On the Bingham Flow in a Thin Y-Like Shaped Structure

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Communicated by K. Pileckas

Abstract. We consider the steady Bingham flow in a two-dimensional thin Y-like shaped structure, with no-slip boundary conditions and under the action of given external forces. After passage to the limit with respect to a small parameter related to the thickness of the domain, we obtain three uncoupled problems. Each of these problems describes an anisotropic flow, corresponding to a lower-dimensional “Bingham-like” constitutive law. These results are in accordance with the engineering models.

Mathematics Subject Classification. 76A05, 76A99, 74K30.

Keywords. Non-Newtonian fluid, junctions, thin structures.

1. Introduction

We consider a steady Bingham flow in a two-dimensional thin Y-like shaped structure, with no-slip boundary conditions and under the action of given external forces.

We take the following multi-structure Ω_n (see Fig. 1), where h_n is a small positive parameter representing the order of the thickness of each branch, $\alpha \in [0, \frac{\pi}{2}]$ and $\beta \in [-\frac{\pi}{2}, 0]$ are two angles describing the deviation of the two oblique branches from the vertical direction, and $\frac{L^r}{\cos \alpha}$, $\frac{L^l}{\cos \beta}$, and L^b are the main lengths of the three branches (from now on, the exponents r , l and b stand for right, left and below, respectively).

The Bingham fluid is a particular non-newtonian fluid, used as a model for viscoplastic materials as the volcanic lava, the mud for the oil extraction, some paints, etc. (for instance, see [6, 16, 17]). Its constitutive law is non linear and is characterized by the presence of a given threshold (see (2.3)). The fluid moves like a rigid body as long as a certain function of the stress tensor is below the threshold and obeys a non linear constitutive law beyond the threshold. In this perspective, also the blood can be modeled as a Bingham fluid in the first approximation, since it has a yield stress as the Bingham fluid, but at high shear stress it behaves as a pseudoplastic fluid. Thus, the Bingham fluid in our geometry can also modelize the microcirculation in blood vessels.

According to [19], the steady flow of a Bingham fluid in Ω_n , with no-slip boundary conditions, corresponds to the following variational inequality,

$$\begin{cases} (u_n, p_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R}), \quad \operatorname{div}(u_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D u_n D(v - u_n) + h_n g |Dv| - h_n g |Du_n|) dX_1 dx_2 \\ \geq \int_{\Omega_n} (f_n(v - u_n) + p_n \operatorname{div}(v - u_n)) dX_1 dx_2 \quad \forall v \in (H_0^1(\Omega_n))^2, \end{cases} \quad (1.1)$$

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FIG. 1. The thin domain Ω_n

where, $h_n^2\mu$ and h_ng , with μ and g two positive constants, are the viscosity and the yield stress, respectively, f_n is the given external force, and the unknown (u_n, p_n) represents the velocity and the pressure of the fluid.

Our aim is to study the asymptotic behavior of problem (1.1) as h_n tends to zero. Indeed, the very small size of h_n could not allow us to solve numerically this problem, since the construction of appropriate meshes could not be possible. Consequently, an asymptotic analysis of this problem is needed.

By the dimensional reduction method introduced in [15], we first rescale the problem (1.1) in a fixed domain, independent on h_n , by using the three different transformations given in (3.2) and we get problem (3.8). Then, assuming suitable convergence conditions on the external forces (see (3.9)), after passage to the limit we obtain three uncoupled problems (see Theorem 4.1). Each of these problems describes an anisotropic flow, corresponding to a lower-dimensional "Bingham-like" constitutive law (see (8.3)). These results are in accordance with the engineering models. Indeed, for the one-dimensional "Bingham-like" constitutive law used in engineering, one may refer, for instance, to [29]. In [29], first the authors give the one-dimensional "Bingham-like" constitutive law (8.3), employed in order to modelize the Bingham flow in a two-dimensional domain with small thickness. Then, in a two-dimensional domain with small thickness they consider a Bingham flow with a two-viscosities constitutive law. Under some constraints on the viscosities, they perform a perturbation analysis of the two-dimensional model and thus they formally derive the previous given one-dimensional "Bingham-like" constitutive law (8.3). In our case, starting by a two-dimensional Bingham flow stated in a thin domain of thickness h_n , considering a viscosity of the fluid of order h_n^2 and a yield stress of order h_n , we rigorously derive the one-dimensional "Bingham-like" constitutive law (8.3), performing a full asymptotic analysis, by passing to the limit as h_n tends to zero in the variational formulation (1.1) of the problem under consideration.

Let us point out that rescaling the problem from a thin oblique branch to a fixed one involves differential operators that are quite sophisticated and more difficult to handle than those obtained in [10–12]. This leads to further difficulties on *a priori* estimates for the velocity, but especially for the pressure. Indeed, unlike in [10–12], *a priori* estimates on the velocity imply that the two components (u_1, u_2) of the limit velocity on each oblique branch are both non-zero and linked by a constraint of the type $u_1 = cu_2$, where the real constant c only depends on the angle (see (5.8)). The latter property will guide the choice of suitable test functions for obtaining *a priori* estimates on the pressure, that will be only local, in the sense that they depend on the branch. This requires working with three different pressures. Consequently,

for passing to the limit in each branch, we first get a local reformulation of our problem. Eventually, test functions satisfying the constraint $u_1 = cu_2$ will be used for passing to the limit and for deducing the variational inequality satisfied by u_2 in each branch. As far as the limit force is concerned, only its projection on the axis of each branch is involved in the limit inequalities (see the first term in the right-hand side of (4.8), (4.9), (4.10), and Remark 4.2). The limit velocity in the vertical branch is classical by now. Indeed, as already observed in [10–12], the component in the transverse direction, i.e. u_1 , is zero, while u_2 is the solution to problem (4.10). One of the novelties of this paper appears in the limit velocities in the two oblique branches. On each oblique branch, the velocity has two non-zero components and its direction coincides with the axis of the branch (see Remark 4.2).

Let us remark that the case corresponding to $\alpha = \frac{\pi}{2}$ and/or $\beta = -\frac{\pi}{2}$ is not included in our study, since in this particular case the initial domain Ω_n degenerates, at least one of the branches becoming a horizontal line. We refer [10] for the study of this problem in a T -structure, which is not a particular case of the present work.

Note that problem (1.1) becomes the system modeling the Stokes flow in a thin multi-structure when $g = 0$. For such problems we refer to [7, 31]. For Navier-Stokes flow in a thin tube or thin layer we refer to [13, 30, 36], while for multi-structures we refer to [23, 32, 33].

As far as the study of Bingham flow in domains depending on small geometrical parameters is concerned, we recall that the first results for the Bingham flow in periodic domains are obtained in [28]. The justification of the convergence for these results is due to [8, 9]. In the recent paper [5], the authors present a model of Bingham flow in a thin periodic domain which contains an array of obstacles modeled as vertical cylinders. Up to our knowledge, the first convergence result on Bingham flow in thin periodic domains is due to [1], followed by [2]. We refer to [18] for Bingham flows in periodic domains of infinite length. We refer to [14] for Bingham flows in a thin domain with periodically oscillating boundary. For other models of non-newtonian fluid in thin multi-structures we refer to [34, 35].

For other problems in thin multi-structures we refer to [3, 4, 20–22, 24–27].

The paper is organized as follows. In Sect. 2 we introduce the domain Ω_n , the physical properties of the Bingham fluid, and the variational inequality which modelizes the flow. In Section 3 the problem is rescaled on a fixed domain. Section 4 contains the main result of the paper, the convergence theorem. *A priori* estimates and convergence results for the velocity and the pressure are obtained in Sect. 5 and in Sect. 6, respectively. These last results will allow us to prove the convergence theorem in Sect. 7. The paper ends with some conclusions, in Sect. 8.

2. The Setting of the Problem

In what follows, a generic element of $\mathbb{R}^2$ is denoted by (X_1, x_2) or (x_1, x_2) .

Let $\alpha \in [0, \frac{\pi}{2}[$ and $\beta \in]-\frac{\pi}{2}, 0]$, $L^r, L^l, L^b \in]0, +\infty[$, for every $n \in \mathbb{N}$ let $h_n \in]0, 1[$ be a small parameter, and let Ω_n be the thin two-dimensional Y-like shaped domain defined by

$$\begin{cases} \Omega_n = \Omega_n^r \cup \Omega_n^l \cup \Omega_n^b, \\ \Omega_n^r = \left\{ (X_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^r, \quad x_2 \tan \alpha < X_1 < x_2 \tan \alpha + \frac{h_n}{2} \right\}, \\ \Omega_n^l = \left\{ (X_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^l, \quad x_2 \tan \beta - \frac{h_n}{2} < X_1 < x_2 \tan \beta \right\}, \\ \Omega_n^b = \left] -\frac{h_n}{2}, \frac{h_n}{2} \right[\times] -L^b, 0]. \end{cases} \quad (2.1)$$

In Ω_n we consider the non linear flow of a Bingham fluid. If $u_n = (u_{n1}, u_{n2})$ and p_n denote its velocity and pressure, respectively, the corresponding stress tensor is defined by

$$\sigma_{ij}(u_n, p_n) = -p_n \delta_{ij} + gh_n \frac{e_{ij}(u_n)}{\sqrt{e_{II}(u_n)}} + 2\mu h_n^2 e_{ij}(u_n), \quad (2.2)$$

where δ_{ij} is the Kronecker symbol, g is a strictly positive constant related to the yield stress of the fluid, μ is a strictly positive constant related to the viscosity of the fluid,

$$e(u_n) = \frac{1}{2} [D(u_n) + D^T(u_n)], \quad e_{II}(u_n) = \frac{1}{2} \sum_{i,j=1}^2 e_{ij}(u_n) e_{ij}(u_n),$$

with $D(u_n) = \begin{pmatrix} \frac{\partial u_{n1}}{\partial X_1} & \frac{\partial u_{n1}}{\partial x_2} \\ \frac{\partial u_{n2}}{\partial X_1} & \frac{\partial u_{n2}}{\partial x_2} \end{pmatrix}$ and $D^T(u_n)$ the transposed of $D(u_n)$. Moreover, we set

$$\sigma_{ij}^e(u_n) = gh_n \frac{e_{ij}(u_n)}{\sqrt{e_{II}(u_n)}} + 2\mu h_n^2 e_{ij}(u_n), \quad \forall i, j \in \{1, 2\}; \quad \sigma_{II}(u_n) = \frac{1}{2} \sum_{i,j=1}^2 \sigma_{ij}^e(u_n) \sigma_{ij}^e(u_n).$$

We remark that (2.2), i.e., the constitutive law of the Bingham fluid, is valid only if $e_{II}(u_n) \neq 0$. In [19] it is shown that this constitutive law is equivalent to the following one

$$\begin{cases} \sqrt{\sigma_{II}(u_n)} \leq gh_n \Leftrightarrow e_{ij}(u_n) = 0, \\ \sqrt{\sigma_{II}(u_n)} > gh_n \Leftrightarrow e_{ij}(u_n) = \frac{1}{2\mu h_n^2} \left(1 - \frac{gh_n}{\sqrt{\sigma_{II}(u_n)}} \right) \sigma_{ij}^e(u_n). \end{cases} \quad (2.3)$$

We point out that this is a threshold law: as long as the shear stress $\sqrt{\sigma_{II}(u_n)}$ is below gh_n , the fluid behaves as a rigid solid. When the value of the shear stress $\sqrt{\sigma_{II}(u_n)}$ exceeds gh_n , then the fluid flows obeying a non linear law.

We also suppose that the fluid is incompressible, that is

$$\operatorname{div} u_n = \frac{\partial u_{n1}}{\partial X_1} + \frac{\partial u_{n2}}{\partial x_2} = 0 \quad \text{in } \Omega_n. \quad (2.4)$$

Moreover, we apply to the fluid a given external force $f_n = (f_{n1}, f_{n2})$ belonging to $(L^2(\Omega_n))^2$, and then we have the following relations

$$\frac{\partial \sigma_{i1}}{\partial X_1} + \frac{\partial \sigma_{i2}}{\partial x_2} = f_{ni} \quad \text{in } \Omega_n, \quad i = 1, 2. \quad (2.5)$$

Furthermore, we assume the no-slip condition to the boundary of the domain, which reads

$$u_n = 0 \quad \text{on } \partial\Omega_n. \quad (2.6)$$

In [19] it is shown that the velocity u_n satisfying (2.2), (2.4), (2.5) and (2.6) solves the following variational inequality

$$\begin{cases} u_n \in (H_0^1(\Omega_n))^2, \quad \operatorname{div}(u_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D u_n D(v - u_n) + h_n g |Dv| - h_n g |D u_n|) dX_1 dx_2 \\ \geq \int_{\Omega_n} f_n (v - u_n) dX_1 dx_2 \quad \forall v \in (H_0^1(\Omega_n))^2 : \operatorname{div}(v) = 0 \text{ in } \Omega_n. \end{cases} \quad (2.7)$$

For each n , this inequality admits a unique solution u_n . According to [19], problem (2.7) is equivalent to the following one

$$\begin{cases} (u_n, p_n) \in (H_0^1(\Omega_n))^2 \times (L^2(\Omega_n)/\mathbb{R}), \quad \operatorname{div}(u_n) = 0 \text{ in } \Omega_n, \\ \int_{\Omega_n} (h_n^2 \mu D u_n D(v - u_n) + h_n g |Dv| - h_n g |D u_n|) dX_1 dx_2 \\ \geq \int_{\Omega_n} (f_n (v - u_n) + p_n \operatorname{div}(v - u_n)) dX_1 dx_2 \quad \forall v \in (H_0^1(\Omega_n))^2. \end{cases} \quad (2.8)$$

which admits a solution (u_n, p_n) , such that u_n is unique, but p_n is not unique.

The aim of this paper is to study the asymptotic behavior, as n diverges, of problem (2.8), under suitable assumption on the given data $\{f_n\}_{n \in \mathbb{N}}$ and on the assumption

$$\lim_n h_n = 0. \quad (2.9)$$

We note that the pressure p_n is not unique. If p_n is a solution of (2.8), then $p_n + c$ is also a solution of (2.8), for every $c \in \mathbb{R}$. In particular, the constant c can be chosen such that the average of $p_n + c$ is zero on a subset of Ω_n .

3. The Rescaled Problem

Set now

$$\begin{cases} \Omega = \Omega^r \cup \Omega^l \cup \Omega^b, \\ \Omega^r = \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^r, \quad x_2 \tan \alpha < x_1 < x_2 \tan \alpha + \frac{1}{2} \right\}, \\ \Omega^l = \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^l, \quad x_2 \tan \beta - \frac{1}{2} < x_1 < x_2 \tan \beta \right\}, \\ \Omega^b =] -\frac{1}{2}, \frac{1}{2}[\times] -L^b, 0]. \end{cases} \quad (3.1)$$

In order to pass to the limit in (2.8), the first step consists in rewriting problems (2.7) and (2.8) on the domain Ω (which is independent of n). This is done as usual, by a domain dilatation technique (see [15]), through the three different maps

$$\begin{cases} x = (x_1, x_2) \in \Omega^r \longrightarrow (h_n(x_1 - x_2 \tan \alpha) + x_2 \tan \alpha, x_2) \in \Omega_n^r, \\ x = (x_1, x_2) \in \Omega^l \longrightarrow (h_n(x_1 - x_2 \tan \beta) + x_2 \tan \beta, x_2) \in \Omega_n^l, \\ x = (x_1, x_2) \in \Omega^b \longrightarrow (h_n x_1, x_2) \in \Omega_n^b. \end{cases} \quad (3.2)$$

More precisely, for every $n \in \mathbb{N}$ we set

$$\begin{cases} D_n^r : v \in (H^1(\Omega^r))^2 \longrightarrow \left(\frac{1}{h_n} \partial_{x_1} v, \partial_{x_2} v - \frac{1-h_n}{h_n} \tan \alpha \partial_{x_1} v \right) \in (L^2(\Omega^r))^2 \times (L^2(\Omega^r))^2, \\ D_n^l : v \in (H^1(\Omega^l))^2 \longrightarrow \left(\frac{1}{h_n} \partial_{x_1} v, \partial_{x_2} v - \frac{1-h_n}{h_n} \tan \beta \partial_{x_1} v \right) \in (L^2(\Omega^l))^2 \times (L^2(\Omega^l))^2, \\ D_n^b : v \in (H^1(\Omega^b))^2 \longrightarrow \left(\frac{1}{h_n} \partial_{x_1} v, \partial_{x_2} v \right) \in (L^2(\Omega^b))^2 \times (L^2(\Omega^b))^2, \\ \operatorname{div}_n^r : v = (v_1, v_2) \in (H^1(\Omega^r))^2 \longrightarrow \frac{1}{h_n} \partial_{x_1} v_1 + \partial_{x_2} v_2 - \frac{1-h_n}{h_n} \tan \alpha \partial_{x_1} v_2 \in L^2(\Omega^r), \\ \operatorname{div}_n^l : v = (v_1, v_2) \in (H^1(\Omega^l))^2 \longrightarrow \frac{1}{h_n} \partial_{x_1} v_1 + \partial_{x_2} v_2 - \frac{1-h_n}{h_n} \tan \beta \partial_{x_1} v_2 \in L^2(\Omega^l), \\ \operatorname{div}_n^b : v = (v_1, v_2) \in (H^1(\Omega^b))^2 \longrightarrow \frac{1}{h_n} \partial_{x_1} v_1 + \partial_{x_2} v_2 \in L^2(\Omega^b), \end{cases} \quad (3.3)$$

$$\begin{cases} f_n^r : x = (x_1, x_2) \in \Omega^r \longrightarrow f_n(h_n(x_1 - x_2 \tan \alpha) + x_2 \tan \alpha, x_2), \\ f_n^l : x = (x_1, x_2) \in \Omega^l \longrightarrow f_n(h_n(x_1 - x_2 \tan \beta) + x_2 \tan \beta, x_2), \\ f_n^b : x = (x_1, x_2) \in \Omega^b \longrightarrow f_n(h_n x_1, x_2), \end{cases} \quad (3.4)$$

and

$$\tilde{\mathcal{V}}_n = \left\{ v \in (H_0^1(\Omega))^2 : \operatorname{div}_n^r(v) = 0 \text{ in } \Omega^r, \operatorname{div}_n^l(v) = 0 \text{ in } \Omega^l, \operatorname{div}_n^b(v) = 0 \text{ in } \Omega^b \right\}. \quad (3.5)$$

If u_n solves (2.7) (or equivalently (u_n, p_n) solves problem (2.8)), then u_n (or equivalently (u_n, p_n)) defined by

$$\begin{cases} u_n(x_1, x_2) = \begin{cases} u_n(h_n(x_1 - x_2 \tan \alpha) + x_2 \tan \alpha, x_2), & \text{in } \Omega^r, \\ u_n(h_n(x_1 - x_2 \tan \beta) + x_2 \tan \beta, x_2), & \text{in } \Omega^l, \\ u_n(h_n x_1, x_2) & \text{in } \Omega^b, \end{cases} \\ p_n(x_1, x_2) = \begin{cases} p_n(h_n(x_1 - x_2 \tan \alpha) + x_2 \tan \alpha, x_2), & \text{in } \Omega^r, \\ p_n(h_n(x_1 - x_2 \tan \beta) + x_2 \tan \beta, x_2), & \text{in } \Omega^l, \\ p_n(h_n x_1, x_2) & \text{in } \Omega^b, \end{cases} \end{cases} \quad (3.6)$$

solves

$$\begin{cases} u_n \in \tilde{\mathcal{V}}_n, \\ \int_{\Omega^r} (h_n^2 \mu D_n^r u_n D_n^r (v - u_n) + h_n g |D_n^r v| - h_n g |D_n^r u_n|) dx_1 dx_2 + \\ + \int_{\Omega^l} (h_n^2 \mu D_n^l u_n D_n^l (v - u_n) + h_n g |D_n^l v| - h_n g |D_n^l u_n|) dx_1 dx_2 + \\ + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n D_n^b (v - u_n) + h_n g |D_n^b v| - h_n g |D_n^b u_n|) dx_1 dx_2 \\ \geq \int_{\Omega^r} f_n^r (v - u_n) dx_1 dx_2 + \int_{\Omega^l} f_n^l (v - u_n) dx_1 dx_2 + \int_{\Omega^b} f_n^b (v - u_n) dx_1 dx_2 \quad \forall v \in \tilde{\mathcal{V}}_n, \end{cases} \quad (3.7)$$

or equivalently

$$\begin{cases} u_n \in \tilde{\mathcal{V}}_n, \quad p_n \in L^2(\Omega)/\mathbb{R}, \\ \int_{\Omega^r} (h_n^2 \mu D_n^r u_n D_n^r (v - u_n) + h_n g |D_n^r v| - h_n g |D_n^r u_n|) dx_1 dx_2 + \\ + \int_{\Omega^l} (h_n^2 \mu D_n^l u_n D_n^l (v - u_n) + h_n g |D_n^l v| - h_n g |D_n^l u_n|) dx_1 dx_2 + \\ + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n D_n^b (v - u_n) + h_n g |D_n^b v| - h_n g |D_n^b u_n|) dx_1 dx_2 \\ \geq \int_{\Omega^r} (f_n^r (v - u_n) + p_n \operatorname{div}_n^r (v - u_n)) dx_1 dx_2 \\ + \int_{\Omega^l} (f_n^l (v - u_n) + p_n \operatorname{div}_n^l (v - u_n)) dx_1 dx_2 \\ + \int_{\Omega^b} (f_n^b (v - u_n) + p_n \operatorname{div}_n^b (v - u_n)) dx_1 dx_2 \quad \forall v \in (H_0^1(\Omega))^2, \end{cases} \quad (3.8)$$

Conversely, if u_n solves (3.7) (or equivalently (u_n, p_n) solves (3.8)), then u_n (or equivalently (u_n, p_n)) defined by

$$\begin{cases} u_n(x_1, x_2) = \begin{cases} u_n \left(\frac{X_1 - x_2 \tan \alpha}{h_n} + x_2 \tan \alpha, x_2 \right) & \text{in } \Omega_n^r, \\ u_n \left(\frac{X_1 - x_2 \tan \beta}{h_n} + x_2 \tan \beta, x_2 \right) & \text{in } \Omega_n^l, \\ u_n \left(\frac{X_1}{h_n}, x_2 \right) & \text{in } \Omega_n^b, \end{cases} \\ p_n(x_1, x_2) = \begin{cases} p_n \left(\frac{X_1 - x_2 \tan \alpha}{h_n} + x_2 \tan \alpha, x_2 \right) & \text{in } \Omega_n^r, \\ p_n \left(\frac{X_1 - x_2 \tan \beta}{h_n} + x_2 \tan \beta, x_2 \right) & \text{in } \Omega_n^l, \\ p_n \left(\frac{X_1}{h_n}, x_2 \right) & \text{in } \Omega_n^b, \end{cases} \end{cases}$$

solves (2.7) (or equivalently (2.8)). Therefore, the goal of this paper becomes to study the asymptotic behavior, as n diverges, of problem (3.8). To this aim, we assume

$$\begin{cases} f_n^r \rightarrow f^r = (f_1^r, f_2^r) \text{ strongly in } (L^2(\Omega^r))^2, \\ f_n^l \rightarrow f^l = (f_1^l, f_2^l) \text{ strongly in } (L^2(\Omega^l))^2, \\ f_n^b \rightarrow f^b = (f_1^b, f_2^b) \text{ strongly in } (L^2(\Omega^b))^2. \end{cases} \quad (3.9)$$

4. The Main Results

In order to give the main result of our paper, according to [11] we introduce the applications

$$T^r : L^2(\Omega^r) \rightarrow L^2(]0, L^r[), \quad T^l : L^2(\Omega^l) \rightarrow L^2(]0, L^l[), \quad T^b : L^2(\Omega^b) \rightarrow L^2(]-L^b, 0[),$$

defined by

$$\begin{cases} T^r(w) : x_2 \in]0, L^r[\mapsto \int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} w(x_1, x_2) dx_1, & \forall w \in L^2(\Omega^r), \\ T^l(w) : x_2 \in]0, L^l[\mapsto \int_{x_2 \tan \beta - \frac{1}{2}}^{x_2 \tan \beta} w(x_1, x_2) dx_1, & \forall w \in L^2(\Omega^l), \\ T^b(w) : x_1 \in]-L^b, 0[\mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} w(x_1, x_2) dx_1, & \forall w \in L^2(\Omega^b), \end{cases} \quad (4.1)$$

respectively, and we set

$$\begin{aligned} W^r &= \left\{ w \in L^2(\Omega^r) : T^r(w) \in H_0^1(]0, L^r[), \quad \partial_{x_1} w \in L^2(\Omega^r), \right. \\ &\quad \left. w = 0 \text{ on } \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^r, \quad x_1 = x_2 \tan \alpha \vee x_1 = x_2 \tan \alpha + \tfrac{1}{2}\} \right\}, \\ W^l &= \left\{ w \in L^2(\Omega^l) : T^l(w) \in H_0^1(]0, L^l[), \quad \partial_{x_1} w \in L^2(\Omega^l), \right. \\ &\quad \left. w = 0 \text{ on } \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^l, \quad x_1 = x_2 \tan \beta \vee x_1 = x_2 \tan \beta - \tfrac{1}{2}\} \right\}, \\ W^b &= \{w \in L^2(\Omega^b) : T^b(w) \in H_0^1(]-L^b, 0[), \partial_{x_1} w \in L^2(\Omega^b), w = 0 \text{ on } \{-\tfrac{1}{2}, \tfrac{1}{2}\} \times]-L^b, 0[\}, \\ &\quad \begin{cases} W_0^r = \{w \in W^r(\Omega^r) : T^r(w) = 0 \text{ in }]0, L^r[\}, \\ W_0^l = \{w \in W^l(\Omega^l) : T^l(w) = 0 \text{ in }]0, L^l[\}, \\ W_0^b = \{w \in W^b(\Omega^b) : T^b(w) = 0 \text{ in }]-L^b, 0[\}. \end{cases} \end{aligned} \quad (4.2)$$

Theorem 4.1. *Assume that (2.9) and (3.9) hold true. For every $n \in \mathbb{N}$ let (u_n, p_n) , (u_n, q_n) , and (u_n, s_n) be three solutions to (3.8) such that*

$$\int_{\Omega^r} p_n dx_1 dx_2 = 0, \quad \int_{\Omega^l} q_n dx_1 dx_2 = 0, \quad \int_{\Omega^b} s_n dx_1 dx_2 = 0 \quad \forall n \in \mathbb{N},$$

and set

$$u_n^r = u_n|_{\Omega^r}, \quad u_n^l = u_n|_{\Omega^l}, \quad u_n^b = u_n|_{\Omega^b}.$$

Let $W_0^r(\Omega^r)$, $W_0^l(\Omega^l)$, and $W_0^b(\Omega^b)$ be defined in (4.2). Then, there exist u^r in $W_0^r(\Omega^r)$, u^l in $W_0^l(\Omega^l)$, and u^b in $W_0^b(\Omega^b)$ such that

$$\begin{cases} u_n^r \rightharpoonup (u^r \tan \alpha, u^r) \text{ weakly in } (L^2(\Omega^r))^2, \\ \partial_{x_1} u_n^r \rightharpoonup (\partial_{x_1} u^r \tan \alpha, \partial_{x_1} u^r) \text{ weakly in } (L^2(\Omega^r))^2, \\ h_n u_n^r \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^r))^2, \end{cases} \quad (4.3)$$

$$\begin{cases} u_n^l \rightharpoonup (u^l \tan \beta, u^l) \text{ weakly in } (L^2(\Omega^l))^2, \\ \partial_{x_1} u_n^l \rightharpoonup (\partial_{x_1} u^l \tan \beta, \partial_{x_1} u^l) \text{ weakly in } (L^2(\Omega^l))^2, \\ h_n u_n^l \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^l))^2, \end{cases} \quad (4.4)$$

$$\begin{cases} u_n^b \rightharpoonup (0, u^b) \text{ weakly in } (L^2(\Omega^b))^2, \\ \partial_{x_1} u_n^b \rightharpoonup (0, \partial_{x_1} u^b) \text{ weakly in } (L^2(\Omega^b))^2, \\ h_n u_n^b \rightharpoonup 0 \text{ weakly in } (H^1(\Omega^b))^2, \end{cases} \quad (4.5)$$

as n diverges. Moreover, there exist an increasing sequence of positive numbers, still denoted by $\{n\}$, and (in possible dependence on the subsequence) three functions independent of x_1 : p in $L^2(\Omega^r)$, q in $L^2(\Omega^l)$, and s in $L^2(\Omega^b)$ such that

$$\int_{\Omega^r} p dx_1 dx_2 = 0, \quad \int_{\Omega^l} q dx_1 dx_2 = 0, \quad \text{and} \quad \int_{\Omega^b} s dx_1 dx_2 = 0, \quad (4.6)$$

$$p_n|_{\Omega^r} \rightharpoonup p \text{ weakly in } L^2(\Omega^r), \quad q_n|_{\Omega^l} \rightharpoonup q \text{ weakly in } L^2(\Omega^l), \quad s_n|_{\Omega^b} \rightharpoonup s \text{ weakly in } L^2(\Omega^b), \quad (4.7)$$

as n diverges. Furthermore, (u^r, p) , (u^l, q) , and (u^b, s) solve

$$\begin{cases} u^r \in W_0^r(\Omega^r), \quad p \in L^2(\Omega^r), \\ \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 \partial_{x_1} u^r \partial_{x_1} (w - u^r) + g (1 + \tan^2 \alpha) (|\partial_{x_1} w| - |\partial_{x_1} u^r|) \right) dx_1 dx_2 \\ \geq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) (w - u^r) dx_1 dx_2 + \int_0^{L^r} p \frac{d(T^r(w - u^r))}{dx_2} dx_2 \quad \forall w \in W^r(\Omega^r), \end{cases} \quad (4.8)$$

$$\begin{cases} u^l \in W_0^l(\Omega^l), \quad q \in L^2(\Omega^l), \\ \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 \partial_{x_1} u^l \partial_{x_1} (w - u^l) + g (1 + \tan^2 \beta) (|\partial_{x_1} w| - |\partial_{x_1} u^l|) \right) dx_1 dx_2 \\ \geq \int_{\Omega^l} (f_1^l \tan \beta + f_2^l) (w - u^l) dx_1 dx_2 + \int_0^{L^l} q \frac{d(T^l(w - u^l))}{dx_2} dx_2 \quad \forall w \in W^l(\Omega^l), \end{cases} \quad (4.9)$$

$$\begin{cases} u^b \in W_0^b(\Omega^b), \quad s \in L^2(\Omega^b), \\ \int_{\Omega^b} \left(\mu \partial_{x_1} u^b \partial_{x_1} (w - u^b) + g (|\partial_{x_1} w| - |\partial_{x_1} u^b|) \right) dx_1 dx_2 \\ \geq \int_{\Omega^b} f_2^b (w - u^b) dx_1 dx_2 + \int_{-L^b}^0 s \frac{d(T^b(w - u^b))}{dx_2} dx_2 \quad \forall w \in W^b(\Omega^b), \end{cases} \quad (4.10)$$

respectively, where (f_1^r, f_2^r) , (f_1^l, f_2^l) , and f_2^b are given by (3.9). Furthermore, u^r , u^l , and u^b are unique.

Remark 4.2. The limit velocity in the right branch is $u^r(\tan \alpha, 1)$ (see (4.3)). We note that α describes the deviation of the right branch from the x_2 -axis, then the direction $(\tan \alpha, 1)$ of the limit velocity is exactly along the axis of the right branch. Moreover, as far as the limit force (f_1^r, f_2^r) is concerned, only its projection on the axis of the right branch is involved in problem (4.8). Indeed,

$$f_1^r \tan \alpha + f_2^r = \pm \left\| P_{|(\tan \alpha, 1)}(f_1^r, f_2^r) \right\| \sqrt{1 + \tan^2 \alpha},$$

where $P_{|(\tan \alpha, 1)}$ denotes the projection operator on the axis $x = y \tan \alpha$.

If $\alpha = 0$, then the domain Ω^r is a vertical rectangle, $\tan \alpha = 0$, and the first limit in (4.3) is $(0, u^r)$. This is in accordance with the results obtained in [10–12].

A similar remark holds true for problem (4.9) stated in the left branch, and for problem (4.10) stated in the vertical branch (which corresponds to the previous cases with $\alpha = 0$).

5. A *Priori* Estimates and Convergence Result for the Velocity

In order to use compactness results for passing to the limit in the terms involving the velocity, we first derive *a priori* estimates for this function.

Proposition 5.1. *Assume (2.9) and (3.9) hold true. For every $n \in \mathbb{N}$ let u_n be the solution to (3.7). Then there exists a positive constant c independent on n such that*

$$h_n \|D_n^r u_n\|_{(L^2(\Omega^r))^4} \leq c, \quad h_n \|D_n^l u_n\|_{(L^2(\Omega^l))^4} \leq c, \quad h_n \|D_n^b u_n\|_{(L^2(\Omega^b))^4} \leq c \quad \forall n \in \mathbb{N}, \quad (5.1)$$

$$\|u_n\|_{(L^2(\Omega))^2} \leq c \quad \forall n \in \mathbb{N}. \quad (5.2)$$

Proof. Choosing $v = 0$ and $v = 2u_n$ as test functions in (3.7), and comparing the variational inequalities which are obtained give that

$$\begin{aligned} & \int_{\Omega^r} \left(h_n^2 \mu |D_n^r u_n|^2 + h_n g |D_n^r u_n| \right) dx_1 dx_2 + \int_{\Omega^l} \left(h_n^2 \mu |D_n^l u_n|^2 + h_n g |D_n^l u_n| \right) dx_1 dx_2 \\ & + \int_{\Omega^b} \left(h_n^2 \mu |D_n^b u_n|^2 + h_n g |D_n^b u_n| \right) dx_1 dx_2 \\ & = \int_{\Omega^r} f_n^r u_n dx_1 dx_2 + \int_{\Omega^l} f_n^l u_n dx_1 dx_2 + \int_{\Omega^b} f_n^b u_n dx_1 dx_2 \quad \forall n \in \mathbb{N}. \end{aligned} \quad (5.3)$$

On the other hand, the homogeneous boundary conditions of u_n provide

$$\begin{cases} \|u_n\|_{(L^2(\Omega^r))^2} \leq \|\partial_{x_1} u_n\|_{(L^2(\Omega^r))^2} \leq \|h_n D_n^r u_n\|_{(L^2(\Omega^r))^4}, \\ \|u_n\|_{(L^2(\Omega^l))^2} \leq \|\partial_{x_1} u_n\|_{(L^2(\Omega^l))^2} \leq \|h_n D_n^l u_n\|_{(L^2(\Omega^l))^4}, \\ \|u_n\|_{(L^2(\Omega^b))^2} \leq \|\partial_{x_1} u_n\|_{(L^2(\Omega^b))^2} \leq \|h_n D_n^b u_n\|_{(L^2(\Omega^b))^4}, \end{cases} \quad \forall n \in \mathbb{N}. \quad (5.4)$$

Applying the Hölder inequality to the right-hand side of (5.3) and using (5.4) imply

$$\begin{aligned} & \mu \int_{\Omega^r} |h_n D_n^r u_n|^2 dx_1 dx_2 + \mu \int_{\Omega^l} |h_n D_n^l u_n|^2 dx_1 dx_2 + \mu \int_{\Omega^b} |h_n D_n^b u_n|^2 dx_1 dx_2 \\ & \leq \|f_n^r\|_{(L^2(\Omega^r))^2} \|h_n D_n^r u_n\|_{(L^2(\Omega^r))^4} + \|f_n^l\|_{(L^2(\Omega^l))^2} \|h_n D_n^l u_n\|_{(L^2(\Omega^l))^4} \\ & + \|f_n^b\|_{(L^2(\Omega^b))^2} \|h_n D_n^b u_n\|_{(L^2(\Omega^b))^4} \quad \forall n \in \mathbb{N}. \end{aligned} \quad (5.5)$$

Eventually, (5.1) follows from (5.5), (3.9), and the following inequality

$$(\gamma + \delta + \eta)^2 \leq 3(\gamma^2 + \delta^2 + \eta^2) \quad \forall \gamma, \delta, \eta \in [0, +\infty[;$$

while (5.2) follows from (5.4) and (5.1). $\square$

Proposition 5.1 implies the following result.

Proposition 5.2. *Let T^r , T^l and T^b be defined in (4.1). Assume (2.9) and (3.9) hold true. For every $n \in \mathbb{N}$ let $u_n = (u_{n1}, u_{n2})$ be the solution to (3.7). Then there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and $u = (u_1, u_2)$ in $(L^2(\Omega))^2$ with $\partial_{x_1} u$ in $(L^2(\Omega))^2$ such that*

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } (L^2(\Omega))^2, & \partial_{x_1} u_n \rightharpoonup \partial_{x_1} u \text{ weakly in } (L^2(\Omega))^2, \\ h_n u_n \rightharpoonup 0 \text{ weakly in } (H^1(\Omega))^2, \end{cases} \quad (5.6)$$

$$\begin{aligned} & u = 0 \text{ on } \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^r, \quad x_1 = x_2 \tan \alpha \vee x_1 = x_2 \tan \alpha + \frac{1}{2}\} \\ & \cup \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < L^l, \quad x_1 = x_2 \tan \beta \vee x_1 = x_2 \tan \beta - \frac{1}{2}\} \\ & \cup \{\pm \frac{1}{2}\} \times]-L^b, 0[), \end{aligned} \quad (5.7)$$

$$u_1 = \tan \alpha u_2 \text{ a.e. in } \Omega^r, \quad u_1 = \tan \beta u_2 \text{ a.e. in } \Omega^l, \quad u_1 = 0 \text{ a.e. in } \Omega^b, \quad (5.8)$$

$$T^r(u_2) = 0 \text{ a.e. in }]0, L^r[, \quad T^l(u_2) = 0 \text{ a.e. in }]0, L^l[, \quad T^b(u_2) = 0 \text{ a.e. in }]-L^b, 0[. \quad (5.9)$$

Proof. Statements in (5.6) and in (5.7) follow from Proposition 5.1.

The first equality in (5.8) is obtained by passing to the limit in

$$\partial_{x_1} u_{n1} + h_n \partial_{x_2} u_{n2} - (1 - h_n) \tan \alpha \partial_{x_1} u_{n2} = h_n \operatorname{div}_n^r(u_n) = 0 \text{ in } \Omega^r, \quad \forall n \in \mathbb{N},$$

and taking into account the last two convergences in (5.6), and (5.7). Similarly, one can prove the last two equalities in (5.8).

As far as the proof of (5.9) is concerned, equality $\operatorname{div}_n^r(u_n) = 0$ in Ω^r provides

$$\int_{\Omega^r} \left(\frac{1}{h_n} \partial_{x_1} u_{n1} + \partial_{x_2} u_{n2} - \frac{1 - h_n}{h_n} \tan \alpha \partial_{x_1} u_{n2} \right) \varphi(x_2) dx_1 dx_2 = 0 \quad (5.10)$$

$$\forall \varphi \in C_0^\infty([0, 1]), \quad \forall n \in \mathbb{N}.$$

On the other side, the boundary conditions of u_n give

$$\begin{cases} \int_{\Omega^r} \partial_{x_1} u_{n1} \varphi(x_2) dx_1 dx_2 = \int_{\partial\Omega^r} u_{n1} \varphi(x_2) \nu_1 dx_1 dx_2 = 0, \\ \int_{\Omega^r} \partial_{x_1} u_{n2} \varphi(x_2) dx_1 dx_2 = \int_{\partial\Omega^r} u_{n2} \varphi(x_2) \nu_1 dx_1 dx_2 = 0, \end{cases} \quad \forall \varphi \in C_0^\infty([0, 1]), \quad \forall n \in \mathbb{N}, \quad (5.11)$$

where ν_1 is the first component of the exterior unit normal on $\partial\Omega^r$. Taking into account again the boundary conditions of u_n and combining (5.10) with (5.11) provide

$$\begin{aligned} \int_0^{L^r} \partial_{x_2} (T^r(u_{n2})) \varphi(x_2) dx_2 &= \int_0^{L^r} \partial_{x_2} \left(\int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} u_{n2}(x_1, x_2) dx_1 \right) \varphi(x_2) dx_2 \\ &= \int_0^{L^r} \left[\tan \alpha u_{n2} \left(x_2 \tan \alpha + \frac{1}{2}, x_2 \right) - \tan \alpha u_{n2} (x_2 \tan \alpha, x_2) \right] \varphi(x_2) dx_2 \\ &+ \int_0^{L^r} \left[\int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} \partial_{x_2} u_{n2}(x_1, x_2) dx_1 \right] \varphi(x_2) dx_2 \\ &= \int_{\Omega^r} \partial_{x_2} u_{n2} \varphi(x_2) dx_1 dx_2 = 0 \quad \forall \varphi \in C_0^\infty([0, L^r]), \quad \forall n \in \mathbb{N}, \end{aligned}$$

which implies

$$T^r(u_{n2}) = 0 \text{ a.e. in }]0, L^r[, \quad \forall n \in \mathbb{N}, \quad (5.12)$$

since $T^r(u_{n2})(1) = 0$. Eventually, combining the first convergence in (5.6) with (5.12) provides the first equality in (5.9), since T^r is weakly continuous, being strongly continuous. Similarly, one can prove the last two equalities in (5.9). $\square$

6. A *Priori* Estimates and Convergence Result for the Pressure

In what follows, we set

$$\begin{aligned} W_{reg} = \Big\{ w = (w_1, w_2) \in (L^2(\Omega))^2 : w_2|_{\Omega^r} \in H_0^1(\Omega^r), \quad w_2|_{\Omega^l} \in H_0^1(\Omega^l), \\ w_2|_{\Omega^b} \in H_0^1(\Omega^b), \quad w_1 = \tan \alpha w_2 \text{ in } \Omega^r, \quad w_1 = \tan \beta w_2 \text{ in } \Omega^l, \quad w_1 = 0 \text{ in } \Omega^b \Big\}. \end{aligned} \quad (6.1)$$

Proposition 6.1. *Assume that (2.9) and (3.9) hold true. For every $n \in \mathbb{N}$ let (u_n, p_n) be a solution to (3.8). Then there exists a positive constant c independent on n such that*

$$\|\partial_{x_1} p_n\|_{H^{-1}(\Omega)} \leq h_n c, \quad (6.2)$$

$$\|\partial_{\nu_\alpha} p_n\|_{H^{-1}(\Omega^r)} \leq c, \quad \|\partial_{\nu_\beta} p_n\|_{H^{-1}(\Omega^l)} \leq c, \quad \|\partial_{x_2} p_n\|_{H^{-1}(\Omega^b)} \leq c, \quad (6.3)$$

for every $n \in \mathbb{N}$, where

$$\nu_\alpha = \left(\frac{\tan \alpha}{\sqrt{\tan^2 \alpha + 1}}, \frac{1}{\sqrt{\tan^2 \alpha + 1}} \right), \quad \nu_\beta = \left(\frac{\tan \beta}{\sqrt{\tan^2 \beta + 1}}, \frac{1}{\sqrt{\tan^2 \beta + 1}} \right). \quad (6.4)$$

Moreover,

$$\begin{cases} \|p_n\|_{L^2(\Omega_r)} \leq c \left(1 + \left| \int_{\Omega_r} p_n dx_1 dx_2 \right| \right), & \|p_n\|_{L^2(\Omega_l)} \leq c \left(1 + \left| \int_{\Omega_l} p_n dx_1 dx_2 \right| \right), \\ \|p_n\|_{L^2(\Omega_b)} \leq c \left(1 + \left| \int_{\Omega_b} p_n dx_1 dx_2 \right| \right), & \forall n \in \mathbb{N}. \end{cases} \quad (6.5)$$

Proof. Choosing $v = w + u_n$ as test functions in (3.8), with w in $(H_0^1(\Omega))^2$, gives

$$\begin{aligned} & \int_{\Omega^r} (h_n^2 \mu D_n^r u_n D_n^r w + h_n g |D_n^r(w + u_n)| - h_n g |D_n^r u_n|) dx_1 dx_2 \\ & + \int_{\Omega^l} (h_n^2 \mu D_n^l u_n D_n^l w + h_n g |D_n^l(w + u_n)| - h_n g |D_n^l u_n|) dx_1 dx_2 \\ & + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n D_n^b w + h_n g |D_n^b(w + u_n)| - h_n g |D_n^b u_n|) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_n^r w + p_n \operatorname{div}_n^r w) dx_1 dx_2 + \int_{\Omega^l} (f_n^l w + p_n \operatorname{div}_n^l w) dx_1 dx_2 \\ & + \int_{\Omega^b} (f_n^b w + p_n \operatorname{div}_n^b w) dx_1 dx_2 \quad \forall w \in (H_0^1(\Omega))^2, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (6.6)$$

Applying the Hölder inequality in (6.6) provides

$$\begin{aligned} & \int_{\Omega^r} p_n \operatorname{div}_n^r w dx_1 dx_2 + \int_{\Omega^l} p_n \operatorname{div}_n^l w dx_1 dx_2 + \int_{\Omega^b} p_n \operatorname{div}_n^b w dx_1 dx_2 \\ & \leq h_n^2 \mu \|D_n^r u_n\|_{(L^2(\Omega^r))^4} \|D_n^r w\|_{(L^2(\Omega^r))^4} + h_n g \|D_n^r w\|_{(L^1(\Omega^r))^4} + \|f_n^r\|_{(L^2(\Omega^r))^2} \|w\|_{(L^2(\Omega^r))^2} \\ & + h_n^2 \mu \|D_n^l u_n\|_{(L^2(\Omega^l))^4} \|D_n^l w\|_{(L^2(\Omega^l))^4} + h_n g \|D_n^l w\|_{(L^1(\Omega^l))^4} + \|f_n^l\|_{(L^2(\Omega^l))^2} \|w\|_{(L^2(\Omega^l))^2} \\ & + h_n^2 \mu \|D_n^b u_n\|_{(L^2(\Omega^b))^4} \|D_n^b w\|_{(L^2(\Omega^b))^4} + h_n g \|D_n^b w\|_{(L^1(\Omega^b))^4} + \|f_n^b\|_{(L^2(\Omega^b))^2} \|w\|_{(L^2(\Omega^b))^2} \\ & \forall w \in (H_0^1(\Omega))^2, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Consequently, using (3.9), (5.1), the Hölder inequality, and the Poincaré inequality ensure the existence of a positive constant c such that

$$\begin{aligned} & \left| \int_{\Omega^r} p_n \operatorname{div}_n^r w dx_1 dx_2 + \int_{\Omega^l} p_n \operatorname{div}_n^l w dx_1 dx_2 + \int_{\Omega^b} p_n \operatorname{div}_n^b w dx_1 dx_2 \right| \leq ch_n \\ & \left(\|D_n^r w\|_{(L^2(\Omega^r))^4} + \|D_n^l w\|_{(L^2(\Omega^l))^4} + \|D_n^b w\|_{(L^2(\Omega^b))^4} \right) \forall w \in (H_0^1(\Omega))^2, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (6.7)$$

Now, in (6.7) choosing $w = (w_1, w_2) \in (H_0^1(\Omega))^2$ with $w_2 = 0$ in Ω gets

$$\left| \int_{\Omega} p_n \partial_{x_1} w_1 dx_1 dx_2 \right| \leq ch_n \|Dw_1\|_{(L^2(\Omega))^2} \quad \forall w_1 \in H_0^1(\Omega), \quad \forall n \in \mathbb{N},$$

with a positive constant c independent of $w_1 \in H_0^1(\Omega)$ and of $n \in \mathbb{N}$, which implies (6.2).

Now, in (6.7) choosing $w = (w_1, w_2)$ in W_{reg} defined in (6.1) gets

$$\begin{aligned} & \left| \int_{\Omega^r} p_n (\partial_{x_2} w_2 + \tan \alpha \partial_{x_1} w_2) dx_1 dx_2 + \int_{\Omega^l} p_n (\partial_{x_2} w_2 + \tan \beta \partial_{x_1} w_2) dx_1 dx_2 + \int_{\Omega^b} p_n \partial_{x_2} w_2 dx_1 dx_2 \right| \\ & \leq c \|Dw_2\|_{(L^2(\Omega))^2} \quad \forall w_2 \in L^2(\Omega) : w_2|_{\Omega^r} \in H_0^1(\Omega^r), w_2|_{\Omega^l} \in H_0^1(\Omega^l), w_2|_{\Omega^b} \in H_0^1(\Omega^b), \quad \forall n \in \mathbb{N}, \end{aligned}$$

with a positive constant c independent of w_2 and of n . Consequently, there exists a positive constant c such that

$$\left\{ \begin{array}{l} \left| \int_{\Omega^r} p_n \partial_{\nu_\alpha} w_2 dx_1 dx_2 \right| \leq c \|Dw_2\|_{(L^2(\Omega^r))^2} \quad \forall w_2 \in H_0^1(\Omega^r), \\ \left| \int_{\Omega^l} p_n \partial_{\nu_\beta} w_2 dx_1 dx_2 \right| \leq c \|Dw_2\|_{(L^2(\Omega^l))^2} \quad \forall w_2 \in H_0^1(\Omega^l), \quad \forall n \in \mathbb{N}, \\ \left| \int_{\Omega^b} p_n \partial_{x_2} w_2 dx_1 dx_2 \right| \leq c \|Dw_2\|_{(L^2(\Omega^b))^2} \quad \forall w_2 \in H_0^1(\Omega^b), \end{array} \right.$$

where ν_α and ν_β are defined in (6.4), which implies the estimates in (6.3).

Let us just prove the first estimate in (6.5). The proof of the other ones is similar.

Thanks to (6.2), one has

$$\|\partial_{x_1} p_n\|_{H^{-1}(\Omega_r)} \leq \|\partial_{x_1} p_n\|_{H^{-1}(\Omega)} \leq h_n c \quad \forall n \in \mathbb{N}, \quad (6.8)$$

with a constant c independent of $n \in \mathbb{N}$. Then, from (6.3), (6.8), and the relation

$$\partial_{x_2} w = \sqrt{\tan^2 \alpha + 1} \partial_{\nu_\alpha} w - \tan \alpha \partial_{x_1} w \quad \forall w \in H_0^1(\Omega_r),$$

it follows that

$$\begin{aligned} \left| \int_{\Omega^r} p_n \partial_{x_2} w dx_1 dx_2 \right| &\leq \sqrt{\tan^2 \alpha + 1} \left| \int_{\Omega^r} p_n \partial_{\nu_\alpha} w dx_1 dx_2 \right| + \tan \alpha \left| \int_{\Omega^r} p_n \partial_{x_1} w dx_1 dx_2 \right| \\ &\leq c \left(\sqrt{\tan^2 \alpha + 1} + h_n \tan \alpha \right) \|Dw\|_{H_0^1(\Omega_r)} \quad \forall w \in H_0^1(\Omega_r), \quad \forall n \in \mathbb{N}, \end{aligned}$$

which implies that

$$\|\partial_{x_2} p_n\|_{H^{-1}(\Omega_r)} \leq c \quad \forall n \in \mathbb{N}, \quad (6.9)$$

with a positive constant c independent of $n \in \mathbb{N}$.

Eventually, the first estimate in (6.5) follows from Lemma 6.1 in [37], (6.8) and (6.9). $\square$

Proposition 5.2 and Proposition 6.1 imply the following result.

Corollary 6.2. *Assume that (2.9) and (3.9) hold true. For every $n \in \mathbb{N}$ let (u_n, p_n) , (u_n, q_n) , and (u_n, s_n) be three solutions to (3.8) such that*

$$\int_{\Omega^r} p_n dx_1 dx_2 = 0, \quad \int_{\Omega^l} q_n dx_1 dx_2 = 0, \quad \int_{\Omega^b} s_n dx_1 dx_2 = 0 \quad \forall n \in \mathbb{N}.$$

Then there exist an increasing sequence of positive numbers, still denoted by $\{n\}$, and (in possible dependence on the subsequence) three functions independent of x_1 : p in $L^2(\Omega^r)$, q in $L^2(\Omega^l)$, and s in $L^2(\Omega^b)$ such that

$$\begin{aligned} \int_{\Omega^r} p dx_1 dx_2 &= 0, \quad \int_{\Omega^l} q dx_1 dx_2 = 0, \quad \int_{\Omega^b} s dx_1 dx_2 = 0, \\ p_n|_{\Omega^r} &\rightharpoonup p \text{ weakly in } L^2(\Omega^r), \quad q_n|_{\Omega^l} \rightharpoonup q \text{ weakly in } L^2(\Omega^l), \quad s_n|_{\Omega^b} \rightharpoonup s \text{ weakly in } L^2(\Omega^b). \end{aligned}$$

7. Proof of Theorem 4.1

Proof. By virtue of Proposition 5.2 and Corollary 6.2, there exists an increasing sequence of positive numbers, still denoted by $\{n\}$, and (in possible dependence on the subsequence) there exist u^r in $W_0^r(\Omega^r)$, u^l in $W_0^l(\Omega^l)$, u^b in $W_0^b(\Omega^b)$, p in $L^2(\Omega^r)$, q in $L^2(\Omega^l)$, and s in $L^2(\Omega^b)$, with p , q , and s independent of x_1 , such that (4.3)-(4.7) hold true. In what follows, n take values in this increasing sequence of positive numbers. It remains to prove that (u^r, p) , (u^l, q) , and (u^b, s) solve (4.8), (4.9), and (4.10), respectively. The uniqueness of u^r , u^l , and u^b can be proved as in [12]. It implies that limits in (4.3)-(4.5) hold true for the whole sequence.

In (3.7) choosing $v = 0$ first, then $v = 2u_n$, and comparing provide

$$\begin{aligned} & \int_{\Omega^r} (h_n^2 \mu |D_n^r u_n|^2 + h_n g |D_n^r u_n|) dx_1 dx_2 + \int_{\Omega^l} (h_n^2 \mu |D_n^l u_n|^2 + h_n g |D_n^l u_n|) dx_1 dx_2 \\ & + \int_{\Omega^b} (h_n^2 \mu |D_n^b u_n|^2 + h_n g |D_n^b u_n|) dx_1 dx_2 \\ & = \int_{\Omega^r} f_n^r u_n dx_1 dx_2 + \int_{\Omega^l} f_n^l u_n dx_1 dx_2 + \int_{\Omega^b} f_n^b u_n dx_1 dx_2 \quad \forall n \in \mathbb{N}. \end{aligned} \quad (7.1)$$

Combining (3.8) and (7.1), and recalling that

$$\operatorname{div}_n^r(u_n) = 0 \text{ in } \Omega^r, \quad \operatorname{div}_n^l(u_n) = 0 \text{ in } \Omega^l, \quad \operatorname{div}_n^b(u_n) = 0 \text{ in } \Omega^b,$$

give

$$\begin{aligned} & \int_{\Omega^r} (h_n^2 \mu D_n^r u_n D_n^r v + h_n g |D_n^r v|) dx_1 dx_2 + \int_{\Omega^l} (h_n^2 \mu D_n^l u_n D_n^l v + h_n g |D_n^l v|) dx_1 dx_2 \\ & + \int_{\Omega^b} (h_n^2 \mu D_n^b u_n D_n^b v + h_n g |D_n^b v|) dx_1 dx_2 \geq \int_{\Omega^r} (f_n^r v + p_n \operatorname{div}_n^r v) dx_1 dx_2 \\ & + \int_{\Omega^l} (f_n^l v + p_n \operatorname{div}_n^l v) dx_1 dx_2 + \int_{\Omega^b} (f_n^b v + p_n \operatorname{div}_n^b v) dx_1 dx_2 \quad \forall v \in (H_0^1(\Omega))^2. \end{aligned} \quad (7.2)$$

Choosing

$$v = (\tan \alpha w, w) \text{ in } \Omega^r, \quad v = 0 \text{ in } \Omega^l, \quad v = 0 \text{ in } \Omega^b,$$

with w in $H_0^1(\Omega^r)$, as test function in (7.2) gets

$$\begin{aligned} & \int_{\Omega^r} (h_n^2 \mu D_n^r u_n D_n^r (\tan \alpha w, w) + h_n g |D_n^r (\tan \alpha w, w)|) dx_1 dx_2 \\ & \geq \int_{\Omega^r} \left((f_{n1}^r \tan \alpha + f_{n2}^r) w + \sqrt{\tan^2 \alpha + 1} p_n \partial_{\nu_\alpha} w \right) dx_1 dx_2 \quad \forall w \in H_0^1(\Omega^r). \end{aligned} \quad (7.3)$$

Then passing to the limit, as n diverges, in (7.3) and using (3.9), (4.3)-(4.7) lead to

$$\begin{aligned} & \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 \partial_{x_1} u^r \partial_{x_1} w + g (1 + \tan^2 \alpha) |\partial_{x_1} w| \right) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) w dx_1 dx_2 + \sqrt{\tan^2 \alpha + 1} \int_{\Omega^r} p \partial_{\nu_\alpha} w dx_1 dx_2 \quad \forall w \in H_0^1(\Omega^r). \end{aligned} \quad (7.4)$$

About the last integral in (7.4), since p is independent of x_1 and $T^r(u^r) = 0$, one has

$$\begin{aligned} & \sqrt{\tan^2 \alpha + 1} \int_{\Omega^r} p \partial_{\nu_\alpha} w dx_1 dx_2 \\ & = \int_0^{L^r} p \left(\tan \alpha \int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} \partial_{x_1} w dx_1 + \int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} \partial_{x_2} w dx_1 \right) dx_2 = \\ & \int_0^{L^r} p \left(\int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} \partial_{x_2} w dx_1 \right) dx_2 = \int_0^{L^r} p \partial_{x_2} \left(\int_{x_2 \tan \alpha}^{x_2 \tan \alpha + \frac{1}{2}} w dx_1 \right) dx_2 \\ & - \int_0^{L^r} p \left[\tan \alpha w \left(x_2 \tan \alpha + \frac{1}{2}, x_2 \right) - \tan \alpha w (x_2 \tan \alpha, x_2) \right] dx_2 \\ & = \int_0^{L^r} p \frac{d}{dx_2} (T^r(w)) dx_2 = \int_0^{L^r} p \frac{d}{dx_2} (T^r(w - u^r)) dx_2 \quad \forall w \in H_0^1(\Omega^r). \end{aligned} \quad (7.5)$$

Combining (7.4) and (7.5) gives

$$\begin{aligned} & \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 \partial_{x_1} u^r \partial_{x_1} w + g (1 + \tan^2 \alpha) |\partial_{x_1} w| \right) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) w dx_1 dx_2 + \int_0^{L^r} p \frac{d}{dx_2} (T^r(w - u^r)) dx_2 \quad \forall w \in H_0^1(\Omega^r). \end{aligned} \quad (7.6)$$

A density argument (see [12]) ensures that (7.6) holds true with w in W^r , i.e.

$$\begin{aligned} & \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 \partial_{x_1} u^r \partial_{x_1} w + g (1 + \tan^2 \alpha) |\partial_{x_1} w| \right) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) w dx_1 dx_2 + \int_0^{L^r} p \frac{d}{dx_2} (T^r(w - u^r)) dx_2 \quad \forall w \in W^r. \end{aligned} \quad (7.7)$$

In particular, choosing $w = u^r \in W_0^r \subset W^r$ in (7.7) gives

$$\begin{aligned} & \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 |\partial_{x_1} u^r|^2 + g (1 + \tan^2 \alpha) |\partial_{x_1} u^r| \right) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) u^r dx_1 dx_2. \end{aligned} \quad (7.8)$$

Arguing as above, but replacing p_n with q_n in (3.8), gives variational inequality (7.2) with p_n replaced by q_n . Then, choosing $v = 0$ in Ω^r , $v = (\tan \beta w, w)$ in Ω^l , $v = 0$ in Ω^b , with w in $H_0^1(\Omega^l)$, as test function in this new variational inequality, and then passing to the limit in it, as $n \rightarrow +\infty$, the same arguments used above lead to

$$\begin{aligned} & \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 \partial_{x_1} u^l \partial_{x_1} w + g (1 + \tan^2 \beta) |\partial_{x_1} w| \right) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_1^l \tan \beta + f_2^l) w dx_1 dx_2 + \int_0^{L^l} q \frac{d}{dx_2} (T^l(w - u^l)) dx_2 \quad \forall w \in W^l. \end{aligned} \quad (7.9)$$

In particular, choosing $w = u^l \in W_0^l \subset W^l$ in (7.9) gives

$$\begin{aligned} & \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 |\partial_{x_1} u^l|^2 + g (1 + \tan^2 \beta) |\partial_{x_1} u^l| \right) dx_1 dx_2 \\ & \geq \int_{\Omega^l} (f_1^l \tan \beta + f_2^l) u^l dx_1 dx_2. \end{aligned} \quad (7.10)$$

Similarly, one can obtain (7.2) with p_n replaced by s_n . Then, choosing $v = 0$ in Ω^r , $v = 0$ in Ω^l , $v = (0, w)$ in Ω^b , with w in $H_0^1(\Omega^b)$, as test function in this new variational inequality, and then passing to the limit in it, as $n \rightarrow +\infty$, the same arguments used above lead to

$$\begin{aligned} & \int_{\Omega^b} (\mu \partial_{x_1} u^b \partial_{x_1} w + g |\partial_{x_1} w|) dx_1 dx_2 \\ & \geq \int_{\Omega^b} f_2^b w dx_1 dx_2 + \int_{-L^b}^0 s \frac{d}{dx_2} (T^b(w - u^b)) dx_2 \quad \forall w \in W^b. \end{aligned} \quad (7.11)$$

In particular, choosing $w = u^b \in W_0^b \subset W^b$ in (7.11) gives

$$\int_{\Omega^b} (\mu |\partial_{x_1} u^b|^2 + g |\partial_{x_1} u^b|) dx_1 dx_2 \geq \int_{\Omega^b} f_2^b u^b dx_1 dx_2. \quad (7.12)$$

Now, adding (7.8), (7.10), and (7.12) provides

$$\begin{aligned} & \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 |\partial_{x_1} u^r|^2 + g (1 + \tan^2 \alpha) |\partial_{x_1} u^r| \right) dx_1 dx_2 \\ & + \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 |\partial_{x_1} u^l|^2 + g (1 + \tan^2 \beta) |\partial_{x_1} u^l| \right) dx_1 dx_2 \\ & + \int_{\Omega^b} (\mu |\partial_{x_1} u^b|^2 + g |\partial_{x_1} u^b|) dx_1 dx_2 \\ & \geq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) u^r dx_1 dx_2 + \int_{\Omega^l} (f_1^l \tan \beta + f_2^l) u^l dx_1 dx_2 + \int_{\Omega^b} f_2^b u^b dx_1 dx_2. \end{aligned} \quad (7.13)$$

On the other hand, passing to the limit in (7.1) and using (3.9), (4.3), (4.4), (4.5), and a l.s.c. argument provide

$$\begin{aligned}
& \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 |\partial_{x_1} u^r|^2 + g (1 + \tan^2 \alpha) |\partial_{x_1} u^r| \right) dx_1 dx_2 \\
& + \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 |\partial_{x_1} u^l|^2 + g (1 + \tan^2 \beta) |\partial_{x_1} u^l| \right) dx_1 dx_2 \\
& + \int_{\Omega^b} (\mu |\partial_{x_1} u^b|^2 + g |\partial_{x_1} u^b|) dx_1 dx_2 \\
& \leq \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) u^r dx_1 dx_2 + \int_{\Omega^l} (f_1^l \tan \beta + f_2^l) u^l dx_1 dx_2 + \int_{\Omega^b} f_2^b u^b dx_1 dx_2.
\end{aligned} \tag{7.14}$$

Combining (7.13) and (7.14) implies

$$\begin{aligned}
& \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 |\partial_{x_1} u^r|^2 + g (1 + \tan^2 \alpha) |\partial_{x_1} u^r| \right) dx_1 dx_2 \\
& + \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 |\partial_{x_1} u^l|^2 + g (1 + \tan^2 \beta) |\partial_{x_1} u^l| \right) dx_1 dx_2 \\
& + \int_{\Omega^b} (\mu |\partial_{x_1} u^b|^2 + g |\partial_{x_1} u^b|) dx_1 dx_2 \\
& = \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) u^r dx_1 dx_2 + \int_{\Omega^l} (f_1^l \tan \beta + f_2^l) u^l dx_1 dx_2 + \int_{\Omega^b} f_2^b u^b dx_1 dx_2.
\end{aligned} \tag{7.15}$$

Now comparing (7.8), (7.10), (7.12), and (7.15) leads to

$$\left\{ \begin{aligned} & \int_{\Omega^r} \left(\mu (1 + \tan^2 \alpha)^2 |\partial_{x_1} u^r|^2 + g (1 + \tan^2 \alpha) |\partial_{x_1} u^r| \right) dx_1 dx_2 \\ & = \int_{\Omega^r} (f_1^r \tan \alpha + f_2^r) u^r dx_1 dx_2, \\ & \int_{\Omega^l} \left(\mu (1 + \tan^2 \beta)^2 |\partial_{x_1} u^l|^2 + g (1 + \tan^2 \beta) |\partial_{x_1} u^l| \right) dx_1 dx_2 \\ & = \int_{\Omega^l} (f_1^l \tan \beta + f_2^l) u^l dx_1 dx_2, \\ & \int_{\Omega^b} (\mu |\partial_{x_1} u^b|^2 + g |\partial_{x_1} u^b|) dx_1 dx_2 = \int_{\Omega^b} f_2^b u^b dx_1 dx_2. \end{aligned} \right. \tag{7.16}$$

Eventually, (7.7), (7.9), (7.11), and (7.16) ensure that (u^r, p) , (u^l, q) , and (u^b, s) solve (4.8), (4.9), and (4.10), respectively. $\square$

8. A Lower-Dimensional “Bingham-like” Law

We analyze limit problem (4.8), stated in the right branch of the domain. By taking as test function $\frac{w}{1 + \tan^2 \alpha}$ and denoting $\tilde{u}^r = (1 + \tan^2 \alpha) u^r$, $\tilde{p} = \frac{p}{1 + \tan^2 \alpha}$, problem (4.8) becomes

$$\left\{ \begin{aligned} & \tilde{u}^r \in W_0^r(\Omega^r), \quad \tilde{p} \in L^2(\Omega^r), \\ & \int_{\Omega^r} \mu \partial_{x_1} \tilde{u}^r \partial_{x_1} (w - \tilde{u}^r) + g (|\partial_{x_1} w| - |\partial_{x_1} \tilde{u}^r|) dx_1 dx_2 \\ & \geq \int_{\Omega^r} \frac{(f_1^r \tan \alpha + f_2^r)}{1 + \tan^2 \alpha} (w - \tilde{u}^r) dx_1 dx_2 + \int_0^L \tilde{p} \frac{d(T^r(w - \tilde{u}^r))}{dx_2} dx_2 \quad \forall w \in W^r(\Omega^r). \end{aligned} \right. \tag{8.1}$$

According to Sect. 5 in [12], the following differential equation in Ω^r is derived from problem (8.1)

$$-\partial_{x_1} [\mu \partial_{x_1} \tilde{u}^r(x_1, x_2) + g \operatorname{sgn}(\partial_{x_1} \tilde{u}^r(x_1, x_2))] = \frac{f_1^r(x_1, x_2) \tan \alpha + f_2^r(x_1, x_2)}{1 + \tan^2 \alpha} - \tilde{p}'(x_2),$$

which is valid if $\partial_{x_1} \tilde{u}^r \neq 0$ and where sgn states for the signum function.

As in [11, 12], we define, up to an additive function of x_2 , the limiting stress tensor

$$\sigma_{21} = -\tilde{p}'(x_2) + \tilde{\sigma}_{21}, \text{ with } \tilde{\sigma}_{21} = \mu \partial_{x_1} \tilde{u}^r + g \operatorname{sgn}(\partial_{x_1} \tilde{u}^r). \quad (8.2)$$

This is exactly the following lower-dimensional “Bingham-like” law describing the limit strain tensor, corresponding to the model used in the engineering literature for the Bingham flow in thin domains (for instance, see [29])

$$|\tilde{\sigma}_{21}| \leq g \Leftrightarrow \mu \partial_{x_1} \tilde{u}^r = 0, \quad |\tilde{\sigma}_{21}| > g \Leftrightarrow \mu \partial_{x_1} \tilde{u}^r = \tilde{\sigma}_{21} - g \operatorname{sgn}(\partial_{x_1} \tilde{u}^r), \quad (8.3)$$

where $\tilde{\sigma}_{21}$ represents the constraint. Indeed, we notice that, if $\partial_{x_1} \tilde{u}^r \neq 0$, then relation (8.2) implies $|\tilde{\sigma}_{21}| > g$. Consequently, if $|\tilde{\sigma}_{21}| \leq g$, we have $\partial_{x_1} \tilde{u}^r = 0$.

We note that the constitutive law (8.3) is the limit of the initial constitutive law (2.3).

Analogous results hold true for the limit problem (4.9), stated in the left branch, as far as for the limit problem (4.10), stated in the vertical branch: in each case one can exhibit a lower-dimensional “Bingham-like” law.

Conflict of interest statement

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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(accepted: December 28, 2021; published online: January 29, 2022)