Modeling of Fluid Flow in a Flexible Vessel with Elastic Walls

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Abstract. We exploit a two-dimensional model (Ghosh et al. in Q J Mech Appl Math 71(3):349–367, 2018; Kozlov and Nazarov in Dokl Phys 56(11):560–566, 2011, J Math Sci 207(2):249–269, 2015) describing the elastic behavior of the wall of a flexible blood vessel which takes interaction with surrounding muscle tissue and the 3D fluid flow into account. We study time periodic flows in an infinite cylinder with such intricate boundary conditions. The main result is that solutions of this problem do not depend on the period and they are nothing else but the time independent Poiseuille flow. Similar solutions of the Stokes equations for the rigid wall (the no-slip boundary condition) depend on the period and their profile depends on time.

Keywords. Blood vessel with elastic walls, Demension reduction procedure, Periodic in time flows, Poiseuille flow.

1. Introduction

The elasticity of the composite walls of the arteries and the muscle material surrounding arteria's bed significantly contributes to the transfer of blood pushed by the heart along the arterial tree. In addition, hardening of the walls of blood vessels caused by calcification or other diseases makes it much more difficult to supply blood to peripheral parts of the system. The fundamental question here is how the elasticity of walls and surrounded muscle tissue supports mechanism of the blood supply system. In numerous publications, modeling the circulatory system, computational or pure theoretical, there are no fundamental differences between the steady flows of viscous fluid in pipes with rigid walls and vessels with elastic walls. Moreover, quite often much attention is ungroundedly paid to nonlinear convective terms in the Navier-Stokes equations, although blood as a multi-component viscoelastic fluid should be described by much more involved integro-differential equations. We also note that for technical reasons, none of the primary one-dimensional models of a single artery or the entire arterial tree obtained using dimension reduction procedures includes the terms generated by these nonlinear terms.

In connection with the foregoing, in this paper we consider the linear non-stationary Stokes equations simulating a heartbeat, we study time-periodic solutions in a straight infinite cylinder with a constant but arbitrary arbitrary cross-section.

In the case of the Dirichlet (no-slip) condition, according to well-known results (see, for example [21] and [3] and references there), one finds many periodic solutions $(\mathbf{v}, p)$, where the velocity $\mathbf{v}$ has only the component directed along the cylinder (z -axis) and the pressure p depends linear on z with coefficients depending only on the time variable t . For elastic walls, there is only one such solution up to a constant factor, proportional to the steady Poiseuille flow which does not depend on the time variable but can be considered as periodic one with any period. This is the main result of our paper, which describes the difference in behaviour of blood flow in the case of elastic and rigid walls. The above property of the flow inside the thin-walled elastic cylinder implies that the flow has the same direction all time and at every point. So there are no local reverse flows in our model.

Our results refer to a model problem in a straight infinite cylinder with an elastic wall. In reality, the arterial tree consists of curved vessels that narrow when moving away from the heart and has many bifurcation nodes. The presence of bifurcations influences to a large extent the one-dimensional model (see works [12] and [16, 17] referring to the case of absolutely rigid walls in steady and no-steady formulations respectively) and, in particular, generates local reverse (towards the heart) currents, the proof of the absence of which in a straight infinite cylinder is performed in this work. Narrowing of blood vessels and their curvature also significantly affects the flow of fluid, see [15] and [2, 7], where the cases of rigid and elastic walls were studied.

Due to the elastic walls of arteries, an increased heart rate only leads to an increase in the speed of blood flow (the flux grows) without changing the structure of the flow as a whole, and only at an ultra-high beat rate, when the wall elasticity is not enough to compensate for the flow pulsation, the body begins to feel heart beating. The fact is that the human arterial system is geometrically arranged in a very complex system, gently conical shape of blood vessels, their curvature and a considerable number of bifurcation nodes. Therefore, the considered model problem of an infinite straight cylinder gives only a basic approximation to the real circulatory system and on some of its elongated fragments, which acquires periodic disturbances found in the wrists, temples, neck and other periphery of the circulatory system. The correctness of such views is also confirmed by a full-scale experiment on watering the garden: a piston pump delivers water with shocks, but with a long soft hose the water jet at the outlet is unchanged, but with a hard short-pulsating one.

A general two-dimensional model describing the elastic behaviour of the wall of a flexible vessel has been presented in the case of a straight vessel in [9, 11], in the case of a curved vessel in [7] and for numerical results see [1, 2]. The wall has a laminate structure consisting of several anisotropic elastic layers of varying thickness and is assumed to be much thinner than the radius of the channel which itself is allowed to vary. This two-dimensional model takes the interaction of the wall with surrounding or supporting elastic material of muscles and the fluid flow into account and is obtained via a dimension reduction procedure. We study the time-periodic flow in the straight flexible vessel with elastic walls.

Compared with the classical works [19–21] of J.R. Womersley (for an alternative description of the above works see [5]), our formulation of problem has much in common, both of them involve momentless shell theory for modeling the elastic wall. In Womersley's works, axisymmetric pulsative blood flow in a vessel with circular isotropic elastic wall is found as a perturbation of the steady Poiseuille flow. Apart from inessential generalizations like arbitrary shape of vessel's cross-section and orthotropic wall, the main difference of our paper is as follows.

First, Womersley's model does not take into account the reaction of the vessel wall to the radial (normal) component of the force acting on the wall from the side of the liquid (compare formula (3.7) in [21] and formulas (4), (7) in our paper). Within the momentless shell theory, elastic walls cannot resist the normal force in the case of weak reaction of the material on stretching and/or very small curvature of the cross-section. Both the cases are meaningful for diseased vessels due to atherosclerotic sequelae or aneurysm swelling but for healthy vessels, inflation of elastic wall is suppressed precisely by its stretching stress. Mathematically, the observed difference between the models in question results in the important fact that our model support standard Green's formula and it has a energy integral, see [11], Sect.4.3.

Second, we include in the model the coefficient $K(s)$ describing the reaction of the surrounding cell material on deformation of the wall. In other words, the vessel is assumed in [19,20] to “hang in air” while in our paper it is placed inside the muscular arteria’s bed as in human and animal bodies intended to compensate for external and internal influences as well as to protect the arterial tree which is the most vulnerable part of the circulatory system. Consider oscillations in the blood circulatory system due to the beating of the human heart. There are different factors which influence on oscillations of the blood flow and stresses and strains of the vessels wall. Elasticity of walls and interaction of walls with surrounded tissue contribute to the laminar part of the flow and low stresses and strains of the wall. Bifurcations in the circulatory system and the curvature of vessels support oscillations and can contribute to local reverse blood flow. Our result shows that in the model situation of the straight infinite cylinder with elastic boundary all periodic flows are laminar and directed along the axis of cylinder.

2. Problem Statement

2.1. Preliminaries

Let Ω be a bounded, simple connected domain in the plane $\mathbb{R}^2$ with $C^{1,1}$ boundary $\gamma = \partial\Omega$ and let us introduce the spatial cylinder

$$\mathcal{C} = \{x = (y, z) \in \mathbb{R}^2 \times \mathbb{R} : y = (y_1, y_2) \in \Omega, z \in \mathbb{R}\}. \quad (1)$$

We assume that the curve γ is parameterised as $(y_1, y_2) = (\zeta_1(s), \zeta_2(s))$, where s is the arc length along γ measured counterclockwise from a certain point and $\zeta = (\zeta_1, \zeta_2)$ is a vector $C^{1,1}$ function. The length of the countour γ is denoted by $|\gamma|$ and its curvature by

$$\kappa = \kappa(s) = \zeta_1''(s)\zeta_2'(s) - \zeta_2''(s)\zeta_1'(s).$$

In a neighborhood of γ , we introduce the natural curvilinear orthogonal coordinates system (n, s) , where n is the oriented distance to γ ($n > 0$ outside Ω).

The boundary of the cylinder $\mathcal{C}$ is denoted by Γ , i.e.

$$\Gamma = \{x = (y, z) : y \in \gamma, z \in \mathbb{R}\}. \quad (2)$$

The flow in the vessel is described by the velocity vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and by the pressure p which are subject to the non-stationary Stokes equations:

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \mathcal{C} \times \mathbb{R} \ni (x, t). \quad (3)$$

Here $\nu > 0$ is the kinematic viscosity related to the dynamic viscosity μ by $\nu = \mu/\rho_b$, where $\rho_b > 0$ is the density of the fluid.

The elastic properties of the 2D boundary are described by the displacement vector $\mathbf{u}$ defined on Γ and they are presented in [10, 20] for a straight cylinder and in [7] for a curve-linear cylinder. If we use the curve-linear coordinates (s, z) on Γ and write the vector $\mathbf{u}$ in the basis $\mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{z}$, where $\mathbf{n}$ is the outward unit normal vector, $\boldsymbol{\tau}$ is the tangent vector to the curve γ and $\mathbf{z}$ is the direction of z axis, then the balance equation has the following form:

$$D(\kappa, -\partial_s, -\partial_z)^T Q(s) D(\kappa, \partial_s, \partial_z) \mathbf{u} + \rho(s) \partial_t^2 \mathbf{u} + K(s) \mathbf{u} + \sigma(s) \mathcal{F} = 0 \quad \text{in } \Gamma \times \mathbb{R}, \quad (4)$$

where $\rho(s)$ is the average density of the vessel wall, $\sigma = \rho_b/h$, $h = h(s) > 0$ is the thickness of the wall, A^T stands for the transpose of a matrix A , $D(\kappa, \partial_s, \partial_z) = D_0 \partial_z + D_1(\partial_s)$, where

$$D_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad D_1(\partial_s) = \begin{pmatrix} \kappa & \partial_s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \partial_s \end{pmatrix} \quad (5)$$

and

$$K(s) \mathbf{u} = (k(s) \mathbf{u}_1, 0, 0). \quad (6)$$

Here $k(s)$ is a scalar function, Q is a 3×3 symmetric positive definite matrix of homogenized elastic moduli (see [7]) and the displacement vector $\mathbf{u}$ is written in the curve-linear coordinates (s, z) in the basis $\mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{z}$. Furthermore $\mathcal{F} = (\mathcal{F}_n, \mathcal{F}_z, \mathcal{F}_s)$ is the hydrodynamical force given by

$$\mathcal{F}_n = -p + 2\nu \frac{\partial v_n}{\partial n}, \quad \mathcal{F}_s = \nu \left(\frac{\partial v_n}{\partial s} + \frac{\partial v_s}{\partial n} - \kappa v_s \right), \quad \mathcal{F}_z = \nu \left(\frac{\partial v_n}{\partial z} + \frac{\partial v_z}{\partial n} \right), \quad (7)$$

where v_n and v_s are the velocity components in the direction of the normal $\mathbf{n}$ and the tangent $\boldsymbol{\tau}$, respectively, whereas v_z is the longitudinal velocity component. The functions ρ , k and the elements of the matrix Q are bounded measurable functions satisfying

$$\rho(s) \geq \rho_0 > 0 \quad \text{and} \quad k(s) \geq k_0 > 0, \quad (8)$$

where k is the function in (6). The elements of the matrix Q are assumed to be Lipschitz continuous and $\langle Q\xi, \xi \rangle \geq q_0 |\xi|^2$ for all $\xi \in \mathbb{R}^3$ with $q_0 > 0$, where $\langle \cdot, \cdot \rangle$ is the cartesian inner product in $\mathbb{R}^3$.

We note that

$$D(\kappa, \partial_s, \partial_z) \mathbf{u}^T = \left(\kappa \mathbf{u}_1 + \partial_s \mathbf{u}_2, \partial_z \mathbf{u}_3, \frac{1}{\sqrt{2}} (\partial_z \mathbf{u}_2 + \partial_s \mathbf{u}_3) \right)^T \quad (9)$$

and one can easily see that

$$\kappa \mathbf{u}_1 + \partial_s \mathbf{u}_2 = \boldsymbol{\varepsilon}_{ss}(\mathbf{u}), \quad \partial_z \mathbf{u}_3 = \boldsymbol{\varepsilon}_{zz}(\mathbf{u}) \quad \text{and} \quad \partial_z \mathbf{u}_2 + \partial_s \mathbf{u}_3 = 2\boldsymbol{\varepsilon}_{sz}(\mathbf{u}) \quad (10)$$

on Γ . Here $\boldsymbol{\varepsilon}_{ss}(\mathbf{u})$, $\boldsymbol{\varepsilon}_{zz}(\mathbf{u})$ and $\boldsymbol{\varepsilon}_{sz}(\mathbf{u})$ are components of the deformation tensor in the basis $\{\mathbf{n}, \boldsymbol{\tau}, \mathbf{z}\}$. In what follows we will write the displacement vector as $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where $\mathbf{u}_1 = \mathbf{u}_n$, $\mathbf{u}_2 = \mathbf{u}_s$ and $\mathbf{u}_3 = \mathbf{u}_z$. For the velocity $\mathbf{v}$ we will use indexes 1, 2 and 3 for the components of $\mathbf{v}$ in y_1 , y_2 and z directions respectively.

Furthermore the vector functions $\mathbf{v}$ and $\mathbf{u}$ are connected on the boundary by the relation

$$\mathbf{v} = \partial_t \mathbf{u} \quad \text{on } \Gamma \times \mathbb{R}. \quad (11)$$

The problem (3)–(11) appears when we deal with a flow in a pipe surrounded by a thin layered elastic wall which separates the flow from the muscle tissue. Since we have in mind an application to the blood flow in the blood circulatory system, we are interested in periodic in time solutions. One of goals of this paper is to describe all periodic solutions to the problem (3), (4), (11) which are bounded in $\mathbb{R} \times \mathcal{C} \ni (t, x)$.

It is reasonable to compare property of solutions to this problem with similar properties of solutions to the Stokes system (3) supplied with the no-slip boundary condition

$$\mathbf{v} = 0 \text{ on } \Gamma. \quad (12)$$

Considering the problem (3), (12) we assume that the boundary γ is Lipschitz only.

The following result about the problem (3), (12) is possibly known (see, for example [3, 4] and references there) but we present a concise proof for reader's convenience.

Theorem 2.1. *Let the boundary γ be Lipschitz and $\Lambda > 0$. There exists $\delta > 0$ such that if $(\mathbf{v}, p)$ are Λ -periodic in time functions satisfying (3), (12) and may admit a certain exponential growth at infinity*

$$\max_{0 \leq t \leq \Lambda} \int_{\mathcal{C}} e^{-2\delta|z|} (|\nabla \mathbf{v}(x, t)|^2 + |\nabla \partial_z \mathbf{v}|^2 + |p(x, t)|^2) dx < \infty. \quad (13)$$

Then

$$p = zp_*(t) + p_0(t), \quad \mathbf{v}(x, t) = (0, 0, \mathbf{v}_3(y, t)), \quad (14)$$

where p and $\mathbf{v}_3$ are Λ -periodic functions in t which satisfy the problem

$$\begin{aligned} \partial_t \mathbf{v}_3 - \nu \Delta_y \mathbf{v}_3 + p_*(t) &= 0 \text{ in } \Omega \times \mathbb{R} \\ \mathbf{v} &= 0 \text{ on } \gamma \times \mathbb{R}. \end{aligned} \quad (15)$$

Thus the dimension of the space of periodic solutions to the problem (3), (12) is infinite and they can be parameterised by a periodic function p_* . In the case of elastic wall situation is quite different

Theorem 2.2. *Let the boundary γ be $C^{1,1}$ and $\Lambda > 0$. Let also $(\mathbf{v}, p, \mathbf{u})$ be a Λ -periodic with respect to t solution to the problem (3)–(11) admitting an arbitrary power growth at infinity*

$$\begin{aligned} \max_{0 \leq t \leq \Lambda} \left(\int_{\mathcal{C}} (1 + |z|)^{-N} (|\mathbf{v}|^2 + |\nabla_x \mathbf{v}|^2 + |\nabla_x \partial_z \mathbf{v}|^2 + |p|^2) dx \right. \\ \left. + \int_{\Gamma} (1 + |z|)^{-N} (|u|^2 + \sum_{k=2}^3 (|\nabla_{sz} u_k|^2 + |\nabla_{sz} \partial_z u_k|^2)) ds dz \right) < \infty \end{aligned} \quad (16)$$

for a certain $N > 0$. Then

$$p = zp_0 + p_1, \quad \mathbf{v}_1 = \mathbf{v}_2 = 0, \quad \mathbf{v}_3 = p_0 \mathbf{v}_*(y), \quad (17)$$

where p_0 and p_1 are constants and $\mathbf{v}_*$ is the Poiseuille profile, i.e.

$$\nu \Delta_y \mathbf{v}_* = 1 \text{ in } \Omega \text{ and } \mathbf{v}_* = 0 \text{ on } \gamma. \quad (18)$$

The boundary displacement vector $\mathbf{u} = \mathbf{u}(s, z)$ satisfies the equation

$$D(\kappa(s), -\partial_s, -\partial_z)^T Q(s) D(\kappa(s), \partial_s, \partial_z) \mathbf{u} + K \mathbf{u} = \sigma(p, 0, p_0 \nu \partial_n \mathbf{v}_3|_{\gamma})^T. \quad (19)$$

If the elements Q_{21} and Q_{31} vanish then the function $\mathbf{u}$ is a polynomial of second degree in z : $\mathbf{u}(s, z) = (0, \alpha, \beta)^T z^2 + \mathbf{u}^{(1)}(s)z + \mathbf{u}^{(2)}(s)$, where α and β are constants.

Thus, in the case of elastic wall all periodic solutions are independent of t and hence are the same for any period. Moreover inside the cylinder the flow takes the Poiseuille form. The above theorems have different requirements on the behavior of solutions with respect to z , compare (13) and (16). This is because of the following reason. In the case of the Dirichlet boundary condition we can prove a resolvent estimate on the imaginary axis ($\lambda = i\omega$, ω is real) with exponential weights independent on ω . In the case of the elastic boundary condition exponential weights depends on ω . Because of that we can not put in (16) the same exponential weight as in (13).

The structure of our paper is the following. In Sect. 3 we treat the Stokes system with the no-slip condition on the boundary of cylinder. Since we are dealing with time-periodic solutions the problem can be reduced to a series of time independent problems with a parameter (frequency). The main result there is Theorem 3.1. Using this assertion it is quite straightforward to prove the main theorem 2.1 for the Dirichlet problem. Parameter dependent problems are studied in Sects. 3.2–3.4. Theorem 3.1 is proved in Sect. 3.5.

Stokes problem in a vessel with elastic walls is considered in Sect. 4. We also reduce the time periodic problem to a series of time independent problems depending on a parameter. The main result there is Theorem 4.1. Using this result we prove our main theorem 2.2 for the case of elastic wall in Sect. 4.1. The parameter depending problem is studied in Sects. 4.2–4.6. The proof of Theorem 4.1 is given in Sect. 4.7. In Sect. 4.8 we consider the case when the parameter in the elastic wall problem is vanishing. This consideration completes the proof of Theorem 2.2.

3. Dirichlet Problem for the Stokes System

The first step in the proof of Theorem 2.1 is the following reduction of the time dependent problem to time independent one. Due to Λ -periodicity of our solution we can represent it in the form

$$\mathbf{v}(x, t) = \sum_{k=-\infty}^{\infty} \mathbf{V}_k(x) e^{2\pi k i t / \Lambda}, \quad p(x, t) = \sum_{k=-\infty}^{\infty} P_k(x) e^{2\pi k i t / \Lambda}, \quad (20)$$

where

$$\mathbf{V}_k(x) = \frac{1}{\Lambda} \int_0^\Lambda \mathbf{v}(x, t) e^{-2\pi k i t / \Lambda} dt, \quad P_k(x) = \frac{1}{\Lambda} \int_0^\Lambda p(x, t) e^{-2\pi k i t / \Lambda} dt. \quad (21)$$

These coefficients satisfy the following time independent problem

$$i\omega \mathbf{V} - \nu \Delta \mathbf{V} + \nabla P = \mathbf{F} \quad \text{and} \quad \nabla \cdot \mathbf{V} = 0 \quad \text{in } \mathcal{C}, \quad (22)$$

with the Dirichlet boundary condition

$$\mathbf{V} = 0 \quad \text{on } \Gamma \quad (23)$$

and with $\omega = 2\pi k / \Lambda$ and $\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3) = 0$ (for further analysis it is convenient to have an arbitrary $\mathbf{F}$).

Theorem 3.1. *There exist a positive number β^* depending only on Ω and ν such that for $\beta \in (0, \beta^*)$ the only solution to problem (22), (23) with $\mathbf{F} = 0$ which may admit a certain exponential growth*

$$\int_{\mathcal{C}} e^{-2\beta|z|} (|\nabla \mathbf{V}|^2 + |\nabla \partial_z \mathbf{V}|^2 + |P|^2) dydz < \infty \quad (24)$$

is

$$\mathbf{V}(x) = p_0(0, 0, \hat{v}(y)) \quad \text{and} \quad P(x) = p_0 z + p_1, \quad (25)$$

where p_0 and p_1 are constants and $\hat{v}$ satisfies

$$i\omega \hat{v} - \nu \Delta \hat{v} + 1 = 0 \quad \text{in } \Omega, \quad \hat{v} = 0 \quad \text{on } \gamma. \quad (26)$$

Remark 3.1. From (26) it follows that

$$\int_{\Omega} \hat{v}(y) dy = i\omega \int_{\Omega} |\hat{v}|^2 du - \nu \int_{\Omega} |\nabla \hat{v}|^2 dy,$$

i.e. the flux does not vanish for this solution.

We postpone the proof of the above theorem to Sect. 3.5 and in the next section we present the proof of Theorem 2.1

3.1. Proof of Theorem 2.1

By (13)

$$\int_{\mathcal{C}} e^{-2\delta|z|} (|\nabla \mathbf{V}_k(x)|^2 + |\mathbf{P}_k(x)|^2) dx < \infty, \quad (27)$$

where $\mathbf{V}_k$ and $\mathbf{P}_k$ are the coefficients in (20).

Applying Theorem 3.1 and assuming $\delta < \beta^*$, we get that

$$V_k = p_{0k}(0, 0, \hat{v}_k(y)), \quad P_k = z p_{0k} + p_{1k},$$

where p_{0k} and p_{1k} are constants. This implies that $\mathbf{v}_1 = \mathbf{v}_2 = 0$, $\mathbf{v}_3$ depends only on y, t and $p = p_0(t)z + p_1(t)$, which proves the required assertion.

3.2. System for Coefficients (22), (23)

To describe the main solvability result for the problem (22), (23), let us introduce some function spaces. For $\beta \in \mathbb{R}$ we denote by $L^2_{\beta}(\mathcal{C})$ the space of functions on $\mathcal{C}$ with the finite norm

$$\|u; L^2_{\beta}(\mathcal{C})\| = \left(\int_{\mathcal{C}} e^{2\beta z} |u(x)|^2 dx \right)^{1/2}.$$

By $W_\beta^{1,2}(\mathcal{C})$ we denote the space of functions in $\mathcal{C}$ with the finite norm

$$\|v; W_\beta^{1,2}(\mathcal{C})\| = \left(\int_{\mathcal{C}} e^{2\beta z} (|\nabla_x v|^2 + |v|^2) dx \right)^{1/2}.$$

We will use the same notation for spaces of vector functions.

Proposition 3.1. *Let the boundary γ is Lipschitz and $\omega \in \mathbb{R}$. There exist $\beta^* > 0$ independent of ω such that the following assertions are valid:*

- (i) *for any $\beta \in (-\beta^*, \beta^*)$, $\beta \neq 0$ and $\mathbf{F} \in L_\beta^2(\mathcal{C})$, the problem (22), (23) has a unique solution $(\mathbf{V}, \mathbf{P})$ in $W_\beta^{1,2}(\mathcal{C})^3 \times L_\beta^2(\mathcal{C})$ satisfying the estimate*

$$\|\mathbf{V}; W_\beta^{1,2}(\mathcal{C})\| + \|\mathbf{P}; L_\beta^2(\mathcal{C})\| \leq C \|\mathbf{F}; L_\beta^2(\mathcal{C})\|. \quad (28)$$

where C may depend on β , ν and Ω . Moreover,

$$\|\partial_z \mathbf{V}; W_\beta^{1,2}(\mathcal{C})\| \leq C \|\mathbf{F}; L_\beta^2(\mathcal{C})\|. \quad (29)$$

- (ii) *Let $\beta \in (0, \beta^*)$ and $\mathbf{F} \in L_\beta^2(\mathcal{C}) \cap L_{-\beta}^2(\mathcal{C})$. Then solutions $(\mathbf{V}_\pm, \mathbf{P}_\pm) \in W_{\pm\beta}^{1,2}(\mathcal{C})^3 \times L_{\pm\beta}^2(\mathcal{C})$ to (22), (23) are connected by*

$$\mathbf{V}_-(x) = \mathbf{V}_+(x) + p_0(0, 0, \hat{v}(y)), \quad \mathbf{P}_-(x) = \mathbf{P}_+(x) + p_0 z + p_1, \quad (30)$$

with certain constants p_0 and p_1 . Here $\hat{v}$ is solution to (26).

Remark 3.2. If γ is $C^{1,1}$ then it follows from [18] that the left-hand side in (28) can be replaced by

$$\|\mathbf{V}; W_\beta^{2,2}(\mathcal{C})\| + \|\nabla \mathbf{P}; L_\beta^2(\mathcal{C})\|.$$

3.3. Operator Pencil, Weak Formulation

We will use the spaces of complex valued functions $H_0^1(\Omega)$, $L^2(\Omega)$ and $H^{-1}(\Omega)$ and the corresponding norms are denoted by $\|\cdot\|_1$, $\|\cdot\|_0$ and $\|\cdot\|_{-1}$ respectively.

Let us introduce an operator pencil by

$$\mathcal{S}(\lambda) \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} \mu v_1 - \nu \Delta_y v_1 + \partial_{y_1} p \\ \mu v_2 - \nu \Delta_y v_2 + \partial_{y_2} p \\ \mu v_3 - \nu \Delta_y v_3 + \lambda p \\ \partial_{y_1} v_1 + \partial_{y_2} v_2 + \lambda v_3 \end{pmatrix}, \quad \mu = i\omega - \nu\lambda^2, \quad (31)$$

where $v = (v_1, v_2, v_3)$ is a vector function and p is a scalar function in Ω . This pencil is defined for vectors (v, p) such that $v = 0$ on γ .

Clearly

$$\mathcal{S}(\lambda) : H_0^1(\Omega)^3 \times L^2(\Omega) \rightarrow (H^{-1}(\Omega))^3 \times L^2(\Omega) \quad (32)$$

is a bounded operator for all $\lambda \in \mathbb{C}$. The following problem is associated with this operator

$$\begin{aligned}\mu v_1 - \nu \Delta_y v_1 + \partial_{y_1} p &= f_1, \\ \mu v_2 - \nu \Delta_y v_2 + \partial_{y_2} p &= f_2, \\ \mu v_3 - \nu \Delta_y v_3 + \lambda p &= f_3 \quad \text{in } \Omega\end{aligned}\tag{33}$$

and

$$\partial_{y_1} v_1 + \partial_{y_2} v_2 + \lambda v_3 = h \quad \text{in } \Omega\tag{34}$$

supplied with the Dirichlet condition

$$v = 0 \quad \text{on } \partial\Omega.\tag{35}$$

The corresponding sesquilinear form is given by

$$\begin{aligned}\mathbb{A}(v, p; \hat{v}, \hat{p}; \lambda) &= \int_{\Omega} \sum_{j=1}^3 (\mu v_j \overline{\hat{v}_j} + \nu \nabla_y v_j \cdot \nabla_y \overline{\hat{v}_j}) dy \\ &\quad - \int_{\Omega} p \overline{(\nabla_y \cdot \hat{v}' - \bar{\lambda} \hat{v}_3)} dy + \int_{\Omega} (\nabla_y \cdot v' + \lambda v_3) \bar{\hat{p}} dy\end{aligned}$$

where $v' = (v_1, v_2)$. This form is well-defined on $H_0^1(\Omega)^3 \times L^2(\Omega)$.

The weak formulation of (33)–(35) can be written as

$$\mathbb{A}(v, p; \hat{v}, \hat{p}; \lambda) = \int_{\Omega} (f \cdot \bar{\hat{v}} + h \bar{\hat{p}}) dy\tag{36}$$

for all $(\hat{v}, \hat{p}) \in H_0^1(\Omega)^3 \times L^2(\Omega)$. As it was shown in the proof of Lemma 3.2(ii) [4] the operator $\mathcal{S}(\lambda)$ is isomorphism for $\lambda = i\xi$, $\xi \in \mathbb{R} \setminus \{0\}$. Since the operator corresponding to the difference of the forms $\mathbb{A}$ for different λ is compact, the operator pencil $\mathcal{S}(\lambda)$ is Fredholm for all $\lambda \in \mathbb{C}$ and its spectrum consists of isolated eigenvalues of finite algebraic multiplicity, see [8].

3.4. Operator Pencil Near the Imaginary Axis $\Re \lambda = 0$

Here we consider the right-hand sides in (33)–(35) as follows $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$

The next assertion is proved in Lemma 3.2(i) [4], after a straightforward modification.

Lemma 3.1. *Let $h \in L^2(\Omega)$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then the equation*

$$\partial_{y_1} w_1 + \partial_{y_2} w_2 + \lambda w_3 = h \quad \text{in } \Omega\tag{37}$$

has a solution $w \in H_0^1(\Omega)$ satisfying the estimate

$$\sum_{j=1}^2 \|w_j\|_1 \leq C(\|h\|_0 + |\alpha|), \quad \|w_3\|_1 \leq C \frac{|\alpha|}{|\lambda|}$$

where

$$\alpha = \int_{\Omega} h dy$$

and C depends only on Ω . The mapping $h \mapsto w$ can be chosen linear.

The proof of the next lemma can be extracted from the proof of Lemma 3.2(ii) [4].

Lemma 3.2. *Let $f \in L_2(\Omega)$, $h = 0$ in (34) and let $\lambda = i\xi$, $0 \neq \xi \in \mathbb{R}$. Then the solution to (33)–(35) admits the estimate*

$$(1 + |\omega| + |\xi|^2) \|v\|_0 + (1 + |\omega| + |\xi|^2)^{1/2} \|\nabla v\|_0 \leq C \|f\|_0, \quad (38)$$

and

$$\|p\|_0 \leq C \frac{1 + |\lambda|}{|\lambda|} \|f\|_0, \quad (39)$$

where the constant C depends only on ν and Ω .

Proof. To estimate the norm of v by the right-hand side in (38), we take $\hat{v} = v$ in (36) and obtain

$$\int_{\Omega} (\mu |v|^2 + \nu |\nabla_y v|^2) dy = \int_{\Omega} f \cdot \bar{v} dy.$$

This implies the inequality (38). To estimate the norm of p , we choose $\nabla_{i\xi} w = p$. Then the relation (36) becomes

$$\int_{\Omega} |p|^2 dy = \int_{\Omega} (\mu v \cdot \bar{w} + \nu \nabla v \cdot \nabla \bar{w} - f \bar{w}) dy.$$

Since by Lemma 3.1

$$\|w\|_1 \leq C \frac{1 + |\lambda|}{|\lambda|} \|p\|_0,$$

we arrive at (39). □

Lemma 3.3. *There exists a positive number δ depending on ν and Ω such that if $f \in L_2(\Omega)$, $h = 0$ and $0 < |\lambda| < \delta$, then the problem (33)–(35) has a unique solution satisfying*

$$(|\omega| + 1) \|v\|_0 + (|\omega| + 1)^{1/2} \|v\|_1 + \|p - p_m\|_0 + |\lambda| \|p\|_0 \leq C \|f\|_0, \quad (40)$$

where the constant C depends only on ν and Ω and

$$p_m = \frac{1}{|\Omega|} \int_{\Omega} p dy.$$

Proof. From (34) it follows that

$$\int_{\Omega} v_3 dy = 0 \quad \text{when } \lambda \neq 0. \quad (41)$$

Taking $\hat{v} = v$ in (36) yields

$$\mu \int_{\Omega} |v|^2 dy + \nu \int_{\Omega} |\nabla_y v|^2 dy = -2\Re\left(\lambda \int_{\Omega} p \bar{v}_3 dy\right) + \Re\left(\int_{\Omega} f \cdot \bar{v} dy\right).$$

For a small $\lambda \neq 0$, this together with (41) implies

$$(1 + |\omega|) \|v\|_0 \leq C(\|f\|_0 + |\lambda| \|p - p_m\|_0)$$

and

$$(1 + |\omega|)^{1/2} \|\nabla v\|_0 \leq C(\|f\|_0 + |\lambda| \|p - p_m\|_0)$$

with C independent of ω and λ . Taking $\hat{v} = w = (w_1, w_2, 0)$ in (36) where $w_k \in H_0^1(\Omega)$ satisfies

$$\partial_{y_1} w_1 + \partial_{y_2} w_2 = p - p_m, \quad \|w\|_1 \leq C \|p - p_m\|_0,$$

we get

$$\|p - p_m\|_0^2 = \mu \int_{\Omega} v \cdot \bar{w} dy + \nu \int_{\Omega} \nabla_y v \cdot \nabla_y \bar{w} dy.$$

Therefore,

$$\|p - p_m\|_0^2 \leq C((1 + |\omega|) \|v\|_0 + \|\nabla v\|_0) \|p - p_m\|_0$$

and, hence,

$$\|p - p_m\|_0 \leq C(\|f\|_0 + |\lambda| \|p - p_m\|_0)$$

which implies

$$\|p - p_m\|_0 \leq C \|f\|_0 \quad \text{and} \quad (1 + |\omega|) \|v\|_0 + (1 + |\omega|)^{1/2} \|\nabla v\|_0 \leq C \|f\|_0. \quad (42)$$

Now taking $\hat{v} = w = (w_1, w_2, w_3)$ in (36), where $w_k \in H_0^1(\Omega)$ is subject to

$$\partial_{y_1} w_1 + \partial_{y_2} w_2 - \bar{\lambda} w_3 = p - p_m, \quad \|w\|_1 \leq C \frac{1 + |\lambda|}{|\lambda|} \|p - p_m\|_0,$$

we obtain

$$\|p\|_0 \leq C \frac{1 + |\lambda|}{|\lambda|} \|f\|_0.$$

The last inequality together with (42) gives (40). $\square$

Now we can describe properties of the pencil $\mathcal{S}$ in a neighborhood of the imaginary axis $\Re\lambda = 0$.

Lemma 3.4. *There exist $\beta^* > 0$ such that the following assertion are valid:*

- (i) *the only eigenvalue of $\mathcal{S}$ in the strip $|\Re\lambda| < \beta^*$ is zero;*
- (ii) *if $\beta \in (-\beta^*, 0) \cup (0, \beta^*)$ then for $f \in L^2(\Omega)$, $h = 0$ and $\Re\lambda = \beta$ the problem (33)–(35) has a unique solution in $H_0^1(\Omega)^3 \times L^2(\gamma)$ and its norm is estimated as follows:*

$$(1 + |\lambda|^2 + |\omega|) \|v\|_0 + (1 + |\lambda|^2 + |\omega|)^{1/2} \|\nabla v\|_0 + \|p\|_0 \leq C \|f\|_0, \quad (43)$$

where the constant C may depend on β , ν and Ω .

Proof. First, we observe that

$$\begin{aligned} & \mathbb{A}(v, p; \hat{v}, \hat{p}; \beta + i\xi) - \mathbb{A}(v, p; \hat{v}, \hat{p}; \xi) \\ &= \nu\beta(\beta + 2\xi i) \int_{\Omega} v \cdot \bar{\hat{v}} dy + \beta \int_{\Omega} (v_3 \bar{\hat{p}} - p \bar{\hat{v}_3}) dy. \end{aligned}$$

Thus the first form is a small perturbation of the second one. Now using Lemmas 3.2 and 3.3 for small β we arrive at the existence of β^* , which satisfies (i). Moreover, the estimates (38) and (39) are true for $\lambda = \beta + i\xi$ for a fixed $\beta \in (-\beta^*, 0) \cup (0, \beta^*)$ and arbitrary real ξ . With this the constants in (38) and (39) may depend now on β , ν and Ω only. $\square$

3.5. Proof of Proposition 3.1 and Theorem 3.1

Proof of Proposition 3.1. The assertion (i) in Proposition 3.1 is obtained from Lemma 3.4(ii) by using the inverse Fourier transform together with Parseval's identity.

To prove (ii) in Proposition 3.1, we observe that

$$(\mathbf{V}_{\pm}, \mathbf{P}_{\pm}) = \frac{1}{2\pi i} \int_{\Re\lambda = \pm\beta} e^{-\lambda z} \mathcal{S}^{-1}(f(y, \lambda), 0) d\lambda$$

and the relation (30) is obtained by applying the residue theorem. $\square$

Now we turn to **Proof of Theorem 3.1.** Let $(\mathbf{V}, \mathbf{P})$ be a solution to (22), (23) satisfying (24). Our first step is to construct a representation of the solution in the form

$$(\mathbf{V}, \mathbf{P}) = (\mathbf{V}^{(+)}, \mathbf{P}^{(+)}) + (\mathbf{V}^{(-)}, \mathbf{P}^{(-)}), \quad (44)$$

where $\mathbf{V}^{(\pm)} \in W_{\pm\beta}^{1,2}(\mathcal{C})$, $\mathbf{P}^{(\pm)} \in L_{\pm\beta}^2(\mathcal{C})$ and they solve the problem (22), (23) with certain $\mathbf{F} = \mathbf{F}^{(\pm)} \in L_{\pm\beta}^2(\mathcal{C})$.

By the second equation in (22) and by (23) the flux

$$\Psi = \int_{\Omega} \mathbf{V}_3(y, z) dy \text{ is constant.} \quad (45)$$

The vector-function $(0, 0, p_0 \hat{v}_3, p_0)$ with a constant p_0 verifies the homogeneous problem (22), (23) and its flux does not vanish in the case $p_0 \neq 0$. So subtracting it with appropriate constant p_0 from $(\mathbf{V}, \mathbf{P})$ we can reduce the proof of theorem to the case $\Psi = 0$. In this way we assume in what follows that this is the case.

Let $\zeta(z)$ be a smooth cut-off function equal 1 for large positive z and 0 for large negative z and let ζ' be its derivative. We choose in (44)

$$\begin{aligned} (\mathbf{V}^{(+)}, \mathbf{P}^{(+)}) &= \zeta(\mathbf{V}, \mathbf{P}) + \zeta'(\mathbf{W}, 0), \quad (\mathbf{V}^{(-)}, \mathbf{P}^{(-)}) \\ &= (1 - \zeta)(\mathbf{V}, \mathbf{P}) - \zeta'(\mathbf{W}, 0) \end{aligned}$$

where the vector function $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, 0)$ is such that

$$\nabla_y \cdot (\mathbf{W}_1, \mathbf{W}_2)(y, z) = \mathbf{V}_3(y, z). \quad (46)$$

We construct solution $\mathbf{W}$ by solving two-dimensional Stokes problem in Ω depending on the parameter z :

$$-\nu \Delta \mathbf{W}_k + \partial_{y_k} \mathbf{Q} = 0, \quad k = 1, 2, \quad \partial_{y_1} \mathbf{W}_1(y) + \partial_{y_2} \mathbf{W}_2(y) = V_3(y, z) \text{ in } \Omega$$

and $\mathbf{W}_k = 0$ on γ , $k = 1, 2$. This problem has a solution in $H_0^1(\Omega)^2 \times L^2(\Omega)$, which is unique if we require $\int_{\Omega} \mathbf{Q} dy = 0$. If we look on the dependence on the parameter z it is the same as in the right-hand side. So

$$\mathbf{W}_k, \partial_z \mathbf{W}_k, \partial_z^2 \mathbf{W}_k \in L_{\text{loc}}^2(\mathbb{R}; H_0^1(\Omega)) \text{ and } \mathbf{Q}, \partial_z \mathbf{Q} \in L^2(\Omega).$$

Therefore,

$$i\omega \mathbf{V}_k^{(+)} - \nu \Delta \mathbf{V}_k^{(+)} + \partial_{y_k} \mathbf{P}^{(+)} = \mathbf{F}_k^{(+)}$$

where

$$\mathbf{F}_k^{(+)} = -\nu \zeta'' \mathbf{V}_k - 2\nu \zeta' \partial_z \mathbf{V}_k + i\omega \zeta' \mathbf{W}_k - \nu \partial_z^2 (\zeta' \mathbf{W}_k) \in (L_{2, \beta_+} \cap L_{2, \beta_-})(\mathcal{C})$$

for $k = 1, 2$ and

$$\mathbf{F}_3^{(+)} = -\nu \zeta'' \mathbf{V}_3 - 2\nu \zeta' \partial_z \mathbf{V}_3 + \zeta' \mathbf{P}^{(+)} \in (L_{2, \beta_+} \cap L_{2, \beta_-})(\mathcal{C})$$

Similar formulas are valid for $(\mathbf{V}^{(-)}, \mathbf{P}^{(-)})$ with

$$\mathbf{F}_k^{(-)} = -\mathbf{F}_k^{(+)}.$$

By Proposition 3.1(ii) this implies

$$(\mathbf{V}^{(+)}, \mathbf{P}^{(+)}) + (\mathbf{V}^{(-)}, \mathbf{P}^{(-)}) = (0, 0, p_0 \hat{v}(y), p_0 z + p_1)$$

for certain constants p_0 and p_1 , which furnishes the proof of the assertion.

4. Stokes Flow in a Vessel with Elastic Walls

This section is devoted to the proof of Theorem 2.2. As in the case of the Dirichlet problem considered in Sect. 3 we represent solutions to the problem (3)–(11) in the form (20) (for the velocity $\mathbf{v}$ and the pressure p) and

$$\mathbf{u}(s, z, t) = \sum_{k=-\infty}^{\infty} e^{2\pi k i t / \Lambda} \mathbf{U}_k(s, z) \quad (47)$$

for the displacements $\mathbf{u}$. The coefficients in (20) are given by (21) and in (47) by

$$\mathbf{U}_k(s, z) = \frac{1}{\Lambda} \int_0^\Lambda e^{-2\pi k i t / \Lambda} \mathbf{u}(s, z, t) dt.$$

The above introduced coefficients satisfy the time independent problem

$$i\omega \mathbf{V} - \nu \Delta \mathbf{V} + \nabla P = \mathbf{F} \quad \text{and} \quad \nabla \cdot \mathbf{V} = 0 \quad \text{in } \mathcal{C}, \quad (48)$$

$$D(\kappa(s), -\partial_s, -\partial_z)^T \bar{Q}(s) D(\kappa(s), \partial_s, \partial_z) \mathbf{U}(s, z) - \bar{\rho}(s) \omega^2 \mathbf{U}(s, z) + K \mathbf{U} + \sigma \hat{\mathcal{F}}(s, z) = \mathbf{G}, \quad (49)$$

$$\mathbf{V} = i\omega \mathbf{U} \quad \text{on } \Gamma \times \mathbb{R} \quad (50)$$

where $\mathbf{F} = 0$, $\mathbf{G} = 0$ and $\omega = 2\pi k / \Lambda$ (for forthcoming analysis it is convenient to have arbitrary right-hand sides in this problem).

Theorem 4.1. *Let $\omega \in \mathbb{R}$ and $\omega \neq 0$. Then there exists $\beta > 0$ depending on ω such that the only solution to the homogeneous ($\mathbf{F} = 0$ and $\mathbf{G} = 0$) problem (48)–(50) subject to*

$$\begin{aligned} & \int_{\mathcal{C}} e^{-\beta|z|} (|\mathbf{v}|^2 + |\nabla_x \mathbf{v}|^2 + |\nabla_x \partial_z \mathbf{v}|^2 + |p|^2) dx \\ & + \int_{\Gamma} e^{-\beta|z|} (|u|^2 + \sum_{k=2}^3 (|\nabla_{sz} u_k|^2 + |\nabla_{sz} \partial_z u_k|^2)) ds dz < \infty \end{aligned} \quad (51)$$

is $\mathbf{V} = 0$, $\mathbf{U} = 0$ and $\mathbf{P} = 0$.

We postpone the proof of the formulated theorem to Sect. 4.7 and in the next section we give the proof of Theorem 2.2

4.1. Proof of Theorem 2.2

By (51),

$$\int_{\mathcal{C}} (1 + |x|^2)^{-N} (|\nabla \mathbf{V}_k|^2 + |P_k|^2) dy dz + \int_{\Gamma} (1 + |x|^2)^{-N} |\mathbf{U}_k|^2 ds dz < \infty.$$

Applying Theorem 4.1 we get $\mathbf{V}_k = 0$, $\mathbf{P}_k = 0$ and $\mathbf{U}_k = 0$ for $k \neq 0$. Now using Theorem 3.1 and consideration in forthcoming Sect. 4.8 for $\omega = 0$ we arrive at the required assertion.

4.2. System for Coefficients (48)–(50)

To formulate the main solvability result for the system (48)–(50), we need the following function spaces

$$\mathcal{Y}_\beta = \{\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3) : \mathbf{U}_1 \in L_2^\beta(\Gamma), \mathbf{U}_2, \mathbf{U}_3 \in W_\beta^{1,2}(\Gamma)\}$$

and

$$\mathcal{Z}_\beta = \{(\mathbf{V}, \mathbf{U}) : \mathbf{V} \in W_\beta^{1,2}(\mathcal{C})^3, \mathbf{U} \in \mathcal{Y}_\beta, i\omega\mathbf{U} = \mathbf{V} \text{ on } \Gamma\}.$$

Proposition 4.1. *Let $\omega \in \mathbb{R}$ and $\omega \neq 0$. There exist a positive number β^* depending on ω , Ω and ν such that for any $\beta \in (-\beta^*, \beta^*)$ the following assertions hold*

- (i) *If $\mathbf{F} \in L_\beta^2(\mathcal{C})$, $\mathbf{G}, \partial_z \mathbf{G} \in L_\beta^2(\Gamma)$ then the problem (48)–(50) has a unique solution $(\mathbf{V}, \mathbf{U}) \in \mathcal{Z}_\beta$, $P \in L_\beta^2(\mathcal{C})$ and this solution satisfies the estimate*

$$\begin{aligned} & \| \mathbf{V}; W_\beta^{1,2}(\mathcal{C}) \| + \| P; L_\beta^2(\mathcal{C}) \| + \| \mathbf{U}; \mathcal{Y}_\beta \| \\ & \leq C \left(\| \mathbf{F}; L_\beta^2(\mathcal{C}) \| + \| \mathbf{G}; L_\beta^2(\Gamma) \| + \| \partial_z \mathbf{G}; L_\beta^2(\Gamma) \| \right), \end{aligned}$$

where C may depend on ω , β , ν and Ω . Moreover,

$$\begin{aligned} & \| \partial_z \mathbf{V}; W_\beta^{1,2}(\mathcal{C}) \| + \| \partial_z \mathbf{U}; \mathcal{Y}_\beta \| \\ & \leq C \left(\| \mathbf{F}; L_\beta^2(\mathcal{C}) \| + \| \mathbf{G}; L_\beta^2(\Gamma) \| + \| \partial_z \mathbf{G}; L_\beta^2(\Gamma) \| \right). \end{aligned}$$

- (ii) *If $\mathbf{F} \in L_{\beta_1}^2(\mathcal{C}) \cap L_{\beta_2}^2(\mathcal{C})$, $\mathbf{G}, \partial_z \mathbf{G} \in L_{\beta_1}^2(\Gamma) \cap L_{\beta_2}^2(\Gamma)$ with $\beta_1, \beta_2 \in (-\beta^*, \beta^*)$ and $(\mathbf{V}^{(k)}, \mathbf{U}^{(k)}, \mathbf{P}^{(k)}) \in \mathcal{Z}_{\beta_k} \times L_{\beta_k}^2(\mathcal{C})$ is the solution from (i) for $k = 1, 2$ respectively, then they coincide.*

4.3. Transformations of the Problem (3), (4), (11)

It is convenient to rewrite the Stokes system (22) in the form

$$i\omega \mathbf{V}_j - \sum_{i=1}^3 \partial_{x_i} T_{ji}(\mathbf{V}) = \mathbf{F}_j, \quad j = 1, 2, 3, \quad \nabla \cdot \mathbf{V} = 0 \quad \text{in } \mathcal{C}. \quad (52)$$

where

$$T_{ji}(\mathbf{V}) = -p\delta_{i,j} + \nu(\partial_{x_i} \mathbf{V}_j + \partial_{x_j} \mathbf{V}_i) \quad (53)$$

and $\delta_{i,j}$ is the Kronecker delta. Moreover, relations (4) and (11) become

$$\begin{aligned} & D(\kappa(s), -\partial_s, -\partial_z)^T Q(s) D(\kappa(s), \partial_s, \partial_z) \mathbf{U}(s, z) \\ & - \rho(s)\omega^2 \mathbf{U}(s, z) + K\mathbf{U} + \sigma \widehat{\mathcal{F}}(s, z) = \mathbf{G}(n, s, z) \end{aligned} \quad (54)$$

and

$$\mathbf{V} = i\omega \mathbf{U} \quad \text{on } \Gamma. \quad (55)$$

Here $\widehat{\mathcal{F}} = e^{-i\omega t} \mathcal{F}$.

Next step is the application of the Fourier transform. We set

$$\mathbf{V}(x) = e^{\lambda z} v(y), P(x) = e^{\lambda z} p(y) \quad \text{and} \quad \mathbf{U}(x) = e^{\lambda z} u(y).$$

As the result we obtain the system

$$i\omega v_j - \sum_{i=1}^2 \partial_{x_i} t_{ji}(v; \lambda) + \lambda t_{j3}(v; \lambda) = f_j \quad j = 1, 2, 3, \quad (56)$$

$$\nabla_y \cdot v' + \lambda v_3 = h \quad \text{in } \Omega, \quad (57)$$

where

$$t_{ij}(v, p; \lambda) = -p\delta_j^i + 2\nu\varepsilon_{ij}(v; \lambda), \quad (58)$$

$$\varepsilon_{ij}(v) = \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i), \quad i, j \leq 2,$$

$$\varepsilon_{i3}(v; \lambda) = \hat{\varepsilon}_{3i}(v; \lambda) = \frac{1}{2}(\lambda v_i + \partial_{x_i} v_3), \quad i = 1, 2,$$

$$\varepsilon_{33}(v; \lambda) = \lambda v_3.$$

The equations (54) and (55) take the form

$$\begin{aligned} D(\kappa(s), -\partial_s, -\lambda)^T Q(s) D(\kappa(s), \partial_s, \lambda) u \\ - \bar{\rho}(s) \omega^2 u + K(s) u + \sigma(s) \Phi(s) = g(s) \end{aligned} \quad (59)$$

and

$$v = i\omega u \quad \text{on } \partial\Omega. \quad (60)$$

Here $\Phi(s) = (\Phi_n, \Phi_z, \Phi_s)$ and

$$\Phi_n = -p + 2\nu \frac{\partial v_n}{\partial n}, \quad \Phi_s = \nu \left(\frac{\partial v_n}{\partial s} + \frac{\partial v_s}{\partial n} - \kappa v_s \right), \quad \Phi_z = \nu \left(\lambda v_n + \frac{\partial v_z}{\partial n} \right). \quad (61)$$

4.4. Weak Formulation and Function Spaces

Let us introduce an energy integral

$$E(\mathbf{v}, \hat{\mathbf{v}}) = \int_{\Omega} \sum_{i,j=1}^2 \varepsilon_{ij}(\mathbf{v}) \overline{\varepsilon_{ij}(\hat{\mathbf{v}})} dy$$

and put

$$a(u, \hat{u}; \lambda) = \int_{\partial\Omega} \langle Q(s) D(\kappa(s), \partial_s, \lambda) u(s), D(\kappa(s), \partial_s, -\bar{\lambda}) \hat{u}(s) \rangle ds,$$

where $\langle \cdot, \cdot \rangle$ is the euclidian inner product in $\mathbb{C}^3$. Since the matrix Q is positive definite

$$a(u, u; i\xi) \geq c_1(|\xi|^2 |u_3|^2 + |\kappa u_1 + \partial_s u_2|^2 + |i\xi u_2 + \partial_s u_3|^2), \quad (62)$$

where $\xi \in \mathbb{R}$ and c_1 is a positive constant independent of ξ . Another useful inequality is the following

$$\int_{\partial\Omega} |v|^2 dy \leq c_2 \|v\|_0 \|v\|_1, \quad (63)$$

or by using Korn's inequality

$$q \int_{\gamma} |v|^2 dy \leq c_3 \left(q^2 \|v\|_0^2 + E(v, v) \right) \quad \text{for } q \geq 1, \quad (64)$$

where c_3 does not depend on q .

To define a weak solution, we introduce the vector function spaces:

$$\begin{aligned} X &= \{v = (v_1, v_2, v_3) : v \in H^1(\Omega)^3\}, \\ Y &= \{u_1 \in H^{1/2}(\gamma) : u_2, u_3 \in H^1(\gamma)\} \end{aligned}$$

and

$$Z = Z_\omega = \{(v, u) : v \in X, u \in Y, v = i\omega u \text{ on } \gamma\}.$$

We supply the space Z with the inner product

$$\begin{aligned} \langle v, u; \hat{v}, \hat{u} \rangle_0 &= \int_{\Omega} (v \cdot \bar{\hat{v}} + \sum_{j=1}^3 \nabla_y v_j \cdot \nabla_y \bar{\hat{v}}_j) dy \\ &\quad + \int_{\gamma} (\partial_s u_2 \partial_s \bar{\hat{u}}_2 + \partial_s u_3 \partial_s \bar{\hat{u}}_3) ds. \end{aligned} \quad (65)$$

Since $\omega \neq 0$, the norm $\|u_1; H^{1/2}(\gamma)\|$ is estimated by the norm of v in the space $H^1(\gamma)$ therefore we do not need a term with u_1 and $\hat{u}_1$ in (65), indeed. Let also

$$\langle v, p, u; \hat{v}, \hat{p}, \hat{u} \rangle_1 = \langle v, u; \hat{v}, \hat{u} \rangle_0 + \int_{\Omega} p \bar{\hat{p}} dx$$

be the inner product in $Z \times L^2(\Omega)$.

We introduce also a sesqui-linear form corresponding to the formulation (56), (57), (59), (60):

$$\begin{aligned} \hat{\mathcal{A}}(v, p, u; \hat{v}, \hat{p}, \hat{u}; \lambda) &= \int_{\Omega} (i\omega v \bar{\hat{v}} + 2\nu \sum \varepsilon_{ij}(v; \lambda) \overline{\varepsilon_{ij}(\hat{v}; -\bar{\lambda})}) dx \\ &\quad + \int_{\Omega} p \overline{(\nabla_y \cdot \hat{v}' - \bar{\lambda} \hat{v}_3)} dx + \int_{\Omega} (\nabla_y \cdot v' + \lambda v_3) \bar{\hat{p}} dx \\ &\quad + i\omega (-\omega^2 \int_{\gamma} u \bar{\hat{u}} + a(u, \hat{u}; \lambda) + k \int_{\gamma} u_1 \bar{\hat{u}}_1 ds). \end{aligned}$$

Clearly, this form is bounded in $Z \times L^2(\Omega)$. For $\mathcal{F} \in Z^*$, $\mathcal{H} \in L^2(\Omega)$ and $h \in L^2(\Omega)$ the weak formulation reads as the integral identity

$$\mathcal{A}(v, p, u; \hat{v}, \hat{p}, \hat{u}; \lambda) = \mathcal{F}(\hat{v}, \hat{u}) + \int_{\Omega} \mathcal{H} \bar{\hat{p}} dy \quad (66)$$

which has to be valid for all $(\hat{v}, \hat{p}, \hat{u}) \in Z \times L^2(\Omega)$ and $\nabla_y \cdot v' + \lambda v_3 = h$ in Ω , where $v' = (v_1, v_2)$.

If $\nabla_y \cdot v' + \lambda v_3 = 0$, it is enough to require that

$$\hat{\mathcal{A}}(v, p, u; \hat{v}, 0, \hat{u}; \lambda) = \mathcal{F}(\hat{v}, \hat{u}) \quad (67)$$

for all $(\hat{v}, \hat{u}) \in Z$.

It will be useful to introduce the operator pencil in the space $\mathcal{Z} \times L^2(\Omega)$ depending on the parameter $\lambda \in \mathbb{C}$ by

$$\widehat{\mathcal{A}}(v, p, u; \hat{v}, \hat{p}, \hat{u}; \lambda) = \langle \Theta(\lambda)(v, u, p); \hat{v}, \hat{p}, \hat{u} \rangle_1. \quad (68)$$

Clearly

$$\Theta(\lambda) : Z \times L^2(\Omega) \mapsto Z \times L^2(\Omega) \quad (69)$$

is a bounded operator pencil, quadratic with respect to λ .

4.5. Properties of the Operator Pencil Θ

We will need the following known lemma on the divergence equation

$$\partial_{y_1} v_1 + \partial_{y_2} v_2 = h \text{ in } \Omega. \quad (70)$$

Lemma 4.1. *There exists a linear operator*

$$L^2(\Omega) \ni h \rightarrow v' = (v_1, v_2) \in (H^1(\Omega))^2$$

such that the equation (70) is satisfied, $v_s = 0$ on γ , $v'|_\gamma \in H^1(\gamma)$ and

$$\|v'; H^1(\Omega)\| + \|v_n|_\gamma; H^1(\gamma)\| \leq C\|h\|_0. \quad (71)$$

Clearly the vector function (v, u) belongs to Z , where $v = (v_1, v_2, 0)$ and $i\omega u = v|_\gamma$. Estimate (71) implies

$$\|(v, u); Z\| \leq C\|h\|_0. \quad (72)$$

Proof. We represent h as

$$h(y) = \check{h}(y) + \tilde{h}, \quad \tilde{h} = \frac{1}{|\Omega|} \int_{\Omega} h(x) dx.$$

Then $v = \check{v} + \tilde{v}$, where $\check{v} \in H_0^1(\Omega)$ solves the problem $\nabla_y \cdot \check{v} = \check{h}$ in Ω and $\tilde{v}$ is a solution to

$$\nabla_y \cdot \tilde{v} = \tilde{h} \text{ and } \tilde{v}_n = \tilde{c} = \frac{|\partial\Omega|}{|\Omega|} \tilde{h}.$$

Both mappings $\check{h} \mapsto \check{v}$ and $\tilde{h} \mapsto \tilde{v}$ can be chosen linear and satisfying

$$\|\check{v}; H^1(\Omega)\| \leq C\|\check{h}; L^2(\Omega)\| \text{ and } \|\tilde{v}; H^2(\Omega)\| \leq C\|\tilde{h}; H^1(\Omega)\|$$

respectively. This implies the required assertion. $\square$

Lemma 4.2. *Let $\omega \in \mathbb{R}$ and $\omega \neq 0$. Then the operator pencil $\mathbb{C} \ni \lambda \mapsto \Theta(\lambda)$ possesses the following properties:*

- (i) $\Theta(\lambda)$ is a Fredholm operator for all $\lambda \in \mathbb{C}$ and its spectrum consists of isolated eigenvalues of finite algebraic multiplicity. The line $\Re \lambda = 0$ is free of the eigenvalues of Θ .
- (ii) Let $\lambda = i\xi$, $\xi \in \mathbb{R}$. Then there exists a positive constant $\rho(|\omega|)$ which may depend on $|\omega|$ such that the solution of problem (67) with $\mathcal{F} = (f, g) \in L^2(\Omega) \times L^2(\gamma)$ and $|\xi| \geq \rho(|\omega|)$ satisfies the estimate

$$\begin{aligned} & |\xi|^2 \int_{\gamma} (|\xi|^2(|u_2|^2 + |u_3|^2) + |\partial_s u_2|^2 + |\partial_s u_3|^2) ds + \int_{\Omega} |p|^2 dy \\ & + |\xi|^2 \int_{\Omega} (|\xi|^2 |v|^2 + |\nabla_y v|^2) dy \leq CN(f, g; \xi)^2, \end{aligned} \quad (73)$$

where

$$N(f, g; \xi) = \left(\|f\|_0 + \|g_2\|_0 + \|g_3\|_0 + |\xi|^{1/2} \|g_1\|_0 \right). \quad (74)$$

The constant c here may depend on ω but it is independent of ξ .

Proof. Let $\lambda = i\xi$ and

$$\mathcal{X} = \mathcal{X}(\lambda) = \{(v, u) \in \mathcal{Z}_{\omega} : \nabla \cdot v' + i\xi v_3 = 0\}.$$

- (i) Consider the integral identity

$$\mathcal{A}(v, 0, u; \hat{v}, \hat{u}; i\xi) = \mathcal{F}(\hat{v}, \hat{u}) \quad \forall (\hat{v}, \hat{u}) \in \mathcal{X}. \quad (75)$$

We want to apply the Lax-Milgram lemma to find solution $(v, u) \in \mathcal{X}$. First, we note that

$$\Re \mathcal{A}(v, 0, u; v, u; i\xi) \geq c(E(v, v) + |\lambda|^2 \int_{\Omega} |v_3|^2 dy + \int_{\Omega} |\lambda v' + \nabla_y v_3|^2 dy) \quad (76)$$

and

$$\begin{aligned} |\Im \mathcal{A}(v, 0, u; v, u; i\xi)| & \leq c|\omega| \left(\int_{\Omega} |v|^2 dy + \int_{\gamma} (k|u_1|^2 - \omega^2 |u|^2) ds \right. \\ & \left. + \int_{\gamma} (|\xi|^2 |u_3|^2 + |\kappa u_1 + \partial_s u_2|^2 + |i\xi u_2 + \partial_s u_3|^2) ds \right), \end{aligned} \quad (77)$$

where the constant c does not depend on ω and ξ .

We use the representation

$$\begin{aligned} |i\xi v + \nabla_y v_3|^2 & = (1 - \tau)|i\xi v + \nabla_y v_3|^2 + \tau|\xi|^2 |v|^2 + \tau|\nabla_y v_3|^2 \\ & + 2\tau \Re(i\xi v \cdot \nabla_y \overline{v_3}), \end{aligned} \quad (78)$$

where $\tau \in (0, 1]$. Let us estimate the last term in (78). We have

$$\begin{aligned} \int_{\Omega} v \cdot \nabla_y \overline{v_3} dx & = - \int_{\Omega} \nabla_y \cdot v \overline{v_3} dx + \int_{\partial\Omega} v_n \overline{v_3} ds \\ & = - \int_{\Omega} (\hat{\varepsilon}_{11}(v) + \hat{\varepsilon}_{22}(v)) \overline{v_3} dx + |\omega|^2 \int_{\partial\Omega} u_n \overline{u_3} ds. \end{aligned}$$

Since

$$\begin{aligned} \left| \xi \int_{\Omega} (\hat{\varepsilon}_{11}(v) + \hat{\varepsilon}_{22}(v)) \overline{v_3} dx \right| &\leq \frac{1}{2} |\xi|^2 \int_{\Omega} |v_3|^2 dx \\ &+ \int_{\Omega} (|\hat{\varepsilon}_{11}(v)|^2 + |\hat{\varepsilon}_{22}(v)|^2) dy, \end{aligned} \quad (79)$$

we derive that

$$2\tau |\Re(i\xi v \cdot \nabla_y \overline{v_3})| \leq C_{\omega} \tau \left(\int_{\gamma} (|\xi|^2 |u_3|^2 + |u|^2) ds + |\xi|^2 \int_{\Omega} |v_3|^2 dx + I(v, v) \right)$$

Using above inequalities together with (64), we arrive at the estimate

$$\begin{aligned} |\mathcal{A}(v, 0, u; v, u; i\xi)| &\geq C_{\omega} \left(\int_{\Omega} (|\xi|^2 |v|^2 + |\nabla_y v|^2) dy \right. \\ &\left. + \int_{\gamma} (|\xi|^2 |u_3|^2 + |\partial_s u_2|^2 + |i\xi u_2 + \partial_s u_3|^2) ds \right), \end{aligned} \quad (80)$$

where C_{ω} is a positive constant which may depend on ω and $|\xi|$ is chosen to be sufficiently large with respect to $|\omega| + 1$. On the basis of

$$\int_{\gamma} |i\xi u_2 + \partial_s u_3|^2 ds = \int_{\gamma} (|\xi u_2|^2 + |\partial_s u_3|^2 - 2\Re(i\xi \partial_s u_2 \bar{u}_3)) ds$$

one can continue the estimation in (80) as follows:

$$\begin{aligned} |\mathcal{A}(v, 0, u; v, u; i\xi)| &\geq C_{\omega} \left(\int_{\Omega} (|\xi|^2 |v|^2 + |\nabla_y v|^2) dy \right. \\ &\left. + \int_{\gamma} (|\xi|^2 |u_3|^2 + |\partial_s u_2|^2 + \xi^2 |u_2|^2 + |\partial_s u_3|^2) ds \right), \end{aligned} \quad (81)$$

with possibly another constant C_{ω} . Application of the Lax-Milgram lemma gives existence of a unique solution in $\mathcal{X}$ and the following estimate for this solution

$$\begin{aligned} &\int_{\Omega} (|\xi|^2 |v|^2 + |\nabla_y v|^2) dy \\ &+ \int_{\partial\Omega} (|\xi|^2 |u_3|^2 + |\partial_s u_2|^2 + \xi^2 |u_2|^2 + |\partial_s u_3|^2) ds \leq C \|\mathcal{F}; \mathcal{Z}^*\|^2 \end{aligned} \quad (82)$$

with a constant C which may depend on ω and ξ . It remains to estimate the function p . We chose the test function $(\hat{v}, \hat{u})$ in the following way: $i\omega \hat{u} = v$ on $\partial\Omega$, $\hat{v}_3 = 0$ and $\hat{v}' \in H^1(\Omega)$ solves the problem

$$\nabla_y \cdot \hat{v}' = \hat{h} \quad \text{in } \Omega$$

where $h \in L^2(\Omega)$. According to Lemma 4.1 the mapping $h \mapsto \hat{v}'$ can be chosen linear and satisfying the estimate (71). The pressure p must satisfy the relation

$$\begin{aligned} \int_{\Omega} p \bar{\hat{h}} dy &= \mathcal{F}(\hat{v}, \hat{u}) - \int_{\Omega} (i\omega v \bar{\hat{v}} + 2\nu \sum \varepsilon_{ij}(v) \overline{\varepsilon_{ij}(\hat{v})}) dx \\ &- i\omega (\omega^2 \int_{\partial\Omega} u \bar{\hat{u}} + a(u, \hat{u}; \lambda)). \end{aligned} \quad (83)$$

One can verify using (82) that the right-hand side of (83) is a linear bounded functional with respect to $h \in L^2(\Omega)$ and therefore there exists $p \in L^2(\Omega)$ solving (83) and estimated by the corresponding norm of $\mathcal{F}$. Thus the operator pencil (69) is isomorphism for large $|\xi|$.

Since the operator $\Theta(\lambda_1) - \Theta(\lambda_2)$ is compact we obtain that the spectrum of the operator pencil Θ consists of isolated eigenvalues of finite algebraic multiplicities, see [8].

Let us show that the kernel of $\Theta(i\xi)$ is trivial for all $\xi \in \mathbb{R}$. Indeed, if $(v, u, p) \in \mathcal{Z}_\omega \times L_2(\Omega)$ is a weak solution with $\mathcal{F} = 0$ then in the case $\xi \neq 0$ inequality (76) implies $v = 0$ and hence $u = 0$ because $i\omega u = v$ on γ . From (68) it follows that

$$\int_{\Omega} p(\nabla_y \cdot \hat{v}' + i\xi \hat{v}_3) dy = 0.$$

By Lemma 4.1 there exists the element $(v_1, v_2, 0) \in \mathcal{X}$ solving $\nabla_y \cdot \hat{v}' = p$ which gives $p = 0$. In the case $\lambda = 0$ we derive from (76) $v_3 = c_3$ and $(v_1, v_2) = (c_1, c_2) + c_0(y, -x)$, where $c_0, \dots, c_3$ are constant. From (33) it follows that

$$v_3 = 0 \text{ and } i\omega v_j + \partial_{y_j} p = 0, \quad j = 1, 2.$$

Since the vector v is a rigid displacement, we have $Du = 0$ due to (10) and (60), where D is the same matrix operator as in (9). Hence relation (59) implies

$$-\rho(s)\omega^2 u(s) + Ku(s) = \sigma(p, 0, 0)^T. \quad (84)$$

Therefore, $u_2 = 0$ and $-\rho(s)\omega^2 u_1 + ku_1 = \sigma p$. By (60) and $u_2 = 0$, we have

$$c_1 \zeta_1' + c_2 \zeta_2' + c_0(\zeta_2 \zeta_1' - \zeta_1 \zeta_2') = 0. \quad (85)$$

Let us show that this equality implies $c_1 = c_2 = c_3 = 0$. Indeed, it is sufficient to prove (85) for real c_0, c_1 and c_2 . Assume that $c_0 = 0$. Then $c_1 \zeta_1(s) + c_2 \zeta_2(s) = c$ and hence the boundary γ belongs to the line or $c_1 = c_2 = 0$. Since the first option is impossible we obtain that both constants are zero.

So it is sufficient to prove that $c_0 = 0$. Assume that it is not. Moving the origin in the (y_1, y_2) plane (replacing ζ_k by $\zeta_k + \alpha_k$, $k = 1, 2$), we arrive at the equation

$$(\zeta_2(s)\zeta_1'(s) - \zeta_1(s)\zeta_2'(s)) = 0 \text{ for } s \in (0, |\gamma|).$$

The last relation means that at each point (y_1, y_2) on γ the corresponding vector is orthogonal to the normal to this curve at the same point, what is impossible. Thus $c_1 = c_2 = c_3 = 0$ and hence $v = 0$. This leads to $u = 0$ and by (84) $p = 0$. Thus, the assertion (i) is proved

(ii) Let

$$\mathcal{F}(\hat{v}, \hat{u}) = \int_{\Omega} f \cdot \bar{\hat{v}} dy + \int_{\gamma} g \cdot \bar{\hat{u}} ds, \quad f \in L_2(\Omega), \quad g \in L_2(\gamma).$$

Then

$$\begin{aligned} |\mathcal{F}(v, u)| &\leq \left(\|f\|_0 + \|g_2\|_0 + \|g_3\|_0 + |\xi|^{1/2} \|g_1\|_0 \right) \\ &\quad \times \left(\|v\|_0 + \|u_2\|_0 + \|u_3\|_0 + |\xi|^{-1/2} \|u_1\|_0 \right). \end{aligned}$$

Using (64) and (829), we get

$$|\xi|^2 \left(\int_{\Omega} |v|^2 dy + \int_{\gamma} (|u_2|^2 + |u_3|^2 + |\xi|^{-1} |u_1|^2) ds \right) \leq C |\mathcal{F}(v, u)|.$$

Therefore,

$$\begin{aligned} & |\xi|^4 \left(\int_{\Omega} |v|^2 dy + \int_{\gamma} (|u_2|^2 + |u_3|^2 + |\xi|^{-1} |u_1|^2) ds \right) \\ & \leq C \left(\|f\|_0^2 + \|g_2\|_0^2 + \|g_3\|_0^2 + |\xi| \|g_1\|_0^2 \right). \end{aligned} \quad (86)$$

Furthermore,

$$\begin{aligned} & \int_{\Omega} |\nabla_y v|^2 dy + \int_{\gamma} (|\partial_s u_2|^2 + |\partial_s u_3|^2) ds \leq C |\mathcal{F}(v, u)| \\ & \leq C \left(\|f\|_0 + \|g_2\|_0 + \|g_3\|_0 + |\xi|^{1/2} \|g_1\|_0 \right) \\ & \quad \times \left(\|v\|_0 + \|u_2\|_0 + \|u_3\|_0 + |\xi|^{-1/2} \|u_1\|_0 \right) \\ & \leq C |\xi|^{-2} \left(\|f\|_0 + \|g_2\|_0 + \|g_3\|_0 + |\xi|^{1/2} \|g_1\|_0 \right)^2 \end{aligned}$$

where we have used (86). The last inequality together with (86) delivers (73) for the vector functions v and u .

To obtain the estimate for p , we proceed as in (i). □

4.6. Solvability of the Problem (52)–(55)

Proposition 4.2. *Let $\omega \in \mathbb{R}$ and $\omega \neq 0$. There exist a positive number β^* depending on ω , Ω and ν such that the following assertions hold:*

- (i) *The strip $|\Re \lambda| < \beta^*$ is free of eigenvalues of the operator pencil Θ .*
- (ii) *For $\Re \lambda = \beta \in (-\beta^*, \beta^*)$ the estimate*

$$\begin{aligned} & (|\lambda|^2 + 1) \int_{\gamma} ((|\lambda|^2 + 1)(|u_2|^2 + |u_3|^2) + |\partial_s u_2|^2 + |\partial_s u_3|^2) ds \\ & + \int_{\Omega} |p|^2 dy + (|\lambda|^2 + 1) \int_{\Omega} ((|\lambda|^2 + 1)|v|^2 + |\nabla_y v|^2) dy \leq CN(f, g; \xi)^2 \end{aligned} \quad (87)$$

is valid, where N is given by (74). The positive constant C here may depend on β , ω , ν and Ω .

Proof. Let $\lambda = \beta + i\xi$. It is straightforward to verify that

$$\Theta(\lambda)(v, u, p) - \Theta(i\xi)(v, u, p) = \beta(A, B, C)^T$$

where

$$\begin{aligned} A &= -\nu(\beta + 2i\xi)v + (0, 0, p); \quad B = v_3; \\ C &= \frac{\nu\sigma}{i\omega}(u_1, 0, 0) - (\beta + 2i\xi)D_0^T Q D_0 u + (D_1(-\partial_s)^T Q D_0 + D_0^T Q D(\partial_s))u. \end{aligned}$$

Therefore,

$$\Theta(\lambda) - \Theta(i\xi) : Z \times L_2(\Omega) \rightarrow L_2(\Omega)^3 \times L_2(\gamma)^3 \times L_2(\Omega)$$

is a small operator for small β . Therefore the estimate (87) for large $|\xi|$ follows from (73). From Lemma 4.2(i) it follows that this can be extended on $\xi \in \mathbb{R}$ if β is chosen sufficiently small. Thus we arrive at both assertions of the lemma. $\square$

4.7. Proof of Proposition 4.1 and Theorem 4.1

Proof. The assertion (i) in Proposition 4.1 is obtained from Proposition 4.2(ii) by using the inverse Fourier transform together with Parseval's identity.

To conclude with (ii) we observe that the same proposition provides

$$(\mathbf{V}_\pm, \mathbf{P}_\pm) = \frac{1}{2\pi i} \int_{\Re \lambda = \pm \beta} e^{-\lambda z} \Theta^{-1}(\lambda)(f(y, \lambda), g(s, \lambda), 0) d\lambda$$

and the assertion (ii) in Proposition 4.1 is obtained by applying the residue theorem. $\square$

Proof of Theorem 4.1. Let $(\mathbf{V}, \mathbf{U}, \mathbf{P})$ be a solution to (48)–(50) satisfying (51). Our first step is to construct a representation of the solution in the form

$$(\mathbf{V}, \mathbf{U}, \mathbf{P}) = (\mathbf{V}^{(+)}, \mathbf{U}^{(+)}, \mathbf{P}^{(+)}) + (\mathbf{V}^{(-)}, \mathbf{U}^{(-)}, \mathbf{P}^{(-)}), \quad (88)$$

where $(\mathbf{V}^{(\pm)}, \mathbf{U}^{(\pm)}) \in \mathcal{Z}_{\pm\beta}(\mathcal{C})$, $\mathbf{P}^{(\pm)} \in L_{\pm\beta}^2(\mathcal{C})$ and they solve the problem (48)–(50) with certain $(\mathbf{F}, \mathbf{G}) = (\mathbf{F}^{(\pm)}, \mathbf{G}^{(\pm)})$ such that $\mathbf{F}^{(\pm)} \in L_{\pm\beta}^2(\mathcal{C})$ and $\mathbf{G}^{(\pm)}, \partial_z \mathbf{G}^{(\pm)} \in L_{\pm\beta}^2(\Gamma)$.

Let ζ be the same cut-off function as in the proof of Theorem 3.1. We choose in (88)

$$\begin{aligned} (\mathbf{V}^{(+)}, \mathbf{U}^{(+)}, \mathbf{P}^{(+)}) &= \zeta(\mathbf{V}, \mathbf{U}, \mathbf{P}) + \zeta'(\tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \tilde{\mathbf{P}}), \\ (\mathbf{V}^{(-)}, \mathbf{U}^{(-)}, \mathbf{P}^{(-)}) &= (1 - \zeta)(\mathbf{V}, \mathbf{U}, \mathbf{P}) - \zeta'(\tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \tilde{\mathbf{P}}), \end{aligned} \quad (89)$$

where the vector function $(\tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \tilde{\mathbf{P}})$ solves the problem (66) for $\lambda = 0$ and with $\mathcal{F} = 0$, $\mathcal{H} = 0$ and $h = \mathbf{V}_3(y, z)$ i.e.

$$\partial_{y_1} \mathbf{W}_1 + \partial_{y_2} \mathbf{W}_2 = \mathbf{V}_3(y, z) \quad \text{for } y \in \Omega. \quad (90)$$

In this problem the variable z is considered as a parameter. In order to apply Lemma 4.2(i) we reduce the above formulation to the case $h = 0$. Applying for this purpose Lemma 4.1 we find a function $\mathcal{V}(y) = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)(y)$ solving (90) and satisfying (71) or (72) where $i\omega\mathcal{U} = \mathcal{V}$. The function $(\mathcal{V}, \mathcal{U}) \in Z$ and the above formulation is reduced to the case $h = 0$ but with some nonzero $\mathcal{F}$. Applying to the new problem Lemma 4.2(i) we find solution satisfying

$$\|(\tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \tilde{\mathbf{P}}); Z \times L^2(\Omega)\| \leq C\|\mathbf{V}_3\|_0,$$

which depends on the parameter z . Since

$$\mathbf{V}_2, \partial_z \mathbf{V}_3, \partial_z^2 \mathbf{V}_3 \in L_{\text{loc}}^2(\mathbb{R}; L_2(\Omega)),$$

we get that

$$\begin{aligned} (\tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \tilde{\mathbf{P}}), \partial_z(\tilde{\mathbf{V}}, \tilde{\mathbf{U}}, \tilde{\mathbf{P}}) &\in L_{\text{loc}}^2(\mathbb{R}; Z \times L_2(\Omega)) \quad \text{and} \\ \partial_z^2(\tilde{\mathbf{V}}, \tilde{\mathbf{U}}) &\in L_{\text{loc}}^2(\mathbb{R}; L_2(\Omega) \times L_2(\gamma)). \end{aligned}$$

Now one can verify that the vector functions (89) satisfy (48)–(50) with certain right-hand sides $(\mathbf{F}^{(\pm)}, \mathbf{G}^{(\pm)})$ having compact supports. Moreover $\mathbf{F}^{(+)} = -\mathbf{F}^{(-)}$ and $\mathbf{G}^{(+)} = -\mathbf{G}^{(-)}$ and

$$\mathbf{F}^{(\pm)} \in L^2_{\pm\beta}(\mathcal{C}), \quad \mathbf{G}^{(\pm)}, \partial_z \mathbf{G}^{(\pm)} \in L^2_{\pm\beta}(\Gamma).$$

Applying Theorem 4.1(ii) we get

$$(\mathbf{V}^{(+)}, \mathbf{U}^{(+)}, \mathbf{P}^{(+)}) = -(\mathbf{V}^{(-)}, \mathbf{U}^{(-)}, \mathbf{P}^{(-)}),$$

which means that $(\mathbf{V}, \mathbf{U}, \mathbf{P}) = 0$. Theorem 4.1 is proved. $\square$

4.8. The Case $\omega = 0$, Homogeneous System

If $\omega = 0$ the system (3)–(11) becomes

$$-\nu \Delta \mathbf{v} + \nabla p = 0 \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \mathcal{C} \times \mathbb{R}. \quad (91)$$

$$v = 0 \quad \text{on } \Gamma \quad (92)$$

and

$$\begin{aligned} D(\kappa(s), -\partial_s, -\partial_z)^T \overline{Q}(s) D(\kappa(s), \partial_s, \partial_z) \mathbf{u}(s, z) \\ + k(s) \mathbf{u}_1(s, z) + \sigma(s) \mathcal{F}(s, z) = 0, \end{aligned} \quad (93)$$

where $\mathcal{F}$ is given by (7). So we see that the system becomes uncoupled with respect to $(\mathbf{v}, p)$ and u . Solutions to (91)–(92) are given by

$$\mathbf{v}_1 = \mathbf{v}_2 = 0, \quad p = p_0 z + p_1 \quad \text{and} \quad \mathbf{v}_3 = p_0 \mathbf{v}_*(y),$$

where p_0, p_1 are constants and $\mathbf{v}_*$ solves the problem (18). In this case the vector function $\mathcal{F}$ is evaluated as

$$\mathcal{F}_1 = \mathcal{F}_n = -(p_0 z + p_1), \quad \mathcal{F}_2 = \mathcal{F}_s = 0 \quad \text{and} \quad \mathcal{F}_3 = \mathcal{F}_z = \nu \partial_n \mathbf{v}_3.$$

Let

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \quad (94)$$

First we consider the case $p_0 = p_1 = 0$. Namely, we want to solve the homogeneous equation

$$(-D_0 \partial_z + D_1(-\partial_s))^T Q(s) (D_0 \partial_z + D_1(\partial_s)) \mathbf{U} + k \mathbf{U}_1(s) = 0, \quad (95)$$

where D_0 and D_1 are defined by (5). First, we are looking for solution independent of z . Then it must satisfy

$$\kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2 = 0, \quad \partial_s \mathbf{U}_3 = 0 \quad \text{and} \quad \mathbf{U}_1 = 0.$$

Thus

$$\mathbf{U}_1 = 0, \quad \mathbf{U}_2 = c_2, \quad \mathbf{U}_3 = c_3,$$

where c_2 and c_3 are constants.

Next let $\mathbf{U}$ be a linear function in z , i.e. $\mathbf{U}(s, z) = z \mathbf{u}^0(s) + \mathbf{u}^1(s)$. Then

$$D_1(-\partial_s)^T Q(s) D_1(\partial_s) \mathbf{u}^0 + K \mathbf{u}^0 = 0,$$

and

$$\begin{aligned} D_1(-\partial_s)^T Q(s) D_1(\partial_s) \mathbf{u}^1 + K \mathbf{u}^1 \\ + \left(D_1(-\partial_s)^T Q(s) D_0 - D_0^T Q(s) D_1(\partial_s) \right) \mathbf{u}^0 = 0. \end{aligned} \quad (96)$$

Since $\mathbf{u}^0 = (0, \alpha, \beta)^T$, α and β are constant, equation (96) takes the form

$$D_1(-\partial_s)^T Q(s) (D_1(\partial_s) \mathbf{u}^1 + D_0 \mathbf{u}^0) + K \mathbf{u}^1 = 0 \quad (97)$$

and it is solvable since the term containing $\mathbf{u}^0$ is orthogonal to constant vectors $(0, a_1, a_2)$. Thus there exists linear in z solutions. Let us find these solutions. We have

$$Q(D_1(\partial_s)\mathbf{u}^1 + D_0\mathbf{u}^0) = (A, B, C)^T, \quad (98)$$

where

$$\begin{aligned} A &= Q_{11}(\kappa\mathbf{u}_1^1 + \partial_s\mathbf{u}_2^1) + Q_{13}\frac{1}{\sqrt{2}}\partial_s\mathbf{u}_3^1 + Q_{12}\beta + \frac{1}{\sqrt{2}}Q_{13}\alpha, \\ B &= Q_{21}(\kappa\mathbf{u}_1^1 + \partial_s\mathbf{u}_2^1) + Q_{23}\frac{1}{\sqrt{2}}\partial_s\mathbf{u}_3^1 + Q_{22}\beta + \frac{1}{\sqrt{2}}Q_{23}\alpha, \\ C &= Q_{31}(\kappa\mathbf{u}_1^1 + \partial_s\mathbf{u}_2^1) + Q_{33}\frac{1}{\sqrt{2}}\partial_s\mathbf{u}_3^1 + Q_{32}\beta + \frac{1}{\sqrt{2}}Q_{33}\alpha. \end{aligned}$$

Now system (97) takes the form

$$\kappa A + k\mathbf{u}_1^1 = 0, \quad \partial_s A = 0, \quad \partial_s C = 0.$$

This implies

$$A = b_1, \quad C = b_2, \quad \kappa b_1 + k\mathbf{u}_1^1 = 0,$$

where b_1 and b_2 are constants. Therefore

$$\mathbf{u}_1^1 = -\frac{\kappa b_1}{k} \quad (99)$$

and

$$(\kappa\mathbf{u}_1^1 + \partial_s\mathbf{u}_2^1, \frac{1}{\sqrt{2}}\partial_s\mathbf{u}_3^1)^T = \mathcal{R}(s)(b_1, b_2)^T - \mathcal{R}(s)\mathcal{S}(\beta, \frac{1}{\sqrt{2}}\alpha)^T, \quad (100)$$

where

$$\mathcal{R}(s) = \begin{pmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q_{33} \end{pmatrix}^{-1}, \quad \mathcal{S} = \begin{pmatrix} Q_{12} & Q_{13} \\ Q_{32} & Q_{33} \end{pmatrix}.$$

Using (99) we can write the compatibility condition for (100) as

$$\int_0^{|\gamma|} \mathcal{R}(s)ds(b_1, b_2)^T + \int_0^{|\gamma|} \frac{\kappa^2}{k}ds(b_1, 0)^T = \int_0^{|\gamma|} \mathcal{R}(s)\mathcal{S}ds\left(\beta, \frac{1}{\sqrt{2}}\alpha\right)^T. \quad (101)$$

Since $\mathcal{R}$ is a positive definite matrix, this system is uniquely solvable with respect to (b_1, b_2) .

Next let us look for solution to (95) in the form

$$\mathbf{U} = \frac{1}{2}z^2\mathbf{u}^0 + z\mathbf{u}^1 + \mathbf{u}^2,$$

where $\mathbf{u}^0$ and $\mathbf{u}^1$ just constructed above vector functions. Then equation for $\mathbf{u}^2$ has the form

$$\begin{aligned} D_1(-\partial_s)^T Q(s)(D_1(\partial_s)\mathbf{u}^2 + D_0\mathbf{u}^1) + K\mathbf{u}^2 - D_0^T Q(s) \\ \times (D_1(\partial_s)\mathbf{u}^1 + QD_0\mathbf{u}^0) = 0. \end{aligned} \quad (102)$$

According to (101) solvability of this system is equivalent to

$$\int_0^{|\gamma|} Bds = 0 \quad \text{and} \quad \int_0^{|\gamma|} Cds = 0.$$

this means that $b_2 = 0$ and

$$\begin{aligned} \int_0^{|\gamma|} \left((Q_{22}, Q_{23}) - (Q_{21}, Q_{23})\mathcal{R}\mathcal{S} \right) ds \left(\beta, \frac{1}{\sqrt{2}}\alpha \right)^T \\ + \int_0^{|\gamma|} (Q_{21}, Q_{23})\mathcal{R}ds(b_1, b_2)^T = 0. \end{aligned} \quad (103)$$

Furthermore, (b_2, b_1) and (α, β) are connected by (98). To simplify calculation we assume from now that

$$Q_{13} = Q_{23} = 0.$$

Then the matrixes $\mathcal{R}$ and $\mathcal{S}$ are diagonal and from (101) it follows that $b_2 = 0$ implies $\alpha = 0$ and

$$\int_0^{|\gamma|} Q_{11}^{-1} ds b_1 + \int_0^{|\gamma|} \frac{\kappa^2}{k} ds b_1 = \int_0^{|\gamma|} Q_{11}^{-1} Q_{12} ds \beta. \quad (104)$$

The relation (103) implies

$$\int_0^{|\gamma|} (Q_{22} - Q_{21} Q_{11}^{-1}) ds \beta + \int_0^{|\gamma|} Q_{21} Q_{11}^{-1} ds b_1 = 0.$$

This relation together with (104) requires that $\beta = 0$ and $b_1 = 0$.

If $\xi \neq 0$ then $\mathbf{U} = 0$. Consider the case $\xi = 0$ and let $p_0 = 0$. Then (93) takes the form

$$\begin{pmatrix} \kappa & 0 & 0 \\ -\partial_s & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \partial_s \end{pmatrix} Q \begin{pmatrix} \kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2 \\ 0 \\ \frac{1}{\sqrt{2}} \partial_s \mathbf{U}_3 \end{pmatrix} + K \mathbf{U} = -p_1 \sigma \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (105)$$

This is equivalent to the following three equations

$$\begin{aligned} \kappa \left(Q_{11} (\kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2) + Q_{13} \frac{1}{\sqrt{2}} \partial_s \mathbf{U}_3 \right) + k \mathbf{U}_1 &= -p_1 \sigma, \\ -\partial_s \left(Q_{11} (\kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2) + Q_{13} \frac{1}{\sqrt{2}} \partial_s \mathbf{U}_3 \right) &= 0, \\ \partial_s \left(Q_{31} (\kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2) + Q_{33} \frac{1}{\sqrt{2}} \partial_s \mathbf{U}_3 \right) &= 0 \end{aligned}$$

This implies

$$\begin{aligned} Q_{11} (\kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2) + Q_{13} \frac{1}{\sqrt{2}} \partial_s \mathbf{U}_3 &= b_1, \\ Q_{31} (\kappa \mathbf{U}_1 + \partial_s \mathbf{U}_2) + Q_{33} \frac{1}{\sqrt{2}} \partial_s \mathbf{U}_3 &= b_2, \end{aligned} \quad (106)$$

where b_1 and b_2 are constants, and

$$\kappa b_1 + k \mathbf{U}_1 = -p_1 \sigma.$$

Hence

$$\mathbf{U}_1 = -\frac{p_1 \sigma + \kappa b_1}{k}. \quad (107)$$

Solving the system

$$Q_{11} B_1 + Q_{13} B_2 = b_1, \quad Q_{31} B_1 + Q_{33} B_2 = b_2,$$

we get

$$(B_1, B_2)^T = \mathcal{R} (b_1, b_2)^T, \quad \mathcal{R} = \begin{pmatrix} Q_{11} & Q_{13} \\ Q_{31} & Q_{33} \end{pmatrix}^{-1}.$$

We write the equations (106) as

$$\partial_s \mathbf{U}_2 = B_1 - \kappa \mathbf{U}_1, \quad \partial_s \mathbf{U}_3 = \sqrt{2} B_2. \quad (108)$$

These equations has periodic solutions if

$$\int_0^{|\gamma|} (B_1 - \kappa \mathbf{U}_1) ds = 0, \quad \int_0^{|\gamma|} B_2 ds = 0.$$

We write these equations as a system with respect to b_1 and b_2

$$|\gamma|\mathcal{R}(b_1, b_2)^T + \int_0^{|\gamma|} \frac{\kappa^2}{k} ds (b_1, 0)^T = - \left(\int_0^{|\gamma|} \kappa \frac{p_1 \sigma}{k} ds, 0 \right)^T. \quad (109)$$

From these relations we can find b_1 and b_2 and then solving (108) we can find $\mathbf{U}_2$ and $\mathbf{U}_3$.

5. Conclusion

In this paper we consider the model describing the fluid flow in infinite cylinder with elastic wall, which takes into account also the interaction of the blood flow with surrounded tissue material. This model was introduced in our papers [10, 11] for a straight cylinder and in [7] for curvilinear cylinder. Our main goal is to show new properties of the fluid flow which appears due to elastic walls and interaction with surrounded tissue in the frame of this model. Having this in mind we have compared two boundary value problems in a cylinder describing periodic in time fluid flow. The first one is the Dirichlet problem and the second one is a boundary value problem for flexible vessel with elastic walls. The second problem was derived in [7, 11] by using asymptotic approach for a pipe with arbitrary cross-section bounded by several thin elastic layers. Periodic in time flows of the Dirichlet problem have only one non-trivial component in the velocity-the longitudinal one. This component can be found from the Dirichlet problem of the heat equation with a periodic flux depending only on time. Periodic solutions of the second boundary value problem for the flexible pipe with elastic walls have also only one non-zero component and it is longitudinal as well. It is also satisfies the heat equation but the right hand side is a constant flux. Thus in this case the longitudinal component of fluid velocity does not depend on time and therefore has the same form for all periods. This means that the pipe with elastic wall supports only classical Poiseuille solutions which depends only on the cross-section variable. This property of elastic wall is quite reasonable, it gives a certain stability of the flow since it does not depend on the period, which however is perturbed in the whole arterial tree of human's body due to numerous bifurcations and conical and curved shape of blood vessels. Moreover there is no local reverse flow in the case of elastic walls, which follows from our result.

Another interesting property is connected with the term in the boundary condition for elastic wall, which is proportional to the normal component of displacement. This term takes care on the surrounded tissue and it stabilizes the stresses and strains. One can see that without this term stresses and strains are essentially larger (see formulas (107) and (109)).

If we compare our model with the model for elastic tube in [21], Sect. III, then they are quite close to each other for circular cylinder. Vormesley's model does not take into account the reaction of the vessel wall to the radial (normal) component of the force acting on the wall from the side of the liquid. This is the reason that Vormesley's model admits periodic in time solutions of the form $e^{\omega t}v$ but our model does not have such solutions. We refer the reader to the end of the introduction where we discussed which factors influence on the laminar and oscillating parts of blood flow.

It should be emphasized that all models, even the Dirichlet one for absolutely rigid walls have certain sense because the state of the circulatory system changes with age and under influence of medications and diseases.

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Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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