular, by using Poincaré inequality, one has

$$\boldsymbol{u}_n \to 0 \;\text{ strongly in } L^2(0,1;\boldsymbol{V}). \tag{29}$$

Moreover, since $\boldsymbol{u}_n$ and $\partial_t \boldsymbol{u}_n$ are bounded in $L^\infty(0,1;\boldsymbol{H})$ and $L^{4/3}(0,1;\boldsymbol{V}')$ respectively, then by using Aubin-Lions compactness,

$$\boldsymbol{u}_n \to 0 \;\text{ strongly in } C([0,1];\boldsymbol{V}').$$

In particular, $\boldsymbol{u}(t_n) = \boldsymbol{u}_n(0) \to 0$ in $\boldsymbol{V}'$, hence $\boldsymbol{u}_\infty = 0$ (owing to (27)). Consequently, the whole trajectory $\boldsymbol{u}(t) \to 0$ as $t \to +\infty$, weakly in $\boldsymbol{H}$.

Furthermore, Q_n is bounded in $L^2(0,1;\mathbb{H}^2) \bigcap L^\infty(0,1;\mathbb{H}^1)$ and $\partial_t Q_n$ is bounded in $L^{4/3}(0,1;\mathbb{L})$. Therefore, by using again Aubin-Lions compactness, there exists a subsequence of Q_n (equally denoted) and a limit function $\overline{Q}$ such that $Q_n \to \overline{Q}$ strongly in $C^0([0,1];\mathbb{L}^2) \cap L^2(0,1;\mathbb{H}^1)$ and weakly in $L^2(0,1;\mathbb{H}^2)$.

In particular, $Q(t_n) = Q_n(0) \to \overline{Q}(0)$ in $\mathbb{L}^2$, hence $\overline{Q}(0) = Q_\infty$ (owing to (27)) in $\mathbb{H}^1$. On the other hand, $\partial_t Q_n$ converges weakly to $\partial_t \overline{Q}$ in $L^{4/3}(0,1;\mathbb{L}^2)$, hence taking limits in the variational formulation:

$$(\partial_t Q_n, \widetilde{Q}) + ((\boldsymbol{u}_n \cdot \nabla) Q_n, \widetilde{Q}) - (S(W(\boldsymbol{u}_n), Q_n), \widetilde{Q}) + (H(Q_n), \widetilde{Q}) = 0$$

for all $\widetilde{Q} \in \mathbb{L}^2$ and taking into account (28) and (29), we have that $\partial_t Q_n \to 0$ in $L^{4/3}(0,1;\mathbb{L}^2)$ weakly. Therefore, $\partial_t \overline{Q} = 0$ and $\overline{Q}(t)$ is a constant function of $\mathbb{L}^1$ for all $t \in [0,1]$, hence since $\overline{Q}(0) = Q_\infty$, we have

$$\overline{Q}(t) = Q_\infty \in \mathbb{H}^1 \quad \text{ for all } t \in [0,1]. \tag{30}$$

Finally, since $f(Q_n)$ converges to $f(\overline{Q})$ weakly* in $L^\infty(0,1;\mathbb{L}^2)$, by taking limit as $n \to +\infty$ in the variational formulation $(H(Q_n), \widetilde{Q}) = \varepsilon(\nabla Q_n, \nabla \widetilde{Q}) + (f(Q_n), \widetilde{Q})$ for all $\widetilde{Q} \in \mathbb{H}^1$, we deduce

$$\varepsilon(\nabla \overline{Q}, \nabla \widetilde{Q}) + (f(\overline{Q}), \widetilde{Q}) = 0, \quad \forall \widetilde{Q} \in \mathbb{H}^1, \text{ a.e. } t \in (0,1).$$

Then, from (30), $Q_\infty \in \mathbb{H}^1$ and $\varepsilon(\nabla Q_\infty, \nabla \widetilde{Q}) + (f(Q_\infty), \widetilde{Q}) = 0$, $\forall \widetilde{Q} \in \mathbb{H}^1$. Finally, by applying H^2-regularity of the Poisson problem:

$$\big\{ -\varepsilon \Delta Q + Q = -f(Q) + Q \quad \text{in } \Omega, \partial_{\boldsymbol{n}} Q|_\Gamma = 0$$

we deduce that $Q_\infty \in \mathbb{H}^2$, hence $Q_\infty \in \mathcal{S}$ and the proof is finished. □

4.2. Convergence of the Tensor

In the next theorem we apply the following Lojasiewicz-Simon's result that can be found in [17].

Lemma 5 (Lojasiewicz-Simon inequality). *Let $Q_* \in \mathcal{S}$ and $K > 0$ fixed. Then, there exists positive constants β_1, β_2 and C and $\theta \in (0,1/2]$, such that for all $Q \in \mathbb{H}^2$ symmetric and traceless with $\|Q\|_{H^1} \le K$, $\|Q - Q_*\|_{L^2} \le \beta_1$ and $|\mathcal{E}_\mu(Q) - \mathcal{E}_\mu(Q_*)| \le \beta_2$, it holds*

$$|\mathcal{E}_\mu(Q) - \mathcal{E}_\mu(Q_*)|^{1-\theta} \le C\,\|H(Q)\|_{H^{-1}}$$

where $H(Q)$ is defined in (4).

Now, we are in position to prove that $Q(t) \to Q_\infty$ as $t \uparrow +\infty$.

Theorem 6. *Assume the following regularity hypothesis of the energy of the weak solution:*

$$\widetilde{\mathcal{E}}(t) = \mathcal{E}(\boldsymbol{u}(t), Q(t)) \text{ a.e. } t \geq 0,$$

where $\widetilde{\mathcal{E}}(t)$ *is defined in Lemma 3. Then, under the hypotheses of Theorem 4, there exists a unique limit* $Q_\infty \in \mathcal{S}$ *such that* $Q(t) \to Q_\infty$ *in* $\mathbb{H}^1$*-weakly as* $t \uparrow +\infty$*, i.e.*

$$\omega(\boldsymbol{u}, Q) = \{(0, Q_\infty)\}.$$

Proof. Let $Q_\infty \in \mathcal{S}$ such that $(0, Q_\infty) \in \omega(\boldsymbol{u}, Q)$, i.e. there exists $t_n \uparrow +\infty$ such that $\boldsymbol{u}(t_n) \to 0$ weakly in $\boldsymbol{L}^2$ and $Q(t_n) \to Q_\infty$ weakly in $\mathbb{H}^1$ (and strongly in $\mathbb{L}^2$).

It can be assumed that $\widetilde{\mathcal{E}}(t) > \mathcal{E}_\mu(Q_\infty)(= \mathcal{E}_\infty)$ for all $t > 0$, because otherwise, if it exists some $\widetilde{t} > 0$ such that $\widetilde{\mathcal{E}}(\widetilde{t}) = \mathcal{E}_\infty$, then the energy inequality (23) implies

$$\widetilde{\mathcal{E}}(t) = \mathcal{E}_\infty, \quad \|\nabla \boldsymbol{u}(t)\|^2_{L^2} = 0 \quad \text{and} \quad \|H(Q(t))\|^2_{L^2} = 0, \quad \forall\, t \geq \widetilde{t}.$$

Therefore, $\boldsymbol{u}(t) = 0$ and $H(Q(t)) = 0$ for all $t \geq \widetilde{t}$, and by using the Q-equation of (1), $\partial_t Q(t) = 0$, hence $Q(t) = Q_\infty$ for all $t \geq \widetilde{t}$. Then, the convergence of the whole Q-trajectory towards Q_∞ is trivial and $\widetilde{\mathcal{E}}(t) > \mathcal{E}_\infty$ is assumed for all $t \geq 0$.

The proof will be divided into three steps.

Step 1 *On the assumption that there exists* $t_1 > 0$ *such that*

$$\|Q(t) - Q_\infty\|_{L^2} \leq \beta_1 \text{ and } |\mathcal{E}_\mu(Q(t)) - \mathcal{E}_\mu(Q_\infty)| \leq \beta_2 \quad \forall t \geq t_1 > 0$$

where $\beta_1 > 0$, $\beta_2 > 0$ *are the constants appearing in Lemma 5, then the following inequalities hold:*

$$\frac{d}{dt}\Big((\mathcal{E}(\boldsymbol{u}(t), Q(t)) - E_\infty)^\theta\Big) + C\,\theta\,(\|\nabla \boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{L^2}) \leq 0, \tag{31}$$

a.e. $t \in (t_1, \infty)$ *and*

$$\int_{t_1}^{t_2} \|\partial_t Q\|_{H^{-1}} \leq \frac{C}{\theta}((\mathcal{E}(\boldsymbol{u}(t_1), Q(t_1)) - E_\infty)^\theta, \tag{32}$$

for all $t_2 \in (t_1, \infty)$*, where* $\theta \in (0, 1/2]$ *is the constant appearing in Lemma 5.*

Proof of Step 1 Since $\mathcal{E}_\infty$ is constant, we can rewrite the energy inequality (24) as

$$\frac{d}{dt}(\widetilde{\mathcal{E}}(t) - \mathcal{E}_\infty) + C\left(\|\nabla \boldsymbol{u}(t)\|^2_{L^2} + \|H(t)\|^2_{L^2}\right) \leq 0, \quad \text{a.e. } t \geq 0,$$

By taking into account that

$$\|\nabla \boldsymbol{u}(t)\|^2_{L^2} + \|H(t)\|^2_{\mathbb{L}^2} \geq \frac{1}{2}\left(\|\nabla \boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{\mathbb{L}^2}\right)^2$$

and

$$\frac{1}{2}(\|\nabla \boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{\mathbb{L}^2}) \geq C(\|\boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{H^{-1}}),$$

we obtain

$$\frac{d}{dt}(\widetilde{\mathcal{E}}(t) - \mathcal{E}_\infty) + C(\|\boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{H^{-1}})\,(\|\nabla \boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{L^2}) \leq 0, \quad \text{a.e. } t \geq 0.$$

Using the time derivative of the $(\widetilde{\mathcal{E}}(t) - \mathcal{E}_\infty)^\theta$, we get

$$\begin{aligned}&\frac{d}{dt}\left((\widetilde{\mathcal{E}}(t) - \mathcal{E}_\infty)^\theta\right)\\ &\quad + \theta(\widetilde{\mathcal{E}}(t) - \mathcal{E}_\infty)^{\theta-1} C(\|\boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{H^{-1}})\,(\|\nabla \boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{L^2}) \leq 0.\end{aligned} \tag{33}$$

almost everywhere $t \geq 0$.

On the other hand, since $|\mathcal{E}_k(\boldsymbol{u}(t))| = \dfrac{1}{2}\|\boldsymbol{u}(t)\|^2_{L^2}$ and $\|\boldsymbol{u}(t)\|_{L^2} \le K$, we have that

$$|\mathcal{E}_k(\boldsymbol{u}(t))|^{1-\theta} = \frac{1}{2^{1-\theta}}\|\boldsymbol{u}(t)\|^{2(1-\theta)}_{L^2} = \frac{1}{2^{1-\theta}}\|\boldsymbol{u}(t)\|^{1-2\theta}_{L^2}\|\boldsymbol{u}(t)\|_{L^2} \le C\|\boldsymbol{u}(t)\|_{L^2} \quad a.e.\ t \ge 0.$$

This estimate together the Lojasiewicz-Simon inequality $|\mathcal{E}_\mu(Q) - \mathcal{E}_\infty|^{1-\theta} \le C\|H(Q)\|_{H^{-1}}$, give

$$\begin{aligned}(\mathcal{E}(u(t), Q(t)) - \mathcal{E}_\infty)^{1-\theta} &\le |\mathcal{E}_k(\boldsymbol{u}(t))|^{1-\theta} + |\mathcal{E}_\mu(Q(t)) - \mathcal{E}_\infty|^{1-\theta} \\ &\le C(\|\boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{H^{-1}}) \quad \text{a.e. } t \ge t_1.\end{aligned}$$

Therefore,

$$(\mathcal{E}(u(t), Q(t)) - \mathcal{E}_\infty)^{\theta-1}(\|\boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{H^{-1}}) \ge C \quad \text{a.e. } t \ge t_1. \tag{34}$$

By applying (34) in (33),

$$\frac{d}{dt}\left((\mathcal{E}(u(t), Q(t)) - \mathcal{E}_\infty)^\theta\right) + C\,\theta\,(\|\nabla\boldsymbol{u}(t)\|_{L^2} + \|H(t)\|_{L^2}) \le 0, \quad \text{a.e. } t \ge t_1$$

and (31) is proved.

In this step, the hypothesis $\mathcal{E}(u(t), Q(t)) = \widetilde{\mathcal{E}}(t)$ for almost every t is a key point. In particular, this hypothesis implies that the integral and differential versions of the energy law (23) and (24) are satisfied by $\mathcal{E}(u(t), Q(t))$ a.e. in time. In fact, energy law (24), changing $\widetilde{\mathcal{E}}(t)$ by $\mathcal{E}(u(t), Q(t))$, is the crucial hypothesis imposed in Remark 2.4 of [17].

Secondly, for any $t_2 \in (t_1, +\infty)$, since $(\mathcal{E}(\boldsymbol{u}(t_2), Q(t_2)) - \mathcal{E}_\infty)^\theta > 0$, integrating (31) into $[t_1, t_2]$ we have

$$\theta\, C\int_{t_1}^{t_2}(\|\nabla\boldsymbol{u}(t)\|_{L^2} + \|H(Q(t))\|_{L^2})dt \le (\mathcal{E}(\boldsymbol{u}(t_1), Q(t_1)) - \mathcal{E}_\infty)^\theta. \tag{35}$$

From (11), by using the weak regularity $Q \in L^\infty(0; +\infty; \mathbb{L}^6(\Omega))$ given in (16), we achieve

$$\begin{aligned}\|\partial_t Q(t)\|_{H^{-1}} &\le C(\|\boldsymbol{u}(t)Q(t)\|_{L^2} + \|W(\boldsymbol{u}(t))\|_{L^2}\|Q(t)\|_{L^3} + \|H(Q(t))\|_{L^2}). \\ &\le C(\|\nabla\boldsymbol{u}(t)\|_{L^2} + \|H(Q(t))\|_{L^2}) \qquad a.e.\ t \ge 0.\end{aligned}$$

By integrating this inequality into $[t_1, t_2]$ and using (35), we attain (32).

Step 2 *There exists a sufficiently large n_0 such that $\|Q(t) - Q_\infty\|_{L^2} \le \beta_1$ and $|\mathcal{E}_\mu(Q(t)) - \mathcal{E}_\mu(Q_*)| \le \beta_2$ for all $t \ge t_{n_0}$ (β_1, β_2 given in Lemma 5).*

Proof of Step 2 Since $Q(t_n) \to Q_\infty$ strongly in $\mathbb{L}^2$ and $\mathcal{E}(\boldsymbol{u}(t_n), Q(t_n)) \searrow \mathcal{E}_\infty = \mathcal{E}_\mu(Q_\infty)$ as $t_n \to +\infty$ (see (26)), then for any $\delta = \delta(\beta_1, \beta_2) > 0$ such that

$$\delta < \beta_1 \quad \text{and} \quad \theta\delta \le \beta_2^\theta,$$

there exists an integer $M(\delta)$ such that, for all $n \ge M(\delta)$,

$$\|Q(t_n) - Q_\infty\|_{L^2} \le \delta \quad \text{and} \quad \frac{1}{\theta}(\mathcal{E}_\mu(Q(t_n)) - \mathcal{E}_\infty)^\theta \le \delta. \tag{36}$$

For each $n \ge M(\delta)$, we define

$$\bar{t}_n := \sup\{t : t > t_n,\ \|Q(s) - Q_\infty\|_{L^2} < \beta_1 \quad \forall s \in [t_n, t)\}.$$

It suffices to prove that $\bar{t}_{n_0} = +\infty$ for some n_0 sufficiently large. Assume by contradiction that $t_n < \bar{t}_n < +\infty$ for all n, hence $\|Q(\bar{t}_n) - Q_\infty\|_{L^2} = \beta_1$ and $\|Q(t) - Q_\infty\|_{L^2} < \beta_1$ for all $t \in [t_n, \bar{t}_n)$.

Observe that the hypothesis $|\mathcal{E}_\mu(Q) - \mathcal{E}_\mu(Q_\infty)| \le \beta_2$ holds owing to the constraint $\theta\delta \le \beta_2^\theta$. By applying Step 1 for all $t \in [t_n, \bar{t}_n]$, from (32) and (36) we obtain,

$$\int_{t_n}^{\bar{t}_n}\|\partial_t Q\|_{H^{-1}} \le C\delta, \quad \forall\, n \ge N(\delta).$$

Therefore,

$$\|Q(\bar{t}_n) - Q_\infty\|_{H^{-1}} \leq \|Q(t_n) - Q_\infty\|_{H^{-1}} + \int_{t_n}^{\bar{t}_n} \|\partial_t Q\|_{H^{-1}} \leq (1+C)\delta,$$

which implies that $\lim_{n\to+\infty} \|Q(\bar{t}_n) - Q_\infty\|_{H^{-1}} = 0$.

On the other hand, $Q(\bar{t}_n)$ is bounded in $\mathbb{H}^1$. Indeed, since $Q \in C_w([0,+\infty);\mathbb{H}^1)$, $Q(s)$ converges weakly to $Q(\bar{t})$ in $\mathbb{H}^1$ as $s \to \bar{t}$. Owing to the weak lower semi-continuity $\|Q(\bar{t})\|_{H^1} \leq \liminf \|Q(s)\|_{H^1} \leq C$. But, since $F_\mu(Q)$ is bounded in $L^\infty(\mathbb{L}^1)$, then $\nabla Q(\bar{t}_n)$ is bounded in $\mathbb{L}^2(\Omega)$ and $Q(\bar{t}_n)$ is bounded in $\mathbb{H}^1$.

Therefore, $Q(\bar{t}_n)$ is relatively compact in $\mathbb{L}^2$. There exists a subsequence of $Q(\bar{t}_n)$, also denoted $Q(\bar{t}_n)$, that converges to Q_∞ in $\mathbb{L}^2$-strong. Hence $\|Q(\bar{t}_n) - Q_\infty\|_{\mathbb{L}^2} < \beta_1$ for a sufficiently large n, which contradicts the definition of $\bar{t}_n$.

Step 3 *There exists a unique Q_∞ such that $Q(t) \to Q_\infty$ weakly in $\mathbb{H}^1$ as $t \uparrow +\infty$.*
Proof of Step 3 By using (32) for any $t_1, t_0 : t_1 > t_0 \geq t_{n_0}$,

$$\|Q(t_1) - Q(t_0)\|_{H^{-1}} \leq \int_{t_0}^{t_1} \|\partial_t Q\|_{H^{-1}} \to 0, \quad \text{as } t_0, t_1 \to +\infty.$$

Therefore, $(Q(t))_{t \geq t_{n_0}}$ is a Cauchy sequence in $\mathbb{H}^{-1}$ as $t \uparrow +\infty$, hence, there exists a unique $Q_\infty \in \mathbb{H}^{-1}$ such that $Q(t) \to Q_\infty$ in $\mathbb{H}^{-1}$ as $t \uparrow +\infty$. Finally, the convergence in $\mathbb{H}^1$-weak by sequences of $Q(t)$ proved in Theorem 4, yields to $Q(t) \to Q_\infty$ in $\mathbb{H}^1$-weak, and the proof is finished. □

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