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On bounds for steady waves with negative vorticity

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Abstract. We prove that no two-dimensional Stokes and solitary waves exist when the vorticity function is negative and the Bernoulli constant is greater than a certain critical value given explicitly. In particular, we obtain an upper bound $F \leq \sqrt{2} + \epsilon$ for the Froude number of solitary waves with a negative constant vorticity, sufficiently large in absolute value.

1. Introduction

We consider the classical water wave problem for two-dimensional steady waves with vorticity on water of finite depth. We neglect effects of the surface tension and consider a fluid of constant (unit) density. Thus, in an appropriate coordinate system moving along with the wave, the stationary Euler equations are given by

$$(u-c)u_x + vu_y = -P_x, \tag{1a}$$

$$(u-c)v_x + vv_y = -P_y - g, \tag{1b}$$

$$u_x + v_y = 0, \tag{1c}$$

and hold true in a two-dimensional fluid domain $D_\eta = \{(x,y) : x \in \mathbb{R},\ 0 < y < \eta(x)\}$. Here (u,v) are components of the velocity field, $y = \eta(x)$ is the surface profile, c is the wave speed, P is the pressure and g is the gravitational constant. The corresponding boundary conditions are

$$v = 0 \quad \text{on } y = 0, \tag{1d}$$

$$v = (u-c)\eta_x \quad \text{on } y = \eta, \tag{1e}$$

$$P = P_{\text{atm}} \quad \text{on } y = \eta. \tag{1f}$$

It is often assumed in the literature that the flow is irrotational, that is $v_x - u_y$ is zero everywhere in the fluid domain. Under this assumption the components of the velocity field are harmonic functions, which allows to apply methods of complex analysis. Being a convenient simplification it forbids modeling of non-uniform currents, commonly occurring in nature. In the present paper we will consider rotational flows, where the vorticity function is defined by

$$\omega = v_x - u_y. \tag{2}$$

Throughout the paper we assume that the flow is unidirectional, that is

$$u - c < 0 \tag{3}$$

everywhere in the fluid. This forbids the presence of stagnation points an gives an advantage of using the partial hodograph transform.

In the two-dimensional setup relation (1c) allows to reformulate the problem in terms of a stream function ψ, defined implicitly by the relations

$$\psi_y = c - u, \quad \psi_x = v.$$

This determines ψ up to an additive constant, while relations (1d), (1d) force ψ to be constant along the boundaries. Thus, by subtracting a suitable constant, we can always assume that

$$\psi = m, \quad y = \eta; \quad \psi = 0, \quad y = 0.$$

Here $m > 0$ is the mass flux, defined by $m = \int_0^\eta (c-u)dy$. In what follows we will use non-dimensional variables proposed by Keady & Norbury [9], where lengths and velocities are scaled by $(m^2/g)^{1/3}$ and $(mg)^{1/3}$ respectively; in new units $m = 1$ and $g = 1$. For simplicity we keep the same notations for η and ψ.

Taking the curl of Euler equations (1a)–(1c) one checks that the vorticity function ω defined by (2) is constant along paths tangent everywhere to the relative velocity field $(c-u, v)$; see [3] for more details. Having the same property by the definition, the stream function ψ is strictly monotone by (3) on every vertical interval inside the fluid region. These observations together show that ω depends only on values of the stream function, that is $\omega = \omega(\psi)$. This property and Bernoulli's law allow to express the pressure P as

$$P - P_{\text{atm}} + \frac{1}{2}|\nabla\psi|^2 + y + \Omega(\psi) - \Omega(1) = const, \tag{4}$$

where $\Omega(\psi) = \int_0^\psi \omega(p)\, dp$ is a primitive of the vorticity function $\omega(\psi)$. Thus, we can eliminate the pressure from equations and obtain the following problem:

$$\Delta\psi + \omega(\psi) = 0 \quad \text{for } 0 < y < \eta \tag{5a}$$

$$\tfrac{1}{2}|\nabla\psi|^2 + y = r \quad \text{on } y = \eta, \tag{5b}$$

$$\psi = 1 \quad \text{on } y = \eta, \tag{5c}$$

$$\psi = 0 \quad \text{on } y = 0, \tag{5d}$$

$$\psi_y > 0 \quad \text{for } 0 \le y \le \eta. \tag{5e}$$

Here $r > 0$ is referred to as Bernoulli's constant. As for the regularity we will assume that $\omega \in C^\gamma([0,1])$, $\psi \in C^{2,\gamma}(\overline{D_\eta})$ and $\eta \in C^{2,\gamma}(\mathbb{R})$ for some $\gamma \in (0,1)$ being fixed throughout the paper.

Beside the Bernoulli constant r, the water wave problem (5) admits another spatial constant of motion known as the flow force, given by

$$\mathcal{S} = \int_0^\eta (\psi_y^2 - \psi_x^2 - y + \Omega(1) - \Omega(\psi) + r)\, dy. \tag{6}$$

This constant is important in several ways; for instance, it plays the role of the Hamiltonian in spatial dynamics; see [1]. The flow force constant is also involved in a classification of steady motions; see [2].

In what follows we will consider Stokes waves, periodic solutions to (5) that are monotone between each neighbouring crest and trough and symmetric around every vertical line passing through crests and troughs; see [4,6] for related results on the symmetry of Stokes waves.

Now it is convenient to define a solitary wave as a Stokes wave with the infinite period. More precisely, each solitary wave (ψ, η) has symmetric profile (around the vertical line passing through the single crest), monotone on sides, and is subject to the asymptotic relation

$$\eta(x) \to d, \quad x \to \pm\infty, \tag{7}$$

where d is the depth of the limiting shear flow. Note that we do not require any decay properties for ψ_x, such as

$$\psi_x \to 0 \quad \text{as} \quad x \to \pm\infty \tag{8}$$

uniformly in y. In fact, (8) follows from our regularity assumptions and relation (5e). A short argumentation is that if (8) is false one can find a family of shifts of the solution (in partial hodograph transform variables) converging over compact subsets to a solution with flat surface. The latter solution must be a stream by the maximum principle, which leads to a contradiction.

Every solitary wave is known to be symmetric and supercritical; see [7,13]. It is worth mentioning that the depth d in (7) is necessarily the same at both infinities, so that no monotone fronts exist for the problem (5); this fact follows from a recent study [13].

The asymptotic depth in (7) supports a stream solution $U = U(y)$ solving (5) with the same Bernoulli constant r. It is essential that this stream and the solitary wave have the same flow force constant, which follows from (8) (by passing to the limit in (6)).

1.1. Stream solutions

Laminar flows or shear currents, for which the vertical component v of the velocity field is zero play an important role in the theory of steady waves. Let us recall some basic facts about stream solutions $\psi = U(y)$ and $\eta = d$, describing shear currents. It is convenient to parameterize the latter solutions by the relative speed at the bottom. Thus, we put $U_y(0) = s$ and find that $U = U(y; s)$ is subject to

$$U'' + \omega(U) = 0, \quad 0 < y < d; \quad U(0) = 0, \quad U(d) = 1. \tag{9}$$

Our assumption (3) implies $U' > 0$ on $[0; d]$, which puts a natural constraint on s. Indeed, multiplying the first equation in (9) by U' and integrating over $[0; y]$, we find $U'^2 = s^2 - 2\Omega(U)$. This shows that the expression $s^2 - 2\Omega(p)$ is positive for all $p \in [0; 1]$, which requires $s > s_0 = \sqrt{\max_{p\in[0,1]} 2\Omega(p)}$. On the other hand, every $s > s_0$ gives rise to a monotonically increasing function $U(y; s)$ solving (9) for some unique $d = d(s)$, given explicitly by

$$d(s) = \int_0^1 \frac{1}{\sqrt{s^2 - 2\Omega(p)}} dp.$$

This formula shows that $d(s)$ monotonically decreases to zero with respect to s and takes values between zero and

$$d_0 = \lim_{s\to s_0+} d(s).$$

The latter limit can be finite or not. For instance, when $\omega = 0$ we have $d(s) = 1/s$ and $s_0 = 0$, so that $d_0 = +\infty$. On the other hand, when $\omega = -b$ for some positive constant $b \neq 0$, then $s_0 = 0$ but $d_0 < +\infty$. We note that our main theorem is concerned with the case $d_0 < +\infty$.

Every stream solution $U(y; s)$ determines the Bernoulli constant $R(s)$, which can be found from the relation (5b). This constant can be computed explicitly as $R(s) = \frac{1}{2}s^2 - \Omega(1) + d(s)$. As a function of s it decreases from R_0 to R_c when s changes from s_0 to s_c and increases to infinity for $s > s_c$. Here the critical value s_c is determined by the relation

$$\int_0^1 \frac{1}{(s^2 - 2\Omega(p))^{3/2}} dp = 1.$$

The constants R_0 and R_c are of special importance for the theory. For example, it is proved in [11] that $r > R_c$ for any steady motion other than a laminar flow. In the present paper we will consider the water wave problem (5) for $r > R_0$, provided $R_0 < +\infty$. The latter is true, for instance, for a negative constant vorticity.

For any $r \in (R_c, R_0]$ there are exactly two solutions $s_-(r) < s_+(r)$ to the equation $R(s) = r$, while for $r > R_0$ one finds only one solution $s = s_+(r)$. The laminar flow corresponding to $s_-(r)$ is called subcritical and it's depth is denoted by $d_+(r) = d(s_-(r))$. The other flow, with $s = s_+(r)$ is called supercritical and it's depth is $d_-(r) = d(s_+(r))$. According to the definition, we have $d_-(r) < d_+(r)$. The flow force constants corresponding to flows with $d = d_\pm$ are denoted by $S_\pm(r)$.

It was recently proved in [13] that all solitary waves are supported by supercritical depths $d_-(r)$ and the corresponding flow force constant equals to $\mathcal{S}_-(r)$; here r is the Bernoulli constant of a solitary wave.

1.2. Formulations of main results.

Just as in [11] we split the set of all vorticity functions into three classes as follows: (i) $\max_{p\in[0,1]}\Omega(p)$ is attained either at an inner point of $(0,1)$ or at an end-point, where ω attains zero value; (ii) $\Omega(p)<0$ for all $p\in(0,1]$ and $\omega(0)\neq 0$; (iii) $\Omega(p)<\Omega(1)$ for all $p\in[0,1)$ (and so $\omega(1)\neq 0$). The first class can be characterized by relations $R_0=+\infty$ and $d_0=+\infty$, while $R_0,d_0<+\infty$ for all vorticity functions that belong to the second and third classes. Our main result states

Theorem 1.1. *Let $\omega\in C^\gamma([0,1])$ be such that $R_0<+\infty$. Then there exist no Stokes waves with $r\geq R_0-\Omega(1)$. Furthermore, there are no solitary waves with $r\geq R_0$.*

A part of the statement, when ω is subject to (iii) was proved in [11], where it was shown that no steady waves exist for $r\geq R_0$ (under condition (iii)). We note that there is no analogues statement for irrotational waves. A typical example of a vorticity function satisfying condition (ii) (for which $R_0<+\infty$) is a negative constant vorticity $\omega(p)=-b$, $b>0$. It is known (see [16]) that vorticity distributions of this type give rise to Stokes waves over flows with internal stagnation points, that exist for all Bernoulli constants $r>R_0$. Furthermore, a recent study [12] shows that there exist continuous families of such Stokes waves that approach a solitary wave in the long wavelength limit. The latter solitary wave has $r>R_0$ and rides a supercritical unidirectional flow (corresponding to one of stream solutions $U(y;s)$ with $s>s_c$) but has a near-bottom stagnation point on a vertical line passing through the crest. Thus, even though there are no unidirectional waves for $r>R_0$, there exist Stokes and solitary waves with $r>R_0$ violating assumption (3). These considerations show that the statement of Theorem 1.1 is sharp in a certain sense. On the other hand, inequality $r\geq R_0-\Omega(1)$ is not sharp and probably can be improved further. However it is not clear if one can omit completely the term $-\Omega(1)$ from the bound on the Bernoulli constant.

Inequality $r\leq R_0$ for solitary waves puts a natural upper bound for the Froude number

$$F^2(s)=\left(\int_0^d (U_y(y;s))^{-2}dy\right)^{-1}.$$

It is well known that for irrotational solitary waves $F<\sqrt{2}$; see [14], [10]. Furthermore, the bound $F<2$ for rotational waves with a negative vorticity was obtained in [17]. For small negative vorticity distributions inequality $1<F(s)<2$ is stronger than $R_c<R(s)<R_0$. However, already for $\omega(p)=-1$ the inequality $R(s)<R_0$ becomes stronger. For $\omega(p)=-b$ with a large $b>0$ we find that inequality $R_c<R(s)<R_0$ is equivalent to $1<F(s)<F(s_0)$, where $F(s_0)\to\sqrt{2}$ as $b\to+\infty$, which is significantly better than $F<2$.

2. Preliminaries

2.1. Reformulation of the problem

Under assumption (3) we can apply the partial hodograph transform introduced by Dubreil-Jacotin [5]. More precisely, we present new independent variables

$$q=x,\quad p=\psi(x,y),$$

while new unknown function $h(q,p)$ (height function) is defined from the identity

$$h(q,p)=y.$$

Note that it is related to the stream function ψ through the formulas

$$\psi_x=-\frac{h_q}{h_p},\quad \psi_y=\frac{1}{h_p}, \tag{10}$$

where $h_p > 0$ throughout the fluid domain by (3). An advantage of using new variables is in that instead of two unknown functions $\eta(x)$ and $\psi(x,y)$ with an unknown domain of definition, we have one function $h(q,p)$ defined in a fixed strip $S = \mathbb{R} \times [0,1]$. An equivalent problem for $h(q,p)$ is given by

$$\left(\frac{1+h_q^2}{2h_p^2} + \Omega\right)_p - \left(\frac{h_q}{h_p}\right)_q = 0 \quad \text{in } S, \tag{11a}$$

$$\frac{1+h_q^2}{2h_p^2} + h = r \quad \text{on } p = 1, \tag{11b}$$

$$h = 0 \quad \text{on } p = 0. \tag{11c}$$

The wave profile η becomes the boundary value of h on $p = 1$:

$$h(q,1) = \eta(q), \quad q \in \mathbb{R}.$$

Using (10) and Bernoulli's law (4) we recalculate the flow force constant $\mathcal{S}$ defined in (6) as

$$\mathcal{S} = \int_0^1 \left(\frac{1-h_q^2}{h_p^2} - h - \Omega + \Omega(1) + r\right) h_p \, dp. \tag{12}$$

Laminar flows defined by stream functions $U(y;s)$ correspond to height functions $h = H(p;s)$ that are independent of horizontal variable q. The corresponding equations are

$$\left(\frac{1}{2H_p^2} + \Omega\right)_p = 0, \quad H(0) = 0, \quad H(1) = d(s), \quad \frac{1}{2H_p^2(1)} + H(1) = R(s).$$

Solving equations for $H(p;s)$ explicitly, we find

$$H(p;s) = \int_0^p \frac{1}{\sqrt{s^2 - 2\Omega(\tau)}} \, d\tau.$$

Given a height function $h(q,p)$ and a stream solution $H(p;s)$, we define

$$w^{(s)}(q,p) = h(q,p) - H(p;s). \tag{13}$$

This notation will be frequently used in what follows. In order to derive an equation for $w^{(s)}$ we first write (11a) in a non-divergence form as

$$\frac{1+h_q^2}{h_p^2} h_{pp} - 2\frac{h_q}{h_p} h_{qp} + h_{qq} - \omega(p) h_p = 0.$$

Now using our ansatz (13), we find

$$\frac{1+h_q^2}{h_p^2} w_{pp}^{(s)} - 2\frac{h_q}{h_p} w_{qp}^{(s)} + w_{qq}^{(s)} - \omega(p) w_p^{(s)} + \frac{(w_q^{(s)})^2 H_{pp}}{h_p^2} - \frac{w_p^{(s)}(h_p + H_p) H_{pp}}{h_p^2 H_p^2} = 0. \tag{14}$$

Thus, $w^{(s)}$ solves a homogeneous elliptic equation in S and is subject to a maximum principle; see [15] for an elliptic maximum principle in unbounded domains. The boundary conditions for $w^{(s)}$ can be obtained directly from (11b) and (11c) by inserting (13) and using the corresponding equations for H. This gives

$$\frac{(w_q^{(s)})^2}{2h_p^2} - \frac{w_p^{(s)}(h_p + H_p)}{2h_p^2 H_p^2} + w^{(s)} = r - R(s) \quad \text{for } p = 1, \tag{15a}$$

$$w^{(s)} = 0 \quad \text{for } p = 0. \tag{15b}$$

Concerning the regularity, we will always assume that $\omega \in C^\gamma([0;1])$ and $h \in C^{2,\gamma}(\overline{S})$, where $C^{2,\gamma}(\overline{S})$ is the usual subspace of $C^2(\overline{S})$ (all partial derivatives up to the second order are bounded and continuous in $\overline{S}$) of functions with Hölder continuous second-order derivatives with a finite Hölder norm, calculated over the whole strip S. The exponent $\gamma \in (0;1)$ will be fixed throughout the paper.

2.2. Auxiliary function σ

For a given $r > R_c$ and $s > s_0$ we define

$$\sigma(s;r) = \int_0^1 \left(\frac{1}{2H_p^2(p;s)} - H(p;s) - \Omega(p) + \Omega(1) + r \right) H_p(p;s)\, dp. \tag{16}$$

This expression coincides with the flow force constant for $H(p;s)$, but with the Bernoulli constant $R(s)$ replaced by r. We note that $\sigma(s_\mp(r);r) = \mathcal{S}_\pm(r)$. The key property of $\sigma(s;r)$ is stated below.

Lemma 2.1. *For a given $r \geq R_0$ the function $s \mapsto \sigma(s;r)$ decreases for $s \in (s_0, s_+(r))$ and increases to infinity for $s \in (s_+(r), +\infty)$.*

Proof. Because $H_p(p;s) = \frac{1}{\sqrt{s^2 - 2\Omega(p)}}$ and $\partial_s H_p(p;s) = -sH_p^3(p;s)$, we can compute the derivative

$$\begin{aligned}
\sigma_s(s;r) &= \int_0^1 \left(\frac{1}{2H_p^2(p;s)} - H(p;s) - \Omega(p) + \Omega(1) + r \right) \partial_s H_p(p;s)\, dp \\
&\quad + \int_0^1 \left(-\frac{\partial_s H_p(p;s)}{H_p^3(p;s)} - \partial_s H(p;s) \right) H_p(p;s)\, dp \\
&= \int_0^1 \left(-\frac{1}{2H_p^2(p;s)} - \Omega(p) + \Omega(1) + r \right) \partial_s H_p(p;s)\, dp - d(s)d'(s) \\
&= \int_0^1 \left(-\tfrac{1}{2}s^2 + \Omega(1) + r \right) \partial_s H_p(p;s)\, dp - d(s) \int_0^1 \partial_s H_p(p;s) \\
&= -s(r - R(s)) \int_0^1 H_p^3(p;s)\, dp.
\end{aligned}$$

Finally, because $R(s) < r$ for $s_0 < s < s_+(r)$ and $R(s) > r$ for $s > s_+(r)$ we obtain the statement of the lemma. Note that since $R(s) = \frac{1}{2}s^2 + O(1)$ and $H_p(p;s) = \frac{1}{s} + O(\frac{1}{s^2})$ we have $\sigma_s(s;r) \sim 1$ as $s \to +\infty$. Therefore, we conclude that $\lim_{s\to+\infty} \sigma(s;r) = +\infty$. □

Our function $\sigma(s;r)$ and it's role is similar to the function $\sigma(h)$ introduced by Keady and Norbury in [8]. The main purpose of the latter is to be used for a comparison with the flow force constant $\mathcal{S}$.

2.3. Flow force flux functions

Our aim is to extract some information by comparing the flow force constant $\mathcal{S}$ (of a given solution with the Bernoulli constant $r \geq R_0$) to $\sigma(s;r)$ for different values of $s > s_0$. For this purpose we first compute the difference

$$\begin{aligned}
\mathcal{S} - \sigma(s;r) &= \int_0^1 \left(\frac{1 - (w_q^{(s)})^2}{2h_p^2} - w^{(s)} - \frac{1}{2H_p^2} \right) H_p\, dp \\
&\quad + \int_0^1 \left(\frac{1 - (w_q^{(s)})^2}{2h_p^2} - h - \Omega + \Omega(1) + r \right) w_p^{(s)}\, dp \\
&= \int_0^1 \left(\frac{(w_p^{(s)})^2}{2h_p H_p^2} - \frac{(w_q^{(s)})^2}{2h_p} - w^{(s)} H_p \right) dp \\
&\quad + \int_0^1 \left(-\frac{1}{2H_p^2} - w^{(s)} - H - \Omega + \Omega(1) + r \right) w_p^{(s)}\, dp.
\end{aligned}$$

Now using the identity

$$-\Omega(p) + \Omega(1) + R(s) = \frac{1}{2H_p^2} + H(1)$$

and integrating first-order terms, we conclude that

$$2(\mathcal{S}-\sigma(s;r))=2(r-R(s))w^{(s)}(q,1)-(w^{(s)}(q,1))^2+\int_0^1\left(\frac{(w_p^{(s)})^2}{h_pH_p^2}-\frac{(w_q^{(s)})^2}{h_p}\right)dp.$$

Let us define the (relative) flow force flux function $\Phi^{(s)}$ by setting

$$\Phi^{(s)}(q,p)=\int_0^p\left(\frac{(w_p^{(s)}(q,p'))^2}{h_p(q,p')(H_p(p';s))^2}-\frac{(w_q^{(s)}(q,p'))^2}{h_p(q,p')}\right)dp'. \tag{17}$$

An analog (partial case with $s=s_+(r)$) of this function was recently introduced in [13]. The same computation as in [13] gives

$$\Phi_q^{(s)}=-w_q^{(s)}\left(\frac{1+(w_q^{(s)})^2}{h_p^2}-\frac{1}{H_p^2}\right),\quad \Phi_p^{(s)}=\frac{(w_p^{(s)})^2}{h_pH_p^2}-\frac{(w_q^{(s)})^2}{h_p}. \tag{18}$$

A surprising fact about $\Phi^{(s)}$ is that it solves a homogeneous elliptic equation as stated in the next proposition.

Proposition 2.2. *There exist functions $b_1,b_2\in L^\infty(S)$ such that*

$$\frac{1+h_q^2}{h_p^2}\Phi_{pp}^{(s)}-2\frac{h_q}{h_p}\Phi_{qp}^{(s)}+\Phi_{qq}^{(s)}+b_1\Phi_q^{(s)}+b_2\Phi_p^{(s)}=0\quad in\quad S. \tag{19}$$

Furthermore, $\Phi^{(s)}$ satisfies the boundary conditions

$$\Phi^{(s)}=2(\mathcal{S}-\sigma(s;r))-2(r-R(s))w^{(s)}(q,1)+(w^{(s)}(q,1))^2\quad for\ p=1, \tag{20a}$$

$$\Phi^{(s)}=0\quad for\ p=0. \tag{20b}$$

In the irrotational case $b_1,b_2=0$ and (19) is equivalent to the Laplace equation in the original physical variables.

For the proof we refer to [13, Proposition 3.1], where a similar statement was proved for the special case $s=s_\pm(r)$. More precisely, it is shown that the function Φ defined by (17) with $s=s_\pm(r)$ solves a homogenous elliptic equation, which only requires the interior relation (11a) from the laminar stream H. Thus, if we replace $H(p;s_\pm)$ by an arbitrary stream solution $H(p;s)$ (still solving the same equation (11a)) the corresponding statement of [13, Proposition 3.1] remains true. Thus, to prove (20a) it is enough to repeat the argument in [13, Proposition 3.1] but for $H(p;s)$ instead of $H(p;s_\pm)$. On the other hand, the boundary relation (20a) is different from the one in [13, Proposition 3.1] and follows directly from the computation given above.

We also note that $\Phi^{(s)}\in C^{2,\gamma}(\overline{S})$, provided $h\in C^{2,\gamma}(\overline{S})$ and $\omega\in C^\gamma([0,1])$.

Proposition 2.3. *Let $h\in C^{2,\gamma}(\overline{S})$ be a solution to (11) with $r>R_c$. Assume that the flow force flux function $\Phi^{(s)}$ for some $s>s_0$ satisfies $\inf_{q\in\mathbb{R}}\Phi^{(s)}(q;1)\le 0$. Then*

$$\inf_{q\in\mathbb{R}}\Phi^{(s)}(q;1)=2(\mathcal{S}-\sigma(s;r))-(r-R(s))^2. \tag{21}$$

Proof. First, we assume that the infimum is attained at some point $(q_0;1)$, where $\Phi_q^{(s)}(q_0;1)=0$. Differentiating the boundary condition (20a), we find

$$\Phi_q^{(s)}(q_0,1)=2w_q^{(s)}(q_0,1)(w^{(s)}(q_0,1)-(r-R(s)))=0. \tag{22}$$

Because $\Phi^{(s)}$ attains it's global minimum at $(q_0,1)$, then the maximum principle and the Hopf lemma give $\Phi_p^{(s)}(q_0,1)<0$. In particular, we find that $w_q^{(s)}(q_0,1)\neq 0$ by the second formula (18). Thus, we necessarily obtain $w^{(s)}(q_0,1)=(r-R(s))$. Using this equality in (20a), we conclude (21) as required.

Now we assume that the infimum is attained over a sequence $\{q_j\}_{j=1}^{\infty}$ accumulating at the positive infinity (without loss of generality). Passing to a subsequence, if necessary, we can assume that

$$\lim_{j\to+\infty} \Phi_q^{(s)}(q_j,1) = 0, \quad \lim_{j\to+\infty} \Phi_p^{(s)}(q_j,1) \le 0. \tag{23}$$

Indeed, if this is not possible, we can consider a sequence of functions $f_j(q,p) = \Phi^{(s)}(q+q_j,p)$ such that for all $j \ge 1$ either $|\partial_q f_j(0,1)| \ge \epsilon$ or $\partial_p f_j(0,1) \ge \epsilon$ for some $\epsilon > 0$ independent of j. Because the norms $\|f_j\|_{C^{2,\gamma}(\overline{S})}$ are uniformly bounded for $j \ge 1$ we can use a compactness argument to find a subsequence f_{j_k} converging to a function $f \in C^{2,\gamma}(\overline{S})$ as $k \to +\infty$. The convergence is in every space $C^2(K)$ over compact all subsets $K \subset \overline{S}$. Thus, the limiting function solves the same elliptic equation, attains global minimum at $q = 0$, while $f_q(0,1) \ne 0$ or $f_p(0,1) > 0$. In both cases we clearly obtain a contradiction with the regularity assumption or the Hopf lemma respectively.

Now there are two possibilities:

$$(i) \ \lim_{j\to+\infty} w_q^{(s)}(q_j,1) = 0 \quad \text{and} \quad (ii) \ \lim_{j\to+\infty} w_q^{(s)}(q_j,1) \ne 0.$$

In the first case relations in (23) give

$$\lim_{j\to+\infty} w_q^{(s)}(q_j,1) = \lim_{j\to+\infty} w_p^{(s)}(q_j,1) = 0,$$

which then require $\lim_{j\to+\infty} w^{(s)}(q_j,1) = r - R(s)$ by the Bernoulli equation (15a). In this case one obtains (21) as before. The remaining option (ii) provides with a subsequence $\{q_{j_k}\}$ such that $\lim_{k\to+\infty} w^{(s)}(q_{j_k},1) = r - R(s)$, which follows from the first relation in (23) and (22). Similarly, this leads to (21). □

3. Proof of Theorem 1.1

Assume that the vorticity function ω satisfies condition (ii) of the theorem. In this case $d_0, R_0 < +\infty$, $s_0 = 0$ and

$$\sup_{s>s_0} H_p(0;s) = +\infty. \tag{24}$$

First we prove the claim about solitary waves. Thus, we assume that there exists a solitary wave solution h with $r \ge R_0$. Choosing $s = s_+(r)$, we put

$$w(q,p) = h(q,p) - H(p; s_+(r)).$$

It follows from Theorem 1 in [11] that $w(q,1) > 0$ for all $q \in \mathbb{R}$. Now because for a supercritical solitary wave $\mathcal{S} = \sigma(s_+(r);r)$ and the relation (20a) is then reduced to $\Phi^{(s_+(r))} = (w^{(s_+(r))})^2$, we find that $\Phi^{(s_+(r))}$ is strictly positive along the top boundary. On the other hand, we can choose $s \in (s_0, s_+(r))$ sufficiently small so that $w_p^{(s)}(q_0,0) = 0$ for some $q_0 \in \mathbb{R}$, which follows from (24). Then the corresponding flow force flux function $\Phi^{(s)}$ must attain negative values somewhere along the top boundary, because otherwise $\Phi_p^{(s)}(q,0) > 0$ for all $q \in \mathbb{R}$ by the Hopf lemma, leading to a contradiction with $w_p^{(s)}(q_0,0) = 0$ in view of the second formula (18). Since $\Phi^{(s)}$ depends smoothly on s, by the continuity we can find $s_\star \in (s_0, s_+(r))$ for which $\inf_{q\in\mathbb{R}} \Phi^{(s_\star)}(q,1) = 0$. By Proposition 2.3 we obtain $2(\mathcal{S} - \sigma(s_\star;r)) - (r - R(s_\star))^2 = 0$ so that $\mathcal{S} > \sigma(s_\star;r)$. Now Lemma 2.1 gives $\sigma(s_\star;r) > \sigma(s_+(r);r)$ and then $\mathcal{S} > \sigma(s_+(r);r) = \mathcal{S}_-(r)$, which is false, since $\mathcal{S} = \mathcal{S}_-(r)$ for any solitary wave.

Now we consider the case of a Stokes wave h for some $r \ge R_0$. Our aim is to show that $r < R_0 - \Omega(1)$. We start by proving

Lemma 3.1. *There exists* $s_\star \in (s_0, s_+(r))$ *such that* $\mathcal{S} < \sigma(s_\star; r)$.

Proof. Let $q_t < q_c$ be coordinates for some adjacent trough and crest respectively, so that $h(q,1)$ is monotonically increasing on the interval (q_t, q_c). By (24) we can choose a stream solution $H(p; s_\star)$ with $s_\star \in (s_0, s_+(r))$ such that $h_p(q_\star, 0) = H_p(0; s_\star)$ for some $q_\star \in (q_t, q_c)$. For the function $w^{(\star)}(q,p) = h(q,p) - H(p; s_\star)$ we consider the zero level set

$$\Gamma = \{(q,p) \in (q_t, q_c) \times (0,1) : \quad w^{(\star)}(q,p) = 0\}$$

inside the rectangle $Q = (q_t, q_c) \times (0,1)$. We claim that Γ is a graph $\{(f(p), p), \quad p \in (0,1)\}$ of some function $f \in C^{2,\gamma}([0,1])$ such that $f(0) = q_\star$ and $f(1) \in (q_t, q_c)$. Thus, the curve Γ connects a point on the bottom with the surface. To explain this fact we need to recall some properties of Stokes waves. Let Q_l, Q_r, Q_t and Q_b be the left, right, top and bottom boundaries of Q, excluding corner points. Then the following properties are true:

(a) $w_q^{(\star)} > 0$ on Q, while $w_q^{(\star)} = 0$ on Q_l, Q_r and Q_b;
(b) $w_{qq}^{(\star)} > 0$ on Q_l and $w_{qq}^{(\star)} < 0$ on Q_r;
(c) $w_{qp}^{(\star)} > 0$ on Q_b.

These properties are true for all Stokes waves and follow from the symmetry (guaranteed by [4]) and the maximum principle applied to the function $w_q^{(\star)}$. First of all, (a) ensures that Γ (if not empty) is locally the graph of a function as desired. We only need to show that it connects Q_t and Q_b. Note that $w^{(\star)}$ attains a unique zero value at some point $(q_\dagger, 1)$ on Q_t. Otherwise, we would find that $w_p^{(\star)}(q,0)$ has a constant sign by the Hopf lemma, contradicting to the equality $w_p^{(\star)}(q_\star, 0) = 0$. Thus, Γ bifurcates locally from $(q_\dagger, 1)$ inside Q. On the other hand, (c) shows that Γ also bifurcates inside Q from $(q_\star, 0)$ on the bottom. Now it is easy to see that theses two curves must be connected with each other. Indeed, relations (b) and inequalities $w_p^{(\star)}(q_t, 0) < 0 < w_p^{(\star)}(q_c, 0)$ guarantee that $w_p^{(\star)}$ has constant sign on the vertical sides Q_l and Q_r. Indeed, taking the difference of (11a) and the corresponding equation for H, we obtain that

$$\left(\frac{1}{2h_p^2} - \frac{1}{2H_p^2}\right)_p = \left(\frac{h_q}{h_p}\right)_q.$$

Therefore, taking into account (b), we conclude that the function

$$\frac{1}{2h_p^2} - \frac{1}{2H_p^2} = -\frac{w_p^{(\star)}(h_p + H_p)}{2h_p^2 H_p^2}$$

is increasing for $p \in [0,1]$ on Q_l and decreasing on Q_r. Now relations $w_p^{(\star)}(q_t, 0) < 0 < w_p^{(\star)}(q_c, 0)$ show that $w_p^{(\star)}$ is strictly negative on Q_l and positive on Q_r. Thus, Γ can not approach sides Q_l and Q_r and must connect Q_t and Q_b as desired.

Now we can prove that $\Phi^{(s_\star)}(q_\dagger, 1) < 0$ and then $\mathcal{S} < \sigma(s_\star; r)$ by (20a), since $w^{(\star)}(q_\dagger, 1) = 0$. For that purpose we compute $\Phi^{(s_\star)}(q_\dagger, 1)$ by changing a contour of integration as follows:

$$\Phi^{(s_\star)}(q_\dagger, 1) = \int_0^1 \Phi_p^{(s_\star)}(q_\dagger, p)\, dp = \int_\Gamma (\Phi_p^{(s_\star)}, -\Phi_q^{(s_\star)}) \cdot \boldsymbol{n}\, \mathrm{dl},$$

where dl is the length element and $\boldsymbol{n} = (n_1, n_2)$ is the unit normal to Γ with $n_1 > 0$ (because Γ is the graph of $f(p)$). Note that $\boldsymbol{n}$ is proportional with $(w_q^{(\star)}, w_p^{(\star)})$ along Γ and is oriented in the same way. Therefore, $(\Phi_p^{(s_\star)}, -\Phi_q^{(s_\star)}) \cdot \boldsymbol{n}$ has the same sign as

$$(\Phi_p^{(s_\star)}, -\Phi_q^{(s_\star)}) \cdot (w_q^{(\star)}, w_p^{(\star)}) = -\left(\frac{(w_p^{(\star)})^2}{h_p^2 H_p} + \frac{(w_q^{(\star)})^2 H_p}{h_p^2}\right) w_q^{(\star)} < 0, \tag{25}$$

which is a matter of a straightforward computation based on (18). To see that we first rewrite $\Phi_q^{(s)}$ as

$$\Phi_q^{(s_\star)} = w_q^{(s_\star)} \left(\frac{\Phi_p^{(s_\star)}}{h_p} + \frac{2w_p^{(s_\star)}}{h_p^2 H_p} \right).$$

Using this formula we compute

$$\begin{aligned}(\Phi_p^{(s_\star)}, -\Phi_q^{(s_\star)}) \cdot (w_q^{(\star)}, w_p^{(\star)}) &= \Phi_p^{(s_\star)} w_q^{(s_\star)} - w_q^{(s_\star)} w_p^{(s_\star)} \left(\frac{\Phi_p^{(s_\star)}}{h_p} + \frac{2w_p^{(s_\star)}}{h_p^2 H_p} \right) \\ &= w_q^{(s_\star)} \left(\frac{H_p \Phi_p^{(s_\star)}}{h_p} - \frac{2(w_p^{(s_\star)})^2}{h_p^2 H_p} \right)\end{aligned}$$

It is left to use formula (18) for $\Phi_p^{(s_\star)}$ to conclude (25). Thus, $(\Phi_p^{(s_\star)}, -\Phi_q^{(s_\star)}) \cdot \boldsymbol{n}$ is negative along Γ and then $\Phi^{(s_\star)}(q_\dagger, 1) < 0$. The lemma is proved. □

Using Lemma 3.1 it is easy to complete the proof of the theorem. Indeed, for all $s \in (s_0, s_\star)$ we have $\mathcal{S} < \sigma(s_\star; r) < \sigma(s; r)$ (see Lemma 2.1), while at the every crest we have $\Phi^{(s)}(q_c, 1) > 0$, because of (17) and that $w_q^{(s)}(q_c, p) = 0$ for all $p \in [0, 1]$. Thus, the boundary condition (20a) then implies $w^{(s)}(q_c, 1) > 2(r - R(s))$, which is true for all $s \in (s_0, s_\star)$. Here we used the fact that $w^{(s)}(q_c, 1) > 0$ that was proved in [11, Proposition 3]. Passing to the limit $s \to s_0$, we find $\eta(q_c) > d_0 + 2(r - R_0)$. Finally, because $\eta(q_c) < r$ by (11b) and $R_0 = d_0 - \Omega(1)$, we obtain $r < R_0 - \Omega(1)$, which finises the proof of the theorem.

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