On the Field-Induced Transport of Magnetic Nanoparticles in Incompressible Flow: Existence of Global Solutions

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Abstract. We prove global-in-time existence of weak solutions to a pde-model for the motion of dilute superparamagnetic nanoparticles in fluids influenced by quasi-stationary magnetic fields. This model has recently been derived in Grün and Weiß (On the field-induced transport of magnetic nanoparticles in incompressible flow: modeling and numerics, Mathematical Models and Methods in the Applied Sciences, in press). It couples evolution equations for particle density and magnetization to the hydrodynamic and magnetostatic equations. Suggested by physical arguments, we consider no-flux-type boundary conditions for the magnetization equation which entails $H(\text{div}, \text{curl})$ -regularity for magnetization and magnetic field. By a subtle approximation procedure, we nevertheless succeed to give a meaning to the Kelvin force $(\mathbf{m} \cdot \nabla)\mathbf{h}$ and to establish existence of solutions in the sense of distributions in two space dimensions. For the three-dimensional case, we suggest two regularizations of the system which each guarantee existence of solutions, too.

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1. Introduction

Given two domains $\Omega \subset \subset \Omega' \subset \subset \mathbb{R}^d$, $d \in \{2, 3\}$, we are concerned with existence results for the model

$$\begin{aligned} \rho_0 \mathbf{u}_t + \rho_0 (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \text{div}(2\eta \mathbf{D}\mathbf{u}) \\ = \mu_0 (\mathbf{m} \cdot \nabla) (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) + \frac{\mu_0}{2} \text{curl}(\mathbf{m} \times \underbrace{(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a)}_{=: \hat{\mathbf{h}}}), \end{aligned} \quad (1.1a)$$

$$\text{div } \mathbf{u} = 0, \quad (1.1b)$$

$$c_t + \mathbf{u} \cdot \nabla c + \text{div}(c \mathbf{V}_{\text{part}}) = 0, \quad (1.1c)$$

$$\mathbf{V}_{\text{part}} = -KD \frac{f_2(c)}{c} \nabla g'(c) + K\mu_0 \frac{f_2(c)}{c^2} (\nabla \underbrace{(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a - \alpha_3 \mathbf{m})}_{=: \hat{\mathbf{b}}})^T \mathbf{m}, \quad (1.1d)$$

$$-\Delta R = \text{div}(\mathbf{m}), \quad (1.1e)$$

$$\mathbf{m}_t + \text{div}(\mathbf{m} \otimes (\mathbf{u} + \mathbf{V}_{\text{part}})) - \sigma \Delta \mathbf{m} = \frac{1}{2} \text{curl } \mathbf{u} \times \mathbf{m} - \frac{1}{\tau_{\text{rel}}} (\mathbf{m} - \chi(c, \mathbf{h}) \mathbf{h}) \quad (1.1f)$$

in $\Omega \times (0, T)$ and

$$-\Delta R = 0 \quad (1.1g)$$

in $(\Omega' \setminus \Omega) \times (0, T)$. Supplemented with boundary conditions

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \times [0, T], \quad (1.2a)$$

$$c \mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (1.2b)$$

$$(\mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu})(\mathbf{m} - (\mathbf{m} \cdot \boldsymbol{\nu})\boldsymbol{\nu}) - \sigma \operatorname{curl} \mathbf{m} \times \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (1.2c)$$

$$(\mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu})(\mathbf{m} \cdot \boldsymbol{\nu}) - \sigma \operatorname{div} \mathbf{m} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (1.2d)$$

transmission conditions

$$[\nabla R + \mathbf{m}] \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (1.2e)$$

$$\nabla R \cdot \boldsymbol{\nu} = \mathbf{h}_a \cdot \boldsymbol{\nu} \quad \text{on } \partial\Omega' \times (0, T], \quad (1.2f)$$

and initial conditions

$$\mathbf{u}(\cdot, 0) = \mathbf{u}^0, \quad (1.3a)$$

$$c(\cdot, 0) = c^0, \quad (1.3b)$$

$$\mathbf{m}(\cdot, 0) = \mathbf{m}^0, \quad (1.3c)$$

this system has been proposed in [17] up to a change of boundary conditions to model the motion of dilute solutions of superparamagnetic nanoparticles influenced by external magnetic fields. Note that in points $(x, t) \in \partial\Omega \times (0, T]$ such that $c(x, t) \neq 0$, the boundary conditions (1.2b)–(1.2d) simplify to become $\mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu} = 0$, $\operatorname{curl} \mathbf{m} \times \boldsymbol{\nu} = 0$, and $\operatorname{div} \mathbf{m} = 0$. These are the boundary conditions which have been used in the section on numerics in [17].

Here, $(\mathbf{u}, p)$ denote the hydrodynamic variables of the carrier fluid, c stands for the number density of the magnetic nanoparticles and $\mathbf{m}$ or $\mathbf{h} = \nabla R$ describe magnetization or magnetic field, respectively. The vector field $\mathbf{V}_{\text{part}}$ denotes the particle's velocity relative to the flow of the carrier fluid. It takes diffusive and magnetic effects into account. The system is driven by the external magnetic field $\mathbf{h}_a$ which satisfies Maxwell's equations in the absence of matter, i.e.

$$\operatorname{curl} \mathbf{h}_a = 0, \quad (1.4a)$$

$$\operatorname{div} \mathbf{h}_a = 0. \quad (1.4b)$$

For further explanation of the parameters in the model, we refer the reader to Sect. 2. The model includes a two-domain-approach, i.e. the magnetic field is defined on a larger domain $\Omega' \supset \Omega$, which has the advantage that one can account for stray field effects at the boundary of the fluid domain.

As already pointed out in [17], in the literature so far two pathways have been pursued to study problems of ferrohydrodynamics. The first one is concerned with phenomena for which the particle distribution can be assumed to be homogeneous in space (and consequently constant in time). For those ferrofluids, pde-models have been derived by Shliomis [25] and Rosensweig [24]. Both models couple evolution equations for momentum and magnetization to Maxwell's equations or to their simplifications from magnetostatics. Rosensweig takes in addition an evolution equation for the angular momentum of the fluid into account. In a series of papers, Amirat and Hamdache [1–6] developed a mathematical existence theory in the framework of the Shliomis model. Just recently, Nochetto, Salgado and Tomas [20] proposed numerical schemes for the Rosensweig model. In a second paper, they [19] considered two-phase flow with one ferrofluid involved to model the famous Rosensweig instability [23], and they provided a convergence proof in a simplified setting. All these publications have in common that there are no pathways suggested how to deal with non-homogeneous, non-steady particle densities.

In a second line of research, mathematical models have been suggested and investigated for the transport of magnetic nanoparticles with particle densities varying in space and time. These models have in common that evolution equations for the magnetization are not considered separately. Instead, authors assume the magnetization to be given explicitly as a function of particle density and magnetic field. Polevikov and Tobiska [22] were interested in a steady-state diffusion problem for particles in a ferrofluid. Most recently, Himmelsbach, Neuss-Radu and Neuß [18] proposed a new model featuring an evolution equation for the particle density coupled to the magnetostatic equations and assuming the macroscopic flow field to be given. In the radial symmetric case they show existence and uniqueness of solutions and provide numerical simulations, too.

In this spirit, model (1.1) is the first model which takes both non-constant particle densities and magnetization fields into account. Note that the boundary conditions (1.2c), (1.2d) are of no-flux-type, combining the no-flux boundary condition (1.2b) of the number density c with the additional flux originating from the diffusive term $-\sigma \Delta \mathbf{m} = -\sigma (\nabla \operatorname{div} \mathbf{m} - \operatorname{curl} \operatorname{curl} \mathbf{m})$. These boundary conditions may be favorable from a physical point of view. They allow for energy estimates in a natural way and they do not prohibit tangential or normal traces of $\mathbf{m}$ to be different from zero on $\partial\Omega$. This is in contrast to the work of [1] where the normal component of the magnetization has been prescribed to vanish and therefore H^1 -regularity holds for $\mathbf{m}$ and $\mathbf{h}$ globally.

In our framework, however, the problem arises that magnetization and magnetic field have only $H(\operatorname{div}, \operatorname{curl})$ -regularity, cf. [8]. This is the more an issue, as the Kelvin force $(\mathbf{m} \cdot \nabla) \mathbf{h}$ requires control of gradients of $\mathbf{h}$ and as it enters all the evolution equations (1.1a), (1.1c), and (1.1f). In this situation, it seems natural to derive H^1_{loc} -regularity for $\mathbf{m}$ and $\mathbf{h}$ and to consider solutions in the sense of distributions. Due to the intricate coupling of the evolution equations for $\mathbf{m}$ and c , estimates on gradients of (appropriate powers of) c depend on the integrability of $(\mathbf{m} \cdot \nabla) \mathbf{h}$ and of $(\mathbf{m} \cdot \nabla) \mathbf{m}$. As a consequence, we expect results only in L^p_{loc} -spaces. As such a localization seems to become indispensable already on the level of Galerkin approximations, the appropriate choice of approximation spaces is a central topic of this paper. This includes in particular a strategy how to deal with the two-domain modeling approach.

The outline of the paper is as follows. In Sect. 2, we explain further features of the model (1.1), we state an energy estimate, and we formulate our hypotheses on the domains Ω and Ω' and on the data.

To cope with the nonlinearities $(\mathbf{m} \cdot \nabla) \mathbf{h}$ and $(\nabla \mathbf{m})^T \mathbf{m}$ in the convective term of the evolution equation for the particle density c , we refrain to local arguments – which are needed already in the first limit passage *Discrete to Regularized Continuous*. This requires a subtle choice of Galerkin approximation spaces as in general the projections $\Pi_{X_n} \varphi$, $n \in \mathbb{N}$, of C^∞_0 -functions onto ansatz spaces X_n do not have compact support. For this reason, some techniques which we use for the passage to the limit *Regularized Continuous to Continuous*, cannot be applied at this earlier stage. The remedy is to choose the approximation spaces in such a way that L^∞ -convergence of the gradients of $\Pi_{X_n} \varphi$ to $\nabla \varphi$ is guaranteed.¹ A sufficient condition is to require H^3 -convergence of $\Pi_{X_n} \varphi$ to φ . Inspired by the discussion of boundary conditions for the magnetization fields in [20], we prefer to take natural boundary conditions $\operatorname{div} \mathbf{m}|_{\partial\Omega} \equiv 0$ and $\operatorname{curl} \mathbf{m}|_{\partial\Omega} \equiv \mathbf{0}$ for our ansatz spaces. It is worth mentioning that the boundary conditions (1.2c), (1.2d) reduce to these conditions in those points $(x, t) \in \partial\Omega \times (0, T)$ where $c(x, t) \neq 0$. In addition, some effort is devoted to guarantee that ansatz functions for R are defined on Ω' , having in addition gradients, the restriction of which to Ω is contained in the approximation space for $\mathbf{m}$. We devote Sect. 3 to construction and decomposition of such ansatz spaces for magnetization and magnetic potential. Then, Sect. 4 collects the ansatz spaces for velocity and particle density.

In Sect. 5, we introduce a (T)ransport and (M)obility (R)egularized model – replacing the usual $c \log c$ -entropy by a strictly convex approximation with quadratic growth, and using a further density cut-off in the transport velocity $\mathbf{V}_{\operatorname{part}}$, see (5.2) and (5.3). For this TMR-model, global existence of discrete solutions is established, and Sections 6 and 7 provide compactness results as well as the limit passage *Discrete to Regularized Continuous*.

In Sect. 8, we show that solutions to the TMR-model converge in the limit of vanishing regularization parameters to weak solutions of model (1.1). For this, it will be essential to choose the nonlinear mobility in the particle evolution in such a way that the flux $\mathbf{V}_{\operatorname{part}}$ has L^2 -regularity. It turns out that this can be achieved by choosing the mobility quadratic in c . Due to a certain regularity gap, we have to confine ourselves to the case of two space dimensions unless additional regularizing terms are considered. For this, see Remark 8.11.

In this paper, we cannot avoid a rather involved notation. For the reader's convenience, the Appendix A.5 explains the notation and provides references on definitions and further properties.

¹For more details, why this convergence is desirable, see (7.9) and the subsequent paragraph.

2. The Model

Let us first give some more details on model (1.1). The function g is a usual mixture energy of $c \log c$ -type. The diffusion of the magnetic particles is described by $f_2(c) := c^m$, where the cases $m = 1$, which yields classical Fickian diffusion, and $m = 2$, which entails finite speed of propagation [13], are the most prominent choices. The susceptibility function χ has been chosen as

$$\mathbf{m}_{\text{eq}} = \chi(c, \mathbf{h}) \mathbf{h} := \left(\tilde{\chi}(c) m_0 \frac{L(\xi|\mathbf{h}|)}{|\mathbf{h}|} \right) \mathbf{h}, \quad (2.1)$$

with the equilibrium magnetization $\mathbf{m}_{\text{eq}}$ given in (2.1) by means of the Langevin formula— $L(x) = \coth(x) - \frac{1}{x}$ is the Langevin function—and $\tilde{\chi}(c)$ is a not necessarily linear function of the particle density c , which we assume to be Lipschitz-continuous within this paper. $\rho_0, \eta, \mu_0, \alpha_1, \alpha_3, \sigma, \tau_{\text{rel}}, m_0$, are positive parameters, $\beta \in \mathbb{R}$. For their physical meaning, we refer to [17].

Formally, the system satisfies the energy estimate

$$\begin{aligned} & \|\mathbf{u}\|_{L^\infty((0,T);L^2(\Omega)^d)}^2 + \|\mathbf{u}\|_{L^2((0,T);H^1(\Omega)^d)}^2 + \|g(c(\cdot, t))\|_{L^\infty((0,T);L^1(\Omega))} \\ & + \|\mathbf{m}\|_{L^\infty((0,T);L^2(\Omega)^d)}^2 + \|\mathbf{h}\|_{L^\infty((0,T);L^2(\Omega)^d)}^2 \\ & + \|\mathbf{m}\|_{L^2((0,T);H(\text{div}, \text{curl})(\Omega))}^2 + \|\mathbf{h}\|_{L^2((0,T);H(\text{div}, \text{curl})(\Omega))}^2 + \int_0^T \int_\Omega \chi(c, h) |\mathbf{h}|^2 \, dx \, dt \\ & + \int_0^T \int_\Omega c^{2-m} |\mathbf{V}_{\text{part}}|^2 \, dx \, dt \leq C. \end{aligned} \quad (2.2)$$

For the reader's convenience, we include a formal derivation of the estimate in the appendix, see Sect. A.1, together with some general reflections on the physical background of the model.

Let us formulate our assumptions on the spatial domains and on the data.

- (H1) $\Omega' \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a simply connected, bounded domain of class $C^{1,1}$. $\Omega \subset \subset \Omega'$ is of class $C^{3,1}$.
- (H2) Let $\mathbf{h}_a \in H^1([0, T]; H^3(\Omega)^d \cap H^1(\Omega') \cap H(\text{div}_0, \text{curl}_0)(\Omega'))$.
- (H3) $\alpha_1, \alpha_3, \beta, K, D, \sigma, \tau_{\text{rel}}$ are positive parameters.
- (H4) The susceptibility χ is bounded.

For the reader's convenience, we give a definition for two-dimensional curl-operator and cross product for vector fields or vectors, respectively.

Definition 2.1. Let $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$ be such that the weak curl of $(\mathbf{u}_x, \mathbf{u}_y, 0)^T$ exists, then—without introducing new notation—the curl-operator of two dimensional vector fields is defined by

$$\text{curl } \mathbf{u} := (\partial_x \mathbf{u}_y - \partial_y \mathbf{u}_x),$$

which is a scalar function. Analogously, the vector product $\times$ will be defined for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and scalar $g \in \mathbb{R}$ as follows.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &:= \mathbf{a}_x \mathbf{b}_y - \mathbf{a}_y \mathbf{b}_x, \\ c \times \mathbf{b} &:= c \begin{pmatrix} -\mathbf{b}_y \\ \mathbf{b}_x \end{pmatrix}. \end{aligned}$$

We have

$$\text{curl } \nabla = 0, \quad (2.3)$$

and if $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^2$, $v \in \mathbb{R}$, then

$$v \times \mathbf{a} = \mathbf{0} \Leftrightarrow v = 0. \quad (2.4)$$

3. Construction of Discrete Spaces for Magnetization and Magnetic Potential

In this section, we introduce the function spaces which will serve for the construction of approximation spaces for magnetization and magnetic field in the Faedo–Galerkin approach of Sect. 5. Our choice is guided by the following criteria.

- (C1) Formal energy estimates, compare (2.2), indicate that the magnetization is contained in $L^2((0, T); H(\operatorname{div}, \operatorname{curl})(\Omega))$.
- (C2) The magnetic field $\mathbf{h}$ is a gradient field on Ω' , satisfying $\mathbf{h}|_{\Omega' \setminus \overline{\Omega}} \in H(\operatorname{div}_0)(\Omega' \setminus \overline{\Omega})$ due to the magnetostatic equations

$$\begin{aligned} \operatorname{curl} \mathbf{h} &= 0, \\ \operatorname{div}(\mathbf{h} + \mathbf{m}) &= 0, \end{aligned}$$

and $\mathbf{h}|_{\Omega} \in H(\operatorname{div})(\Omega)$ due to the formal energy estimate (2.2).

- (C3) Approximation functions for the magnetization should satisfy the boundary conditions

$$\begin{cases} \operatorname{curl} \mathbf{m} \times \boldsymbol{\nu}|_{\partial\Omega} = 0 & \text{if } d = 3, \\ \operatorname{curl} \mathbf{m}|_{\partial\Omega} = 0 & \text{if } d = 2 \end{cases} \quad (3.1)$$

and

$$\operatorname{div} \mathbf{m}|_{\partial\Omega} = 0 \quad (3.2)$$

which allow for stability estimates.

- (C4) We require the sequence of approximation spaces for the magnetization to be dense in a closed subspace of $H^3(\Omega)^d$.

This leads us to consider the space

$$\mathcal{M} := \left\{ \boldsymbol{\Psi} \in H^3(\Omega)^d \left| \begin{cases} \operatorname{curl} \boldsymbol{\Psi} \times \boldsymbol{\nu}|_{\partial\Omega} = \mathbf{0} & \text{if } d = 3, \\ \operatorname{curl} \boldsymbol{\Psi}|_{\partial\Omega} = 0 & \text{if } d = 2, \end{cases} \operatorname{div} \boldsymbol{\Psi}|_{\partial\Omega} = 0 \right. \right\} \subset H^3(\Omega)^d, \quad (3.3)$$

as the function space related to the magnetization and the space

$$\mathcal{R} := \{S \in H_{\text{mean}}^1(\Omega') | \nabla S|_{\Omega} \in H(\operatorname{div})(\Omega), \nabla S|_{\Omega' \setminus \overline{\Omega}} \in H(\operatorname{div}_0)(\Omega' \setminus \Omega)\} \quad (3.4)$$

as the function space for the scalar potentials of the magnetic field $\mathbf{h}$.

Note that

- $\mathcal{M}$ is a dense subset of $L^2(\Omega)^d$ as it contains all $C_0^\infty(\Omega)^d$ -functions.
- $\mathcal{M}$ is a closed subspace of $H^3(\Omega)^d$ as it is the preimage of closed sets under continuous mappings.
- $\mathcal{R}$ is complete with respect to the norm

$$\|\cdot\|_{\mathcal{R}} := \|\nabla(\cdot)\|_{L^2(\Omega')^d} + \|\Delta(\cdot)\|_{L^2(\Omega)} + \overbrace{\|\Delta(\cdot)\|_{L^2(\Omega' \setminus \overline{\Omega})}}^{=0}.$$

The latter is true as the norm $\|\cdot\|_{\mathcal{R}}$ dominates $\|\cdot\|_{H^1(\Omega')}$ (by using Poincaré's inequality for mean value free functions) and $\|\Delta(\cdot)\|_{L^2(\Omega)}$. Hence, if $\{r_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{R}$, there are functions $r \in H_{\text{mean}}^1(\Omega')$ and $s \in L^2(\Omega)$ such that $r_k \rightarrow r$ in $H_{\text{mean}}^1(\Omega')$ and $\Delta r_k \rightarrow s$ in $L^2(\Omega)$. The identification $s = \Delta r$ is done in a standard way via integration by parts. The fact that Δr vanishes on $\Omega' \setminus \overline{\Omega}$ follows analogously.

3.1. Construction of a Basis of $\mathcal{M}$

The goal of this subsection is to construct a basis of $\mathcal{M}$ which is orthonormal with respect to the L^2 -scalar product. In particular, we aim at substructures which can be exploited to be the starting point to construct a basis of the set $\mathcal{R}$ as well. For consistency reasons with respect to the Galerkin procedure to be studied later on, we require that approximation spaces $\mathcal{R}_n \subset \mathcal{R}$ and $\mathcal{M}_n \subset \mathcal{M}$ satisfy $\nabla \mathcal{R}_n|_{\Omega} \subset \mathcal{M}_n$.

Let us first introduce two subspaces of $\mathcal{M}$.

(i) Gradient fields,

$$\mathcal{H} := \nabla[H_{\text{mean}}^1(\Omega) \cap H^4(\Omega)] \cap \{\Psi \in H^3(\Omega)^d \mid \operatorname{div} \Psi|_{\partial\Omega} = 0\} =: \mathcal{A}_0 \cap \mathcal{A}_1 \subset H^3(\Omega)^d. \quad (3.1.1)$$

(ii) Gradient fields with potentials having a constant trace on $\partial\Omega$,

$$\mathcal{S} := \{\mathfrak{h} \in \mathcal{H} \mid \mathfrak{h} = \nabla S \text{ for a } S \in H_0^1(\Omega)\}. \quad (3.1.2)$$

Note that $\mathcal{S}$ is not trivial as all gradients of homogeneous Dirichlet–Laplace eigenfunctions are elements of this space. This is due to the fact that those functions have a constant trace and that their gradient is invariant under subtraction of the mean-value.

Lemma 3.1.1. (i) $\mathcal{H}$ is a closed subspace of $\mathcal{M}$ with respect to the H^3 -norm.

(ii) $\mathcal{S}$ is a closed subspace of $\mathcal{H}$.

Proof. Ad i): Closedness of $\mathcal{A}_1 \subset H^3(\Omega)^d$ is obvious. The range of the gradient operator on the domain $H_{\text{mean}}^1(\Omega) \cap H^4(\Omega)$ is closed as well, as can be seen by standard arguments. One easily checks $\mathcal{H} \subset \mathcal{M}$ by definition and using $\operatorname{curl} \nabla = \mathbf{0}$ (in the case $d = 2$, analogously $\operatorname{curl} \nabla = 0$, cf. (2.3)). Hence, $\mathcal{H}$ is a closed subspace of the Hilbert space $\mathcal{M}$ (equipped with H^3 -scalar product).

Ad ii): By the closedness of the gradient operator on $H_0^1(\Omega)$ the result immediately follows. $\square$

Lemma 3.1.2. The following identities hold true.

$$(i) \quad \mathcal{S} = \overline{\mathcal{S}}^{L^2(\Omega)^d} \cap \mathcal{H}.$$

$$(ii) \quad \mathcal{H} = \overline{\mathcal{H}}^{L^2(\Omega)^d} \cap \mathcal{M}.$$

Proof. By the completeness of $H_0^1(\Omega)$ and Poincaré’s inequality, the L^2 -closure of $\mathcal{S}$ is a subset of $\nabla[H_0^1(\Omega)]$. Hence, its intersection with $\mathcal{H}$ is a subset of $\mathcal{S}$ by definition, see (3.1.2). The converse inclusion is obvious. By a similar reasoning, the L^2 -closure of $\mathcal{H}$ is a subset of $\nabla[H_{\text{mean}}^1(\Omega)]$ and its intersection with $\mathcal{M}$ guarantees H^3 -regularity as well as vanishing of the divergence on the boundary. Again, the converse inclusion is obvious. $\square$

Now, we formulate a general decomposition lemma, which might be of independent interest, and which yields the desired basis with a gradient substructure. We postpone its proof to the appendix.

Lemma 3.1.3. Let U, X, Y be separable Hilbert spaces that satisfy the following assumptions.

- $X \subset Y$, $X \hookrightarrow Y$.
- $U \subset X$ is a closed subspace, and we have

$$U = \overline{U}^Y \cap X. \quad (3.1.3)$$

Then, the following holds true:

- (i) The space $V = (\overline{U}^Y)^\perp \cap X$ is a closed subspace of X .
- (ii) There are sets $\{u_i\}_{i \in N_1}$, $\{v_i\}_{i \in N_2}$, $N_1, N_2 \subset \mathbb{N}$ such that

$$U = \overline{\operatorname{span}\{u_i\}_{i \in N_1}}^X, \quad V = \overline{\operatorname{span}\{v_i\}_{i \in N_2}}^X,$$

and $\{u_i\}_{i \in N_1}, \{v_i\}_{i \in N_2}$ form orthonormal sets in Y and orthogonal sets in X , respectively.

- (iii) We have the decomposition

$$X = \overline{\operatorname{span}\{u_i\}_{i \in N_1} \cup \operatorname{span}\{v_i\}_{i \in N_2}}^X \quad (3.1.4)$$

and for each $i \in N_1$ and $j \in N_2$, we have

$$u_i \perp^Y v_j. \quad (3.1.5)$$

- (iv) For any given basis $\{\tilde{u}_i\}_{i \in N_1}$ of U , a basis $\{v_i\}_{i \in N_2}$ of V can be found, such that the basis $\{v_i\}_{i \in N_2}$ is orthogonal in X , orthonormal in Y and (3.1.4) and (3.1.5) hold for $\{u_i\}_{i \in N_1}$ replaced by $\{\tilde{u}_i\}_{i \in N_1}$.

For a simple example of spaces U, X, Y which satisfy assumption (3.1.3) of Lemma 3.1.3, we refer to Remark A.2.2 in the appendix.

Note that according to Lemma 3.1.2 the triples $(\mathcal{S}, \mathcal{H}, L^2(\Omega)^d)$ and $(\mathcal{H}, \mathcal{M}, L^2(\Omega)^d)$ satisfy the assumptions of Lemma 3.1.3. This gives rise to the following decomposition result for $\mathcal{M}$.

Lemma 3.1.4. *There exist spaces $\mathcal{S}^o \subset \mathcal{H}$ and $\mathcal{V} \subset \mathcal{M}$ such that*

- (i) $\mathcal{S}^o := (\overline{\mathcal{S}}^{L^2(\Omega)^d})^\perp \cap \mathcal{H}$ is a closed subspace of $\mathcal{H}$.
- (ii) $\mathcal{V} := (\overline{\mathcal{H}}^{L^2(\Omega)^d})^\perp \cap \mathcal{M}$ is a closed subspace of $\mathcal{M}$.

Moreover, there are sets $\{\mathbf{s}_i\}_{i \in \mathbb{N}}$, $\{\mathbf{s}_i^\perp\}_{i \in \mathbb{N}}$, $\{\mathbf{m}_i\}_{i \in \mathbb{N}}$, such that

$$\mathcal{S} = \overline{\text{span}\{\mathbf{s}_i\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d}, \quad \mathcal{S}^o = \overline{\text{span}\{\mathbf{s}_i^\perp\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d}, \quad (3.1.6)$$

$$\mathcal{H} = \overline{\text{span}\{\mathbf{s}_i\}_{i \in \mathbb{N}} \cup \text{span}\{\mathbf{s}_i^\perp\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d}, \quad (3.1.7)$$

$$\mathcal{V} = \overline{\text{span}\{\mathbf{m}_i\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d}, \quad (3.1.8)$$

$$\mathcal{M} = \overline{\text{span}\{\mathbf{s}_i\}_{i \in \mathbb{N}} \cup \text{span}\{\mathbf{s}_i^\perp\}_{i \in \mathbb{N}} \cup \text{span}\{\mathbf{m}_i\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d}. \quad (3.1.9)$$

In particular, the family $\{\mathbf{s}_i\}_{i \in \mathbb{N}} \cup \{\mathbf{s}_j^\perp\}_{j \in \mathbb{N}} \cup \{\mathbf{m}_k\}_{k \in \mathbb{N}}$ is orthonormal in $L^2(\Omega)^d$. The families $\{\mathbf{s}_i\}_{i \in \mathbb{N}}$, $\{\mathbf{s}_j^\perp\}_{j \in \mathbb{N}}$, $\{\mathbf{m}_k\}_{k \in \mathbb{N}}$ are each orthogonal in $H^3(\Omega)^d$.

Proof. By Lemma 3.1.3—applied to $\mathcal{S} \subset \mathcal{H}$ —we have bases

$$\{\mathbf{s}_i\}_{i \in \mathbb{N}} \text{ of } \mathcal{S}, \quad \{\mathbf{s}_i^\perp\}_{i \in \mathbb{N}} \text{ of } \mathcal{S}^o \quad (3.1.10)$$

that are L^2 -orthonormal and orthogonal with respect to the inner product of $H^3(\Omega)^d$. Their union $\{\mathbf{h}_i\}_{i \in \mathbb{N}}$,

$$\mathbf{h}_{2i} := \mathbf{s}_i, \quad \mathbf{h}_{2i-1} := \mathbf{s}_i^\perp, \quad \forall i \in \mathbb{N}, \quad (3.1.11)$$

generates the space $\mathcal{H}$, cf. (3.1.4) in Lemma 3.1.3. By another application of Lemma 3.1.3 to $\mathcal{H} \subset \mathcal{M}$ we get an additional set $\{\mathbf{m}_i\}_{i \in \mathbb{N}}$ which is a basis of

$$\mathcal{V} := (\overline{\mathcal{H}}^{L^2(\Omega)^d})^\perp \cap \mathcal{M} \quad (3.1.12)$$

and the union of all three bases generates the space $\mathcal{M}$. The orthogonality properties are evident. $\square$

For later use, we rename and relabel the basis functions of $\mathcal{M}$.

$$\mathcal{M} := \overline{\text{span}\{\Psi_i^{\mathbf{m}}\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d}, \quad \Psi_{2i}^{\mathbf{m}} := \mathbf{h}_i, \quad \Psi_{2i-1}^{\mathbf{m}} := \mathbf{m}_i \quad \forall i \in \mathbb{N}. \quad (3.1.13)$$

By construction we have the following properties,

$$\begin{aligned} &\Psi_{2i-1}^{\mathbf{m}} \perp \Psi_{2j-1}^{\mathbf{m}} \text{ for } i, j \in \mathbb{N}, i \neq j, \text{ with respect to the } L^2 \text{ and } H^3 \text{ scalar products,} \\ &\Psi_{4i-2}^{\mathbf{m}} \perp \Psi_{4j-2}^{\mathbf{m}} \text{ for } i, j \in \mathbb{N}, i \neq j, \text{ with respect to the } L^2 \text{ and } H^3 \text{ scalar products,} \\ &\Psi_{4i}^{\mathbf{m}} \perp \Psi_{4j}^{\mathbf{m}} \text{ for } i, j \in \mathbb{N}, i \neq j, \text{ with respect to the } L^2 \text{ and } H^3 \text{ scalar products,} \\ &\Psi_i^{\mathbf{m}} \perp \Psi_j^{\mathbf{m}} \text{ for } i, j \in \mathbb{N}, i \neq j, \text{ with respect to the } L^2 \text{ scalar product.} \end{aligned} \quad (3.1.14)$$

3.2. Construction of a Basis of $\mathcal{R}$

Recalling criterion (C2) as well as our requirement that approximation spaces $\mathcal{R}_n$ and $\mathcal{M}_n$ should satisfy $\nabla \mathcal{R}_n|_\Omega \subset \mathcal{M}_n$, it is natural to extend in a first step functions in $\mathcal{H} \subset \mathcal{A}_0$ to the whole of Ω' , consistent with the definition of $\mathcal{R}$. For this, we choose the uniquely determined mean-value-free potentials

$$\phi_i^\Omega \in H_{\text{mean}}^1(\Omega) \cap H^4(\Omega) \text{ such that } \nabla \phi_i^\Omega = \mathfrak{h}_i = \begin{cases} \mathbf{s}_{i/2}, & i \text{ even}, \\ \mathbf{s}_{(i+1)/2}^\perp, & i \text{ odd} \end{cases}, \quad \forall i \in \mathbb{N}. \quad (3.2.1)$$

To explain our extension procedure, we begin with some considerations valid on a general bounded $C^{1,1}$ -domain V . Note that these ideas later on shall be applied both to $\Omega' \setminus \Omega$ and to Ω .

Definition 3.2.1. On a bounded $C^{1,1}$ -domain $V \subset \mathbb{R}^d$, let $L_V^{-1} : H^{\frac{1}{2}}(\partial V) \rightarrow H^1(V)$ be defined for $f \in H^{1/2}(\partial V)$ to be the unique solution of the inhomogeneous Dirichlet–Laplace problem

$$\begin{aligned} -\Delta L_V^{-1} f &= 0 \\ L_V^{-1} f|_{\partial V} &= f. \end{aligned} \quad (3.2.2)$$

Also, denote by $\{u_i^V\}_{i \in \mathbb{N}}$ the eigenfunctions to positive eigenvalues $(\lambda_i^V)_{i \in \mathbb{N}}$ of the homogeneous Neumann–Laplace problem,

$$\begin{aligned} -\Delta u_i^V &= \lambda_i^V u_i^V \quad \text{in } V, \\ \nabla u_i^V \cdot \boldsymbol{\nu}|_{\partial V} &= 0, \\ \oint_V u_i^V \, dx &= 0. \end{aligned} \quad (3.2.3)$$

Remark 3.2.2. The following properties of the operator L_V^{-1} and of functions $\{u_i^V\}_{i \in \mathbb{N}}$ introduced in Definition 3.2.1 are well known:

(i)

$$\|\nabla L_V^{-1} f\|_{L^2(V)^d} \leq C \|f\|_{H^{\frac{1}{2}}(\partial V)}. \quad (3.2.4)$$

(ii) Augmented by the constant function $u_0^V := |V|^{-\frac{1}{2}}$, the set of all functions $\{u_i^V\}_{i \in \mathbb{N}_0}$ is dense in $H^1(V)$ and their traces $\{u_i^V|_{\partial V}\}_{i \in \mathbb{N}_0}$ are dense in $H^{\frac{1}{2}}(\partial V)$.

Using Definition 3.2.1, we can extend and augment the basis $\{\mathfrak{h}_i\}_{i \in \mathbb{N}}$. Define

$$\begin{aligned} \tilde{R}_{2i+1} &:= \begin{cases} 0 & \text{in } \Omega \\ L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} 0 & \text{on } \partial\Omega \\ u_i^{\Omega'}|_{\partial\Omega'} & \text{on } \partial\Omega' \end{cases} & \text{in } \Omega' \setminus \overline{\Omega} \end{cases} \quad \forall i \in \mathbb{N}_0, \\ \tilde{R}_{2i} &:= \begin{cases} \phi_i^\Omega - c_i & \text{in } \Omega \\ L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} \phi_i^\Omega|_{\partial\Omega} - c_i & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} & \text{in } \Omega' \setminus \overline{\Omega} \end{cases} \quad \forall i \in \mathbb{N}, \\ c_i &:= \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \phi_i^\Omega \, d\sigma \end{aligned} \quad (3.2.5)$$

and normalize them by

$$\mathbf{R}_i := \tilde{R}_i - \oint_{\Omega'} \tilde{R}_i \, dx \quad \forall i \in \mathbb{N}. \quad (3.2.6)$$

Note that by this choice we have $\mathbf{R}_{4i}|_{\Omega' \setminus \overline{\Omega}} \equiv \text{const.}$ for all $i \in \mathbb{N}$. The reason for this is the fact that $\mathfrak{h}_{2i} = \mathbf{s}_i$, cf. (3.1.11), admits a trace-free potential, hence its mean-value-free potential is constant on $\partial\Omega$. This constant, see c_{2i} above, has been subtracted from ϕ_{2i}^Ω and therefore on $\Omega' \setminus \overline{\Omega}$ the potential $\tilde{R}_{4i}$ is equal to $L_{\Omega' \setminus \overline{\Omega}}^{-1} 0$, which is constant zero.

As a first step to find suitable approximation spaces for the magnetic potential we choose

$$\mathcal{R}_{\text{temp}} := \overline{\text{span}\{\mathbf{R}_i\}_{i \in \mathbb{N}}}^{\mathcal{R}}. \quad (3.2.7)$$

Lemma 3.2.3. *We have*

$$\mathcal{R}_{temp} = \mathcal{R}. \quad (3.2.8)$$

Proof. We assume by contradiction there exists a function $S \in \mathcal{R} \setminus \mathcal{R}_{temp}$. Without loss of generality S can be chosen orthogonal to $\mathcal{R}_{temp}$, hence

$$\langle \nabla S, \nabla R \rangle_{L^2(\Omega')^d} + \langle \Delta S, \Delta R \rangle_{L^2(\Omega)} \stackrel{!}{=} 0 \quad \forall R \in \mathcal{R}_{temp}.$$

Let us show S to be constant. For this, we consider three types of testfunctions.

We start with the following. Let $\{\psi_i^{\text{dir}}\}_{i \in \mathbb{N}}$ be the eigenfunctions associated with positive eigenvalues $(\mu_i)_{i \in \mathbb{N}}$ of the homogenous Dirichlet Laplace problem,

$$\begin{aligned} -\Delta \psi_i^{\text{dir}} &= \mu_i \psi_i^{\text{dir}}, \\ \psi_i^{\text{dir}}|_{\partial\Omega} &= 0. \end{aligned} \quad (3.2.9)$$

Due to (H1), those functions are H^4 -regular. Then define, for all $i \in \mathbb{N}$,

$$\tilde{p}_i := \begin{cases} \psi_i^{\text{dir}} & \text{in } \Omega, \\ 0 & \text{in } \Omega' \setminus \overline{\Omega} \end{cases}, \quad p_i := \tilde{p}_i - \oint_{\Omega'} \tilde{p}_i \, dx \in \mathcal{R}. \quad (3.2.10)$$

According to Lemma A.3.1 in the appendix, the functions in (3.2.10) are admissible testfunctions, i.e. $p_i \in \mathcal{R}_{temp}$ for all $i \in \mathbb{N}$.

The next class of functions we wish to consider is defined as follows by means of Definition 3.2.1 and Remark 3.2.2. Define

$$\begin{aligned} \tilde{q}_i &:= \begin{cases} L_{\Omega}^{-1} u_i^{\Omega}|_{\partial\Omega} & \text{in } \Omega \\ L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} u_i^{\Omega}|_{\partial\Omega} & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} & \text{in } \Omega' \end{cases} \quad \forall i \in \mathbb{N}_0, \\ q_i &:= \tilde{q}_i - \oint_{\Omega'} \tilde{q}_i \, dx \in \mathcal{R} \quad \forall i \in \mathbb{N}_0. \end{aligned} \quad (3.2.11)$$

By Lemma A.3.2 those functions are admissible testfunctions, i.e. elements of $\mathcal{R}_{temp}$.

The last set of functions we consider is $\{\mathbf{R}_{2i+1}\}_{i \in \mathbb{N}_0}$. Obviously, those functions are in $\mathcal{R}_{temp}$.

Now, plugging in those testfunctions, starting with (3.2.10), we get for all $i \in \mathbb{N}$

$$\begin{aligned} 0 &= \langle \nabla S, \nabla p_i \rangle_{L^2(\Omega')^d} + \langle \Delta S, \Delta p_i \rangle_{L^2(\Omega)} \\ &\stackrel{(3.2.10)}{=} \langle \nabla S, \nabla \psi_i^{\text{dir}} \rangle_{L^2(\Omega)^d} + \langle \Delta S, \Delta \psi_i^{\text{dir}} \rangle_{L^2(\Omega)} \\ &\stackrel{(3.2.9)}{=} -(1 + \mu_i) \langle \Delta S, \psi_i^{\text{dir}} \rangle_{L^2(\Omega)}, \end{aligned}$$

which implies $\nabla S \in H(\text{div}_0, \text{curl}_0)(\Omega)$. And consequently, we get for all $i \in \mathbb{N}_0$, using the functions (3.2.11),

$$\begin{aligned} 0 &= \langle \nabla S, \nabla q_i \rangle_{L^2(\Omega')^d} + 0 = \langle \nabla S, \nabla \tilde{q}_i \rangle_{L^2(\Omega')^d} \\ &= -0 + \langle \nabla S \cdot \boldsymbol{\nu}, u_i^{\Omega} \rangle_{(H^{\frac{1}{2}}(\partial\Omega))' \times H^{\frac{1}{2}}(\partial\Omega)} \\ &\quad - 0 + \left\langle \nabla S \cdot \boldsymbol{\nu}, \begin{cases} u_i^{\Omega} & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} \right\rangle_{(H^{\frac{1}{2}}(\partial(\Omega' \setminus \overline{\Omega})))' \times H^{\frac{1}{2}}(\partial(\Omega' \setminus \overline{\Omega}))} \\ &= \langle [\nabla S] \cdot \boldsymbol{\nu}, u_i^{\Omega} \rangle_{(H^{\frac{1}{2}}(\partial\Omega))' \times H^{\frac{1}{2}}(\partial\Omega)}, \end{aligned}$$

where $[\nabla S] \cdot \boldsymbol{\nu}$ denotes the normal jump of ∇S on $\partial\Omega$. As the functions $\{u_i^{\Omega}|_{\partial\Omega}\}_{i \in \mathbb{N}_0}$, cf. (3.2.3) and Remark 3.2.2, generate $H^{\frac{1}{2}}(\partial\Omega)$, this implies that ∇S is in $H(\text{div})(\Omega')$ globally and therefore $\nabla S \in H(\text{div}_0, \text{curl}_0)(\Omega')$. By the last class of functions we similarly get

$$0 = \langle \nabla S, \nabla \mathbf{R}_{2i+1} \rangle_{L^2(\Omega')^d} = \langle \nabla S, \nabla \tilde{\mathbf{R}}_{2i+1} \rangle_{L^2(\Omega')^d} = -0 + \langle \nabla S \cdot \boldsymbol{\nu}, u_i^{\Omega'} \rangle_{(H^{\frac{1}{2}}(\partial\Omega'))' \times H^{\frac{1}{2}}(\partial\Omega')}$$

for all $i \in \mathbb{N}_0$. We finally conclude (on simply connected Ω' with $C^{1,1}$ -boundary [8]) that $\nabla S \in H_{n0}(\operatorname{div}_0, \operatorname{curl}_0)(\Omega') = \{\mathbf{0}\}$ and therefore $S \in \mathcal{R}$ is constant with zero mean, i.e. $S = 0$. $\square$

In Sect. 5 we will use Galerkin approximate solutions to obtain a solution to our model. The usual approach is to define a space given as the linear hull of only finitely many elements of our complete sets for $\mathcal{M}$ and $\mathcal{R}$. For this, it is crucial that those sets are linearly independent. The generating set of $\mathcal{M}$ already is a basis. In the case of $\mathcal{R}$ linear independency is not evident. Moreover, for later purposes, we want to orthogonalize parts of the already established generating set $\{\mathbf{R}_i\}_{i \in \mathbb{N}}$. Those issues are addressed in the following lemma.

Lemma 3.2.4. *There exists a basis $\{\psi_i^R\}_{i \in \mathbb{N}}$ of $\mathcal{R}$, i.e.*

$$\mathcal{R} = \overline{\operatorname{span}\{\psi_k^R\}_{k \in \mathbb{N}}}^{\mathcal{R}}, \quad (3.2.12)$$

which satisfies the following.

- (i) The set $\{\nabla \psi_{2i}^R|_{\Omega}\}_{i \in \mathbb{N}}$ is orthonormal in $L^2(\Omega)^d$.
- (ii) The sets $\{\nabla \psi_{4i}|_{\Omega}\}_{i \in \mathbb{N}}$ and $\{\nabla \psi_{4i-2}\}_{i \in \mathbb{N}}$ are orthogonal in $H^3(\Omega)^d$.
- (iii) The set $\{\nabla \psi_{2i-1}^R\}_{i \in \mathbb{N}}$ is orthonormal in $L^2(\Omega')^d$.

Proof. The basis is a linearly independent selection of functions from $\{\mathbf{R}_i\}_{i \in \mathbb{N}}$. Therefore, by construction we will have i) and ii), see (3.2.1) and Lemma 3.1.4.

Consider the set $\{\mathbf{R}_{2i}\}_{i=1,\dots,n}$. This set is linearly independent as it is linearly independent on Ω due to $\nabla \mathbf{R}_{2i}|_{\Omega} = \nabla \phi_i^{\Omega} = \mathbf{h}_i$. From the set $\{\mathbf{R}_{2i-1}\}_{i \in \mathbb{N}}$ we can pick n elements by induction. We start with $\mathbf{R}_{i_1} := \mathbf{R}_1$. If $1 \leq j \leq n-1$ elements $\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_j}$ have been picked, then add $\mathbf{R}_{i_{j+1}} := \mathbf{R}_K$, where $K \geq i_j$ is the lowest odd integer such that $\{\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_j}, \mathbf{R}_K\}$ is linearly independent. If this procedure fails for some $n \in \mathbb{N}$, the

range $L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{Bmatrix} 0 & \text{on } \partial\Omega \\ H^{\frac{1}{2}}(\partial\Omega') & \text{on } \partial\Omega' \end{Bmatrix}$ was finite dimensional. Then $L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{Bmatrix} 0 & \text{on } \partial\Omega \\ H^{\frac{1}{2}}(\partial\Omega') & \text{on } \partial\Omega' \end{Bmatrix}|_{\partial(\Omega' \setminus \overline{\Omega})} \cong H^{\frac{1}{2}}(\partial\Omega')$ is finite dimensional, which is a contradiction. Let

$$\psi_{2k}^R := \mathbf{R}_{2k}, \quad \tilde{\psi}_{2k-1}^R := \mathbf{R}_{i_k} \quad \forall k \in \mathbb{N}. \quad (3.2.13)$$

It remains to find functions ψ_{2i-1}^R , $i \in \mathbb{N}$, such that iii) is satisfied.

By applying Gram-Schmidt-orthogonalization to the functions $\{\tilde{\psi}_{2i-1}^R\}_{i \in \mathbb{N}}$, we get new functions $\{\psi_{2i-1}^R\}_{i \in \mathbb{N}}$, of which we require

$$\langle \nabla \psi_{2i-1}^R, \nabla \psi_{2j-1}^R \rangle_{L^2(\Omega')^d} = \delta_{ij} \quad \forall i, j \in \mathbb{N}. \quad (3.2.14)$$

Due to Poincaré's inequality the bilinear form above is an inner product and our procedure is well-posed. $\square$

Lemma 3.2.5. *The basis $\{\psi_i^R\}_{i \in \mathbb{N}}$ from Lemma 3.2.4, cf. (3.2.12), satisfies*

$$\begin{aligned} \nabla \psi_{4i}^R|_{\Omega} &= \mathbf{s}_i = \mathbf{h}_{2i}, \\ \nabla \psi_{4i-2}^R|_{\Omega} &= \mathbf{s}_i^{\perp} = \mathbf{h}_{2i-1}, \\ \nabla \psi_{2i-1}^R|_{\Omega} &= 0, \\ \nabla \psi_{4i}^R|_{\Omega' \setminus \overline{\Omega}} &= 0. \end{aligned} \quad (3.2.15)$$

Proof. The application of Gram-Schmidt-orthogonalization in the proof of Lemma 3.2.4 only orthogonalizes the basis functions with odd indices, therefore the properties of $\nabla \mathbf{R}_{2i}|_{\Omega} = \mathbf{h}_i$ do not change. Also, $\psi_{2i-1}^R|_{\Omega}$ is still constant.

A review of the basis of $\mathcal{R}$ yields

$$\psi_{4i}^R = \mathbf{R}_{4i} = \underbrace{\tilde{R}_{4i}}_{\Pi} - \oint_{\Omega'} \tilde{R}_{4i} \, dx$$

$$\left\{ \begin{array}{ll} \phi_{2i}^\Omega - c_{2i} & \text{in } \Omega, \\ L_{\Omega' \setminus \overline{\Omega}}^{-1} \left\{ \begin{array}{ll} \phi_{2i}^\Omega|_{\partial\Omega} - c_{2i} & \text{on } \partial\Omega, \\ 0 & \text{on } \partial\Omega', \end{array} \right\} & \text{in } \Omega' \setminus \overline{\Omega}. \end{array} \right\} = \left\{ \begin{array}{ll} \phi_{2i}^\Omega - c_{2i} & \text{in } \Omega, \\ 0 & \text{in } \Omega' \setminus \overline{\Omega}, \end{array} \right.$$

because $\nabla \psi_{4i}^R|_\Omega = \nabla \tilde{R}_{4i}|_\Omega = \nabla \phi_{2i}^\Omega = \mathbf{h}_{2i} = \mathbf{s}_i$ and therefore a potential in $H_0^1(\Omega)$ exists implying $\phi_{2i}^\Omega|_{\partial\Omega} \equiv c_{2i}$ and $\psi_{4i}^R|_{\Omega' \setminus \overline{\Omega}} \equiv -f_{\Omega'} \mathbf{R}_{4i} \, dx$. In conclusion, the claim follows. $\square$

4. Construction of Discrete Spaces for Velocity Field and Particle Density

In this section, we briefly specify our choice of Galerkin functions to approximate particle density and flow field.

Observing that

$$\mathcal{U} := H^3(\Omega)^d \cap H(\operatorname{div}_0)(\Omega) \cap H_0^1(\Omega)^d, \quad (4.1)$$

is a dense subset of $H_{n0}(\operatorname{div}_0)(\Omega) \subset L^2(\Omega)^d$ (see [15, Theorem 3.4]) and a closed subset of $H^3(\Omega)^d$, Lemma A.2.1 implies the existence of a basis $\{\Psi_i^u\}_{i \in \mathbb{N}}$ such that

$$\langle \Psi_i^u, \Psi_j^u \rangle_{H^3(\Omega)^d} = \delta_{ij} \quad \forall i, j \in \mathbb{N}, \quad (4.2)$$

$$\langle \Psi_i^u, \Psi_j^u \rangle_{H^3(\Omega)^d} = 0 \quad \forall i, j \in \mathbb{N} \text{ with } i \neq j. \quad (4.3)$$

$$\mathcal{U} = \overline{\operatorname{span}\{\Psi_i^u\}_{i \in \mathbb{N}}}^{H^3(\Omega)^d} \quad (4.4)$$

For the particle density, we follow the usual approach and take the complete set $\{\psi_i^c\}_{i \in \mathbb{N}}$ of eigenfunctions of the Laplacian on Ω subjected to homogeneous Neumann boundary conditions. By standard results, they are H^2 -regular, H^1 -orthogonal and form a basis of $L^2(\Omega)^d$, i.e.

$$\langle \psi_i^c, \psi_j^c \rangle_{L^2(\Omega)} = \delta_{ij} \quad \forall i, j \in \mathbb{N}, \quad (4.5)$$

$$\langle \nabla \psi_i^c, \nabla \psi_j^c \rangle_{L^2(\Omega)^d} = 0 \quad \forall i, j \in \mathbb{N} \text{ with } i \neq j, \quad (4.6)$$

$$L^2(\Omega) = \overline{\operatorname{span}\{\psi_i^c\}_{i \in \mathbb{N}}}^{L^2(\Omega)}. \quad (4.7)$$

5. Existence of Discrete Solutions

In this section, we prove the existence of global solutions to a discretization of an appropriate transport and mobility regularized version of model (1.1). We call this regularization the *TMR-model*. It differs from model (1.1) by a new viscosity term in (1.1c) and a cut-off near zero applied to c -terms in the denominator of $\mathbf{V}_{\text{part}}$. In addition, we use a regularized entropy g_s^L as well. It reads

$$-1 \leq g_s^L(c) := \begin{cases} \frac{c^2}{2s} + (\log s - 1)c - \frac{s}{2} & \text{for } c \leq s, \\ c \log c - c & \text{for } s < c < L, \\ \frac{c^2}{2L} + (\log L - 1)c - \frac{L}{2} & \text{for } L \leq c. \end{cases} \quad (5.1)$$

Obviously,

$$(g_s^L)'(c) = \begin{cases} \frac{c}{s} + \log s - 1 & \text{for } c \leq s, \\ \log c & \text{for } s < c < L, \\ \frac{c}{L} + \log L - 1 & \text{for } L \leq c, \end{cases} \quad (g_s^L)''(c) = \begin{cases} \frac{1}{s} & \text{for } c \leq s, \\ \frac{1}{c} & \text{for } s < c < L, \\ \frac{1}{L} & \text{for } L \leq c. \end{cases}$$

We allow for the choice $L = \infty$ where impossible conditions—e.g. $\infty \leq c$ —are skipped. Without loss of generality, we assume $0 < s < e < L$, where e denotes the Euler number. Altogether, for the TMR-model, equations (1.1c) and (1.1d) are replaced by

$$\mathbf{V}_{\text{part}} = -KD \frac{f_2(c)}{c_s} \nabla (g_s^L)'(c) + K\mu_0 \frac{f_2(c)}{(c_s)^2} (\nabla(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a - \alpha_3 \mathbf{m}))^T \mathbf{m}, \quad (5.2)$$

$$c_t + \mathbf{u} \cdot \nabla c + \operatorname{div}(c_s \mathbf{V}_{\text{part}}) = \sigma_c \Delta c, \quad (5.3)$$

Here,

$$(\cdot)_s := \max\{s, (\cdot)\}. \quad (5.4)$$

The boundary condition (1.2b) is adapted accordingly, i.e. $c \mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu}|_{\partial\Omega} = \sigma_c \nabla c \cdot \boldsymbol{\nu}|_{\partial\Omega}$. In our weak solution concept the pressure vanishes by the use of solenoidal testfunctions in the Navier-Stokes equations. We will make use of the space

$$H_*^2(\Omega) := \{\psi \in H^2(\Omega) | \nabla \psi \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0\}. \quad (5.5)$$

Also recall the definitions of $\mathcal{M}$ and $\mathcal{U}$ from (3.3) and (4.1). We define

$$\tilde{\mathcal{M}} \text{ to be identical to } \mathcal{M} \text{ but equipped with the norm of the sum } \mathcal{M} = \mathcal{S} \oplus \mathcal{S}^\circ \oplus \mathcal{V}, \quad (5.6)$$

which is the sum of all H^3 -norms of the individual summands.

Definition 5.1. Let initial data

$$\mathbf{u}^0 \in H_{n0}(\operatorname{div}_0)(\Omega), \quad c^0 \in L^2(\Omega; \mathbb{R}_0^+), \quad \mathbf{m}^0 \in L^2(\Omega)^d$$

be given and take $0 < s < L < \infty$ arbitrarily, but fixed.

Also let $\mathbf{h}_a \in H^1(I; H^1(\Omega')^d \cap H(\operatorname{div}_0, \operatorname{curl}_0)(\Omega'))$. We call the functions

$$\mathbf{u} \in L^2(I; H_0^1(\Omega)^d \cap H(\operatorname{div}_0)(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap W^{1,2}(I; (W^{1,\infty}(\Omega)^d)'),$$

$$c \in L^2(I; H^1(\Omega)) \cap L^\infty(I; L^2(\Omega)) \cap W^{1,5/4}(I; (W^{1,5}(\Omega))')$$

$$R \in L^2(I; \mathcal{R}) \cap L^\infty(I; H^1(\Omega)) \cap L^2(I; H_{\text{loc}}^2(\Omega))$$

$$\mathbf{m} \in L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H_{\text{loc}}^1(\Omega)^d) \cap W^{1,2}(I; (W^{1,\infty}(\Omega)^d)'),$$

a weak solution of (1.1a)–(1.1b), (5.3), (5.2), (1.1e)–(1.1g), (1.2), (1.3) with $g(\cdot)$ replaced by its regularization $g_s^L(\cdot)$, if for all testfunctions

$$\mathbf{v} \in L^2(I; (H(\operatorname{div}_0)(\Omega) \cap H_0^3(\Omega)^d)),$$

$$\psi \in L^5(I; H_*^2(\Omega)),$$

$$S \in \mathcal{R},$$

$$\boldsymbol{\theta} \in L^2(I; H_0^3(\Omega)^d),$$

the equations

$$\begin{aligned} & \rho_0 \int_0^T \langle \mathbf{u}_t, \mathbf{v} \rangle_{\mathcal{U}' \times \mathcal{U}} dt + \int_0^T \int_\Omega 2\eta \mathbf{D}\mathbf{u} \cdot \mathbf{D}\mathbf{v} dx dt \\ & + \frac{\rho_0}{2} \int_0^T \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} dx dt - \frac{\rho_0}{2} \int_0^T \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{u} dx dt \\ & = -\underbrace{\mu_0 \int_0^T \int_\Omega \left((\mathbf{m} \cdot \nabla) \mathbf{v} \cdot (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_a) + \operatorname{div} \mathbf{m} \mathbf{v} \cdot (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_a) \right) dx dt}_{= \mu_0 \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla) (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{m} dx dt} \\ & + \frac{\mu_0}{2} \int_0^T \int_\Omega (\mathbf{m} \times (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_a)) \cdot \operatorname{curl} \mathbf{v} dx dt, \end{aligned} \quad (5.7a)$$

$$\begin{aligned} & \int_0^T \langle c_t, \psi \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} dt - \int_0^T \int_{\Omega} c_s \mathbf{u} \cdot \nabla \psi \, dx \, dt + \sigma_c \int_0^T \int_{\Omega} \nabla c \cdot \nabla \psi \, dx \, dt \\ & - \int_0^T \int_{\Omega} \frac{K f_2(c)}{c_s} \left(-D c_s \nabla (g_s^L)'(c) + \mu_0 (\nabla (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_a - \alpha_3 \mathbf{m}))^T \mathbf{m} \right) \cdot \nabla \psi \, dx \, dt = 0, \end{aligned} \quad (5.7b)$$

$$\int_{\Omega'} \nabla R \cdot \nabla S \, dx = \int_{\Omega'} \mathbf{h}_a \cdot \nabla S \, dx - \int_{\Omega} \mathbf{m} \cdot \nabla S \, dx \quad \text{for almost all } t \in [0, T], \quad (5.7c)$$

$$\begin{aligned} & \int_0^T \langle \mathbf{m}_t, \boldsymbol{\theta} \rangle_{\tilde{\mathcal{M}}' \times \tilde{\mathcal{M}}} dt \\ & - \int_0^T \int_{\Omega} \left(\left(\mathbf{u} + \frac{K f_2(c)}{(c_s)^2} [-D c_s \nabla (g_s^L)'(c) + \right. \right. \\ & \quad \left. \left. \mu_0 (\nabla (\alpha_1 \nabla R + \frac{\beta}{2} \mathbf{h}_a - \alpha_3 \mathbf{m}))^T \mathbf{m} \right] \right) \cdot \nabla \right) \boldsymbol{\theta} \cdot \mathbf{m} \, dx \, dt \\ & + \sigma \int_0^T \int_{\Omega} \operatorname{div} \mathbf{m} \cdot \operatorname{div} \boldsymbol{\theta} \, dx \, dt + \sigma \int_0^T \int_{\Omega} \operatorname{curl} \mathbf{m} \cdot \operatorname{curl} \boldsymbol{\theta} \, dx \, dt \\ & = \frac{1}{2} \int_0^T \int_{\Omega} (\mathbf{m} \times \boldsymbol{\theta}) \cdot \operatorname{curl} \mathbf{u} \, dx \, dt - \frac{1}{\tau_{\text{rel}}} \int_0^T \int_{\Omega} (\mathbf{m} - \chi(c, \nabla R) \nabla R) \cdot \boldsymbol{\theta} \, dx \, dt \end{aligned} \quad (5.7d)$$

hold and

$$\begin{aligned} \langle \mathbf{u}(0), \mathbf{v} \rangle_{\mathcal{U}' \times \mathcal{U}} &= \int_{\Omega} \mathbf{u}^0 \cdot \mathbf{v} \, dx \quad \forall \mathbf{v} \in \mathcal{U}, \\ \langle c(0), \psi \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} &= \int_{\Omega} c^0 \cdot \psi \, dx \quad \forall \psi \in H_*^2(\Omega), \\ \langle \mathbf{m}(0), \boldsymbol{\theta} \rangle_{\tilde{\mathcal{M}}' \times \tilde{\mathcal{M}}} &= \int_{\Omega} \mathbf{m}^0 \cdot \boldsymbol{\theta} \, dx \quad \forall \boldsymbol{\theta} \in \tilde{\mathcal{M}}. \end{aligned}$$

Remark 5.2. Our choice of spaces of test functions for the momentum equation (5.7a) and the magnetization equation (5.7d) is a consequence of taking closures of the corresponding ansatz spaces $C([0, T]; C_0^\infty(\Omega)^d \cap \mathcal{U})$ and $C([0, T]; C_0^\infty(\Omega)^d \cap \mathcal{M})$ in $L^2((0, T); H^3(\Omega)^d)$. Recall that by construction (cf. (3.3)) $C_0^\infty(\Omega)^d \subset \mathcal{M}$. Therefore, the closure of $C_0^\infty(\Omega)^d \cap \mathcal{M}$ in $H^3(\Omega)^d$ is given by $H_0^3(\Omega)^d$.

To identify the closure of $C_0^\infty(\Omega)^d \cap \mathcal{U}$ in $H^3(\Omega)^d$, we first note that the former space is equal to $H(\operatorname{div}_0)(\Omega) \cap C_0^\infty(\Omega)^d$ by definition of $\mathcal{U}$, cf. (4.1). To show that this closure is just $H(\operatorname{div}_0)(\Omega) \cap H_0^3(\Omega)^d$, we adapt the corresponding result for the closure in $H^1(\Omega)^d$ which can be found in [12, p. 149]. The only difference is that (in the notation of [12]) we need an H^3 -estimate and an compactness result for solutions w_k of the problem

$$\begin{cases} \operatorname{div} w_k = -\operatorname{div} v_k, \\ w_k \in H_0^3(\Omega)^d, \\ \int_{\Omega} \operatorname{div} v_k \, dx = 0, \end{cases}$$

for data $v_k \in C_0^\infty(\Omega)$. This is a consequence of Theorem 3.2 in [12].

Galerkin approximation

In the following we fix $\sigma_c > 0$ and $0 < s < e < L < \infty$, where e is Euler's number.

Definition 5.3. Let

$$\begin{aligned} \mathcal{U}_n &:= \operatorname{span}\{\boldsymbol{\Psi}_1^{\mathbf{u}}, \dots, \boldsymbol{\Psi}_{2n}^{\mathbf{u}}\}, \\ \mathcal{C}_n &:= \operatorname{span}\{\psi_1^c, \dots, \psi_{2n}^c\}, \\ \mathcal{R}_n &:= \operatorname{span}\{\psi_1^R, \dots, \psi_{2n}^R\}, \\ \mathcal{M}_n &:= \operatorname{span}\{\boldsymbol{\Psi}_1^{\mathbf{m}}, \dots, \boldsymbol{\Psi}_{2n}^{\mathbf{m}}\}, \end{aligned}$$

$$\mathcal{H}_n := \text{span}\{\Psi_{2i}^{\mathbf{m}}\}_{i=1,\dots,n},$$

with the functions $\Psi_i^{\mathbf{u}}$ from (4.4), ψ_i^c from (4.7), ψ_i^R from (3.2.12), $\Psi_i^{\mathbf{m}}$ from (3.1.13). Let

$$\Pi_{\mathcal{H}_n} : \nabla[\mathcal{R}] \rightarrow \mathcal{H}_n \subset \mathcal{M}_n$$

be the (L^2) -orthogonal projection onto $\mathcal{M}_n$ defined by

$$\Pi_{\mathcal{H}_n} \mathbf{h} = \Pi_{\mathcal{H}_n} \left(\sum_{i=1}^{\infty} \alpha_i \nabla \psi_i^R \right) := \sum_{i=1}^{2n} \alpha_i \nabla \psi_i^R|_{\Omega} \stackrel{(3.2.15), (3.1.13)}{=} \sum_{i=1}^n \alpha_{2i} \widehat{\Psi_{2i}^{\mathbf{m}}}^{\mathfrak{h}_i}. \quad (5.8)$$

Moreover let

$$\Pi_{\mathcal{C}_n} : H^1(\Omega) \rightarrow \mathcal{C}_n,$$

$$\Pi_{\mathcal{U}_n} : \mathcal{U} \rightarrow \mathcal{U}_n,$$

$$\Pi_{\mathcal{R}_n} : \mathcal{R} \rightarrow \mathcal{R}_n,$$

$$\Pi_{\mathcal{M}_n} : \mathcal{M} \rightarrow \mathcal{M}_n$$

be the projections defined by

$$\begin{aligned} \Pi_{\mathcal{C}_n} g &= \Pi_{\mathcal{C}_n} \left(\sum_{i=1}^{\infty} \alpha_i \psi_i^c \right) := \sum_{i=1}^{2n} \alpha_i \psi_i^c, \\ \Pi_{\mathcal{U}_n} \mathbf{v} &= \Pi_{\mathcal{U}_n} \left(\sum_{i=1}^{\infty} \alpha_i \Psi_i^{\mathbf{u}} \right) := \sum_{i=1}^{2n} \alpha_i \Psi_i^{\mathbf{u}}, \\ \Pi_{\mathcal{R}_n} S &= \Pi_{\mathcal{R}_n} \left(\sum_{i=1}^{\infty} \alpha_i \psi_i^R \right) := \sum_{i=1}^{2n} \alpha_i \psi_i^R, \\ \Pi_{\mathcal{M}_n} \Phi &= \Pi_{\mathcal{M}_n} \left(\sum_{i=1}^{\infty} \alpha_i \Psi_i^{\mathbf{m}} \right) := \sum_{i=1}^{2n} \alpha_i \Psi_i^{\mathbf{m}}. \end{aligned} \quad (5.9)$$

Let

$$\mathcal{X}_n := C^1([0, T]; \mathcal{U}_n) \times C^1([0, T]; \mathcal{C}_n) \times H^1([0, T]; \mathcal{R}_n) \times C^1([0, T]; \mathcal{M}_n).$$

Remark 5.4. (i) The projection $\Pi_{\mathcal{H}_n}$, defined in (5.8), is well-defined, as can be seen by the following facts. A function $\psi \in \mathcal{R}$ can be written in terms of the basis (3.2.12) with convergence of infinite sums in the norm of $\mathcal{R}$, which dominates the $H^1(\Omega')$ -norm. Hence, ∇ and the infinite sum commute. Also restriction to Ω and the infinite sum commute. The projection is orthogonal due to L^2 -orthogonality, cf. (3.1.14), of the basis in (3.1.13).

- (ii) The projections $\Pi_{\mathcal{C}_n}$, $\Pi_{\mathcal{U}_n}$, $\Pi_{\mathcal{M}_n}$ are (at least L^2 -)orthogonal projections, see (4.5), (4.6) and (4.2), (4.3) and (3.1.14).
- (iii) For $\mathbf{u} \in C^1([0, T]; \mathcal{U}_n)$ we have the representation

$$\mathbf{u}(t, \mathbf{x}) = \sum_{i=1}^{2n} \alpha_i^{\mathbf{u}}(t) \Psi_i^{\mathbf{u}}(\mathbf{x}) \quad \forall t \in [0, T], \quad \mathbf{x} \in \Omega,$$

with coefficients $\alpha_i^{\mathbf{u}}$ in $C^1([0, T])$. A similar statement holds for $C^1([0, T]; \mathcal{U})$.

- (iv) Obviously, analogous statements as in iii) hold for the spaces related to $\mathcal{C}_n$ and $\mathcal{M}_n$, too. Concerning the space $\mathcal{R}_n$ the argument is more involved. One needs to distinguish between two cases. On Ω , L^2 -orthogonality of the gradients of the basis functions proves the analogous claim for coefficients associated to ψ_{2i}^R , $i \in \mathbb{N}$, see (3.2.15) and L^2 -orthogonality of (3.1.10) combined with (3.1.11). For basis functions with odd indices one can use the orthogonality given in (3.2.14).

Assume (H2). We know the following convergence behaviour of the projectors defined in (5.9),

$$\begin{aligned}
 \forall c \in H^1(\Omega) : \Pi_{\mathcal{C}_n} c &\xrightarrow{n \rightarrow \infty} c \text{ in } H^1(\Omega), \\
 \forall c \in H_*^2(\Omega) : \Pi_{\mathcal{C}_n} c &\xrightarrow{n \rightarrow \infty} c \text{ in } H^2(\Omega), \text{ see [16],} \\
 \forall \mathbf{u} \in \mathcal{U} : \Pi_{\mathcal{U}_n} \mathbf{u} &\xrightarrow{n \rightarrow \infty} \mathbf{u} \text{ in } H^3(\Omega)^d, \\
 \forall R \in \mathcal{R} : \Pi_{\mathcal{R}_n} R &\xrightarrow{n \rightarrow \infty} R \text{ in } \mathcal{R}, \\
 \forall \mathbf{m} \in \mathcal{M} : \Pi_{\mathcal{M}_n} \mathbf{m} &\xrightarrow{n \rightarrow \infty} \mathbf{m} \text{ in } H^3(\Omega)^d.
 \end{aligned} \tag{5.10}$$

Moreover, for orthogonal projections one has stability estimates in a standard way. Therefore independently of $n \in \mathbb{N}$,

$$\begin{aligned}
 \|\Pi_{\mathcal{C}_n} c\|_{H^k(\Omega)} &\stackrel{k=0:(4.5)}{\leq} \|c\|_{H^k(\Omega)} \quad \forall c \in H^1(\Omega), \quad k \in \{0, 1\}, \\
 \|\Pi_{\mathcal{U}_n} \mathbf{u}\|_{H^3(\Omega)^d} &\stackrel{(4.3)}{\leq} \|\mathbf{u}\|_{H^3(\Omega)^d} \quad \forall \mathbf{u} \in \mathcal{U}, \\
 \|\Pi_{\mathcal{M}_n} \mathbf{m}\|_{H^3(\Omega)^d} &\stackrel{\text{Lemma 3.1.4}}{\leq} \|\mathbf{m}^s\|_{H^3(\Omega)^d} + \|\mathbf{m}^{s^\perp}\|_{H^3(\Omega)^d} + \|\mathbf{m}^{h^\perp}\|_{H^3(\Omega)^d} \\
 \forall \mathbf{m} = \mathbf{m}^s + \mathbf{m}^{s^\perp} + \mathbf{m}^{h^\perp} &\in \mathcal{M} = \mathcal{S} \oplus \mathcal{S}^o \oplus \mathcal{V}.
 \end{aligned} \tag{5.11}$$

For the last stability estimate one has to use Minkowski's inequality first, splitting the various components of the basis functions into the three groups that belong either to $\mathcal{S}$, $\mathcal{S}^o$ or $\mathcal{V}$, respectively. One additionally has the stability estimate

$$\|\Pi_{\mathcal{C}_n} \psi\|_{H^2(\Omega)} \leq C \|\psi\|_{H^2(\Omega)} \quad \forall \psi \in H_*^2(\Omega), \text{ see [16],} \tag{5.12}$$

where C is independent of $n \in \mathbb{N}$. Moreover, one has

$$\begin{aligned}
 \|\Pi_{\mathcal{H}_n} \mathbf{h}\|_{L^2(\Omega)^d} &= \left\| \sum_{i=1}^n \alpha_{2i} \Psi_{2i}^{\mathbf{m}} \right\|_{L^2(\Omega)^d} \leq \|\mathbf{h}|_\Omega\|_{L^2(\Omega)^d} \quad \forall \mathbf{h} \in \nabla[\mathcal{R}], \\
 \|\Pi_{\mathcal{H}_n} \mathbf{h}\|_{H^3(\Omega)^d} &\stackrel{(3.1.14),(3.1.13)}{=} \left\| \sum_{i=1}^n \alpha_{2i} \Psi_{2i}^{\mathbf{m}} \right\|_{H^3(\Omega)^d} \leq \|\mathbf{h}^s|_\Omega\|_{H^3(\Omega)^d} + \|\mathbf{h}^{s^\perp}|_\Omega\|_{H^3(\Omega)^d} \\
 \forall \mathbf{h} &\in \nabla[\mathcal{R}] \cap (\mathcal{S} \oplus \mathcal{S}^o), \text{ where } \mathbf{h}|_\Omega := \mathbf{h}^s|_\Omega + \mathbf{h}^{s^\perp}|_\Omega, \\
 &\text{for uniquely determined } \mathbf{h}^s|_\Omega \in \mathcal{S}, \mathbf{h}^{s^\perp}|_\Omega \in \mathcal{S}^o,
 \end{aligned} \tag{5.13}$$

for all $n \in \mathbb{N}$.

Applying Lemma A.4.1 to all our projection operators, we get for arbitrary $p \in [1, \infty)$

$$\begin{aligned}
 \forall c \in L^p(I; H^1(\Omega)) : \Pi_{\mathcal{C}_n} c &\xrightarrow{n \rightarrow \infty} c \text{ in } L^p(I; H^1(\Omega)), \\
 \forall c \in L^p(I; H_*^2(\Omega)) : \Pi_{\mathcal{C}_n} c &\xrightarrow{n \rightarrow \infty} c \text{ in } L^p(I; H^2(\Omega)), \\
 \forall \mathbf{u} \in L^p(I; \mathcal{U}) : \Pi_{\mathcal{U}_n} \mathbf{u} &\xrightarrow{n \rightarrow \infty} \mathbf{u} \text{ in } L^p(I; H^3(\Omega)^d), \\
 \forall \mathbf{m} \in L^p(I; \mathcal{M}) : \Pi_{\mathcal{M}_n} \mathbf{m} &\xrightarrow{n \rightarrow \infty} \mathbf{m} \text{ in } L^p(I; H^3(\Omega)^d), \\
 \forall \mathbf{h} \in L^p(I; \nabla[\mathcal{R}] \cap H^3(\Omega)^d) : \Pi_{\mathcal{H}_n} \mathbf{h} &\xrightarrow{n \rightarrow \infty} \mathbf{h}|_\Omega \text{ in } L^p(I; H(\text{div})(\Omega)).
 \end{aligned} \tag{5.14}$$

We now introduce our Galerkin scheme for approximate solutions. For this, we make the ansatz

$$\mathbf{u}_n(t, x) := \sum_{i=1}^{2n} \alpha_i^{\mathbf{u}}(t) \Psi_i^{\mathbf{u}}(x) \quad \forall (t, x) \in I \times \Omega,$$

for c_n , R_n , $\mathbf{m}_n$ similarly. As an example, we write down the Galerkin scheme of the Navier-Stokes equations in detail. We look for $\alpha^{\mathbf{u}} := (\alpha_i^{\mathbf{u}}, \dots, \alpha_{2n}^{\mathbf{u}})$ such that for all $j = 1, \dots, 2n$, the equations

$$\begin{aligned} & \rho_0 \int_{\Omega} \sum_{i=1}^{2n} \partial_t \alpha_i^{\mathbf{u}}(t) \Psi_i^{\mathbf{u}} \cdot \Psi_j^{\mathbf{u}} \, dx + \frac{\rho_0}{2} \int_{\Omega} \sum_{k,i=1}^{2n} (\alpha_k^{\mathbf{u}}(t) \Psi_k^{\mathbf{u}} \cdot \nabla) \alpha_i^{\mathbf{u}}(t) \Psi_i^{\mathbf{u}} \cdot \Psi_j^{\mathbf{u}} \, dx \\ & - \frac{\rho_0}{2} \int_{\Omega} \sum_{k,i=1}^{2n} (\alpha_k^{\mathbf{u}}(t) \Psi_k^{\mathbf{u}} \cdot \nabla) \Psi_j^{\mathbf{u}} \cdot (\alpha_i^{\mathbf{u}}(t) \Psi_i^{\mathbf{u}}) \, dx + \int_{\Omega} \sum_{i=1}^{2n} 2\eta \alpha_i^{\mathbf{u}}(t) \mathbf{D} \Psi_i^{\mathbf{u}} \cdot \mathbf{D} \Psi_j^{\mathbf{u}} \, dx \\ & = \mu_0 \int_{\Omega} \sum_{k=1}^{2n} (\Psi_j^{\mathbf{u}} \cdot \nabla) (\alpha_1 \sum_{i=1}^{2n} \alpha_i^R(t) \nabla \psi_i^R + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a) \cdot (\alpha_k^{\mathbf{m}}(t) \Psi_k^{\mathbf{m}}) \, dx \\ & + \frac{\mu_0}{2} \int_{\Omega} \sum_{i=1}^{2n} \alpha_i^{\mathbf{m}}(t) \left(\Psi_i^{\mathbf{m}} \times \left(\alpha_1 \sum_{k=1}^{2n} \alpha_k^R(t) \nabla \psi_k^R + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a(t) \right) \right) \cdot \operatorname{curl} \Psi_j^{\mathbf{u}} \, dx \\ & - D \int_{\Omega} \left(\sum_{i=1}^{2n} \alpha_i^c(t) \psi_i^c \right)_s \nabla \Pi_{\mathcal{C}_n} (g_s^L)' \left(\sum_{k=1}^{2n} \alpha_k^c(t) \psi_k^c \right) \cdot \Psi_j^{\mathbf{u}} \, dx \end{aligned} \quad (5.15)$$

hold. The mass matrix $(M_{ij})_{i,j=1}^n$ with $M_{ij} := \langle \Psi_i^{\mathbf{u}}, \Psi_j^{\mathbf{u}} \rangle_{L^2(\Omega)^d}$ is invertible. Therefore, (5.15) can be written as a system of n ordinary differential equations in explicit form. We prefer a short notation for the full system using $\mathbf{u}_n$, c_n , R_n , $\mathbf{m}_n$. This way, we are looking for $(\mathbf{u}_n, c_n, R_n, \mathbf{m}_n) \in \mathcal{X}_n$ such that

$$\begin{aligned} & \rho_0 \int_{\Omega} \partial_t \mathbf{u}_n(t) \cdot \Psi_j^{\mathbf{u}} \, dx + \int_{\Omega} 2\eta \mathbf{D} \mathbf{u}_n(t) \cdot \mathbf{D} \Psi_j^{\mathbf{u}} \, dx \\ & + \frac{\rho_0}{2} \int_{\Omega} (\mathbf{u}_n(t) \cdot \nabla) \mathbf{u}_n(t) \cdot \Psi_j^{\mathbf{u}} \, dx - \frac{\rho_0}{2} \int_{\Omega} (\mathbf{u}_n(t) \cdot \nabla) \Psi_j^{\mathbf{u}} \cdot \mathbf{u}_n(t) \, dx \\ & = \mu_0 \int_{\Omega} (\Psi_j^{\mathbf{u}} \cdot \nabla) (\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a(t)) \cdot \mathbf{m}_n(t) \, dx \end{aligned} \quad (5.16a)$$

$$\begin{aligned} & + \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m}_n(t) \times (\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a(t))) \cdot \operatorname{curl} \Psi_j^{\mathbf{u}} \, dx \\ & - D \int_{\Omega} (c_n(t))_s \nabla \Pi_{\mathcal{C}_n} (g_s^L)' (c_n(t)) \cdot \Psi_j^{\mathbf{u}} \, dx, \\ & \int_{\Omega} \partial_t c_n(t) \psi_j^c \, dx - \int_{\Omega} (c_n(t))_s \mathbf{u}_n(t) \cdot \nabla \psi_j^c \, dx + \sigma_c \int_{\Omega} \nabla c_n(t) \cdot \nabla \psi_j^c \, dx \\ & - \int_{\Omega} K f_2(c_n(t)) \left(-D \nabla \Pi_{\mathcal{C}_n} (g_s^L)' (c_n(t)) \right) \end{aligned} \quad (5.16b)$$

$$\begin{aligned} & + \frac{\mu_0}{(c_n(t))_s} \left(\nabla (\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a(t) - \alpha_3 \mathbf{m}_n(t)) \right)^T \mathbf{m}_n(t) \cdot \nabla \psi_j^c \, dx = 0, \\ & \int_{\Omega'} \nabla R_n(t) \cdot \nabla \psi_j^R \, dx = \int_{\Omega'} \mathbf{h}_a(t) \cdot \nabla \psi_j^R \, dx - \int_{\Omega} \mathbf{m}_n(t) \cdot \nabla \psi_j^R \, dx, \end{aligned} \quad (5.16c)$$

$$\begin{aligned}
& \int_{\Omega} \partial_t \mathbf{m}_n(t) \cdot \Psi_j^{\mathbf{m}} dx \\
& - \int_{\Omega} \left(\left(\mathbf{u}_n(t) + \frac{K f_2(c_n(t))}{(c_n(t))_s} \left[-D \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n(t)) \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{\mu_0}{(c_n(t))_s} \left(\nabla (\alpha_1 \nabla R_n(t) + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a(t) - \alpha_3 \mathbf{m}_n(t)) \right)^T \mathbf{m}_n(t) \right] \right) \cdot \nabla \right) \Psi_j^{\mathbf{m}} \cdot \mathbf{m}_n(t) dx \\
& + \sigma \int_{\Omega} \operatorname{div} \mathbf{m}_n(t) \cdot \operatorname{div} \Psi_j^{\mathbf{m}} dx + \sigma \int_{\Omega} \operatorname{curl} \mathbf{m}_n(t) \cdot \operatorname{curl} \Psi_j^{\mathbf{m}} dx dt \\
& = \frac{1}{2} \int_{\Omega} (\mathbf{m}_n(t) \times \Psi_j^{\mathbf{m}}) \cdot \operatorname{curl} \mathbf{u}_n(t) dx \\
& - \frac{1}{\tau_{\text{rel}}} \int_{\Omega} (\mathbf{m}_n(t) - \chi(c_n(t), \nabla R_n(t)) \nabla R_n(t)) \cdot \Psi_j^{\mathbf{m}} dx,
\end{aligned} \tag{5.16d}$$

for all $j = 1, \dots, 2n$.

Remark 5.5. In contrast to (1.1a) an additional term $-D \int_{\Omega} (c_n(t))_s \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n(t)) \cdot \Psi_j^{\mathbf{u}} dx$ appears on the right-hand side of (5.16a). This term is required to obtain stability estimates. In the limit $n \rightarrow \infty$, it becomes a gradient. Hence it vanishes as the test functions are assumed to be solenoidal.

Note that the stiffness matrix in (5.16c) is invertible as well due to Poincaré's inequality and (5.16c) is not part of the ordinary differential equation. Instead its solution is just a function of the other unknowns and therefore weak differentiability of $\mathbf{h}_a$ and, consequently, $\mathbf{h}_n := \nabla R_n$ is sufficient. Moreover, the susceptibility (2.1), and the operator $(\cdot)_s$ (5.4) are Lipschitz-continuous. All other nonlinear terms are obviously locally Lipschitz-continuous.

As our space dimension is at most $d \leq 3$, by Sobolev's embedding all terms are well-defined. In detail, the regularity of our basis functions implies that the unknowns $\nabla \mathbf{u}_n$, c_n , $\nabla \mathbf{m}_n$, $\nabla \nabla R_n|_{\Omega}$ are L^∞ -functions. Therefore, integrability of the terms in (5.16) is evident. Naturally, due to the density results $\overline{U}^{L^2(\Omega)^d} = H_{n0}(\operatorname{div}_0)(\Omega)^d$, $\overline{H^1(\Omega)}^{L^2(\Omega)} = L^2(\Omega)$, $\overline{\mathcal{M}}^{L^2(\Omega)^d} = L^2(\Omega)^d$ we can choose discrete initial data such that

$$\mathbf{u}_n \ni \mathbf{u}_n^0 \rightarrow \mathbf{u}^0 \text{ in } L^2(\Omega)^d, \quad \mathcal{C}_n \ni c_n^0 \rightarrow c^0 \text{ in } L^2(\Omega), \quad \mathcal{M}_n \ni \mathbf{m}_n^0 \rightarrow \mathbf{m}^0 \text{ in } L^2(\Omega)^d$$

for $n \rightarrow \infty$.

Lemma 5.6. *For any initial data $\mathbf{u}_n^0 \in \mathcal{U}_n$, $c_n^0 \in \mathcal{C}_n$, $\mathbf{m}_n^0 \in \mathcal{M}_n$, system (5.16) has a global solution.*

Proof. By the Picard-Lindelöf Theorem, the system above has a unique local solution that attains any prescribed initial data $\mathbf{u}_n^0 \in \mathcal{U}_n$, $c_n^0 \in \mathcal{C}_n$, $\mathbf{m}_n^0 \in \mathcal{M}_n$. The local solution is indeed a global solution on $[0, T]$ due to a priori estimates which we will prove next.

Step 1: Basic integral estimates.

For fixed $t \in [0, T]$, we multiply (5.16) by the coefficient functions $\alpha_j^{(\cdot)}(t)$ at time t and sum up over all $j = 1, \dots, 2n$. Note that we have $(\nabla \mathcal{R}_n)|_{\Omega} \subset \mathcal{M}_n$, which makes it possible to take the magnetic field $\mathbf{h} = \nabla R|_{\Omega}$, $R \in \mathcal{R}_n$ as testfunction for the magnetization $\mathbf{m} \in \mathcal{M}_n$.

As a first step we test

- (5.16a) with $\mathbf{u}_n$,
- (5.16b) with $D \Pi_{\mathcal{C}_n} (g_s^L)'(c_n)$,
- (5.16c) with $\frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} R_n$,
- (5.16d) with $-\mu_0 (\alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_3 \mathbf{m}_n)$
- and the weak time derivative—note that the coefficients $\alpha_j^{(\cdot)}$ are weakly differentiable and $\mathbf{h}_a$ is weakly differentiable in time—of (5.16c) with $\mu_0 \alpha_1 R_n$.

For the ease of presentation we use the abbreviations

$$\tilde{\mathbf{h}}_n = \alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a$$

and

$$\tilde{\mathbf{b}}_n = \tilde{\mathbf{h}}_n - \alpha_3 \mathbf{m}_n.$$

A rather involved computation is related to the nonlinear testfunction $g'_s(c_n)$ which has to be projected onto $\mathbb{C}_n$. By H^1 -regularity of c_n and linear growth of $(g_s^L)'(c_n)$, the term $\Pi_{\mathbb{C}_n}(g_s^L)'(c_n)$ is well defined. One gets,

$$\begin{aligned} & D \int_{\Omega} \underbrace{\partial_t c_n}_{\in \mathbb{C}_n} \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \, dx - D \int_{\Omega} (c_n)_s \mathbf{u}_n \cdot \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \, dx \\ & + D \sigma_c \int_{\Omega} \nabla c_n \cdot \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \, dx + K D^2 \int_{\Omega} f_2(c_n) |\nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n)|^2 \, dx \\ & - K D \mu_0 \int_{\Omega} \frac{f_2(c_n)}{(c_n)_s} (\nabla \tilde{\mathbf{b}}_n)^T \mathbf{m}_n \cdot \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \, dx \\ & \stackrel{\text{orthogonality}}{=} D \partial_t \int_{\Omega} g_s^L(c_n) \, dx - D \int_{\Omega} (c_n)_s \mathbf{u}_n \cdot \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \, dx \\ & \stackrel{\text{see (4.7)}}{=} D \partial_t \int_{\Omega} g_s^L(c_n) \, dx + D \sigma_c \int_{\Omega} (g_s^L)''(c_n) |\nabla c_n|^2 \, dx + K D^2 \int_{\Omega} f_2(c_n) |\nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n)|^2 \, dx \\ & - K D \mu_0 \int_{\Omega} \frac{f_2(c_n)}{(c_n)_s} (\nabla \tilde{\mathbf{b}}_n)^T \mathbf{m}_n \cdot \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \, dx \\ & = 0. \end{aligned} \tag{5.17}$$

The other computations are straightforward and we easily get

$$\begin{aligned} & \frac{\rho_0}{2} \partial_t \int_{\Omega} |\mathbf{u}_n|^2 \, dx + 2\eta \int_{\Omega} |\mathbf{D} \mathbf{u}_n|^2 \, dx \\ & = \mu_0 \int_{\Omega} (\mathbf{u}_n \cdot \nabla) \tilde{\mathbf{h}}_n \cdot \mathbf{m}_n \, dx + \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m}_n \times \tilde{\mathbf{h}}_n) \cdot \text{curl } \mathbf{u}_n \, dx \\ & - D \int_{\Omega} (c_n)_s \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) \cdot \mathbf{u}_n \, dx \end{aligned} \tag{5.18}$$

and

$$\frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 \, dx = \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h}_a \cdot \nabla R_n \, dx - \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{m}_n \cdot \nabla R_n \, dx \tag{5.19}$$

and

$$\begin{aligned} & - \mu_0 \int_{\Omega} \partial_t \mathbf{m}_n \cdot \tilde{\mathbf{b}} \, dx \\ & + \mu_0 \int_{\Omega} ((\mathbf{u}_n + \frac{f_2(c_n)K}{(c_n)_s}) [-D \nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n) + \mu_0 \frac{1}{(c_n)_s} (\nabla \tilde{\mathbf{b}}_n)^T \mathbf{m}_n]) \cdot \nabla \tilde{\mathbf{b}}_n \cdot \mathbf{m}_n \, dx \\ & - \sigma \mu_0 \int_{\Omega} \text{div } \mathbf{m}_n \text{div } \tilde{\mathbf{b}}_n \, dx - \sigma \mu_0 \int_{\Omega} \text{curl } \mathbf{m}_n \cdot \text{curl } \tilde{\mathbf{b}}_n \, dx \\ & = - \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m}_n \times \tilde{\mathbf{b}}_n) \cdot \text{curl } \mathbf{u}_n \, dx + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} (\mathbf{m}_n - \chi(c_n, \nabla R_n) \nabla R_n) \cdot \tilde{\mathbf{b}}_n \, dx \end{aligned} \tag{5.20}$$

and

$$\frac{\mu_0 \alpha_1}{2} \partial_t \int_{\Omega'} |\nabla R_n|^2 \, dx = \mu_0 \alpha_1 \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \nabla R_n \, dx - \mu_0 \alpha_1 \int_{\Omega'} \partial_t \mathbf{m}_n \cdot \nabla R_n \, dx. \tag{5.21}$$

Summing up equations (5.17)–(5.21) and using $\mathbf{m}_n \times \mathbf{m}_n = \mathbf{0}$ and $\text{div } \mathbf{u}_n = 0$, one arrives—as an intermediate step—as

$$D \partial_t \int_{\Omega} g_s^L(c_n) \, dx + D \sigma_c \int_{\Omega} (g_s^L)''(c_n) |\nabla c_n|^2 \, dx + K D^2 \int_{\Omega} f_2(c_n) |\nabla \Pi_{\mathbb{C}_n}(g_s^L)'(c_n)|^2 \, dx$$

$$\begin{aligned}
& -2KD\mu_0 \int_{\Omega} \frac{f_2(c_n)}{(c_n)_s} (\nabla \tilde{\mathbf{b}}_n)^T \mathbf{m}_n \cdot \nabla \Pi_{c_n} (g_s^L)'(c_n) dx + \mu_0^2 \int_{\Omega} \frac{f_2(c_n)K}{(c_n)_s^2} |\nabla \tilde{\mathbf{b}}_n^T \mathbf{m}_n|^2 dx \\
& + \frac{\rho_0}{2} \partial_t \int_{\Omega} |\mathbf{u}_n|^2 dx + 2\eta \int_{\Omega} |\mathbf{D}\mathbf{u}_n|^2 dx + \frac{\mu_0\alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 dx + \frac{\mu_0\alpha_3}{2} \partial_t \int_{\Omega} |\mathbf{m}_n|^2 dx \\
& - \sigma\mu_0 \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \tilde{\mathbf{b}}_n dx - \sigma\mu_0 \int_{\Omega} \operatorname{curl} \mathbf{m}_n \cdot \operatorname{curl} \tilde{\mathbf{b}}_n dx + \frac{\mu_0\alpha_1}{2} \partial_t \int_{\Omega'} |\nabla R_n|^2 dx \\
& = \mu_0\alpha_1 \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \nabla R_n dx + \frac{\mu_0\beta}{2} \int_{\Omega} \partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a dx \\
& + \frac{\mu_0\alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h}_a \cdot \nabla R_n dx - \frac{\mu_0\alpha_1}{\tau_{\text{rel}}} \int_{\Omega} \mathbf{m}_n \cdot \nabla R_n dx \\
& + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} (\mathbf{m}_n - \chi(c_n, \nabla R_n) \nabla R_n) \cdot \tilde{\mathbf{b}}_n dx.
\end{aligned}$$

Note that $\Pi_{\mathcal{H}_n} \mathbf{h}_a \in \nabla[H^1(\Omega)]$ and therefore $\operatorname{curl} \Pi_{\mathcal{H}_n} \mathbf{h}_a = 0$, see (5.8), (3.1.13) and (3.1.1) for further details. Consequently, further simplification yields

$$\begin{aligned}
& \frac{\rho_0}{2} \partial_t \|\mathbf{u}_n\|_{L^2(\Omega)^d}^2 + 2\eta \|\mathbf{D}\mathbf{u}_n\|_{L^2(\Omega)^{d \times d}}^2 + D\partial_t \int_{\Omega} g_s^L(c_n) dx + D\sigma_c \|\sqrt{(g_s^L)''(c_n)} \nabla c_n\|_{L^2(\Omega)^d}^2 \\
& + K \|\sqrt{f_2(c_n)} [D\nabla \Pi_{c_n} (g_s^L)'(c_n) - \mu_0 \frac{1}{(c_n)_s} (\alpha_1 \nabla \nabla R_n + \frac{\beta}{2} \nabla \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_3 \nabla \mathbf{m}_n)^T \mathbf{m}_n]\|_{L^2(\Omega)^d}^2 \\
& + \frac{\mu_0\alpha_1}{\tau_{\text{rel}}} \|\nabla R_n\|_{L^2(\Omega')^d}^2 + \frac{\mu_0\alpha_3}{2} \partial_t \|\mathbf{m}_n\|_{L^2(\Omega)^d}^2 + \frac{\mu_0\alpha_3}{\tau_{\text{rel}}} \|\mathbf{m}_n\|_{L^2(\Omega)^d}^2 \\
& + \sigma\alpha_3\mu_0 \|\operatorname{div} \mathbf{m}\|_{L^2(\Omega)}^2 + \sigma\alpha_3\mu_0 \|\operatorname{curl} \mathbf{m}\|_{L^2(\Omega)^d}^2 + \frac{\mu_0\alpha_1}{2} \partial_t \|\nabla R_n\|_{L^2(\Omega')}^2 \\
& - \sigma\alpha_1\mu_0 \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \nabla R_n dx - \frac{\sigma\beta\mu_0}{2} \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \Pi_{\mathcal{H}_n} \mathbf{h}_a dx \\
& + \frac{\mu_0\alpha_1}{\tau_{\text{rel}}} \|\sqrt{\chi(c_n, \nabla R_n)} \nabla R_n\|_{L^2(\Omega)^d}^2 + \frac{\mu_0\beta}{2\tau_{\text{rel}}} \int_{\Omega} \chi(c_n, \mathbf{h}_n) \nabla R_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a dx \\
& = \mu_0\alpha_1 \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \nabla R_n dx + \frac{\mu_0\beta}{2} \int_{\Omega} \partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a dx + \frac{\mu_0\alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h}_a \cdot \nabla R_n dx \\
& + \frac{\mu_0\alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot \mathbf{m}_n dx + \frac{\mu_0\beta}{2\tau_{\text{rel}}} \int_{\Omega} \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a dx.
\end{aligned}$$

Step 2: We verify the identity

$$- \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \nabla R_n dx = \|\operatorname{div} \nabla R_n\|_{L^2(\Omega' \setminus \partial\Omega)}^2. \quad (5.22)$$

First, we need more information about the term $(\operatorname{div} \nabla R_n - (\operatorname{div} \nabla R_n)_{\Omega'}) \in \mathcal{R}$, where $(\operatorname{div} \nabla R_n)_{\Omega'}$ is the mean value of $\operatorname{div} \nabla R_n$ on Ω' . We quickly check that $\operatorname{div} \nabla R_n|_{\Omega' \setminus \bar{\Omega}} \equiv 0$ and $\operatorname{div} \nabla R_n|_{\Omega} \in H_0^1(\Omega)$ which is a consequence of the choice of the basis functions for $\mathcal{R}$, (3.2.12)². Their gradients on Ω are associated with the basis of $\mathcal{H}$, see (3.2.1), and the divergence of those functions vanishes at the boundary $\partial\Omega$. Hence, we can easily deduce weak differentiability on the whole domain Ω' . In preparation, we consider the part of the basis (3.2.12) whose gradients generate the space $\mathcal{S}$. The gradients of eigenfunctions $\{\psi_i^{\text{dir}}\}_{i \in \mathbb{N}}$ from the homogeneous Dirichlet–Laplace operator, see (3.2.9), are contained in $\mathcal{S}$. Obviously, $\bar{\mathcal{S}}^{H(\operatorname{div})(\Omega)} \subset \nabla[H_0^1(\Omega)] \cap H(\operatorname{div})(\Omega)$. If there was an element that could not be approximated, we would find $\nabla S \in \nabla[H_0^1(\Omega)] \cap H(\operatorname{div})(\Omega)$ such that

$$\forall i \in \mathbb{N} : 0 = \langle \nabla \psi_i^{\text{dir}}, \nabla S \rangle_{L^2(\Omega)^d} + \underbrace{\langle \Delta \psi_i^{\text{dir}}, \Delta S \rangle_{L^2(\Omega)}}_{>0} = - \langle \psi_i^{\text{dir}}, \Delta S \rangle_{L^2(\Omega)}.$$

²Retrospectively, this guarantees $\Delta R_n \in L^2(\Omega')$ and hence the mean value is well-defined.

With $S|_{\partial\Omega} = 0$ and $\Delta S = 0$ we get $S = 0$, hence

$$\bar{S}^{H(\operatorname{div})(\Omega)} = \nabla[H_0^1(\Omega)] \cap H(\operatorname{div})(\Omega).$$

Therefore, $\nabla(\operatorname{div} \nabla R_n - (\operatorname{div} \nabla R_n)_{\Omega'})|_{\Omega} \in \nabla[H_0^1(\Omega)] \cap H(\operatorname{div})(\Omega)$ can be written in terms of the basis functions $\nabla \psi_{4i}^R|_{\Omega} = \mathbf{s}_i$, see (3.2.15) and (3.1.10), with convergence of the infinite sum in the $H(\operatorname{div})(\Omega)$ -norm. We can easily deduce that $\operatorname{div} \nabla R_n - (\operatorname{div} \nabla R_n)_{\Omega'}$ can be written in terms of the basis functions ψ_{4i}^R , see (3.2.13), (3.2.5), (3.1.11), which just extend from their constant trace on $\partial\Omega$ constantly to $\Omega' \setminus \bar{\Omega}$ —cf. (3.2.15)—just as $\operatorname{div} \nabla R_n - (\operatorname{div} \nabla R_n)_{\Omega'}$ does. With the aforementioned basis functions ψ_{4i}^R , we represent $\operatorname{div} \nabla R_n - (\operatorname{div} \nabla R_n)_{\Omega'}$ with convergence of the infinite sum in the norm of $\mathcal{R}$. We now know that there is a sequence $(\psi_k)_{k \in \mathbb{N}}$,

$$\psi_k = \sum_{i=1}^{2k} a_i \psi_i^R, \quad \text{for given } (a_j)_{j \in \mathbb{N}} \subset \mathbb{R}, \text{ where } a_j = 0 \ \forall \mathbb{N} \ni j \notin 4\mathbb{N}, \quad (5.23)$$

such that

$$\begin{aligned} \mathcal{R}_k \ni \psi_k &\rightarrow \operatorname{div} \nabla R_n - (\operatorname{div} \nabla R_n)_{\Omega'} \text{ in } H^1(\Omega'), \\ \mathcal{H}_k \ni \nabla \psi_k|_{\Omega} &\rightarrow \nabla \operatorname{div} \nabla R_n|_{\Omega} \text{ in } L^2(\Omega)^d. \end{aligned}$$

Due to the boundary conditions of (3.2.12), the L^2 -orthogonality of (3.1.13) and the considerations above, we get

$$\begin{aligned} & -\sigma\alpha_1\mu_0 \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \nabla R_n \, dx \\ &= \sigma\alpha_1\mu_0 \int_{\Omega} \mathbf{m}_n \cdot \nabla \operatorname{div} \nabla R_n \, dx = \sigma\alpha_1\mu_0 \int_{\Omega} \mathbf{m}_n \cdot \lim_{k \rightarrow \infty} \nabla \psi_k \, dx \\ &= \sigma\alpha_1\mu_0 \lim_{k \rightarrow \infty} \int_{\Omega} \mathbf{m}_n \cdot \nabla \psi_k \, dx = \sigma\alpha_1\mu_0 \int_{\Omega} \mathbf{m}_n \cdot \nabla \psi_n \, dx \\ &\stackrel{(5.16c)}{=} \sigma\alpha_1\mu_0 \int_{\Omega'} \mathbf{h}_a \cdot \nabla \psi_n \, dx - \sigma\alpha_1\mu_0 \int_{\Omega'} \nabla R_n \cdot \nabla \psi_n \, dx \\ &\stackrel{\operatorname{div} \mathbf{h}_a = 0}{=} \sigma\alpha_1\mu_0 \int_{\partial\Omega'} \mathbf{h}_a \cdot \boldsymbol{\nu} \psi_n \, d\sigma - \sigma\alpha_1\mu_0 \int_{\Omega'} \nabla R_n \cdot \nabla \psi_n \, dx \\ &=: J_1 + J_2. \end{aligned}$$

In order to proceed, we need more information about ψ_n . We easily obtain $\psi_n|_{\Omega' \setminus \bar{\Omega}} \equiv \text{const.} =: C$ as the basis functions ψ_{4i}^R are constant on $\Omega' \setminus \bar{\Omega}$. Therefore,

$$J_1 = \sigma\alpha_1\mu_0 C \int_{\partial\Omega'} \mathbf{h}_a \cdot \boldsymbol{\nu} \, d\sigma = \sigma\alpha_1\mu_0 C \int_{\Omega'} \operatorname{div} \mathbf{h}_a \, dx = 0.$$

and

$$\begin{aligned} J_2 &= -\sigma\alpha_1\mu_0 \int_{\Omega} \nabla R_n \cdot \nabla \psi_n \, dx \\ &= -\sigma\alpha_1\mu_0 \lim_{k \rightarrow \infty} \int_{\Omega} \nabla R_n \cdot \nabla \psi_k \, dx \\ &= -\sigma\alpha_1\mu_0 \int_{\Omega} \nabla R_n \cdot \nabla (\operatorname{div} \nabla R_n) \, dx \\ &= \sigma\alpha_1\mu_0 \int_{\Omega} |\operatorname{div} \nabla R_n|^2 \, dx - \sigma\alpha_1\mu_0 \int_{\partial\Omega} \nabla R_n \cdot \boldsymbol{\nu} \underbrace{(\operatorname{div} \nabla R_n)}_{=0} \, d\sigma \\ &= \sigma\alpha_1\mu_0 \int_{\Omega} |\operatorname{div} \nabla R_n|^2 \, dx + \underbrace{\sigma\alpha_1\mu_0 \int_{\Omega' \setminus \bar{\Omega}} |\operatorname{div} \nabla R_n|^2 \, dx}_{=0}. \end{aligned}$$

Combining the computations we get (5.22).

Step 3: Estimates on terms containing $\mathbf{h}_a$.

Next, we will integrate in time and apply Young's inequality on terms of the type

$$\mathbf{h}_a \cdot \nabla R_n, \partial_t \mathbf{h}_a \cdot \nabla R_n, \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a.$$

In detail,

$$\begin{aligned} \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h}_a \cdot \nabla R_n \, dx &\leq \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \int_{\Omega'} |\mathbf{h}_a|^2 \, dx + \frac{\mu_0 \alpha_1}{4\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 \, dx, \\ \mu_0 \alpha_1 \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \nabla R_n \, dx &\leq \mu_0 \alpha_1 \tau_{\text{rel}} \int_{\Omega'} |\partial_t \mathbf{h}_a|^2 \, dx + \frac{\mu_0 \alpha_1}{4\tau_{\text{rel}}} \int_{\Omega'} |\nabla R_n|^2 \, dx, \\ \frac{\mu_0 \beta}{2} \int_{\Omega} \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx &\leq \frac{\mu_0 \alpha_3}{2\tau_{\text{rel}}} \int_{\Omega} |\mathbf{m}_n|^2 \, dx + \frac{\mu_0 \beta^2}{8\tau_{\text{rel}} \alpha_3} \int_{\Omega} |\Pi_{\mathcal{H}_n} \mathbf{h}_a|^2 \, dx. \end{aligned}$$

Note that integration in time is possible due to continuity of the solutions $t \mapsto \alpha_{(\cdot)}^{(\cdot)}(t)$. Actually, we can only integrate in time until some $t^* \in (0, T)$, but we can later deduce that we were able to integrate until T as the solutions will be bounded in time and therefore still exist for even larger times. We arrive at

$$\begin{aligned} &\frac{\rho_0}{2} \|\mathbf{u}_n(T)\|_{L^2(\Omega)^d}^2 + D \int_{\Omega} g_s^L(c_n(T)) \, dx + \frac{\mu_0 \alpha_3}{2} \|\mathbf{m}_n(T)\|_{L^2(\Omega)^d}^2 + \frac{\mu_0 \alpha_1}{2} \|\nabla R_n(T)\|_{L^2(\Omega')}^2 \\ &\quad + 2\eta \|\mathbf{D}\mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}}^2 + D\sigma_c \|\sqrt{(g_s^L)''(c_n)} \nabla c_n\|_{L^2(I \times \Omega)^d}^2 \\ &\quad + K \|\sqrt{f_2(c_n)} [D\nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) - \mu_0 \frac{1}{(c_n)_s} (\alpha_1 \nabla \nabla R_n + \frac{\beta}{2} \nabla \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_3 \nabla \mathbf{m}_n)^T \mathbf{m}_n]\|_{L^2(I \times \Omega)^d}^2 \\ &\quad + \sigma \alpha_3 \mu_0 \|\operatorname{div} \mathbf{m}\|_{L^2(I \times \Omega)}^2 + \sigma \alpha_3 \mu_0 \|\operatorname{curl} \mathbf{m}\|_{L^2(I \times \Omega)^d}^2 + \sigma \alpha_1 \mu_0 \|\operatorname{div} \nabla R_n\|_{L^2(I \times (\Omega' \setminus \partial \Omega))}^2 \\ &\quad + \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \|\sqrt{\chi(c_n, \nabla R_n)} \nabla R_n\|_{L^2(I \times \Omega)^d}^2 + \frac{\mu_0 \alpha_1}{2\tau_{\text{rel}}} \|\nabla R_n\|_{L^2(I \times \Omega')^d}^2 + \frac{\mu_0 \alpha_3}{2\tau_{\text{rel}}} \|\mathbf{m}_n\|_{L^2(I \times \Omega)^d}^2 \\ &\leq \frac{\sigma \beta \mu_0}{2} \int_0^T \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx \, dt + \frac{\mu_0 \beta}{2} \int_0^T \int_{\Omega} \partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx \, dt \\ &\quad + \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_0^T \int_{\Omega} \chi(c_n, \nabla R_n) \nabla R_n \cdot \mathbf{m}_n \, dx \, dt - \frac{\mu_0 \beta}{2\tau_{\text{rel}}} \int_0^T \int_{\Omega} \chi(c_n, \mathbf{h}_n) \nabla R_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx \, dt \\ &\quad + \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \|\mathbf{h}_a\|_{L^2(I \times \Omega')^d}^2 + \mu_0 \alpha_1 \tau_{\text{rel}} \|\partial_t \mathbf{h}_a\|_{L^2(I \times \Omega')^d}^2 + \frac{\mu_0 \beta^2}{8\tau_{\text{rel}} \alpha_3} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I \times \Omega)^d}^2 \\ &\quad + \frac{\rho_0}{2} \|\mathbf{u}_n^0\|_{L^2(\Omega)^d}^2 + D \int_{\Omega} g_s^L(c_n^0) \, dx + \frac{\mu_0 \alpha_3}{2} \|\mathbf{m}_n^0\|_{L^2(\Omega)^d}^2 + \frac{\mu_0 \alpha_1}{2} \|\nabla R_n(0)\|_{L^2(\Omega')^d}^2. \end{aligned}$$

In the following, we first describe how to proceed, then give the intermediate steps. Recall, the solution of (5.16c) is just a function of the other unknowns, i.e. $R_n(0) \in \mathcal{R}_n$ is defined as mean value free solution of

$$\int_{\Omega'} \nabla R_n(0) \cdot \nabla \psi_j^R \, dx = \int_{\Omega'} \mathbf{h}_a(0) \cdot \nabla \psi_j^R \, dx - \int_{\Omega} \mathbf{m}_n^0 \cdot \nabla \psi_j^R \, dx \quad \forall j = 1, \dots, 2n.$$

Therefore, for any $\varepsilon > 0$,

$$(1 - \varepsilon) \|\nabla R_n(0)\|_{L^2(\Omega')}^2 \leq \frac{1}{2\varepsilon} (\|\mathbf{h}_a(0)\|_{L^2(\Omega)^d}^2 + \|\mathbf{m}_n^0\|_{L^2(\Omega)^d}^2).$$

Due to the convergence $\mathbf{m}_n^0 \rightarrow \mathbf{m}^0$ in $L^2(\Omega)^d$ one can easily bound $\mathbf{m}_n^0$ in $L^2(\Omega)^d$. Analogously, $\mathbf{u}_n^0$ is bounded in $L^2(\Omega)^d$. The regularised entropy can be bounded by a quadratic function, therefore analogously $g_s^L(c_n^0)$ is bounded in $L^2(\Omega)$. The projection $\Pi_{\mathcal{H}_n} \mathbf{h}_a$ is bounded in $L^2(I \times \Omega)^d$, as a consequence of the stability (5.13) of $\Pi_{\mathcal{H}_n}$ and regularity of the external magnetic field $\mathbf{h}_a$, (H2).

It remains to deal with the first four terms of the right-hand side. Therefore, we compute

$$\frac{\sigma \beta \mu_0}{2} \int_0^T \int_{\Omega} \operatorname{div} \mathbf{m}_n \operatorname{div} \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx \, dt \leq \frac{\sigma \alpha_3 \mu_0}{2} \|\operatorname{div} \mathbf{m}_n\|_{L^2(I \times \Omega)}^2 + \frac{\sigma \beta^2 \mu_0}{8\alpha_3} \|\operatorname{div} \Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I \times \Omega)}^2.$$

The $\mathbf{h}_a$ -term is bounded due to convergence (5.14) and $\mathbf{h}_a|_\Omega \in L^2(I; H^3(\Omega))$, cf. (H2). Hence, absorption is possible. The term

$$\int_0^T \int_\Omega \chi(c_n, \nabla R_n) \nabla R_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx \, dt$$

can be estimated analogously where we exploit the boundedness of χ , (H4). For the term $\partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a$ we rewrite

$$\int_\Omega \partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx = \partial_t \int_\Omega \mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx - \int_\Omega \mathbf{m}_n \cdot \partial_t \Pi_{\mathcal{H}_n} \mathbf{h}_a \, dx. \quad (5.24)$$

Adding $\frac{\mu_0 \beta^2}{4\tau_{\text{rel}} \alpha_3} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a(T)\|_{L^2(\Omega)^d}^2$ to both sides of the final inequality, one can estimate

$$\begin{aligned} & \frac{\mu_0 \alpha_3}{2} \|\mathbf{m}_n(T)\|_{L^2(\Omega)^d}^2 + \frac{\mu_0 \beta^2}{4\alpha_3} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a(T)\|_{L^2(\Omega)^d}^2 - \frac{\mu_0 \beta}{2} \int_\Omega (\mathbf{m}_n \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a)(T) \, dx \\ & \geq \frac{\mu_0 \alpha_3}{4} \|\mathbf{m}_n(T)\|_{L^2(\Omega)^d}^2 > 0. \end{aligned}$$

On the right-hand side the newly added term is bounded due to convergence $\Pi_{\mathcal{H}_n} \mathbf{h}_a(T) \rightarrow \mathbf{h}_a(T)|_\Omega$ in $L^2(\Omega)^d$. On the right-hand side the term $-\frac{\mu_0 \beta}{2\tau_{\text{rel}}} \int_\Omega \mathbf{m}_n^0 \cdot \Pi_{\mathcal{H}_n} \mathbf{h}_a(0) \, dx$ appears but can be estimated by Young's inequality and the same arguments as before. Note that the terms $\mathbf{h}_a(0), \mathbf{h}_a(T)$ are well-defined, as the space of the time variable is one-dimensional and $\mathbf{h}_a$ is weakly differentiable. The second term on the right-hand side of (5.24) can be dealt with by absorption and boundedness of $\|\partial_t \Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I \times \Omega)^d}$. Indeed, for any function $\psi \in C_0^\infty((0, T))$ and the basis representations $\mathbf{h}_a|_\Omega = \sum_{i=1}^\infty \alpha_{2i} \Psi_{2i}^{\mathbf{m}}$, $\partial_t \mathbf{h}_a|_\Omega = \sum_{i=1}^\infty \beta_{2i} \Psi_{2i}^{\mathbf{m}}$, one has

$$\begin{aligned} - \int_0^T \psi' \alpha_{2i} \, dt &= - \int_0^T \psi' \langle \mathbf{h}_a, \Psi_{2i}^{\mathbf{m}} \rangle_{H^3(\Omega)^d} \, dt = - \left\langle \int_0^T \psi' \mathbf{h}_a \, dt, \Psi_{2i}^{\mathbf{m}} \right\rangle_{H^3(\Omega)^d} \\ &= \left\langle \int_0^T \partial_t \mathbf{h}_a \psi \, dt, \Psi_{2i}^{\mathbf{m}} \right\rangle_{H^3(\Omega)^d} = \int_0^T \langle \partial_t \mathbf{h}_a, \Psi_{2i}^{\mathbf{m}} \rangle_{H^3(\Omega)^d} \psi' \, dt = \int_0^T \beta_{2i} \psi \, dt. \end{aligned}$$

Hence, $\partial_t \alpha_{2i} = \beta_{2i}$ and therefore $\partial_t \Pi_{\mathcal{H}_n} \mathbf{h}_a = \Pi_{\mathcal{H}_n} \partial_t \mathbf{h}_a$. The boundedness of that term follows analogously as before. Young's inequality applied to the term of the type

$$\chi(c_n, \nabla R_n) \nabla R_n \cdot \mathbf{m}_n$$

combined with boundedness (H4) of χ makes it possible to achieve an estimate using Gronwall's inequality later on. We end up with

$$\begin{aligned}
& \frac{\rho_0}{2} \|\mathbf{u}_n(T)\|_{L^2(\Omega)^d}^2 + D \int_{\Omega} g_s^L(c_n(T)) \, dx + \frac{\mu_0 \alpha_3}{4} \|\mathbf{m}_n(T)\|_{L^2(\Omega)^d}^2 \\
& + \frac{\mu_0 \alpha_1}{2} \|\nabla R_n(T)\|_{L^2(\Omega')^d}^2 + 2\eta \|\mathbf{D}\mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}}^2 + D\sigma_c \|\sqrt{(g_s^L)''(c_n)} \nabla c_n\|_{L^2(I \times \Omega)^d}^2 \\
& + K \|\sqrt{f_2(c_n)} [D\nabla \Pi \mathbf{e}_n (g_s^L)'(c_n) - \mu_0 \frac{1}{(c_n)_s} (\alpha_1 \nabla \nabla R_n + \frac{\beta}{2} \nabla \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_3 \nabla \mathbf{m}_n)^T \mathbf{m}_n]\|_{L^2(I \times \Omega)^d}^2 \\
& + \frac{\sigma \alpha_3 \mu_0}{2} \|\operatorname{div} \mathbf{m}\|_{L^2(I \times \Omega)} + \sigma \alpha_3 \mu_0 \|\operatorname{curl} \mathbf{m}\|_{L^2(I \times \Omega)^d} + \sigma \alpha_1 \mu_0 \|\operatorname{div} \nabla R_n\|_{L^2(I \times (\Omega' \setminus \partial \Omega))}^2 \\
& + \frac{\mu_0 \alpha_1}{2\tau_{\text{rel}}} \|\sqrt{\chi(c_n, \nabla R_n)} \nabla R_n\|_{L^2(I \times \Omega)}^2 + \frac{\mu_0 \alpha_1}{4\tau_{\text{rel}}} \|\nabla R_n\|_{L^2(I \times \Omega')^d}^2 + \frac{\mu_0 \alpha_3}{4\tau_{\text{rel}}} \|\mathbf{m}_n\|_{L^2(I \times \Omega)^d}^2 \\
& \leq \frac{\sigma \beta^2 \mu_0}{8\alpha_3} \|\operatorname{div} \Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I \times \Omega)}^2 + \frac{\mu_0 \beta^2}{4\alpha_3} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a(T)\|_{L^2(\Omega)^d}^2 \\
& + \frac{\mu_0 \beta}{4} \|\mathbf{m}_n^0\|_{L^2(\Omega)^d}^2 + \frac{\mu_0 \beta}{4} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a(0)\|_{L^2(\Omega)^d}^2 + \frac{\mu_0 \beta^2 \tau_{\text{rel}}}{4\alpha_3} \|\Pi_{\mathcal{H}_n} \partial_t \mathbf{h}_a\|_{L^2(I \times \Omega)^d}^2 \\
& + \frac{\mu_0 \alpha_3^2 \|\chi\|_{\infty}^2}{\tau_{\text{rel}} \alpha_1} \|\mathbf{m}_n\|_{L^2(I \times \Omega)^d}^2 + \frac{\mu_0 \beta^2 \|\chi\|_{\infty}}{2\tau_{\text{rel}} \alpha_1} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I \times \Omega)^d}^2 \\
& + \frac{\mu_0 \alpha_1}{\tau_{\text{rel}}} \|\mathbf{h}_a\|_{L^2(I \times \Omega')^d}^2 + \mu_0 \alpha_1 \tau_{\text{rel}} \|\partial_t \mathbf{h}_a\|_{L^2(I \times \Omega')^d}^2 + \frac{\mu_0 \beta^2}{8\tau_{\text{rel}} \alpha_3} \|\Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I \times \Omega)^d}^2 \\
& + \frac{\rho_0}{2} \|\mathbf{u}_n^0\|_{L^2(\Omega)^d}^2 + D \int_{\Omega} g_s^L(c_n^0) \, dx + \frac{\mu_0 \alpha_3}{2} \|\mathbf{m}_n^0\|_{L^2(\Omega)^d}^2 \\
& + \mu_0 \alpha_1 (\|\mathbf{h}_a(0)\|_{L^2(\Omega')^d}^2 + \|\mathbf{m}_n^0\|_{L^2(\Omega)^d}^2) \\
& \leq C(\|\Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^2(I; H(\operatorname{div})(\Omega))}^2 + \|\mathbf{h}_a(T)\|_{L^2(\Omega)^d}^2 + \|\mathbf{h}_a(0)\|_{L^2(\Omega)^d}^2 \\
& + \|\partial_t \mathbf{h}_a\|_{L^2(I \times \Omega)^d}^2 + \|\mathbf{h}_a\|_{H^1(I; L^2(\Omega')^d)}^2 + \|\mathbf{h}_a(0)\|_{L^2(\Omega')^d}^2) \\
& + C(\|\mathbf{u}_n^0\|_{L^2(\Omega)^d}^2 + \|\mathbf{m}_n^0\|_{L^2(\Omega)^d}^2 + \int_{\Omega} g_s^L(c_n^0) \, dx) \\
& + \frac{\mu_0 \alpha_3}{4} C \|\mathbf{m}_n\|_{L^2(I \times \Omega)^d}^2 \\
& =: C_{\mathbf{h}_a, \text{initial}} + \frac{\mu_0 \alpha_3}{4} C \|\mathbf{m}_n\|_{L^2(I \times \Omega)}^2
\end{aligned} \tag{5.25}$$

Note that from the considerations before it follows that the $\mathbf{h}_a$ -terms and the initial data are bounded. By Gronwall's inequality,

$$\frac{\mu_0 \alpha_3}{4} \|\mathbf{m}_n(T)\|_{L^2(\Omega)^d}^2 \leq C_{\mathbf{h}_a, \text{initial}} (1 + e^{CT}).$$

Note that the final time T is arbitrary and in a standard way L^∞ -in-time-estimates for $\mathbf{u}_n, c_n, \mathbf{m}_n$ can be achieved. We note, that easily one estimates all pure $\mathbf{h}_a$ -terms (without projector) by the $H^1(I; L^2(\Omega')^d)$ -norm of $\mathbf{h}_a$.

Therefore, the solution exists on the whole time interval $[0, T]$. $\square$

6. Compactness Results

In this section, we establish compactness in time and in space necessary for the limit procedures in the discrete model and in the TMR-model. The starting point is the estimate

$$\begin{aligned}
& \|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{u}_n\|_{L^2(I; H^1(\Omega))} + \|g_s^L(c_n)\|_{L^\infty(I; L^1(\Omega))} + \frac{\sigma_c}{L} \|\nabla c_n\|_{L^2(I; L^2(\Omega)^d)} \\
& \|\frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n\|_{L^2(I; L^2(\Omega)^d)} + \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{h}_n\|_{L^\infty(I; L^2(\Omega)^d)} \\
& + \|\mathbf{m}_n\|_{L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega))} + \|\mathbf{h}_n\|_{L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega' \setminus \partial \Omega))} \leq C
\end{aligned} \tag{6.1}$$

uniformly satisfied by the Galerkin solutions of (5.16), where $\mathbf{h}_n := \nabla R_n$ and

$$(\mathbf{V}_{\text{part}})_n := -\frac{f_2(c_n)K}{(c_n)_s} [D\nabla\Pi_{\mathcal{C}_n}(g_s^L)'(c_n) + \mu_0 \frac{1}{(c_n)_s} (\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_3 \nabla \mathbf{m}_n)^T \mathbf{m}_n] \quad (6.2)$$

and the constant $C > 0$ does not depend on σ_c, L, s but only on $\mathbf{h}_a$, initial data and T .

Compactness in time

In this paragraph we establish estimates for $\partial_t \mathbf{u}_n$, $\partial_t c_n$ and $\partial_t \mathbf{m}_n$.

Lemma 6.1. *Let $(c_n, \mathbf{u}_n, \frac{(c_n)_s}{\sqrt{f_2(c_n)}}(\mathbf{V}_{\text{part}})_n)_{n \in \mathbb{N}}$ be a bounded sequence in*

$$L^{10/3}(I \times \Omega) \cap L^2(I; H^1(\Omega)) \times L^{10/3}(I \times \Omega)^d \times L^2(I \times \Omega)^d$$

such that additionally

$$\|\sqrt{f_2(c_n)}\|_{L^{10/3}(I \times \Omega)} \leq \tilde{C}$$

uniformly, for some $\tilde{C} > 0$. Then, there is a constant $C > 0$ such that

$$\|\partial_t c_n\|_{L^{5/4}(I; (H_*^2(\Omega))')} \leq C$$

uniformly in $n \in \mathbb{N}$.

Proof. Let us use the L^2 -orthogonality of the eigenfunctions from (4.7). This gives

$$\begin{aligned} \int_0^T \int_{\Omega} \partial_t c_n \psi \, dx \, dt &= \int_0^T \int_{\Omega} \partial_t c_n \Pi_{\mathcal{C}_n} \psi \, dx \, dt \\ &\stackrel{(5.16b)}{\leq} \int_0^T \int_{\Omega} |(c_n)_s| |\mathbf{u}_n| |\nabla \Pi_{\mathcal{C}_n} \psi| \, dx \, dt + \int_0^T \int_{\Omega} |(c_n)_s| (\mathbf{V}_{\text{part}})_n |\nabla \Pi_{\mathcal{C}_n} \psi| \, dx \, dt \\ &\quad + \sigma_c \int_0^T \int_{\Omega} |\nabla c_n| |\nabla \Pi_{\mathcal{C}_n} \psi| \, dx \, dt \\ &\leq \|(c_n)_s\|_{L^{10/3}(I \times \Omega)} \|\mathbf{u}_n\|_{L^{10/3}(I \times \Omega)^d} \|\nabla \Pi_{\mathcal{C}_n} \psi\|_{L^{5/2}(I \times \Omega)^d} \\ &\quad + \|\sqrt{f_2(c_n)}\|_{L^{10/3}(I \times \Omega)} \left\| -\frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \right\|_{L^2(I \times \Omega)^d} \|\nabla \Pi_{\mathcal{C}_n} \psi\|_{L^5(I \times \Omega)^d} \\ &\quad + \|\nabla c_n\|_{L^2(I \times \Omega)^d} \|\nabla \Pi_{\mathcal{C}_n} \psi\|_{L^2(I \times \Omega)^d} \end{aligned}$$

From this the result follows easily using $H^2(\Omega) \hookrightarrow W^{1,5}(\Omega)$ and (5.12). $\square$

Remark 6.2. For any cut-off function $\phi \in C_0^\infty(\Omega; \mathbb{R}_0^+)$ one can prove analogously to Lemma 6.1 that $(\partial_t(\phi c_n))_{n \in \mathbb{N}}$ is bounded in $L^{5/4}(I; (H_*^2(\Omega))')$ and the weak limit of a converging subsequence is $\partial_t(c\phi)$.

From Remark 6.2 it is obvious that

$$\begin{aligned} \int_0^T \langle \partial_t(\phi c), \psi \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} \, dt &= \int_0^T \langle \partial_t c, \phi \psi \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} \, dt \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ \int_0^T \int_{\Omega} \partial_t(\phi c_n) \psi \, dx \, dt &= \int_0^T \int_{\Omega} \partial_t c_n (\phi \psi) \, dx \, dt. \end{aligned} \quad (6.3)$$

For the proof of compactness in time of the magnetization, it turns out to be advantageous to equip $\mathcal{M}$ with a slightly different norm. By $\tilde{\mathcal{M}}$ we denote the space identical to $\mathcal{M}$ if equipped with the norm of the sum $\mathcal{M} = \mathcal{S} \oplus \mathcal{S}^o \oplus \mathcal{V}$, which is the sum of all H^3 -norms of the individual summands.

Lemma 6.3. *Let $(\mathbf{u}_n, \frac{(c_n)_s}{\sqrt{f_2(c_n)}}(\mathbf{V}_{\text{part}})_n, \mathbf{m}_n)_{n \in \mathbb{N}}$ be a bounded sequence in*

$$L^{10/3}(I \times \Omega)^d \cap L^2(I; H^1(\Omega)^d) \times L^2(I \times \Omega)^d \times L^\infty(I; L^2(\Omega)^d) \cap H(\text{div}, \text{curl})(\Omega)$$

such that additionally

$$\left\| \frac{\sqrt{f_2(c_n)}}{(c_n)_s} \right\|_{L^\infty(I \times \Omega)} \leq \tilde{C}$$

uniformly, for some $\tilde{C} > 0$. Then, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\|\partial_t \mathbf{m}_n\|_{L^2(I; \tilde{\mathcal{M}}')} \leq C.$$

Proof. Let $\Psi \in L^2(I; \tilde{\mathcal{M}})$. Then, we have

$$\int_0^T \int_\Omega \partial_t \mathbf{m}_n \cdot \Psi \, dx \, dt = \int_0^T \int_\Omega \partial_t \mathbf{m}_n \cdot \Pi_{\mathcal{M}_n} \Psi \, dx \, dt =: J$$

due to L^2 -orthogonality of (3.1.13), see (3.1.14).

$$\begin{aligned} |J| &\stackrel{(5.16d)}{\leq} \int_0^T \int_\Omega (|\mathbf{u}_n| + |(\mathbf{V}_{\text{part}})_n|) |\nabla \Pi_{\mathcal{M}_n} \Psi| |\mathbf{m}_n| \, dx \, dt \\ &\quad + \sigma \int_0^T \int_\Omega |\operatorname{div} \mathbf{m}_n| |\operatorname{div} \Pi_{\mathcal{M}_n} \Psi| \, dx \, dt + \sigma \int_0^T \int_\Omega |\operatorname{curl} \mathbf{m}_n| |\operatorname{curl} \Pi_{\mathcal{M}_n} \Psi| \, dx \, dt \\ &\quad + \frac{C}{2} \int_0^T \int_\Omega |\mathbf{m}_n| |\Pi_{\mathcal{M}_n} \Psi| |\operatorname{curl} \mathbf{u}_n| \, dx \, dt + \frac{1}{\tau_{\text{rel}}} \int_0^T \int_\Omega (|\mathbf{m}_n| + \|\chi\|_\infty |\mathbf{h}_n|) |\Pi_{\mathcal{M}_n} \Psi| \, dx \, dt \\ &\leq \|\mathbf{u}_n\|_{L^{10/3}(I \times \Omega)^d} \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega))^d} \|\nabla \Pi_{\mathcal{M}_n} \Psi\|_{L^{10/7}(I; L^5(\Omega))^{d \times d}} \\ &\quad + \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega))^d} \left\| \frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \right\|_{L^2(I \times \Omega)^d} \left\| \frac{\sqrt{f_2(c_n)}}{(c_n)_s} \right\|_{L^\infty(I \times \Omega)} \|\nabla \Pi_{\mathcal{M}_n} \Psi\|_{L^2(I; L^\infty(\Omega))^{d \times d}} \\ &\quad + \sigma \|\mathbf{m}_n\|_{L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega))} \|\Pi_{\mathcal{M}_n} \Psi\|_{L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega))} \\ &\quad + \frac{\tilde{C}}{2} \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega))^d} \|\nabla \mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}} \|\Pi_{\mathcal{M}_n} \Psi\|_{L^2(I; L^\infty(\Omega))^d} \\ &\quad + \frac{1}{\tau_{\text{rel}}} (\|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega))^d} + \|\chi\|_\infty \|\mathbf{h}_n\|_{L^\infty(I; L^2(\Omega))^d}) \|\Pi_{\mathcal{M}_n} \Psi\|_{L^1(I; L^2(\Omega))^d} \end{aligned}$$

The claim follows as $\tilde{\mathcal{M}} \hookrightarrow H^3(\Omega)^d \hookrightarrow W^{1,\infty}(\Omega)^d$ and

$$\|\Pi_{\mathcal{M}_n} \Psi\|_{H^3(\Omega)^d} \leq \|\Pi_{\mathcal{M}_n} \Psi\|_{\tilde{\mathcal{M}}} \leq \|\Psi\|_{\tilde{\mathcal{M}}}$$

due to H^3 -orthogonality (hence stability) of the bases of $\mathcal{S}$, $\mathcal{S}^o$ and $\mathcal{V}$, cf. Lemma 3.1.4. $\square$

Lemma 6.4. Let $(\mathbf{u}_n, \frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n, \mathbf{m}_n, \mathbf{h}_n)_{n \in \mathbb{N}}$ be a bounded sequence in

$$L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1(\Omega)^d) \times L^2(I \times \Omega) \times L^\infty(I; L^2(\Omega)^d) \times L^\infty(I; L^2(\Omega)^d)$$

such that additionally

$$\left\| \frac{(c_n)_s}{\sqrt{f_2(c_n)}} \right\|_{L^\infty(I; L^2(\Omega))} \leq \tilde{C}$$

uniformly, for some $\tilde{C} > 0$. Then, there is a constant $C > 0$ such that

$$\|\partial_t \mathbf{u}_n\|_{L^2(I; \mathcal{U}')} \leq C$$

uniformly in $n \in \mathbb{N}$.

Proof. We use the stability of the projection operator $\Pi_{\mathcal{U}_n}$ and standard estimates to get the result. Let $\Psi \in L^2(I; \mathcal{U})$. Also note that

$$\int_0^T \int_\Omega \partial_t \mathbf{u}_n \cdot \Psi \, dx \, dt = \int_0^T \int_\Omega \partial_t \mathbf{u}_n \cdot \Pi_{\mathcal{U}_n} \Psi \, dx \, dt =: J$$

due to (4.2). First, we notice that

$$\begin{aligned}
& \mu_0 \int_0^T \int_{\Omega} (\Pi_{\mathcal{U}_n} \Psi \cdot \nabla) (\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a) \cdot \mathbf{m}_n \, dx \, dt \\
& - D \int_0^T \int_{\Omega} (c_n)_s \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \cdot \Pi_{\mathcal{U}_n} \Psi \, dx \, dt \\
& \stackrel{\text{div } \Pi_{\mathcal{U}_n} \Psi = 0}{=} \mu_0 \int_0^T \int_{\Omega} (\Pi_{\mathcal{U}_n} \Psi \cdot \nabla) (\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_3 \mathbf{m}_n) \cdot \mathbf{m}_n \, dx \, dt \\
& - D \int_0^T \int_{\Omega} (c_n)_s \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \cdot \Pi_{\mathcal{U}_n} \Psi \, dx \, dt \\
& \stackrel{(6.2)}{=} \int_0^T \int_{\Omega} \frac{(c_n)_s^2}{K f_2(c_n)} (\mathbf{V}_{\text{part}})_n \cdot \Pi_{\mathcal{U}_n} \Psi \, dx \, dt.
\end{aligned}$$

Then, we compute

$$\begin{aligned}
|J| & \stackrel{(5.16d)}{\leq} \int_0^T \int_{\Omega} 2\eta |\mathbf{D}\mathbf{u}_n| |\mathbf{D}\Pi_{\mathcal{U}_n} \Psi| \, dx \, dt + \frac{\rho_0}{2} \int_0^T \int_{\Omega} |\mathbf{u}_n| |\nabla \mathbf{u}_n| |\Pi_{\mathcal{U}_n} \Psi| \, dx \, dt \\
& + \frac{\rho_0}{2} \int_0^T \int_{\Omega} |\mathbf{u}_n| |\nabla \Pi_{\mathcal{U}_n} \Psi| |\mathbf{u}_n| \, dx \, dt \\
& + \int_0^T \int_{\Omega} \left| \frac{(c_n)_s}{K \sqrt{f_2(c_n)}} \right| \left| \frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \right| |\Pi_{\mathcal{U}_n} \Psi| \, dx \, dt \\
& + \frac{\mu_0}{2} \int_0^T \int_{\Omega} |\mathbf{m}_n| |\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a| |\text{curl } \Pi_{\mathcal{U}_n} \Psi| \, dx \, dt \\
& \leq 2\eta \|\mathbf{D}\mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}} \|\mathbf{D}\Pi_{\mathcal{U}_n} \Psi\|_{L^2(I \times \Omega)^{d \times d}} \\
& + \frac{\rho_0}{2} \|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)} \|\nabla \mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}} \|\Pi_{\mathcal{U}_n} \Psi\|_{L^2(I; L^\infty(\Omega)^d)} \\
& + \frac{\rho_0}{2} \|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)} \|\nabla \Pi_{\mathcal{U}_n} \Psi\|_{L^1(I; L^\infty(\Omega)^{d \times d})} \|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)} \\
& + \left\| \frac{(c_n)_s}{K \sqrt{f_2(c_n)}} \right\|_{L^\infty(I; L^2(\Omega))} \left\| \frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \right\|_{L^2(I \times \Omega)^d} \|\Pi_{\mathcal{U}_n} \Psi\|_{L^2(I; L^\infty(\Omega)^d)} \\
& + \frac{\mu_0}{2} \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega)^d)} \|\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a\|_{L^\infty(I; L^2(\Omega)^d)} \|\nabla \Pi_{\mathcal{U}_n} \Psi\|_{L^1(I; L^\infty(\Omega)^d)}.
\end{aligned}$$

Due to H^3 -stability of $\Pi_{\mathcal{U}_n}$, cf. (5.11), the claim follows. $\square$

Next, we are concerned with time-compactness of the magnetic field. Denote

$$\mathcal{H}_{\text{special}} := \{\nabla \psi \in \nabla[\mathcal{R}] \mid \nabla \psi|_{\Omega} \in \mathcal{S}, \nabla \psi|_{\Omega' \setminus \bar{\Omega}} \equiv 0\}. \quad (6.4)$$

As norm of this space we choose $\|\cdot\|_{\mathcal{H}_{\text{special}}} := \|\cdot\|_{H^3(\Omega)^d}$. Note that the $\Omega' \setminus \bar{\Omega}$ -part of those functions does not contribute to the norm of $\nabla[\mathcal{R}]$ and on Ω the $\nabla[\mathcal{R}]$ -norm is bounded by the chosen norm. This space is nontrivial. All gradients of homogeneous Dirichlet–Laplace eigenfunctions—extended by zero on $\Omega' \setminus \bar{\Omega}$ —are elements of this space.

Lemma 6.5. *Let $\mathbf{h}_a \in H^1(I; L^2(\Omega')^d)$ and $(\partial_t \mathbf{m}_n)_{n \in \mathbb{N}}$ be bounded in $L^2(I; \tilde{\mathcal{M}}')$. Then there exists a constant $C > 0$ such that*

$$\|\partial_t \mathbf{h}_n\|_{L^2(I; (\mathcal{H}_{\text{special}})')} \leq C$$

for all $n \in \mathbb{N}$.

Proof. Equation (5.16c) can be differentiated with respect to time, so we get

$$\int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla \psi_i^R \, dx = \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \nabla \psi_i^R \, dx - \int_{\Omega} \partial_t \mathbf{m}_n \cdot \nabla \psi_i^R \, dx \quad \forall i = 1, \dots, 2n.$$

Let $\nabla\psi \in L^2(I; \mathcal{H}_{\text{special}})$, then $\nabla\psi(t)|_{\Omega} \in \mathcal{S}$ for almost all $t \in I$ and therefore from the three types of basis functions within (3.2.12) one only needs those with even index, and from those only every second element—those whose gradients generate $\mathcal{S}$. This implies that $\nabla\Pi_{\mathcal{R}_n}\psi|_{\Omega' \setminus \bar{\Omega}} \equiv 0$, cf. (3.2.15).

$$\begin{aligned}
\int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla\psi \, dx \, dt &= \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla \left(\sum_{i=1}^{\infty} \alpha_i \psi_i^R \right) \, dx \, dt \\
&= \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla \left(\sum_{i=1}^{\infty} \alpha_{4i} \psi_{4i}^R \right) \, dx \, dt \\
&= \int_0^T \int_{\Omega} \underbrace{\partial_t \mathbf{h}_n}_{\in \mathcal{H}_n} \cdot \left(\sum_{i=1}^{\infty} \alpha_{4i} \underbrace{\Psi_{4i}^{\mathbf{m}}}_{\in \mathcal{H}_n} \right) \, dx \, dt \\
&= \int_0^T \int_{\Omega} \partial_t \mathbf{h}_n \cdot \Pi_{\mathcal{H}_n} \nabla\psi \, dx \, dt \\
&= \int_0^T \int_{\Omega} \partial_t \mathbf{h}_n \cdot \nabla \Pi_{\mathcal{R}_n} \psi \, dx \, dt \\
&= \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla \Pi_{\mathcal{R}_n} \psi \, dx \, dt =: J.
\end{aligned}$$

Recall the L^2 -orthogonality of (3.1.13), which has been used above and also recall the relation between the bases of $\mathcal{M}$, $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{R}$ from Section 3. In particular note that on Ω the potentials of the basis functions of $\mathcal{H}$ are used to define the basis functions of $\mathcal{R}$, see (3.2.1). We use (5.16c) and estimate

$$\begin{aligned}
|J| &= \left| \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \nabla \Pi_{\mathcal{R}_n} \psi \, dx \, dt - \int_0^T \int_{\Omega} \partial_t \mathbf{m}_n \cdot \nabla \Pi_{\mathcal{R}_n} \psi \, dx \, dt \right| \\
&\leq \|\partial_t \mathbf{h}_a\|_{L^2(I \times \Omega')^d} \|\nabla \Pi_{\mathcal{R}_n} \psi\|_{L^2(I \times \Omega')^d} + \|\partial_t \mathbf{m}_n\|_{L^2(I; \tilde{\mathcal{M}}')} \|(\nabla \Pi_{\mathcal{R}_n} \psi)|_{\Omega}\|_{L^2(I; \tilde{\mathcal{M}})} \\
&\leq C(\|\nabla \Pi_{\mathcal{R}_n} \psi\|_{L^2(I \times \Omega)^d} + \|\nabla \Pi_{\mathcal{R}_n} \psi\|_{L^2(I; H^3(\Omega)^d)}) \\
&= C(\|\Pi_{\mathcal{H}_n} \nabla\psi\|_{L^2(I \times \Omega)^d} + \|\Pi_{\mathcal{H}_n} \nabla\psi\|_{L^2(I; H^3(\Omega)^d)}).
\end{aligned}$$

The first term on the right-hand side is bounded by the other one and the proof is finished by using stability of the projections due to H^3 -orthogonality of the corresponding basis functions of $\mathcal{S}$. $\square$

Let us specify an assumption on $f_p(c_n)$ such that the conditions in Lemmas 6.1, 6.3 and 6.4 are satisfied. For this, we first state (without proof) an immediate consequence of our assumption of $g_s^L(c)$.

Lemma 6.6. *Let $g_s^L(c) \in L^\infty(I; L^1(\Omega))$ with $\|g_s^L(c)\|_{L^\infty(I; L^1(\Omega))} \leq C$, $s < e < L$, then c is in $L^\infty(I; L^2(\Omega))$ as well, bounded by a constant depending on C, L and $|\Omega|$. Moreover, $g_s(c) + c \approx c \log(c)$ and c are in $L^\infty(I; L^1(\Omega))$, bounded by a constant depending only on C and $|\Omega|$.*

Remark 6.7. Based on the ansatz

$$f_2(c) = (c_s)^m \quad (\text{for fixed } s \text{ and } L),$$

we infer the condition $m \in [0, 2]$. Note that if we do not use the $L^\infty(I; L^2(\Omega))$ -bound of c but the $L^\infty(I; L^1(\Omega))$ -bound which is independent of L , then we arrive at the condition $m \in [1, 2]$ and a slightly less regular time derivative $\partial_t c_n$.

As the gradients of $\mathbf{m}_n$ and $\mathbf{h}_n$ are only locally bounded in Ω , cf. [9], for an application of the Aubin-Lions lemma we will consider the functions $\mathbf{m}_n \phi$ and $\mathbf{h}_n|_{\hat{\Omega}}$, where $\phi \in C_0^\infty(\Omega)$ is a cut-off function and $\hat{\Omega} \subset \subset \Omega$. We have global estimates for the time derivatives of $\mathbf{m}_n$ and $\mathbf{h}_n$. Therefore, we expect to obtain estimates of $\partial_t(\mathbf{m}_n \phi)$ and $\partial_t(\mathbf{h}_n|_{\hat{\Omega}})$ as well. This is done in the subsequent corollaries.

Corollary 6.8. *Under the assumptions of Lemma 6.3, for any cut-off function $\phi \in C_0^\infty(\Omega)$ there is a constant $C > 0$ such that*

$$\|\partial_t(\mathbf{m}_n\phi)\|_{L^2(I;\tilde{\mathcal{M}}')} \leq C$$

for all $n \in \mathbb{N}$.

Proof. Note that for a function $F \in H^3(\Omega)^d$ we have $\|F\phi\|_{H^3(\Omega)^d} \leq C(\phi)\|F\|_{H^3(\Omega)^d}$. Therefore, we try to recycle as much as possible from the original computations in Lemma 6.3 where we put ϕ besides the testfunction. Let $\hat{\Psi} \in \tilde{\mathcal{M}}$. Then we obviously have $(\hat{\Psi}\phi) \in \tilde{\mathcal{M}}$. Therefore, we can just plug in the new testfunction $(\Psi\phi)$, where $\Psi \in L^2(I;\tilde{\mathcal{M}})$ and proceed like in the proof of Lemma 6.3. $\square$

Corollary 6.9. *Under the assumptions of Lemma 6.5, for any $V \subset \Omega$ there is a constant $C > 0$ such that*

$$\|\partial_t \mathbf{h}_n|_V\|_{L^2(I;(\nabla[H_0^4(V)]))'} \leq C$$

Proof. The idea is to estimate $\partial_t \mathbf{h}_n|_V$ by $\partial_t \mathbf{h}_n$. Consider

$$H(V) := \{\nabla\psi : \Omega' \rightarrow \mathbb{R}^d | \psi|_V \in H_0^4(V), \psi|_{\Omega' \setminus V} \equiv 0\}. \quad (6.5)$$

Obviously, $H(V) \subset \mathcal{H}_{\text{special}}$, hence we get

$$\begin{aligned} \sup_{\substack{\nabla\psi \in L^2(I;\nabla[H_0^4(V)]) \\ \|\nabla\psi\|_{L^2(I;H^3(V)^d)}=1}} \int_0^T \int_V \partial_t \mathbf{h}_n|_V \cdot \nabla\psi \, dx \, dt &= \sup_{\substack{\nabla\psi \in L^2(I;H(V)) \\ \|\nabla\psi\|_{L^2(I;H^3(\Omega)^d)}=1}} \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla\psi \, dx \, dt \\ &= \underbrace{\|\nabla\psi\|_{L^2(I;\mathcal{H}_{\text{special}})}}_{=1} \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla\psi \, dx \, dt \\ &\leq \sup_{\substack{\nabla\psi \in L^2(I;\mathcal{H}_{\text{special}}) \\ \|\nabla\psi\|_{L^2(I;\mathcal{H}_{\text{special}})}=1}} \int_0^T \int_{\Omega'} \partial_t \mathbf{h}_n \cdot \nabla\psi \, dx \, dt \stackrel{\text{Lemma 6.5}}{\leq} C \end{aligned}$$

and the result follows. $\square$

7. Existence for the TMR-Model

We proceed taking the limit and identifying the terms of (5.16) with the terms of (5.7). Our strategy is to use sufficiently regular testfunctions, which will be projected onto the finite dimensional Galerkin approximation spaces as testfunctions in (5.16). The kind of testfunctions we are going to consider are

$$\left. \begin{aligned} &\bullet \Pi_{\mathcal{U}_n} \mathbf{v} \quad \text{with } \mathbf{v} \in C^0([0, T]; C_0^\infty(\Omega)^d \cap \mathcal{U}) \text{ for (5.16a),} \\ &\bullet \Pi_{\mathcal{C}_n} \psi \quad \text{with } \psi \in C^0([0, T]; H_*^2(\Omega)) \text{ for (5.16b),} \\ &\bullet \Pi_{\mathcal{R}_n} S \quad \text{with } S \in \mathcal{R} \text{ for (5.16c),} \\ &\bullet \Pi_{\mathcal{M}_n} \boldsymbol{\theta} \quad \text{with } \boldsymbol{\theta} \in C^0([0, T]; C_0^\infty(\Omega)^d \cap \mathcal{M}) \text{ for (5.16d).} \end{aligned} \right\} \quad (7.1)$$

Here, we used a notation like $C_0^\infty(\Omega)^d \cap \mathcal{M}$ to emphasize that such a space is considered under the norm of $\mathcal{M}$.

We label all the terms of (5.16a) from left to right with $\mathcal{L}_1^{\mathbf{u}}, \dots, \mathcal{L}_4^{\mathbf{u}}$ for the terms of the left-hand side and $\mathcal{R}_1^{\mathbf{u}}, \dots, \mathcal{R}_3^{\mathbf{u}}$ for the terms of the right-hand side. The terms of the other equations will be labeled analogously, i.e. they will be labeled by $\mathcal{L}_1^c, \dots, \mathcal{L}_4^c$, $\mathcal{L}_1^R, \mathcal{R}_1^R, \mathcal{R}_2^R$, $\mathcal{L}_1^{\mathbf{m}}, \dots, \mathcal{L}_4^{\mathbf{m}}$, $\mathcal{R}_1^{\mathbf{m}}, \mathcal{R}_2^{\mathbf{m}}$. The term $\mathcal{R}_3^{\mathbf{u}}$ does not exist in (5.7a) and is supposed to vanish. All other terms from (5.16) are in the same order as in (5.7). Moreover, we assume f_2 to be continuous with growth

$$\begin{aligned} 0 < a_0 \leq f_2(c) \leq a_1|c|^m + a_2, \quad m \in [0, 2] \text{ for some } a_0, a_1, a_2 > 0 \\ \text{and } |\chi(\cdot, \cdot)| \leq \chi_{\max} < \infty. \end{aligned} \quad (7.2)$$

We abbreviate $\mathbf{h}_n := \nabla R_n$ and $\mathbf{h} := \nabla R$ for the limit R of R_n . From (6.1) we get the following convergence results for subsequences, which will not be relabeled for the ease of notation, and where s, L, σ_c are fixed, in a standard way, see explanations below.

$$\left\{ \begin{array}{ll} \mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u} & \text{in } L^\infty(I; L^2(\Omega)^d), \\ \mathbf{u}_n \rightharpoonup \mathbf{u} & \text{in } L^2(I; H^1(\Omega)^d), \\ \mathbf{u}_n \rightarrow \mathbf{u} & \text{in } L^{q_d-}(I; L^{q_d-}(\Omega)^d). \end{array} \right\} \left\{ \begin{array}{ll} c_n \overset{*}{\rightharpoonup} c & \text{in } L^\infty(I; L^2(\Omega)), \\ c_n \rightharpoonup c & \text{in } L^2(I; H^1(\Omega)), \\ c_n \rightarrow c & \text{in } L^{q_d-}(I; L^{q_d-}(\Omega)). \end{array} \right\} \quad (7.3)$$

$$\left\{ \begin{array}{ll} \mathbf{h}_n \overset{*}{\rightharpoonup} \mathbf{h} & \text{in } L^\infty(I; L^2(\Omega')^d), \\ \mathbf{h}_n \rightharpoonup \mathbf{h} & \text{in } L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega' \setminus \partial\Omega)), \\ \mathbf{h}_n \rightharpoonup \mathbf{h} & \text{in } L^2(I; H_{\operatorname{loc}}^1(\Omega)^d), \\ \mathbf{h}_n \rightarrow \mathbf{h} & \text{in } L^{q_d-}(I; L_{\operatorname{loc}}^{q_d-}(\Omega)^d). \end{array} \right\} \left\{ \begin{array}{ll} \mathbf{m}_n \overset{*}{\rightharpoonup} \mathbf{m} & \text{in } L^\infty(I; L^2(\Omega)^d), \\ \mathbf{m}_n \rightharpoonup \mathbf{m} & \text{in } L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega)), \\ \mathbf{m}_n \rightharpoonup \mathbf{m} & \text{in } L^2(I; H_{\operatorname{loc}}^1(\Omega)^d), \\ \mathbf{m}_n \rightarrow \mathbf{m} & \text{in } L^{q_d-}(I; L_{\operatorname{loc}}^{q_d-}(\Omega)^d). \end{array} \right\},$$

where

$$2 < q_d := \begin{cases} \frac{10}{3} & \text{if } d = 3, \\ 4 & \text{if } d = 2. \end{cases} \quad (7.4)$$

For the weak-* convergence in $L^\infty(I; L^2(\Omega)^d)$ of c_n , one can use Lemma 6.6. For the determination of the exponent q_d , one uses Hölder's inequality as in the following Lemma 7.1 in the case $d = 3$ or a slightly better result from [11] in the case $d = 2$ in combination with the Aubin-Lions lemma [26] and Vitali's convergence theorem.

Lemma 7.1. *Let $c \in L^\infty(I; L^p(\Omega)) \cap L^q(I; L^r(\Omega))$, $r > q$. Then $c \in L^{q+\frac{p(r-q)}{r}}(I \times \Omega)$.*

For application of the Aubin-Lions lemma estimates of the time derivatives need to be used, see Lemma 6.1, Lemma 6.3 and Corollary 6.8, Lemma 6.4, Lemma 6.5 and Corollary 6.9. We have a look at $\partial_t c_n$, $\partial_t \mathbf{u}_n$ first. From the time-compactness estimates we get converging subsequences in some $L^a(I; L_{(\operatorname{loc})}^b(\Omega)^{(d)})$ -spaces, e.g. $a = 2$ in case of $\partial_t \mathbf{u}_n$ or $a = \frac{5}{4}$ in the case of $\partial_t c_n$, $b = 2$, that converge pointwise almost everywhere. By uniform estimates and Vitali's convergence theorem the convergence in $L^{q_d-}(I; L^{q_d-}(\Omega))$ or $L^{q_d-}(I; L^{q_d-}(\Omega)^d)$, respectively, can be achieved. For the convergence of $\mathbf{m}_n$ we have to use local estimates. The local weak H^1 -convergence of the magnetic variables comes from the well-known result

$$H_{t0}(\operatorname{curl})(\Omega) \cap H_{n0}(\operatorname{div})(\Omega) = H_0^1(\Omega)^d, \quad (7.5)$$

see [14, Lemma 2.5]. Using the formulas

$$\begin{aligned} \operatorname{div}(\phi \mathbf{m}) &= \nabla \phi \cdot \mathbf{m} + \phi \operatorname{div} \mathbf{m}, \\ \operatorname{curl}(\phi \mathbf{m}) &= \nabla \phi \times \mathbf{m} + \phi \operatorname{curl} \mathbf{m}, \end{aligned} \quad (7.6)$$

we find $(\phi \mathbf{m}) \in L^2(I; H_0^1(\Omega))$ for any scalar $\phi \in C_0^\infty(\Omega)$ and $\mathbf{m} \in H(\operatorname{div}, \operatorname{curl})(\Omega)$, according to (7.5). With Corollary 6.8 we obtain a strongly converging subsequence of $(\mathbf{m}_n \phi)$ that converges pointwise almost everywhere. By uniform (local) estimates we get the convergence of $\mathbf{m}_n$ in $L^{q_d-}(I; L_{\operatorname{loc}}^{q_d-}(\Omega)^d)$. For $\partial_t \mathbf{h}_n$ we use the fact that on any $V \subset\subset \Omega$ we have $\partial_t \mathbf{h}_n|_V$ and $\mathbf{h}_n|_V$ bounded in $L^2(I; (\nabla[H_0^1(V)])')$ or $L^2(I; H^1(V)^d)$, respectively, and obtain by the same methods a strongly converging subsequence in $L^{q_d-}(I; L^{q_d-}(V)^d)$ and therefore the local strong convergence.

With those convergence results—(7.3) and (5.14)—at hand, we can easily identify the limits in the linear terms $\mathcal{L}_2^{\mathbf{u}}, \mathcal{L}_3^c, \mathcal{L}_3^{\mathbf{m}}, \mathcal{L}_4^{\mathbf{m}}$ and the nonlinear terms $\mathcal{L}_3^{\mathbf{u}}, \mathcal{L}_4^{\mathbf{u}}$.

For the terms $\mathcal{L}_1^R, \mathcal{R}_1^R, \mathcal{R}_2^R$ we proceed as follows. Let $S \in \mathcal{R}$, then from (5.16c) we infer

$$\int_{\Omega'} \nabla R_n(t) \cdot \nabla \Pi_{\mathcal{R}_n} S \, dx = \int_{\Omega'} \mathbf{h}_a(t) \cdot \nabla \Pi_{\mathcal{R}_n} S \, dx - \int_{\Omega} \mathbf{m}_n(t) \cdot \nabla \Pi_{\mathcal{R}_n} S \, dx. \quad (7.7)$$

We multiply (7.7) with a function $\varphi \in C_0^\infty((0, T))$ and integrate in time. As φ does not depend on the spatial variable, we can write

$$\begin{aligned} & \int_0^T \int_{\Omega'} \nabla R_n \cdot \nabla \Pi_{\mathcal{R}_n}(S\varphi) \, dx \, dt \\ &= \int_0^T \int_{\Omega'} \mathbf{h}_a \cdot \nabla \Pi_{\mathcal{R}_n}(S\varphi) \, dx \, dt - \int_0^T \int_{\Omega} \mathbf{m}_n \cdot \nabla \Pi_{\mathcal{R}_n}(S\varphi) \, dx \, dt. \end{aligned}$$

We take the limit by exploiting weak convergence of ∇R_n , $\mathbf{m}_n$ in $L^2(I \times \Omega)^d$ and strong convergence of the gradients of the projections in $L^2(I \times \Omega)^d$. Indeed, the projections $\nabla \Pi_{\mathcal{R}_n}(S\varphi) = \varphi \nabla \Pi_{\mathcal{R}_n} S$ converge pointwise in time and $\varphi \|\nabla \Pi_{\mathcal{R}_n} S - \nabla S\|_{L^2(\Omega)^d}$ is dominated (in time) by a constant, which is integrable on $[0, T]$. By Lebesgue's dominated convergence theorem the projections converge in $L^2(I \times \Omega)^d$ and we end up with

$$\begin{aligned} & \int_0^T \int_{\Omega'} \nabla R \cdot \nabla S \, dx \, \varphi \, dt \\ &= \int_0^T \int_{\Omega'} \mathbf{h}_a \cdot \nabla S \, dx \, \varphi \, dt - \int_0^T \int_{\Omega} \mathbf{m} \cdot \nabla S \, dx \, \varphi \, dt. \end{aligned}$$

By the fundamental lemma of calculus of variations, for almost all $t \in I$ the equations

$$\int_{\Omega'} \nabla R(t) \cdot \nabla S \, dx = \int_{\Omega'} \mathbf{h}_a(t) \cdot \nabla S \, dx - \int_{\Omega} \mathbf{m}(t) \cdot \nabla S \, dx \quad \forall S \in \mathcal{R}$$

hold.

From (6.1) we also get a subsequence (which will not be relabeled for the ease of notation) such that

$$\frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \rightharpoonup \mathbf{W} \text{ in } L^2(I \times \Omega)^d.$$

As c_n converges strongly in $L^{q^-}(I \times \Omega)$, there is a subsequence (not relabeled) that converges pointwise almost everywhere, and so does

$$\frac{(c_n)_s}{\sqrt{f_2(c_n)}} \rightarrow \frac{(c)_s}{\sqrt{f_2(c)}} \text{ pointwise almost everywhere in } I \times \Omega,$$

because $(\cdot)_s$ and f_2 are continuous. The same argument holds for the other terms occurring below. By uniform bounds—based on (7.2) and (6.1)—and Vitali's convergence theorem one can deduce the strong convergences

$$\begin{aligned} \frac{(c_n)_s}{\sqrt{f_2(c_n)}} &\rightarrow \frac{c_s}{\sqrt{f_2(c)}} \quad \text{and} \quad \sqrt{f_2(c_n)} \rightarrow \sqrt{f_2(c)} \quad \text{in } L^{qd^-}(I \times \Omega), \\ \frac{\sqrt{f_2(c_n)}}{(c_n)_s} &\rightarrow \frac{\sqrt{f_2(c)}}{c_s} \quad \text{in } L^r(I \times \Omega) \text{ for any } 1 \leq r < \infty, \\ (g_s^L)'(c_n) &\rightarrow (g_s^L)'(c) \quad \text{in } L^{qd^-}(I \times \Omega). \end{aligned}$$

From this we get $(\mathbf{V}_{\text{part}})_n \rightharpoonup \frac{\sqrt{f_2(c)}}{c_s} \mathbf{W}$ a priori in $L^{2^-}(I \times \Omega)^d$ but weak convergence in $L^2(I \times \Omega)^d$ can be achieved, because the fluxes

$$(\mathbf{V}_{\text{part}})_n = \underbrace{\frac{\sqrt{f_2(c_n)}}{(c_n)_s}}_{\text{bounded}} \left[\frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \right]$$

are bounded uniformly in $L^2(I \times \Omega)^d$. We are going to identify $\mathbf{W}$ now. Testing with smooth and compactly supported testfunctions $\Phi \in C^\infty([0, T]; C_0^\infty(\Omega)^d)$ we get

$$\begin{aligned} & \int_0^T \int_\Omega \frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \cdot \Phi \, dx \, dt \\ &= -KD \int_0^T \int_\Omega \sqrt{f_2(c_n)} \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \cdot \Phi \, dx \, dt \\ & \quad + \int_0^T \int_\Omega \frac{\mu_0 K \sqrt{f_2(c_n)}}{(c_n)_s} (\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_2 \nabla \mathbf{m}_n)^T \mathbf{m}_n \cdot \Phi \, dx \, dt. \end{aligned} \quad (7.8)$$

We consider the first term of the right-hand side.

$$\begin{aligned} & \left| \int_0^T \int_\Omega \sqrt{f_2(c_n)} \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \cdot \Phi \, dx \, dt - \int_0^T \int_\Omega \sqrt{f_2(c)} \nabla (g_s^L)'(c) \cdot \Phi \, dx \, dt \right| \\ & \leq \left| \int_0^T \int_\Omega (\sqrt{f_2(c_n)} - \sqrt{f_2(c)}) \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \cdot \Phi \, dx \, dt \right| \\ & \quad + \left| \int_0^T \int_\Omega \sqrt{f_2(c)} (\nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) - \nabla (g_s^L)'(c)) \cdot \Phi \, dx \, dt \right| \\ & =: J_1 + J_2. \end{aligned}$$

The term J_1 tends to zero because of the strong convergence of $\sqrt{f_2(c_n)}$ and the L^2 -boundedness of $\nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n)$. The latter follows easily from the H^1 -stability of the projector $\Pi_{\mathcal{C}_n}$ and the computation $\nabla (g_s^L)'(c_n) = (g_s^L)''(c_n) \nabla c_n$, where $(g_s^L)''$ is a bounded function and ∇c_n is bounded in $L^2(I \times \Omega)^d$ (while σ_c, L fixed).

For the term J_2 we will prove the weak convergence $\nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \rightharpoonup \nabla (g_s^L)'(c)$ (for a subsequence without relabeling) in $L^2(I \times \Omega)^d$. From the boundedness of the term $\nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n)$ we also get $\nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \rightharpoonup \mathbf{w}$ in $L^2(I \times \Omega)^d$. We identify the limit by testing with $\Psi \in C^\infty([0, T]; C_0^\infty(\Omega)^d)$,

$$\begin{aligned} \int_0^T \int_\Omega \nabla \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \cdot \Psi \, dx \, dt &= - \int_0^T \int_\Omega \Pi_{\mathcal{C}_n} (g_s^L)'(c_n) \operatorname{div} \Psi \, dx \, dt \\ &\quad \downarrow \qquad \qquad \qquad \downarrow \\ \int_0^T \int_\Omega \mathbf{w} \cdot \Psi \, dx \, dt &= - \int_0^T \int_\Omega (g_s^L)'(c) \operatorname{div} \Psi \, dx \, dt, \end{aligned}$$

where we used the convergence of $(g_s^L)'(c_n)$ in $L^2(I \times \Omega)$ and

$$\begin{aligned} & \|\Pi_{\mathcal{C}_n} (g_s^L)'(c_n) - (g_s^L)'(c)\|_{L^2(I \times \Omega)^d} \\ & \leq \|\Pi_{\mathcal{C}_n} (g_s^L)'(c_n) - \Pi_{\mathcal{C}_n} (g_s^L)'(c)\|_{L^2(I \times \Omega)^d} + \|\Pi_{\mathcal{C}_n} (g_s^L)'(c) - (g_s^L)'(c)\|_{L^2(I \times \Omega)^d} \end{aligned}$$

combined with the L^2 -stability of $\Pi_{\mathcal{C}_n}$, cf. (5.11). Hence, $J_2 \rightarrow 0$. For the second term on the right-hand side of (7.8) we make use of the compact support of Φ and the thereby applicable higher regularity/convergence results for the magnetic variables. All three magnetic contributions are of the same kind, so we will only look at $(\nabla \mathbf{h}_n)^T \mathbf{m}_n$ as an example. The factor $\frac{\sqrt{f_2(c_n)}}{(c_n)_s}$ converges strongly in any $L^r(I \times \Omega)$, $r \in [1, \infty)$, the magnetic field part $\nabla \mathbf{h}_n$ converges weakly in $L_{\text{loc}}^2(I \times \Omega)^{d \times d}$ and the magnetization part $\mathbf{m}_n$ converges strongly in $L^{q_d-}(I; L_{\text{loc}}^{q_d-}(\Omega)^d)$, note that $q_d > 2$. Then, the identification of the limit is straight forward. Hence,

$$\mathbf{W} = -KD \sqrt{f_2(c)} \nabla (g_s^L)'(c) + \frac{\mu_0 K \sqrt{f_2(c)}}{c_s} (\alpha_1 \nabla \mathbf{h} + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_2 \nabla \mathbf{m})^T \mathbf{m}$$

and consequently

$$(\mathbf{V}_{\text{part}})_n \rightharpoonup -KD \frac{f_2(c)}{c_s} \nabla (g_s^L)'(c) + \frac{\mu_0 K f_2(c)}{c_s^2} (\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_2 \nabla \mathbf{m})^T \mathbf{m} =: \mathbf{V}_{\text{part}}$$

in $L^2(I \times \Omega)^d$.

Exploiting the local regularity of the magnetic variables we prove convergence of the convective term in (5.16d), $\mathcal{L}_2^{\mathbf{m}}$.

$$\begin{aligned} \int_0^T \mathcal{L}_2^{\mathbf{m}} dt &\doteq - \int_0^T \int_{\Omega} \left(\left(\mathbf{u}_n + \frac{K f_2(c_n)}{(c_n)_s} \left[-D \nabla \Pi_{c_n} (g_s^L)'(c_n) \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{\mu_0}{(c_n)_s} \left(\nabla (\alpha_1 \nabla R_n + \frac{\beta}{2} \Pi_{\mathcal{H}_n} \mathbf{h}_a - \alpha_2 \mathbf{m}_n) \right)^T \mathbf{m}_n \right] \right) \cdot \nabla \right) \Pi_{\mathcal{M}_n} \boldsymbol{\theta} \cdot \mathbf{m}_n dx dt \\ &= - \int_0^T \int_{\Omega} ((\mathbf{u}_n + (\mathbf{V}_{\text{part}})_n) \cdot \nabla) \Pi_{\mathcal{M}_n} \boldsymbol{\theta} \cdot \mathbf{m}_n dx dt. \end{aligned}$$

It suffices to only consider the less regular part

$$\begin{aligned} &\left| \int_0^T \int_{\Omega} ((\mathbf{V}_{\text{part}})_n \cdot \nabla) \Pi_{\mathcal{M}_n} \boldsymbol{\theta} \cdot \mathbf{m}_n dx dt - \int_0^T \int_{\Omega} ((\mathbf{V}_{\text{part}}) \cdot \nabla) \boldsymbol{\theta} \cdot \mathbf{m} dx dt \right| \\ &\leq \left| \int_0^T \int_{\Omega} (((\mathbf{V}_{\text{part}})_n - \mathbf{V}_{\text{part}}) \cdot \nabla) \boldsymbol{\theta} \cdot \mathbf{m} dx dt \right| \\ &\quad + \left| \int_0^T \int_{\Omega} ((\mathbf{V}_{\text{part}})_n \cdot \nabla) \boldsymbol{\theta} \cdot (\mathbf{m}_n - \mathbf{m}) dx dt \right| \\ &\quad + \left| \int_0^T \int_{\Omega} ((\mathbf{V}_{\text{part}})_n \cdot \nabla) (\Pi_{\mathcal{M}_n} \boldsymbol{\theta} - \boldsymbol{\theta}) \cdot \mathbf{m}_n dx dt \right|. \end{aligned} \tag{7.9}$$

The first term of the right-hand side converges to zero due to weak convergence of $(\mathbf{V}_{\text{part}})_n$ in $L^2(I \times \Omega)^d$ and the sufficient integrability of the smooth and compactly supported testfunction $\boldsymbol{\theta}$ and $\mathbf{m} \in L^{q_d-}(I; L_{\text{loc}}^{q_d-}(\Omega))$. In the second term higher regularity of $\mathbf{m}_n$ will be used,

hence by boundedness of $((\mathbf{V}_{\text{part}})_n)_{n \in \mathbb{N}}$ in $L^2(I \times \Omega)^d$ and strong (local) convergence of $\mathbf{m}_n \rightarrow \mathbf{m}$ in $L^{q_d-}(I; L_{\text{loc}}^{q_d-}(\Omega)^d)$ this term converges to zero. The local strong convergence of $\mathbf{m}_n$ was applicable due to the compact support of $\boldsymbol{\theta}$. In the last term we use boundedness of $((\mathbf{V}_{\text{part}})_n)_{n \in \mathbb{N}}$ in $L^2(I \times \Omega)^d$ and $(\mathbf{m}_n)_{n \in \mathbb{N}}$ in $L^\infty(I; L^2(\Omega)^d)$. For convergence it is sufficient that $\nabla \Pi_{\mathcal{M}_n} \boldsymbol{\theta} \rightarrow \nabla \boldsymbol{\theta}$ in $L^2(I; L^\infty(\Omega)^d)$. But as $\Pi_{\mathcal{M}_n} \boldsymbol{\theta}(t) \rightarrow \boldsymbol{\theta}(t)$ in $H^3(\Omega)^d \hookrightarrow W^{1,\infty}(\Omega)^d$, this is true, see (5.14). The terms $\mathcal{R}_1^{\mathbf{u}}$ and $\mathcal{R}_2^{\mathbf{u}}$ can be dealt with analogously (note that in the setting of the Navier-Stokes equations $\Pi_{\mathcal{U}_n} \mathbf{v}(t) \rightarrow \mathbf{v}(t)$ in $H^3(\Omega)^d$ as well) and for $\mathcal{R}_2^{\mathbf{m}}$ only the part

$$\int_0^T \int_{\Omega} \chi(c_n, \mathbf{h}_n) \mathbf{h}_n \cdot \Pi_{\mathcal{M}_n} \boldsymbol{\theta} dx dt$$

needs to be considered. For this, we extract a pointwise almost everywhere in $I \times \Omega$ converging subsequence of $\mathbf{h}_n$. The susceptibility is a continuous function (it has a continuous extension when the second argument is zero), hence for a pointwise almost everywhere in $I \times \Omega$ converging subsequence (not relabeled) of c_n and $\mathbf{h}_n$ one easily gets pointwise convergence almost everywhere in $I \times \Omega$ for $\chi(c_n, \mathbf{h}_n)$. By assumption (7.2) the susceptibility χ is bounded, hence $\chi(c_n, \mathbf{h}_n)$ converges in any $L^r(I \times \Omega)$, $r \in [1, \infty)$. Then the convergence of this term is an easy consequence.

The sum $\mathcal{R}_1^{\mathbf{u}} + \mathcal{R}_3^{\mathbf{u}}$ is linked to the convective velocity $(\mathbf{V}_{\text{part}})_n$. We have, see e.g. along the lines of the proof of Lemma 6.4,

$$\int_0^T (\mathcal{R}_1^{\mathbf{u}} + \mathcal{R}_3^{\mathbf{u}}) dt \doteq \int_0^T \int_{\Omega} \frac{(c_n)_s}{K \sqrt{f_2(c_n)}} \left[\frac{(c_n)_s}{\sqrt{f_2(c_n)}} (\mathbf{V}_{\text{part}})_n \right] \cdot \Pi_{\mathcal{U}_n} \mathbf{v} dx dt.$$

Note that we also used $\int_{\Omega} (\nabla \mathbf{m}_n)^T \mathbf{m}_n \cdot \Pi_{\mathcal{U}_n} \mathbf{v} \, dx = 0$ for this, which is true as we have $\operatorname{div} \Pi_{\mathcal{U}_n} \mathbf{v} = 0$. The first factor converges strongly in $L^{q_d^-}(I \times \Omega)$, the second factor—with brackets around—converges weakly in $L^2(I \times \Omega)^d$ and the projection of the testfunction converges strongly in $L^5(I \times \Omega)^d$. Note that $\frac{1}{2} + \frac{1}{q_d} \geq \frac{1}{5}$. Hence, we have convergence towards the term

$$\begin{aligned} & \int_0^T \int_{\Omega} \frac{c_s}{K \sqrt{f_2(c)}} \mathbf{W} \cdot \mathbf{v} \, dx \, dt \\ &= \int_0^T \int_{\Omega} (-Dc_s \nabla (g_s^L)'(c) + \mu_0 (\alpha_1 \nabla \mathbf{h} + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_3 \nabla \mathbf{m})^T \mathbf{m}) \cdot \mathbf{v} \, dx \, dt \\ &= \mu_0 \int_0^T \int_{\Omega} (\mathbf{v} \cdot \nabla) (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{m} \, dx \, dt \\ &\stackrel{\text{symmetry}}{=} \mu_0 \int_0^T \int_{\Omega} (\mathbf{m} \cdot \nabla) (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \cdot \mathbf{v} \, dx \, dt \\ &= -\mu_0 \int_0^T \int_{\Omega} \left((\mathbf{m} \cdot \nabla) \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) + \operatorname{div} \mathbf{m} \, \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) \right) \, dx \, dt, \end{aligned}$$

where we used $\operatorname{div} \mathbf{v} = 0$ and $(\nabla \mathbf{m})^T \mathbf{m} = \frac{1}{2} \nabla |\mathbf{m}|^2$ and

$$\begin{aligned} c_s \nabla (g_s^L)'(c) &= c_s (g_s^L)''(c) \nabla c = \begin{cases} 1 & \text{if } c < L, \\ \frac{c}{L} & \text{if } L \leq c \end{cases} \nabla c =: \nabla \tilde{g}(c), \\ \tilde{g}(c) &:= \begin{cases} c - \frac{L}{2} & \text{if } c < L, \\ \frac{c^2}{2L} & \text{if } L \leq c. \end{cases} \end{aligned}$$

Note that due to the compact support of $\mathbf{v}$ in Ω the terms $(\nabla \mathbf{m})^T \mathbf{m}$, $(\nabla \mathbf{h})^T \mathbf{m}$ and $c_s \nabla (g_s^L)'(c)$ are sufficiently regular to be separated from each other. Rewriting term $\mathcal{L}_4^c$ with the definition of $(\mathbf{V}_{\text{part}})_n$, (6.2), one gets a term similar to $\mathcal{L}_2^c$ but with less regularity. Hence it suffices to consider

$$\int_0^T \mathcal{L}_4^c \, dt \doteq \int_0^T \int_{\Omega} (c_n)_s (\mathbf{V}_{\text{part}})_n \cdot \nabla \Pi_{\mathcal{C}_n} \psi \, dx \, dt.$$

The first factor converges strongly in $L^{q_d^-}(I \times \Omega)$, the second converges weakly in $L^2(I \times \Omega)^d$, so the last factor has to converge strongly in $L^5(I \times \Omega)^d$. Concerning the convergence with respect to the spatial variable we use the embedding $H^2(\Omega)^d \hookrightarrow L^6(\Omega)^d$ (for $d \leq 3$) and $L^5(I; H^2(\Omega))$ -convergence of $\Pi_{\mathcal{C}_n} \psi$, see (5.14). Hence, the term $\mathcal{L}_4^c$ converges.

The convergence of the terms $\mathcal{L}_1^{\mathbf{u}}$, $\mathcal{L}_1^c$ and $\mathcal{L}_1^{\mathbf{m}}$ are trivial consequences of Lemma 6.4, Lemma 6.1 and Lemma 6.3. Also, the convergence of solutions at time $t = 0$ towards initial data in the weak sense is obvious.

As the limit functions obey an energy estimate corresponding to weak lower semi-continuity of norms and sufficient weak convergence of all terms on the left-hand side of (6.1), one could prove time-compactness again without the necessity of projectors. Therefore, the stability results related to the H^3 -norm are not needed and a better time-regularity could be achieved. However, to keep it simple, we do not pursue such an approach. However, for easier accessibility, instead of $\partial_t \mathbf{h} \in L^2(I; (\mathcal{H}_{\text{special}})')$, see (6.4), we write $\partial_t \mathbf{h}|_{\Omega} \in L^2(I; (\mathcal{H} \cap \nabla[H_0^1(\Omega)])')$.

Theorem 7.2. *Under the assumptions (H1), (H2), $\sigma_c > 0$ and $0 < s < e < L < \infty$,*

$$\begin{aligned} 0 < a_0 \leq f_2(c) \leq a_1 |c|^m + a_2, \quad m \in [0, 2] \text{ for some } a_0, a_1, a_2 > 0, \\ f_2 \text{ continuous,} \end{aligned}$$

as well as $\|\chi\|_\infty < \infty$ (see (7.2)) and $d \in \{2, 3\}$ there exists a weak solution $(\mathbf{u}, c, R, \mathbf{m})$ as specified in Definition 5.1. Moreover, the weak solution satisfies the energy estimate

$$\begin{aligned} & \|\mathbf{u}\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{u}\|_{L^2(I; H^1(\Omega)^d)} + \|g_s^L(c)\|_{L^\infty(I; L^1(\Omega))} + \frac{\sigma_c}{L} \|\nabla c\|_{L^2(I; L^2(\Omega)^d)} \\ & + \left\| \frac{c_s}{\sqrt{f_2(c)}} \mathbf{V}_{\text{part}} \right\|_{L^2(I; L^2(\Omega)^d)} + \|\mathbf{m}\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{h}\|_{L^\infty(I; L^2(\Omega)^d)} \\ & + \|\mathbf{m}\|_{L^2(I; H(\text{div}, \text{curl})(\Omega))} + \|\mathbf{h}\|_{L^2(I; H(\text{div}, \text{curl})(\Omega' \setminus \partial\Omega))} \\ & + \|\partial_t \mathbf{u}\|_{L^2(I; \mathcal{U}')} + C_{L, \sigma_c} \|\partial_t c\|_{L^{5/4}(I; (H_*^2(\Omega))')} \\ & + \|\partial_t \mathbf{m}\|_{L^2(I; \tilde{\mathcal{M}}')} + \|\partial_t \mathbf{h}|_\Omega\|_{L^2(I; (\mathcal{H} \cap \nabla[H_0^1(\Omega)])')} \\ & + \|\mathbf{u}\|_{L^{q_d}(I \times \Omega)^d} + \|\mathbf{m}\|_{L^{q_d}(I; L_{\text{loc}}^{q_d}(\Omega)^d)} + \|\mathbf{h}\|_{L^{q_d}(I; L_{\text{loc}}^{q_d}(\Omega)^d)} + C_{L, \sigma_c} \|c\|_{L^{q_d}(I \times \Omega)} \leq C, \end{aligned} \quad (7.10)$$

where $q_d = \begin{cases} \frac{10}{3}, & \text{if } d = 3, \\ 4, & \text{if } d = 2 \end{cases}$ and C_{L, σ_c} depends on L, σ_c while C does not. In detail, for some $\hat{C} > 0$, one has

$$C \leq \hat{C} (\|\mathbf{h}_a\|_{H^1(I; L^2(\Omega')^d)}^2 + \|\mathbf{u}^0\|_{L^2(\Omega)^d}^2 + \|\mathbf{m}^0\|_{L^2(\Omega)^d}^2 + \|c^0\|_{L^2(\Omega)}^2) \quad (7.11)$$

Proof. This follows from the considerations made so far. The convergence of the terms in the Galerkin scheme to the terms of the weak formulation has been discussed before this theorem. The energy estimate follows from the weak lower semi-continuity of norms and the convergences of initial data and projections of various $\mathbf{h}_a$ -terms. For further details, recall the right-hand side (5.25), from which the $\liminf$ needs to be considered. Also, for non-negative initial data c^0 one easily obtains the estimate $\int_\Omega g_s^L(c^0) dx \leq K \|c^0\|_{L^2(\Omega)}^2$ for some $K > 0$ and for terms of the kind $\|\mathbf{h}_a(s)\|_{L^2(\Omega)}^2, s \in \{0, T\}$, one can use Sobolev's embedding (with respect to time variable) in order to estimate those terms by the $H^1(I; L^2(\Omega'))$ -norm. The $L^2(I; H(\text{div})(\Omega))$ -norm of $\mathbf{h}_a$ can be estimated by the $H^1(I; L^2(\Omega'))$ -norm as well due to $\text{div } \mathbf{h}_a = 0$. Hence, we obtain the estimate (7.11) of the constant on the right-hand side. $\square$

8. The Non-regularized Case

In this section, we study the limit problem $(s, L^{-1}, \sigma_c) = (0, 0, 0)$. This requires a different approach to obtain regularity of particle densities c_n . To fix further notation, we choose sequences $\sigma_c = \sigma_c(n) := \frac{1}{n} \rightarrow 0, s = s(n) := \frac{1}{n} \rightarrow 0, L = L(n) := 3n \rightarrow \infty$.

We write $\{\mathbf{u}_n, c_n, R_n, \mathbf{m}_n\}$ for the solutions of the regularized system that exist according to the Theorem 7.2. In this section we confine ourselves to the special case $d = 2$ and $f_2(c) \sim c^2$, in detail we approximatively choose for any $n \in \mathbb{N}$,

$$f_2^n(c_n) := (c_n)_{s(n)} (c_n)_{s(n)}^{L(n)}, \quad (8.1)$$

where

$$(\cdot)^L := \min\{L, (\cdot)\}. \quad (8.2)$$

This choice clearly satisfies the assumptions of Theorem 7.2. Recall the definitions of $\mathcal{U}$ and $\tilde{\mathcal{M}}$ in (4.1), (3.3) and (5.6). We have the uniform bound

$$\begin{aligned} & \|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^2)} + \|\mathbf{u}_n\|_{L^2(I; H^1(\Omega)^2)} + \|c_n\|_{L^\infty(I; L^1(\Omega))} + \|g_{s(n)}(c_n) + c_n\|_{L^\infty(I; L^1(\Omega))} \\ & + \|(\mathbf{V}_{\text{part}})_n\|_{L^2(I \times \Omega)^2} + \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega)^2)} + \|\mathbf{h}_n\|_{L^\infty(I; L^2(\Omega)^2)} \\ & + \|\mathbf{m}_n\|_{L^2(I; H(\text{div}, \text{curl})(\Omega))} + \|\mathbf{h}_n\|_{L^2(I; H(\text{div}, \text{curl})(\Omega' \setminus \partial\Omega))} \\ & + \|\partial_t \mathbf{u}_n\|_{L^2(I; \mathcal{U})} + \|\partial_t \mathbf{m}_n\|_{L^2(I; \tilde{\mathcal{M}}')} + \|\partial_t \mathbf{h}_n|_\Omega\|_{L^2(I; (\mathcal{H} \cap \nabla[H_0^1(\Omega)])')} \\ & + \|\mathbf{u}_n\|_{L^4(I \times \Omega)^2} + \|\mathbf{m}_n\|_{L^4(I; L_{\text{loc}}^4(\Omega)^2)} + \|\mathbf{h}_n\|_{L^4(I; L_{\text{loc}}^4(\Omega)^2)} \leq C, \end{aligned} \quad (8.3)$$

where $g_s := g_s^\infty$ and the bounds of c_n follow from Lemma 6.6. Moreover, the particle velocity field is redefined as

$$\begin{aligned} (\mathbf{V}_{\text{part}})_n &= -KD \frac{f_2^n(c_n)}{(c_n)_{s(n)}} \nabla (g_{s(n)}^{L(n)})'(c_n) + \frac{\mu_0 K f_2^n(c_n)}{(c_n)_{s(n)}^2} (\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_2 \nabla \mathbf{m}_n)^T \mathbf{m}_n \\ &= -KD \underbrace{(c_n)_{s(n)}^{L(n)} (g_{s(n)}^{L(n)})''(c_n)}_{=1} \nabla c_n + \mu_0 K \underbrace{\frac{(c_n)_{s(n)}^{L(n)}}{(c_n)_{s(n)}^2}}_{\in (0,1]} (\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_2 \nabla \mathbf{m}_n)^T \mathbf{m}_n. \end{aligned} \quad (8.4)$$

From the estimate $\| \frac{(c_n)_{s(n)}}{\sqrt{f_2^n(c_n)}} (\mathbf{V}_{\text{part}})_n \|_{L^2(I \times \Omega)^d} \leq C$, which is a consequence of (7.10), we deduced the L^2 -bound of $(\mathbf{V}_{\text{part}})_n$ because of the fact $1 \leq \frac{(c_n)_{s(n)}^2}{f_2^n(c_n)} = \frac{(c_n)_{s(n)}}{(c_n)_{s(n)}^{L(n)}}$. Terms like $\frac{1}{3n^2} \|\nabla c_n\|_{L^2(I \times \Omega)}$ on the left-hand side have been omitted as they are not controlled uniformly. The remaining generic constant C on the right-hand side of (8.3) does not depend on s, σ_c nor L . It only depends on $\mathbf{h}_a$ and other given data.

As $H(\text{div}, \text{curl})(\Omega)$ is not compactly embedded into $L^2(\Omega)^d$ (cf. [7]) and both $\mathbf{m}$ and $\mathbf{h}$ enter the system in a nonlinear way, we work with localized spaces—this way guaranteeing applicability of Aubin-Lion-type arguments to deduce strong convergence results. Analytically, we formulate our results both for $d = 2$ and $d = 3$ space dimensions. For a passage to the limit in three space-dimensions, however, apparently an additional regularization is needed. We discuss some options in Remark 8.11. Let us now improve regularity.

Lemma 8.1. *Assume (8.3) to hold and let $\mathbf{h}_a \in L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1(\Omega)^d)$. Then the following holds true.*

(1) *Case $d = 2$.*

$$\begin{aligned} (c_n)_{n \in \mathbb{N}} &\text{ is bounded in} \\ L^{4/3}(I; W_{\text{loc}}^{1,4/3}(\Omega)) &\cap L^2(I; L_{\text{loc}}^2(\Omega)), \\ (\mathbf{m}_n, \mathbf{h}_n)_{n \in \mathbb{N}} &\text{ is bounded in} \\ L^2(I; H_{\text{loc}}^1(\Omega)^2) &\cap L^4(I; L_{\text{loc}}^4(\Omega)^2). \end{aligned}$$

(2) *Case $d = 3$.*

$$\begin{aligned} (c_n)_{n \in \mathbb{N}} &\text{ is bounded in} \\ L^{5/4}(I; W_{\text{loc}}^{1,5/4}(\Omega)) &\cap L^{5/3}(I; L_{\text{loc}}^{5/3}(\Omega)), \\ (\mathbf{m}_n, \mathbf{h}_n)_{n \in \mathbb{N}} &\text{ is bounded in} \\ L^2(I; H_{\text{loc}}^1(\Omega)^3) &\cap L^{10/3}(I; L_{\text{loc}}^{10/3}(\Omega)^3). \end{aligned}$$

Proof. By boundedness of $(\mathbf{V}_{\text{part}})_n$ in $L^2(I \times \Omega)^d$ and the identity (8.4) we infer that ∇c_n has at least the regularity the three terms

$$(\nabla \mathbf{m}_n)^T \mathbf{m}_n, (\nabla \mathbf{h}_n)^T \mathbf{m}_n, (\nabla \mathbf{h}_a)^T \mathbf{m}_n$$

come along with (or regularity of $(\mathbf{V}_{\text{part}})_n$ if the latter is worse). Comparing regularity of $\mathbf{m}_n, \mathbf{h}_n, \mathbf{h}_a$ it suffices to consider the term $(\nabla \mathbf{m}_n)^T \mathbf{m}_n$, only. According to (7.5), (7.6) we have $\mathbf{m}_n \in L^2(I; H_{\text{loc}}^1(\Omega)^d)$. Together with $\mathbf{m}_n \in L^\infty(I; L^2(\Omega)^d)$ we obtain

$$\mathbf{m}_n \in L^q(I; L_{\text{loc}}^q(\Omega)^d) \quad \forall q \in [1, \frac{2d+4}{d}], \quad \text{where } \frac{2d+4}{d} = \begin{cases} 4 & d = 2, \\ 10/3 & d = 3. \end{cases}$$

From

$$\int_{\Omega} (ab)^\gamma dx \leq \left(\int_{\Omega} a^2 dx \right)^{\frac{\gamma}{2}} \left(\int_{\Omega} b^{\frac{2\gamma}{2-\gamma}} dx \right)^{\frac{2-\gamma}{2}}$$

for $\gamma \in (0, 2)$ we infer—setting $a := |\nabla \mathbf{m}_n|$, $b := |\mathbf{m}_n|$ and choosing $\gamma = \frac{d+2}{d+1}$ —that

$$\nabla c_n \in L^\gamma(I; L_{\text{loc}}^\gamma(\Omega)^d).$$

By Sobolev's embedding,

$$c_n \in L^\gamma(I; L_{\text{loc}}^{\gamma'}(\Omega)), \quad \text{where } \gamma' = \begin{cases} 4 & d = 2, \\ 15/7 & d = 3. \end{cases}$$

Using $c_n \in L^\infty(I; L^1(\Omega))$, see Lemma 6.6, the claim follows by Lemma 7.1. Note that we used the estimate in Lemma 6.6 which is independent of $L = L(n)$. $\square$

In order to identify the limit of the fluxes $(c_s)_n(\mathbf{V}_{\text{part}})_n$ via strong convergence of c_n and weak convergence of $(\mathbf{V}_{\text{part}})_n$ in $L^2(I \times \Omega)^d$ higher regularity for c_n is needed.

Lemma 8.2. *Let $d = 2$, $(c_n)_{n \in \mathbb{N}}$ be bounded in $L^\infty(I; L^1(\Omega))$ and $L^{4/3}(I; L_{\text{loc}}^4(\Omega))$. Then, for all $\hat{\Omega} \subset\subset \Omega$ there exists $C > 0$ such that*

$$\int_0^T \int_{\hat{\Omega}} G(c_n) \, dx \, dt \leq C$$

uniformly in $n \in \mathbb{N}$, where

$$G(c) = \begin{cases} e^2 & c \leq e \\ c^2 |\log(c)|^{\frac{2}{3}} & c > e. \end{cases} \quad (8.5)$$

Proof. We split the integration into two parts.

$$\begin{aligned} \int_0^T \int_{\hat{\Omega}} G(c_n) \, dt &= \int_0^T \int_{[c_n > e] \cap \hat{\Omega}} c_n^2 |\log(c_n)|^{\frac{2}{3}} \, dt + \int_0^T \int_{[c_n \leq e] \cap \hat{\Omega}} e^2 \, dt \\ &\leq \int_0^T \int_{[c_n > e] \cap \hat{\Omega}} c_n^2 |\log(c_n)|^{\frac{2}{3}} \, dt + T|\hat{\Omega}|e^2. \end{aligned}$$

We compute

$$\begin{aligned} &\int_0^T \int_{[c_n > e] \cap \hat{\Omega}} c_n^\gamma |\log(c_n)|^\mu \, dt \\ &\leq \int_0^T \left(\int_{[c_n > e] \cap \hat{\Omega}} |c_n| |\log(c_n)|^{\frac{\mu}{\alpha}} \, dx \right)^\alpha \left(\int_{[c_n > e] \cap \hat{\Omega}} |c_n|^{\frac{\gamma-\alpha}{1-\alpha}} \, dx \right)^{1-\alpha} \, dt. \end{aligned}$$

We set $\alpha = \mu = \frac{2}{3}$, $\gamma = 2$ and obtain

$$\int_0^T \int_{[c_n > e] \cap \hat{\Omega}} c_n^2 |\log(c_n)|^{\frac{2}{3}} \, ds \leq \|g_s(c_n) + c_n\|_{L^\infty(I; L^1(\Omega))}^{\frac{2}{3}} \|c_n\|_{L^{4/3}(I; L_{\text{loc}}^4(\Omega))}^{\frac{4}{3}}.$$

The result follows immediately. $\square$

Next, we are concerned with compactness in time for c_n .

Lemma 8.3. *Let $d = 2$ and $\phi \in C_0^\infty(\Omega, \mathbb{R}_0^+)$ be an arbitrary cut-off function, $\hat{\Omega} := \text{supp } \phi \subset\subset \Omega$, $d = 2$ and $(c_n, \mathbf{u}_n, (\mathbf{V}_{\text{part}})_n)_{n \in \mathbb{N}}$ be bounded in*

$$L^2(I; L_{\text{loc}}^2(\Omega)) \times L^4(I \times \Omega)^2 \times L^2(I \times \Omega)^2$$

Then, there is a constant $0 < \overline{C} < \infty$ depending on ϕ , such that

$$\|\partial_t(\phi c_n)\|_{L^1(I; (H^2(\Omega) \cap W^{1,\infty}(\Omega))')} \leq \overline{C}$$

uniformly in $n \in \mathbb{N}$.

Proof. First, we observe that the weak formulation (5.7b) holds pointwise in time for almost all $t \in [0, T]$. This can be achieved by using testfunctions of the type $\psi = \psi_1(t)\psi_2(x)$ for $\psi_1 \in C_0^\infty((0, T))$ and $\psi_2 \in H_*^2(\Omega)$. Then, the first term becomes

$$\int_0^T \psi_1(t) \langle \partial_t c_n(t), \psi_2 \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} dt =: \int_0^T \psi_1(t) a_n(t) dt.$$

The other terms can be written—for some $\mathbf{f}_n \in L^{5/4}(I \times \Omega)^d$ —altogether as

$$\int_0^T \psi_1(t) \int_\Omega \mathbf{f}_n(t) \cdot \nabla \psi_2 dx dt =: \int_0^T \psi_1(t) b_n(t) dt.$$

Hence, $\forall \psi_1 \in C_0^\infty((0, T))$ we have

$$\int_0^T \psi_1(t) (a_n(t) + b_n(t)) dt = 0. \quad (8.6)$$

As countable unions of sets of measure zero have measure zero, too, locality is established, i.e. for almost all $t \in [0, T]$ we have $a_n(t) + b_n(t) = 0$ for all $n \in \mathbb{N}$.

Now, take $\psi \in H_*^2(\Omega) \cap W^{1,\infty}(\Omega)$ arbitrarily. Choose $\psi_2 := (\phi\psi)$ as testfunction in above formulation without time-integrals,

$$\begin{aligned} & \langle \partial_t c_n, (\phi\psi) \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} - \int_\Omega (c_n)_{s(n)} \mathbf{u}_n \cdot \nabla(\phi\psi) dx \\ & - \int_\Omega (c_n)_{s(n)} (\mathbf{V}_{\text{part}})_n \cdot \nabla(\phi\psi) dx + \underbrace{\sigma_c(n)}_{=\frac{1}{n}} \int_\Omega \nabla c_n \cdot \nabla(\phi\psi) dx = 0 \end{aligned}$$

implying

$$\begin{aligned} & \left| \langle \partial_t c_n, (\phi\psi) \rangle_{(H_*^2(\Omega))' \times H_*^2(\Omega)} \right| \\ & \leq \int_\Omega [|(c_n)_{s(n)}| (|\mathbf{u}_n| + |(\mathbf{V}_{\text{part}})_n|) + \sigma_c(n) |\nabla c|] (|\nabla \phi| |\psi| + |\phi| |\nabla \psi|) dx \\ & \leq C_1(\phi) \int_{\hat{\Omega}} [|(c_n)_{s(n)}| (|\mathbf{u}_n| + |(\mathbf{V}_{\text{part}})_n|) + \sigma_c(n) |\nabla c_n|] (|\psi| + |\nabla \psi|) dx \\ & \leq C_1(\phi) \left[\|(c_n)_{s(n)}\|_{L^2(\hat{\Omega})} (\|\mathbf{u}_n\|_{L^4(\Omega)^d} + \|(\mathbf{V}_{\text{part}})_n\|_{L^2(\Omega)^d}) \right. \\ & \quad \left. + \|\nabla c_n\|_{L^{4/3}(\Omega)^d} \right] \|\psi\|_{W^{1,\infty}(\Omega) \cap H_*^2(\Omega)}. \end{aligned}$$

Taking the supremum over all $\phi \in H_*^2(\Omega) \cap W^{1,\infty}(\Omega)$ with norm equal to 1, we have bound the dual norm of $\partial_t c_n$ in the sense

$$\begin{aligned} & \|\psi \mapsto \langle \partial_t c_n, \phi\psi \rangle_{(H_*^2(\Omega) \cap W^{1,\infty}(\Omega))' \times H_*^2(\Omega) \cap W^{1,\infty}(\Omega)}\|_{(H_*^2(\Omega) \cap W^{1,\infty}(\Omega))'} \\ & \leq C_1(\phi) \left[\|(c_n)_{s(n)}\|_{L^2(\hat{\Omega})} (\|\mathbf{u}_n\|_{L^4(\Omega)^d} + \|(\mathbf{V}_{\text{part}})_n\|_{L^2(\Omega)^d}) + \|\nabla c_n\|_{L^{4/3}(\Omega)^d} \right]. \end{aligned}$$

The integrability in time of the right-hand side is determined by the least regular term which is $\|(c_n)_{s(n)}\|_{L^2(\hat{\Omega})} \|(\mathbf{V}_{\text{part}})_n\|_{L^2(\Omega)^d}$ and which is L^1 -integrable. By (6.3) the proof is finished. $\square$

Combining Simon's compactness theorem [26, Section 8, Corollary 4] with the uniform regularity of $(c_n)_{n \in \mathbb{N}}$ in $L^{4/3}(I; W_{\text{loc}}^{1,4/3}(\Omega))$ established in Lemma 8.1 and an appropriate exhaustion argument, we obtain the following lemma.

Lemma 8.4. *Under the assumptions of Lemma 8.3 combined with uniform boundedness of $(c_n)_{n \in \mathbb{N}}$ in $L^{4/3}(I; W_{\text{loc}}^{1,4/3}(\Omega))$, there is a subsequence $(c_{n_k})_{k \in \mathbb{N}}$ and a function $c \in L^{4/3}(I; W_{\text{loc}}^{4/3}(\Omega))$ such that*

- (i) $c_{n_k} \rightarrow c$ strongly in $L^{4/3}(I; L_{\text{loc}}^p(\Omega))$ for any $1 \leq p < 4$.
- (ii) $c_{n_k} \rightarrow c$ pointwise almost everywhere in $I \times \Omega$.

Corollary 8.5. *Let $(c_n)_{n \in \mathbb{N}}$ as in Lemma 8.4 and assume that for any fixed $\hat{\Omega} \subset \subset \Omega$ there exists a constant $C > 0$ (depending on $\hat{\Omega}$) such that G from (8.5) satisfies*

$$\int_0^T \int_{\hat{\Omega}} G(c_n) \, dx \, ds \leq C$$

uniformly in $n \in \mathbb{N}$. Then, there exists a subsequence $(c_{n_k})_{k \in \mathbb{N}}$ which converges pointwise almost everywhere in $I \times \Omega$ and strongly in $L^2(I; L^2_{\text{loc}}(\Omega))$.

Proof. By [10, Chapter 2] and $\frac{G(x)}{x^2} \xrightarrow{x \rightarrow \infty} \infty$ we obtain equi-integrability of $(c_n)_{n \in \mathbb{N}}$ in $L^2(I \times \hat{\Omega})$ for any $\hat{\Omega} \subset \subset \Omega$. Together with pointwise convergence almost everywhere the result follows from Vitali's convergence theorem. $\square$

Note that time-compactness of the Galerkin solutions $\mathbf{m}_n$ established before in the case $d = 3$ carries over to this setting. The reason is the fact that $\sigma_c(n), L(n), s(n)$ occur in the magnetization equation (5.7d) only as part of $(\mathbf{V}_{\text{part}})_n$. But $(\mathbf{V}_{\text{part}})_n$ is bounded in $L^2(I \times \Omega)^d$, see (8.3), as it was in the Galerkin setting, too. Also, it is possible to carry over the estimates for $\partial_t \mathbf{h}_n$. The only reason for the restriction to the case $d = 2$ is the regularity of the particle density c_n . Sufficient time-compactness of $\mathbf{u}_n$ can be proven in dimension $d = 3$ as well. Note that we cannot use the same time-compactness estimates as in the Galerkin-setting for the velocity field right away, because the weak formulation and the Galerkin scheme are not analogous to each other. We are going to estimate globally in Ω , and therefore do not gain any improvements in the case $d = 2$.

Lemma 8.6. *Let $d = 3$ and let $(\mathbf{u}_n, \mathbf{m}_n, \mathbf{h}_n)_{n \in \mathbb{N}}$ be bounded in*

$$L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1(\Omega)^d) \times H(\text{div})(\Omega) \cap L^\infty(I; L^2(\Omega)^d) \times L^\infty(I; L^2(\Omega)^d)$$

and assume $\mathbf{h}_a \in L^\infty(I; L^2(\Omega)^d)$. Then there is a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\|\partial_t \mathbf{u}_n\|_{L^2(I; \mathcal{U})} \leq C.$$

Proof. Take $\mathbf{v} \in L^2(I; \mathcal{U})$, then

$$\begin{aligned} & \left| \int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{v} \rangle_{\mathcal{U}' \times \mathcal{U}} \, dt \right| \\ & \leq C (\|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)} \|\nabla \mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}} \|\mathbf{v}\|_{L^2(I; L^\infty(\Omega)^d)} \\ & \quad + \|\mathbf{u}_n\|_{L^\infty(I; L^2(\Omega)^d)}^2 \|\nabla \mathbf{v}\|_{L^1(I; L^\infty(\Omega)^{d \times d})} \\ & \quad + C \|\nabla \mathbf{u}_n\|_{L^2(I \times \Omega)^{d \times d}} \|\nabla \mathbf{v}\|_{L^2(I \times \Omega)^{d \times d}} \\ & \quad + \left| \int_0^T \int_{\Omega} (\mathbf{m}_n \cdot \nabla) \mathbf{v} \cdot (\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \mathbf{h}_a) \, dx \, dt \right| \\ & \quad + \left| \int_0^T \int_{\Omega} \text{div} \mathbf{m}_n \, \mathbf{v} \cdot (\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \mathbf{h}_a) \, dx \, dt \right| \\ & \quad + \left| \int_0^T \int_{\Omega} (\mathbf{m}_n \times (\alpha_1 \mathbf{h}_n + \frac{\beta}{2} \mathbf{h}_a)) \cdot \text{curl} \, \mathbf{v} \, dx \, dt \right|. \end{aligned}$$

So, we can conclude

$$\begin{aligned} & \left| \int_0^T \langle \partial_t \mathbf{u}_n, \mathbf{v} \rangle_{\mathcal{U}' \times \mathcal{U}} \, dt \right| \leq C' (\|\mathbf{v}\|_{L^2(I; L^\infty(\Omega)^d)} + \|\nabla \mathbf{v}\|_{L^1(I; L^\infty(\Omega)^{d \times d})}) \\ & \quad + \hat{C} \|\text{div} \mathbf{m}_n\|_{L^2(I \times \Omega)} (\|\mathbf{h}_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{h}_a\|_{L^\infty(I; L^2(\Omega)^d)}) \|\mathbf{v}\|_{L^2(I; L^\infty(\Omega)^d)} \\ & \quad + \hat{C} \|\mathbf{m}_n\|_{L^\infty(I; L^2(\Omega)^d)} (\|\mathbf{h}_n\|_{L^\infty(I; L^2(\Omega)^d)} + \|\mathbf{h}_a\|_{L^\infty(I; L^2(\Omega)^d)}) \|\nabla \mathbf{v}\|_{L^1(I; L^\infty(\Omega)^{d \times d})}. \end{aligned}$$

From this and $\mathcal{U} \subset H^3(\Omega)^d \hookrightarrow W^{1,\infty}(\Omega)^d$ the claim follows easily. $\square$

If we combine the Lemmas and Corollaries 8.1–8.6 of this paragraph, we get the following statement.

Corollary 8.7. *Let $(\mathbf{u}_n, c_n, (\mathbf{V}_{\text{part}})_n, \mathbf{m}_n, \mathbf{h}_n)_{n \in \mathbb{N}}$ satisfy (8.3) and let the field $\mathbf{h}_a$ be in $L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H^1(\Omega)^d)$. Then in the case $d = 3$,*

$\mathbf{h}_n, \mathbf{m}_n$ are uniformly bounded in

$$L^\infty(I; L^2(\Omega)^3) \cap L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^2(I; H_{\text{loc}}^1(\Omega)^3) \cap L^{10/3}(I; L_{\text{loc}}^{10/3}(\Omega)^3),$$

c_n are uniformly bounded in

$$L^{5/4}(I; W_{\text{loc}}^{1,5/4}(\Omega)) \cap L^{5/4}(L_{\text{loc}}^{15/7}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \cap L^{5/3}(I; L_{\text{loc}}^{5/3}(\Omega)).$$

In the case $d = 2$, we have additionally that

$\mathbf{h}_n, \mathbf{m}_n$ are uniformly bounded in

$$L^4(I; L_{\text{loc}}^4(\Omega)^2),$$

c_n are uniformly bounded in

$$L^{4/3}(I; W_{\text{loc}}^{1,4/3}(\Omega)) \cap L^{4/3}(L_{\text{loc}}^4(\Omega)) \cap L^2(I; L_{\text{loc}}^2(\Omega)).$$

Moreover, in the case $d = 3$,

$\partial_t \mathbf{m}_n$ are uniformly bounded in

$$L^2(I; \tilde{\mathcal{M}}'),$$

$\partial_t \mathbf{u}_n$ are uniformly bounded in

$$L^2(I; \mathcal{U}'),$$

$\partial_t \mathbf{h}_n|_\Omega$ are uniformly bounded in

$$L^2(I; (\mathcal{H} \cap \nabla[H_0^1(\Omega)])').$$

In the case $d = 2$ there exists a subsequence $n_k \rightarrow \infty$ such that for some limit function $c \in L^2(I; L_{\text{loc}}^2(\Omega))$

$$c_{n_k} \rightarrow c$$

$$\text{in } L^2(I; L_{\text{loc}}^2(\Omega)),$$

$$c_{n_k} \rightarrow c$$

pointwise almost everywhere in Ω .

Note that the time-compactness estimate of $\partial_t \mathbf{h}_n|_V$, for $V \subset\subset \Omega$, carries over as well. The convergence behavior of the Galerkin solutions carries over to our sequences $\mathbf{u}_n, \mathbf{m}_n$ and $\mathbf{h}_n$ due to analogous energy estimates. For the functions c_n new results have been obtained in this paragraph. Therefore as a starting point we use the convergences

$$\left\{ \begin{array}{l} \mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(I; L^2(\Omega)^d), \\ \mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^2(I; H^1(\Omega)^d), \\ \mathbf{u}_n \rightarrow \mathbf{u} \quad \text{in } L^{4-}(I; L^{4-}(\Omega)^d). \end{array} \right\} \left\{ \begin{array}{l} c_n \overset{*}{\rightharpoonup} c \quad \text{in } L^\infty(I; L^1(\Omega)), \\ c_n \rightharpoonup c \quad \text{in } L^{4/3}(I; W^{1,4/3}(\Omega)), \\ c_n \rightarrow c \quad \text{in } L^2(I; L^2_{\text{loc}}(\Omega)). \end{array} \right\} \quad (8.7)$$

$$\left\{ \begin{array}{l} \mathbf{h}_n \overset{*}{\rightharpoonup} \mathbf{h} \quad \text{in } L^\infty(I; L^2(\Omega')^d), \\ \mathbf{h}_n \rightharpoonup \mathbf{h} \quad \text{in } L^2(I; H(\text{div}, \text{curl})(\Omega' \setminus \partial\Omega)), \\ \mathbf{h}_n \rightharpoonup \mathbf{h} \quad \text{in } L^2(I; H^1_{\text{loc}}(\Omega)^d), \\ \mathbf{h}_n \rightarrow \mathbf{h} \quad \text{in } L^{4-}(I; L^{4-}_{\text{loc}}(\Omega)^d). \end{array} \right\} \left\{ \begin{array}{l} \mathbf{m}_n \overset{*}{\rightharpoonup} \mathbf{m} \quad \text{in } L^\infty(I; L^2(\Omega)^d), \\ \mathbf{m}_n \rightharpoonup \mathbf{m} \quad \text{in } L^2(I; H(\text{div}, \text{curl})(\Omega)), \\ \mathbf{m}_n \rightharpoonup \mathbf{m} \quad \text{in } L^2(I; H^1_{\text{loc}}(\Omega)^d), \\ \mathbf{m}_n \rightarrow \mathbf{m} \quad \text{in } L^{4-}(I; L^{4-}_{\text{loc}}(\Omega)^d). \end{array} \right\},$$

Based on the stability estimate (8.3), we obtain a non-negativity result for the limit c of the particle density functions.

Lemma 8.8. *The function c from (8.7) which is a limit of the sequence of regularised solutions $\{c_n\}_{n \in \mathbb{N}}$, satisfying (8.9), is non-negative.*

Proof. We can deduce from (8.3) that $\|g_{s(n)}(c_n)\|_{L^\infty(I; L^1(\Omega))}$ is bounded, hence

$$\|(c_n)_-\|_{L^\infty(I; L^2(\Omega))} \leq 2s(n)\|g_{s(n)}(c_n)\|_{L^\infty(I; L^1(\Omega))} + s(n)^2|\Omega| \leq Cs(n) \rightarrow 0.$$

Therefore $c \geq 0$. □

Additionally, we have the following straight forward pointwise convergence results.

Lemma 8.9. *Let $(c_n)_{n \in \mathbb{N}} \subset L^1(I \times \Omega)$ be a sequence that converges pointwise almost everywhere to a function $c \in L^1(I \times \Omega)$ with $c \geq 0$ almost everywhere. Then,*

$$(c_n)_{s(n)} \rightarrow c, \quad (c_n)_{s(n)}^{L(n)} \rightarrow c, \quad \frac{(c_n)_{s(n)}^{L(n)}}{(c_n)_{s(n)}} \rightarrow 1,$$

almost everywhere in $I \times \Omega$.

The regularised functions $(c_n)_s$ satisfy the same estimates from Lemma 8.2, as the function G that was used in the lemma cannot distinguish between c_n and $(c_n)_{s(n)}$. Moreover the $L^\infty(I; L^1(\Omega))$ -estimate and the $L^{\frac{4}{3}}(I; L^4_{\text{loc}}(\Omega))$ -estimate of c_n can be carried over to $(c_n)_s$ trivially. Hence,

$$(c_n)_{s(n)} \rightarrow c \text{ in } L^2(I; L^2_{\text{loc}}(\Omega)), \quad (8.8)$$

too.

We will now pass to the limit. We will plug in $C_0^\infty([0, T]; C_0^\infty(\Omega))^2$ -testfunctions $\mathbf{v}$ for the Navier-Stokes equations and easily identify the left-hand side of the equations. Exploiting the compact spatial support of the testfunction, we have no problems with the right-hand side, either. As an example let us consider the term

$$\int_0^T \int_\Omega (\mathbf{m}_n \times \mathbf{h}_n) \cdot \text{curl } \mathbf{v} \, dx \, dt.$$

As we can use $\mathbf{h}_n \rightarrow \mathbf{h}$ in $L^q(I; L^q_{\text{loc}}(\Omega)^d)$ and $\mathbf{m}_n \rightarrow \mathbf{m}$ in $L^q(I; L^q_{\text{loc}}(\Omega)^d)$, where $q > 2$ in both cases $d = 2$ or $d = 3$, there is no obstacle in taking the limit. For (5.7b) we analogously take smooth and compactly supported testfunctions.

Many considerations are identical to the case when the Galerkin solutions converged to the weak solution of the regularized system. Note that the convergence behavior in two dimensions is at least as good as in three dimensions and therefore we only need to concentrate on the terms with c_n in it. First, we will restore the weak convergence result of $(\mathbf{V}_{\text{part}})_n$ in $L^2(I \times \Omega)^d$. We easily get $(\mathbf{V}_{\text{part}})_n \rightharpoonup \mathbf{W}$

in $L^2(I \times \Omega)^d$ for some $\mathbf{W} \in L^2(I \times \Omega)^d$, see (8.3). We identify $\mathbf{W}$ in the same way as done in the Galerkin-setting. Let $\Phi \in C_0^\infty(I \times \Omega)^d$, then

$$\begin{aligned} & \int_0^T \int_\Omega (\mathbf{V}_{\text{part}})_n \cdot \Phi \, dx \, dt \\ &= -KD \int_0^T \int_\Omega \nabla c_n \cdot \Phi \, dx \, dt \\ &+ \int_0^T \int_\Omega \mu_0 K \underbrace{\frac{(c_n)_{s(n)}^{L(n)}}{(c_n)_{s(n)}}}_{\rightarrow 1} (\alpha_1 \nabla \mathbf{h}_n + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_2 \nabla \mathbf{m}_n)^T \mathbf{m}_n \cdot \Phi \, dx \, dt. \end{aligned} \quad (8.9)$$

First we note that the two terms that had been combined into one term before can be separated into individual terms due to the regularity of ∇c_n . The second term of the right-hand side converges as the gradients converge weakly and locally with respect to $L^2(I \times \Omega)^d$ and $\mathbf{m}_n$ converges strongly and locally with respect to $L^{2+}(I \times \Omega)^d$ (in any case $d \in \{2, 3\}$) and the bounded quotient $0 < \frac{(c_n)_{s(n)}^{L(n)}}{(c_n)_{s(n)}} \leq 1$ converges strongly in any $L^p(I \times \Omega)$ -norm, $p \in [1, \infty)$, due to pointwise convergence (for a non-relabelled subsequence), see Lemma 8.9, and Vitali's convergence theorem. Those convergences are sufficient to identify the limit. The first term converges, obviously, and we arrive at

$$\mathbf{W} = -KD \nabla c + \mu_0 K (\alpha_1 \nabla \mathbf{h} + \frac{\beta}{2} \nabla \mathbf{h}_a - \alpha_2 \nabla \mathbf{m})^T \mathbf{m}.$$

With this at hand, we can proceed as in the Galerkin case for any term except for those terms of the particle density equation (5.7b). We can use our bound on $\nabla c_n \in L^{5/4}(I; L^{5/4}(\Omega)^d)$ (in both cases $d \in \{2, 3\}$) and the smoothness of the testfunction in order to prove that the third term of (5.7b)—the regularizing term—vanishes.

Now, consider the first term. Integration by parts gives

$$\int_0^T \langle \partial_t c_n, \psi \rangle_{(H_*^2(\Omega)) \times H_*^2(\Omega)} \, dt = - \int_0^T \int_\Omega c_n \partial_t \psi \, dx \, dt + \int_\Omega c^0 \psi(0) \, dx$$

with testfunctions $\psi \in C_0^1([0, T]; C_0^2(\Omega))$. This can be justified already on the level of the Galerkin approximation. Taking the limit is straightforward. The fourth term in (5.7b) simplifies to

$$\int_0^T \int_\Omega (c_n)_s (\mathbf{V}_{\text{part}})_n \cdot \nabla \psi \, dx \, dt,$$

which converges as $(c_n)_{s(n)} \rightarrow c$ in $L^2(I; L_{\text{loc}}^2(\Omega))$ and $(\mathbf{V}_{\text{part}})_n \rightharpoonup \mathbf{V}_{\text{part}}$ in $L^2(I \times \Omega)^d$. This step was the only one where we needed to restrict ourselves to the case $d = 2$. The second term is easier and analogous to the fourth term (no restriction to $d = 2$ needed). Hence, the limit has been taken and we obtain the following result.

Theorem 8.10. *Assume (H1), (H2) as well as $\|\chi\|_\infty < \infty$ and $d = 2$. Let initial data be given as specified in Definition 5.1. Then, there are functions*

$$\begin{aligned} \mathbf{u} &\in L^2(I; H_0^1(\Omega)^d \cap H(\text{div}_0)(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap W^{1,2}(I; \mathcal{U}), \\ c &\in L^{4/3}(I; W_{\text{loc}}^{1,4/3}(\Omega)) \cap L^\infty(I; L^1(\Omega)) \cap L^2(I; L_{\text{loc}}^2(\Omega)) \\ R &\in L^2(I; \mathcal{R}) \cap L^\infty(I; H^1(\Omega)) \cap L^2(I; H_{\text{loc}}^2(\Omega)) \\ \mathbf{m} &\in L^2(I; H(\text{div}, \text{curl})(\Omega)) \cap L^\infty(I; L^2(\Omega)^d) \cap L^2(I; H_{\text{loc}}^1(\Omega)^d) \cap W^{1,2}(I; \tilde{\mathcal{M}}) \end{aligned}$$

such that for all

$$\begin{aligned} \mathbf{v} &\in L^2(I; (H(\text{div}_0)(\Omega) \cap H_0^3(\Omega)^2)), \\ \psi &\in C_0^1([0, T]; C_0^2(\Omega)), \end{aligned}$$

$$S \in \mathcal{R},$$

$$\Psi \in L^2(I; H_0^3(\Omega)^2),$$

the weak formulation

$$\begin{aligned} & \rho_0 \int_0^T \langle \mathbf{u}_t, \mathbf{v} \rangle_{\mathcal{U}' \times \mathcal{U}} dt + \int_0^T \int_{\Omega} 2\eta \mathbf{D}\mathbf{u} \cdot \mathbf{D}\mathbf{v} dx dt \\ & + \frac{\rho_0}{2} \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} dx dt - \frac{\rho_0}{2} \int_0^T \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{u} dx dt \end{aligned} \quad (8.10)$$

$$\begin{aligned} & = -\mu_0 \int_0^T \int_{\Omega} (\mathbf{m} \cdot \nabla) \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) dx dt - \mu_0 \int_0^T \int_{\Omega} \operatorname{div} \mathbf{m} \mathbf{v} \cdot (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a) dx dt \\ & + \frac{\mu_0}{2} \int_0^T \int_{\Omega} (\mathbf{m} \times (\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a)) \cdot \operatorname{curl} \mathbf{v} dx dt, \\ & - \int_0^T \int_{\Omega} c \partial_t \psi dx dt + \int_{\Omega} c^0 \psi(0) dx - \int_0^T \int_{\Omega} c \mathbf{u} \cdot \nabla \psi dx dt \end{aligned} \quad (8.11)$$

$$\int_{\Omega'} \nabla R \cdot \nabla S dx = \int_{\Omega'} \mathbf{h}_a \cdot \nabla S dx - \int_{\Omega} \mathbf{m} \cdot \nabla S dx \quad \text{for almost all } t \in [0, T], \quad (8.12)$$

$$\begin{aligned} & \int_0^T \langle \mathbf{m}_t, \Psi \rangle_{\tilde{\mathcal{M}}' \times \tilde{\mathcal{M}}} dt \\ & - \int_0^T \int_{\Omega} \left(\left(\mathbf{u} + K \left[-D\nabla c + \mu_0 (\nabla(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a - \alpha_2 \mathbf{m}))^T \mathbf{m} \right] \right) \cdot \nabla \right) \Psi \cdot \mathbf{m} dx dt \\ & + \sigma \int_0^T \int_{\Omega} \operatorname{div} \mathbf{m} \cdot \operatorname{div} \Psi dx dt + \sigma \int_0^T \int_{\Omega} \operatorname{curl} \mathbf{m} \cdot \operatorname{curl} \Psi dx dt \\ & = \frac{1}{2} \int_0^T \int_{\Omega} (\mathbf{m} \times \Psi) \cdot \operatorname{curl} \mathbf{u} dx dt - \frac{1}{\tau_{\text{rel}}} \int_0^T \int_{\Omega} (\mathbf{m} - \chi(c, \mathbf{h}) \mathbf{h}) \cdot \Psi dx dt \end{aligned} \quad (8.13)$$

is satisfied and the initial data is attained in the sense

$$\langle \mathbf{u}(0), \Psi \rangle_{\mathcal{U}' \times \mathcal{U}} = \int_{\Omega} \mathbf{u}^0 \cdot \Psi dx \quad \forall \Psi \in \mathcal{U},$$

$$\langle \mathbf{m}(0), \Psi \rangle_{\tilde{\mathcal{M}}' \times \tilde{\mathcal{M}}} = \int_{\Omega} \mathbf{m}^0 \cdot \Psi dx \quad \forall \Psi \in \tilde{\mathcal{M}}.$$

This setting correlates to the case $f_2(c) = c^2$ of (1.1). Moreover,

$$\begin{aligned} & \|\mathbf{u}\|_{L^\infty(I; L^2(\Omega)^2)} + \|\mathbf{u}\|_{L^2(I; H^1(\Omega)^2)} + \|c\|_{L^\infty(I; L^1(\Omega))} \\ & + \|\mathbf{V}_{\text{part}}\|_{L^2(I \times \Omega)^2} + \|\mathbf{m}\|_{L^\infty(I; L^2(\Omega)^2)} + \|\mathbf{h}\|_{L^\infty(I; L^2(\Omega)^2)} \\ & + \|\mathbf{m}\|_{L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega))} + \|\mathbf{h}\|_{L^2(I; H(\operatorname{div}, \operatorname{curl})(\Omega' \setminus \partial\Omega))} \\ & + \|\partial_t \mathbf{u}\|_{L^2(I; \mathcal{U}')} + \|\partial_t \mathbf{m}\|_{L^2(I; \tilde{\mathcal{M}}')} + \|\partial_t \mathbf{h}\|_{L^2(I; (\mathcal{H} \cap \nabla[H_0^1(\Omega)])')} \\ & + \|\mathbf{u}\|_{L^4(I \times \Omega)^2} + \|\mathbf{m}\|_{L^4(I; L_{\text{loc}}^4(\Omega)^2)} + \|\mathbf{h}\|_{L^4(I; L_{\text{loc}}^4(\Omega)^2)} \leq C, \end{aligned} \quad (8.14)$$

Remark 8.11. It would be desirable to have existence of weak solutions in the three-dimensional setting, too. Recall that the bottleneck is the space-time integrability of c . Due to the intricate coupling between the evolution equations for c and $\mathbf{m}$, this integrability is in a subtle way related to the regularity of $(\nabla \mathbf{m})^T \mathbf{m}$ or $(\nabla \mathbf{h})^T \mathbf{m}$ and hence as a consequence related to the regularity of $\mathbf{m}$.

Let us discuss two methods of regularization to overcome this issue.

(A) In the evolution equation (1.1c) for c , we add an additional diffusion term

$$\sigma_c \operatorname{div}(c^\alpha \nabla c)$$

with $\alpha \geq \frac{1}{3}$ to the right-hand side. As the energy estimate is based on testing this equation by $g'(c) \sim \log c$ we obtain a uniform estimate of $c^{\frac{\alpha+1}{2}}$ in $L^2(I; H^1(\Omega))$. This is sufficient to deduce $c \in L^{\alpha+1}(I; L^{3(\alpha+1)}(\Omega))$ in space dimension $d = 3$ which, together with the $L^\infty(I; L^1(\Omega))$ -bound on $g(c)$, entails $c \in L^2(I \times \Omega)$.

However, this new regularization formally contributes to the velocity, the particles are transported with, which is not reflected unless $\mathbf{V}_{\text{part}}$ is changed accordingly.

(B) In the evolution equation (1.1f) for $\mathbf{m}$, one might introduce a viscous relaxation $-\partial_t \Delta \mathbf{m}$. As a consequence, $\mathbf{m}$ would be controlled in $L^\infty(I; H_{\text{loc}}^1(\Omega)^d)$, which would entail $(\mathbf{m} \cdot \nabla) \mathbf{m} \in L^\infty(I; L_{\text{loc}}^{\frac{3}{2}}(\Omega)^d)$.

Hence, mimicking the argument in Lemma 8.1, ∇c is contained in $L^2(I; L_{\text{loc}}^{\frac{3}{2}}(\Omega)^d)$, too, which implies $c \in L^2(I; L_{\text{loc}}^3(\Omega))$.

However, it is not clear how to justify such an regularization from a physics perspective.

Remark 8.12. Assuming sufficient regularity for the weak solutions constructed in Theorem 8.10, they can in a standard way be shown to satisfy the classical solution concept (1.1a), (1.1b), (1.1e), (1.1f), (1.1g). Note in particular that c solves

$$c_t + \nabla c \cdot \mathbf{u} - \operatorname{div}(K D c \nabla c - K \mu_0 c (\nabla(\alpha_1 \mathbf{h} + \frac{\beta}{2} \mathbf{h}_a - \alpha_2 \mathbf{m}))^T \mathbf{m}) = 0$$

which is (1.1c), (1.1d) for $f_2(c) = c^2$. For the hydrodynamic equations, we pursued the usual pathway to prove existence of a pressure, see e.g. [12, Lemma III.1.1].

Boundary conditions, however, can be identified for the velocity field, but not for density c or magnetization $\mathbf{m}$, as the latter are solutions only in the sense of distributions. This is due to the fact that the integral estimates derived so far do not provide more than $H(\operatorname{div}, \operatorname{curl})(\Omega)$ -regularity for $\mathbf{m}$ and $\mathbf{h}$. As a consequence, spatial gradients of c or of $\mathbf{m}$ have only L_{loc}^p -type integrability due to the coupling between terms in ∇c and in $\nabla \mathbf{m}$ expressed by equation (1.1d). This requires test functions in the weak formulation to be compactly supported.

In the case that non-compactly supported test functions, e.g. of class H^1 with respect to the spatial variables, were permitted, the identity

$$\int_{\partial\Omega} c \mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu} \psi \, d\sigma = 0 \quad \forall \psi \in H^{1/2}(\partial\Omega)$$

would be a direct consequence, entailing (1.2b) for positive t . In case of the magnetization equation, one would get the identity

$$\int_{\partial\Omega} [(\mathbf{J} \cdot \boldsymbol{\nu})(\mathbf{m} \cdot \boldsymbol{\theta}) - \sigma \operatorname{div} \mathbf{m} \boldsymbol{\theta} \cdot \boldsymbol{\nu} - \sigma \operatorname{curl} \mathbf{m} \times \boldsymbol{\nu} \cdot \boldsymbol{\theta}] \, d\sigma = 0 \quad \forall \boldsymbol{\theta} \in H^{1/2}(\partial\Omega)^2 \quad (8.15)$$

for positive t . Assuming the normal vector field $\boldsymbol{\nu}$ to be sufficiently regular as well, by surjectivity of the trace operator we would find $\boldsymbol{\theta}$ such that

$$\boldsymbol{\theta}|_{\partial\Omega} = ((\mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu})(\mathbf{m} \cdot \boldsymbol{\nu}) - \sigma \operatorname{div} \mathbf{m}) \boldsymbol{\nu}.$$

Inserting this into (8.15) would entail

$$\int_{\partial\Omega} |(\mathbf{V}_{\text{part}} \cdot \boldsymbol{\nu})(\mathbf{m} \cdot \boldsymbol{\nu}) - \sigma \operatorname{div} \mathbf{m}|^2 \, d\sigma = 0,$$

hence (1.2d). Condition (1.2c) would follow easily now.

We would like to make a last comment about numerics, see [17].

Remark 8.13. As our existence result suggests, a convergence result based on conforming elements might require H^3 -regular finite elements. However, in order to reduce complexity, a non-conforming approach was used in [17], which is based on the previous works [19, 20]. In fact, if the magnetization is discretized

by discontinuous elements, the requirement $\nabla \mathcal{R}_h|_{\Omega} \subset \mathcal{M}_h$ for suitable discrete finite element spaces $\mathcal{R}_h$ and $\mathcal{M}_h$ may be satisfied. Take e.g. $\mathcal{R}_h$ as continuous piecewise quadratic elements and $\mathcal{M}_h$ as discontinuous piecewise linear elements. In [20], strategies have been presented how to prove convergence in the (partially) discontinuous setting of a finite element scheme in the case that $\mathbf{h} = \mathbf{h}_a$ is given and σ in (1.1f) is chosen to be zero. In our previous work [17], we were able to find an energy stable scheme in case of $\sigma > 0$ by introducing divergence and curl operators defined by duality. However, we did not yet succeed in transferring the local H^1 -regularity to the discrete setting. This is just another reason why, instead of proving existence by showing convergence of a numerical scheme, we confined ourselves to the continuous setting which turned out to be already rather intricate concerning the regularity of the particle density.

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Compliance with ethical standards

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Appendix

A.1. Further Modeling Aspects

In this section, we provide some additional information about energetic aspects of model (1.1). As presented in detail in [17], the model has been derived under the assumption that the magnetic energy of the system consists of three components,

- (i) the classical magnetic energy

$$\mathcal{E}_{class} := \frac{\alpha_0}{2} \int_{\Omega'} \mathbf{b} \cdot \mathbf{h} dx = \alpha_0 \frac{\mu_0}{2} \int_{\Omega'} (|\mathbf{h}|^2 + \mathbf{h} \cdot \mathbf{m}) dx,$$

- (ii) a Zeeman-type energy

$$\mathcal{E}_{Zeeman} := -\alpha_1 \frac{\mu_0}{2} \int_{\Omega} (\mathbf{h} - \mathbf{h}_a) \cdot \mathbf{m} dx - \alpha_2 \frac{\mu_0}{2} \int_{\Omega} \mathbf{h}_a \cdot \mathbf{m} dx$$

which is to capture the interaction of the magnetizable particles with the field,

- (iii) and a magnetization energy

$$\mathcal{E}_{\mathbf{m}} := \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 dx$$

which is to model the fact that the particles' magnetization decays in the absence of (external) magnetic fields.

Here, $\alpha_0, \dots, \alpha_3$ are positive parameters.

Using the weak formulation of equation (1.1e), see, e.g. (5.7c), the sum

$$\mathcal{E}_{mag} := \mathcal{E}_{class} + \mathcal{E}_{Zeeman} + \mathcal{E}_{\mathbf{m}}$$

can be rewritten to become

$$\begin{aligned} \mathcal{E}_{mag} = & \alpha_1 \frac{\mu_0}{2} \int_{\Omega'} |\mathbf{h}|^2 dx + (\alpha_0 - 2\alpha_1 + \alpha_2) \frac{\mu_0}{2} \int_{\Omega'} \mathbf{h}_a \cdot \mathbf{h} dx \\ & + \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 dx + (\alpha_1 - \alpha_2) \frac{\mu_0}{2} \int_{\Omega'} |\mathbf{h}_a|^2 dx. \end{aligned}$$

In the special case $\alpha_0 = \alpha_1 = \alpha_2 = 1$, we get

$$\mathcal{E}_{mag} = \frac{\mu_0}{2} \int_{\Omega'} |\mathbf{h}|^2 dx + \alpha_3 \frac{\mu_0}{2} \int_{\Omega} |\mathbf{m}|^2 dx.$$

This corresponds to the choice $\beta = 0$ in system (1.1) (cf. (2.32) in [17]).

Choosing $g = c(\log c - 1)$, we get

$$c_t + \operatorname{div}(c\mathbf{u}) + K \operatorname{div} \left(-D \frac{f_2(c)}{c} \nabla c + \mu_0 \frac{f_2(c)}{c} (\mathbf{m} \cdot \nabla)(\mathbf{h} - \alpha_3 \mathbf{m}) \right) = 0$$

as the evolution equation for the number density c .

For the choice $f_2(c) = c^2$, we end up with the equation

$$c_t + \operatorname{div}(c\mathbf{u}) + K \operatorname{div} (-Dc \nabla c + \mu_0 c (\mathbf{m} \cdot \nabla)(\mathbf{h} - \alpha_3 \mathbf{m})) = 0,$$

i.e. the convective term resembles – up to the term $\alpha_3 \mathbf{m}$ reflecting the decay of magnetization in the absence of external fields – the convective term studied in previous publications, see e.g. [21] and the references therein. Note that we find the diffusion to be nonlinear with a diffusivity linear in c . It is worth mentioning that we would get both linear diffusion and the aforementioned magnetic convection term if we chose $g(c) = c - \log c$ this way a priori excluding the particle density c to be compactly supported.

Finally, to provide a shortcut to the energy estimate (2.2), we present a formal derivation assuming $\alpha_1 = 1, \beta = 0$. For the general case, we refer to the rigorous derivation in the framework of the Galerkin approximation – see the proof of Lemma 5.6. For the ease of presentation, let

$$\hat{\mathbf{b}} := \mathbf{h} - \alpha_3 \mathbf{m}.$$

Multiplying (1.1a) by $\mathbf{u}$, integrating with respect to space, using the symmetry of $\nabla \mathbf{h}$, and integration by parts give

$$\frac{\rho_0}{2} \partial_t \int_{\Omega} |\mathbf{u}|^2 dx + 2\eta \int_{\Omega} |\mathbf{D}\mathbf{u}|^2 dx = \mu_0 \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} dx + \frac{\mu_0}{2} \int_{\Omega} (\mathbf{m} \times \mathbf{h}) \cdot \operatorname{curl} \mathbf{u} dx. \quad (\text{A.1.1})$$

Multiplying (1.1c) by $Dg'(c) = D \log(c)$ and integration by parts gives

$$\begin{aligned} 0 = & \partial_t D \int_{\Omega} g(c) dx + \overbrace{D \int_{\Omega} \nabla g(c) \cdot \mathbf{u} dx}^{=0 \text{ due to (1.1b)}} - D \int_{\Omega} c \mathbf{V}_{\text{part}} \cdot \nabla g'(c) dx \\ \stackrel{(1.1d)}{=} & \partial_t D \int_{\Omega} g(c) dx + \int_{\Omega} \frac{c^2}{K f_2(c)} |\mathbf{V}_{\text{part}}|^2 dx - \mu_0 \int_{\Omega} (\mathbf{V}_{\text{part}} \cdot \nabla) \hat{\mathbf{b}} \cdot \mathbf{m} dx. \end{aligned} \quad (\text{A.1.2})$$

We obtain a weak formulation of the system (1.1e), (1.1g) by integration by parts and exploiting the transmission conditions (1.2e) and the boundary condition (1.2f), i.e.

$$\int_{\Omega'} \nabla R \cdot \nabla S dx = \int_{\Omega'} \mathbf{h}_a \cdot \nabla S dx - \int_{\Omega} \mathbf{m} \cdot \nabla S dx \quad \forall S \in H^1(\Omega). \quad (\text{A.1.3})$$

A detailed computation can be found in [17]. Testing the above with $\frac{\mu_0}{\tau_{\text{rel}}}R$ yields

$$\frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} |\mathbf{h}|^2 dx = \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h}_a \cdot \mathbf{h} dx - \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} \mathbf{m} \cdot \mathbf{h} dx. \quad (\text{A.1.4})$$

Testing the time derivative of (A.1.3) with $\mu_0 R$ yields

$$\frac{\mu_0}{2} \partial_t \int_{\Omega'} |\mathbf{h}|^2 dx = \mu_0 \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \mathbf{h} dx - \mu_0 \int_{\Omega} \mathbf{m}_t \cdot \mathbf{h} dx. \quad (\text{A.1.5})$$

Moreover, by using (1.1e), (1.1g) we get

$$-\operatorname{div}(\mathbf{h}|_{\Omega}) = \operatorname{div} \mathbf{m} \text{ and } \operatorname{div}(\mathbf{h}|_{\Omega' \setminus \bar{\Omega}}) = 0. \quad (\text{A.1.6})$$

Testing the magnetization equation (1.1f) by $-\mu_0 \hat{\mathbf{b}} = -\mu_0 \mathbf{h} + \alpha_3 \mu_0 \mathbf{m}$, using the decomposition $-\Delta = -\nabla \operatorname{div} + \operatorname{curl} \operatorname{curl}$ as well as the boundary conditions (1.2c) and (1.2d) yields

$$\begin{aligned} & -\mu_0 \int_{\Omega} \mathbf{m}_t \cdot \mathbf{h} dx + \frac{\alpha_3 \mu_0}{2} \partial_t \int_{\Omega} |\mathbf{m}|^2 dx + \mu_0 \int_{\Omega} ((\mathbf{u} + \mathbf{V}_{\text{part}}) \cdot \nabla) \hat{\mathbf{b}} \cdot \mathbf{m} dx \\ & - \sigma \mu_0 \int_{\Omega} \operatorname{div} \mathbf{m} \operatorname{div} \mathbf{h} dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\operatorname{div} \mathbf{m}|^2 dx \\ & - \sigma \mu_0 \int_{\Omega} \operatorname{curl} \mathbf{m} \cdot \operatorname{curl} \mathbf{h} dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\operatorname{curl} \mathbf{m}|^2 dx \\ & = -\frac{\mu_0}{2} \int_{\Omega} \operatorname{curl} \mathbf{u} \times \mathbf{m} \cdot \hat{\mathbf{b}} dx + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} (\mathbf{m} - \chi(c, \mathbf{h}) \mathbf{h}) \cdot \hat{\mathbf{b}} dx. \end{aligned}$$

Exploiting that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is an alternating trilinear form, $\mathbf{m} \times \mathbf{m} = \mathbf{0}$, (1.4a), (1.4b) and the computation (A.1.6), we arrive—expanding the abbreviation $\hat{\mathbf{b}}$ —at

$$\begin{aligned} & -\mu_0 \int_{\Omega} \mathbf{m}_t \cdot \mathbf{h} dx + \frac{\alpha_3 \mu_0}{2} \partial_t \int_{\Omega} |\mathbf{m}|^2 dx + \mu_0 \int_{\Omega} ((\mathbf{u} + \mathbf{V}_{\text{part}}) \cdot \nabla) \hat{\mathbf{b}} \cdot \mathbf{m} dx \\ & + \sigma \mu_0 \int_{\Omega' \setminus \partial \Omega} |\operatorname{div} \mathbf{h}|^2 dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\operatorname{div} \mathbf{m}|^2 dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\operatorname{curl} \mathbf{m}|^2 dx \\ & + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} \chi(c, \mathbf{h}) |\mathbf{h}|^2 dx + \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} |\mathbf{m}|^2 dx \\ & = -\frac{\mu_0}{2} \int_{\Omega} (\mathbf{m} \times \mathbf{h}) \cdot \operatorname{curl} \mathbf{u} dx + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} \mathbf{m} \cdot \mathbf{h} dx + \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \chi(c, \mathbf{h}) \mathbf{h} \cdot \mathbf{m} dx. \end{aligned} \quad (\text{A.1.7})$$

Next, exploiting $(\nabla \mathbf{m})^T \mathbf{m} = \frac{1}{2} \nabla(|\mathbf{m}|^2)$ and $\operatorname{div} \mathbf{u} = 0$, we identify $\int_{\Omega} (\mathbf{u} \cdot \nabla) \hat{\mathbf{b}} \cdot \mathbf{m} dx = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{h} \cdot \mathbf{m} dx$.

Adding up (A.1.1)–(A.1.2), (A.1.4)–(A.1.5) and (A.1.7) yields

$$\begin{aligned} & \frac{\rho_0}{2} \partial_t \int_{\Omega} |\mathbf{u}|^2 dx + 2\eta \int_{\Omega} |\mathbf{D}\mathbf{u}|^2 dx + \partial_t D \int_{\Omega} g(c) dx + \int_{\Omega} \frac{c^2}{K f_2(c)} |\mathbf{V}_{\text{part}}|^2 dx \\ & + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} |\mathbf{h}|^2 dx + \frac{\mu_0}{2} \partial_t \int_{\Omega'} |\mathbf{h}|^2 dx + \frac{\alpha_3 \mu_0}{2} \partial_t \int_{\Omega} |\mathbf{m}|^2 dx \\ & + \sigma \mu_0 \int_{\Omega' \setminus \partial \Omega} |\operatorname{div} \mathbf{h}|^2 dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\operatorname{div} \mathbf{m}|^2 dx + \sigma \mu_0 \alpha_3 \int_{\Omega} |\operatorname{curl} \mathbf{m}|^2 dx \\ & + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega} \chi(c, \mathbf{h}) |\mathbf{h}|^2 dx + \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} |\mathbf{m}|^2 dx \\ & = \frac{\mu_0 \alpha_3}{\tau_{\text{rel}}} \int_{\Omega} \chi(c, \mathbf{h}) \mathbf{h} \cdot \mathbf{m} dx + \mu_0 \int_{\Omega'} \partial_t \mathbf{h}_a \cdot \mathbf{h} dx + \frac{\mu_0}{\tau_{\text{rel}}} \int_{\Omega'} \mathbf{h}_a \cdot \mathbf{h} dx. \end{aligned} \quad (\text{A.1.8})$$

Integrating this identity with respect to time, we end up with (2.2) due to Young's inequality, absorption, and Gronwall's lemma. For more details, the reader is referred to the detailed derivation in the framework of the Galerkin approximation—see the proof of Lemma 5.6.

A.2. Generic Results on Decomposition of Hilbert Spaces

Lemma A.2.1. *Let X, Y be separable Hilbert spaces and assume X to be densely and compactly embedded into Y . Then there exists a basis $\{b_i\}_{i \in N}$, $N \subset \mathbb{N}$, such that*

$$\begin{aligned} Y &= \overline{\text{span}\{b_i\}_{i \in N}}^Y, & X &= \overline{\text{span}\{b_i\}_{i \in N}}^X \\ \langle b_i, b_j \rangle_Y &= \delta_{ij} \forall i, j \in N & \langle b_i, b_j \rangle_X &= 0 \forall i, j \in N \text{ with } i \neq j. \end{aligned}$$

Proof. Define $T : Y \rightarrow Y$ by $Tf := x$, where $x \in X$ is the solution of

$$\langle x, \phi \rangle_X = \langle f, \phi \rangle_Y \quad \forall \phi \in X.$$

We check for well-posedness. On the left-hand side we have an X -elliptic bilinear form and f is in $Y' \subset X'$. By an application of the Lax-Milgram Theorem, the operator T is well-defined and linear. It is self-adjoint due to symmetry of the scalar product on the left-hand side and its image is in $X \hookrightarrow Y$, hence T is compact.

By the spectral theorem we get an orthonormal set $\{b_i\}_{i \in N}$, $N \subset \mathbb{N}$, of eigenvectors to eigenvalues $\{\lambda_i\}_{i \in N} \subset \mathbb{R}$ such that

$$Y = \overline{\text{span}\{b_i\}_{i \in N}}^Y \oplus N(T), \quad \langle b_i, b_j \rangle_Y = \delta_{ij}.$$

Easily one deduces $N(T) = (\overline{X}^Y)^\perp = Y^\perp = \{0\}$. Similarly, if there was an element

$$x \in (\overline{\text{span}\{b_i\}_{i \in N}}^X)^\perp$$

then $0 = \langle \lambda_i b_i, x \rangle_X = \langle b_i, x \rangle_Y$ for all $i \in N$ and therefore $x = 0$. Hence,

$$X = \overline{\text{span}\{b_i\}_{i \in N}}^X, \quad \lambda_i \langle b_i, b_j \rangle_X = \delta_{ij}.$$

From this the result follows. $\square$

Proof of Lemma 3.1.3:

Proof. ad i): Let $(v_n)_{n \in \mathbb{N}} \subset V$, be a converging sequence, $v_n \rightarrow v$ in X , then for all $u \in U$,

$$|\langle v, u \rangle_Y| \leq |\langle v - v_n, u \rangle_Y| + |\langle v_n, u \rangle_Y| \leq \|v - v_n\|_Y \|u\|_Y \leq \|v_n - n\|_X \|u\|_Y \rightarrow 0.$$

Therefore, $v \in X$ and $v \perp^Y u$ for all $u \in U$ and by density $v \perp^Y u$ for all $u \in \overline{U}^Y$. Hence, $v \in (\overline{U}^Y)^\perp \cap X$.

ad ii): This is a direct consequence of Lemma A.2.1 applied to the pairs of spaces $(U, \overline{U}^Y)$ and $(V, \overline{V}^Y)$.

ad iii): Assume by contradiction the existence of $x \in X$ such that

$$0 = \langle x, \varphi \rangle_X \quad \forall \varphi \in U \oplus V.$$

Now, the identity

$$U \oplus V \stackrel{(3.1.3)}{=} (\overline{U}^Y \cap X) \oplus V \stackrel{V \subseteq X}{=} (\overline{U}^Y \oplus V) \cap X = Y \cap X = X$$

implies $x = 0$, the desired contradiction.

Note that no basis function in (3.1.4) can be approached by an infinite linear combination of others in the X -norm as this would contradict the orthogonality of $\{u_i\}_{i \in N_1} \cup \{v_i\}_{i \in N_2}$ in Y due to the continuous embedding $X \hookrightarrow Y$.

Moreover, (3.1.5) follows by our definition $V = (\overline{U}^Y)^\perp \cap X$.

ad iv): Finally, we observe that the results ii) and iii) do not depend on the specific choice of a basis in U . $\square$

Remark A.2.2. We give an example of spaces U, X, Y that satisfy the assumptions formulated in Lemma 3.1.3, in particular (3.1.3). We set

$$U := \nabla[H_{\text{mean}}^2(\Omega)] \subset X := H^1(\Omega)^d \hookrightarrow Y := L^2(\Omega)^d.$$

By standard arguments U is a closed subspace of X and the inclusion $U \subset \bar{U}^Y \cap X$ is obvious. Let $\Psi \in \bar{U}^Y \cap X$, then there exists a sequence $(\nabla\phi_n)_{n \in \mathbb{N}}$ in $U = \nabla[H_{\text{mean}}^2(\Omega)]$ and a function $\phi \in H_{\text{mean}}^2(\Omega)$ such that

$$\begin{aligned} \nabla\phi_n &\rightarrow \Psi = \nabla\phi \text{ in } L^2(\Omega)^d, \\ \phi_n &\rightarrow \phi \text{ in } H^1(\Omega). \end{aligned}$$

As $\Psi \in X$ we have $\nabla\Psi = D^2\phi \in L^2(\Omega)^{d \times d}$ which implies $\phi \in H^2(\Omega)$ and finally

$$\Psi = \nabla\phi \in \nabla[H_{\text{mean}}^2(\Omega)] = U.$$

Note that using the notation of Lemma 3.1.3, the space V is given by $H^1(\Omega)^d \cap H_{n0}(\text{div}_0)(\Omega)$, see [9].

A.3. Auxiliary Results for the Proof of Lemma 3.2.3

Lemma A.3.1. *For all $i \in \mathbb{N}$, $p_i \in \mathcal{R}_{\text{temp}}$, where p_i are defined in (3.2.10) and $\mathcal{R}_{\text{temp}}$ is defined in (3.2.7).*

Proof. We have to prove that every function p_i , $i \in \mathbb{N}$, is an element of $\mathcal{R}_{\text{temp}}$. Recall the Dirichlet–Laplace eigenfunctions from (3.2.9). As $\Delta\psi_i^{\text{dir}}|_{\partial\Omega} = -\mu_i\psi_i^{\text{dir}}|_{\partial\Omega} = 0$ and furthermore $(\psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} dx) \in H_{\text{mean}}^1(\Omega) \cap H^4(\Omega)$ —implying the gradient to be in $\mathcal{H}$ —there exists a sequence $(\tilde{\mathbf{h}}_n)_{n \in \mathbb{N}} \subset \text{span}\{\mathbf{h}_i\}_{i \in \mathbb{N}}$ such that

$$\tilde{\mathbf{h}}_n \rightarrow \nabla(\psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} dx) = \nabla p_i|_{\Omega} \text{ in } H^3(\Omega)^d. \quad (\text{A.3.1})$$

We can write

$$\tilde{\mathbf{h}}_n = \sum_{i=1}^{N_n} \alpha_i^n \nabla\phi_i^{\Omega}, \quad \text{for some } N_n \in \mathbb{N}, \alpha_i^n \in \mathbb{R}, \quad \forall n \in \mathbb{N}, i = 1, \dots, N_n. \quad (\text{A.3.2})$$

We now choose a sequence

$$\begin{aligned} Q_n &:= \sum_{i=1}^{N_n} \alpha_i^n \underbrace{\mathbf{R}_{2i}}_{\equiv \phi_i^{\Omega} - c_i \text{ on } \Omega} + \alpha_n \mathbf{R}_1 \in \text{span}\{\mathbf{R}_i\}_{i \in \mathbb{N}}, \\ \alpha_n &:= -|\Omega'|^{\frac{1}{2}} \left(\int_{\Omega} \psi_i^{\text{dir}} dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right) \in \mathbb{R}. \end{aligned}$$

We estimate

$$\|Q_n - p_i\|_{\mathcal{R}} \leq \|\nabla Q_n - \nabla p_i\|_{H(\text{div})(\Omega)} + \|\nabla Q_n - \nabla p_i\|_{L^2(\Omega' \setminus \bar{\Omega})^d} =: I_n + II_n.$$

On Ω we have $Q_n = \sum_{i=1}^{N_n} \alpha_i^n (\phi_i^{\Omega} - c_i)$, hence the gradient converges in $H^3(\Omega)^d$ towards ∇p_i , see (A.3.2) and (A.3.1). This implies $I_n \rightarrow 0$. For the second term we expand all definitions and use $\nabla(V \ni x \mapsto C) \equiv \mathbf{0}$ as well as $L_V^{-1}(\partial V \ni x \mapsto C) = (V \ni x \mapsto C)$ for any constant $C \in \mathbb{R}$ and bounded domain $V \subset \mathbb{R}^d$. We end up with

$$\begin{aligned} II_n &= \|\nabla Q_n - \nabla p_i\|_{L^2(\Omega' \setminus \bar{\Omega})^d} = \|\nabla Q_n\|_{L^2(\Omega' \setminus \bar{\Omega})^d} \\ &= \left\| \sum_{i=1}^{N_n} \alpha_i^n \nabla L_{\Omega' \setminus \bar{\Omega}}^{-1} \begin{cases} \phi_i^{\Omega}|_{\partial\Omega} - c_i & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} \right\| \end{aligned}$$

$$\begin{aligned}
& -|\Omega'|^{\frac{1}{2}} \left(\int_{\Omega} \psi_i^{\text{dir}} dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right) \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \left\{ \begin{array}{ll} 0 & \text{on } \partial\Omega \\ |\Omega'|^{-\frac{1}{2}} & \text{on } \partial\Omega' \end{array} \right\} \Big\|_{L^2(\Omega' \setminus \overline{\Omega})^d} \\
&= \left\| \sum_{i=1}^{N_n} \alpha_i^n \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \left\{ \begin{array}{ll} \phi_i^{\Omega} |_{\partial\Omega} - c_i & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{array} \right\} \right. \\
&\quad \left. - \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \left\{ \begin{array}{ll} 0 & \text{on } \partial\Omega \\ \left(\int_{\Omega} \psi_i^{\text{dir}} dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right) & \text{on } \partial\Omega' \end{array} \right\} + \overbrace{\nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \left(\int_{\Omega} \psi_i^{\text{dir}} dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right)}^{=0} \right\|_{L^2(\Omega' \setminus \overline{\Omega})^d} \\
&= \left\| \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \left\{ \begin{array}{ll} \sum_{i=1}^{N_n} \alpha_i^n (\phi_i^{\Omega} |_{\partial\Omega} - c_i) + \left(\int_{\Omega} \psi_i^{\text{dir}} dx \right) + \sum_{i=1}^{N_n} \alpha_i^n c_i & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{array} \right\} \right\|_{L^2(\Omega' \setminus \overline{\Omega})^d} \\
&\stackrel{(3.2.4)}{\leq} C \left\{ \left\| \sum_{i=1}^{N_n} \alpha_i^n \phi_i^{\Omega} |_{\partial\Omega} + \left(\int_{\Omega} \psi_i^{\text{dir}} dx \right) \right\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|0\|_{H^{\frac{1}{2}}(\partial\Omega')} \right\} \\
&= C \left\| \sum_{i=1}^{N_n} \alpha_i^n \phi_i^{\Omega} |_{\partial\Omega} - \underbrace{\left(\psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} dx \right)}_{=0 \text{ on } \partial\Omega} \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\leq \overline{C} \left\| \sum_{i=1}^{N_n} \alpha_i^n \phi_i^{\Omega} - \left(\psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} dx \right) \right\|_{H^1(\Omega)} \\
&\leq \tilde{C} \|\tilde{\mathbf{h}}_n - \nabla \left(\psi_i^{\text{dir}} - \int_{\Omega} \psi_i^{\text{dir}} dx \right)\|_{L^2(\Omega)} \stackrel{(A.3.1)}{\rightarrow} 0,
\end{aligned}$$

where we used the trace theorem and Poincaré's inequality for mean value free functions in the last two steps. Therefore, p_i can be approximated by $\text{span}\{\mathbf{R}_i\}_{i \in \mathbb{N}}$ with respect to the norm of $\mathcal{R}$ which implies $p_i \in \mathcal{R}_{\text{temp}}$. $\square$

Lemma A.3.2. *For all $i \in \mathbb{N}_0$ we have $q_i \in \mathcal{R}_{\text{temp}}$, where q_i are defined in (3.2.11) and $\mathcal{R}_{\text{temp}}$ is defined in (3.2.7).*

Proof. We use the definition of the Neumann-Laplace eigenfunctions in (3.2.3), which occur in the definition of the functions q_i .

Formally, the weak formulation of (3.2.2) reads as

$$\begin{cases} \text{for given } f \in H^{\frac{1}{2}}(\partial V) \text{ find } u \in H_0^1(V) \text{ such that} \\ \forall \varphi_0 \in H_0^1(V) : \int_V \nabla u \cdot \nabla \varphi_0 dx = \int_V \Delta(Ef) \varphi_0 dx, \end{cases} \quad (\text{A.3.3})$$

where $E : H^{\frac{1}{2}}(\partial V) \rightarrow H^1(V)$ is one of the continuous right-inverses of the trace operator and $L_V^{-1}f := u + (Ef)$. By standard higher regularity results u and $L_V^{-1}f$ are H^4 -regular if $\Delta(Ef)$ is H^2 -regular. Therefore, we require $f \in H^{\frac{7}{2}}(\partial V)$ which can be achieved if the solutions to the Neumann-Laplace problem (3.2.3) are H^4 -regular. Indeed, they are $H^4(\Omega)$ -regular due to (H1). By definition, $\Delta(q_i|_{\Omega}) \equiv 0$ and $(\tilde{q}_i - \int_{\Omega} \tilde{q}_i dx) \in H_{\text{mean}}^1(\Omega) \cap H^4(\Omega)$, hence $\nabla q_i \in \mathcal{H}$. Similarly as before, by using the same notation, there exists a sequence

$$\tilde{\mathbf{h}}_n \rightarrow \nabla \left(\tilde{q}_i - \int_{\Omega} \tilde{q}_i dx \right) = \nabla q_i|_{\Omega} \text{ in } H^3(\Omega)^d, \quad (\text{A.3.4})$$

which can be written as in (A.3.2). We now choose a sequence

$$Q_n := \sum_{i=1}^{N_n} \alpha_i^n \underbrace{\mathbf{R}_{2i}}_{\equiv \phi_i^{\Omega} - c_i \text{ on } \Omega} + \alpha_n \mathbf{R}_1 \in \text{span}\{\mathbf{R}_i\}_{i \in \mathbb{N}},$$

$$\alpha_n := -|\Omega'|^{\frac{1}{2}} \left(\oint_{\Omega} \tilde{q}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right).$$

We compute in a similar fashion as before,

$$\begin{aligned} \|\nabla Q_n - \nabla q_i\|_{\mathcal{R}} &= \|\nabla Q_n - \nabla q_i\|_{L^2(\Omega')} + \|\Delta Q_n - \Delta q_i\|_{L^2(\Omega)} \\ &\leq \underbrace{\|\tilde{\mathbf{h}}_n - \nabla q_i\|_{H(\operatorname{div})(\Omega)}}_{\leq \|\tilde{\mathbf{h}}_n - \nabla q_i\|_{H^3(\Omega)^d} \xrightarrow{(A.3.4)} 0} + \|\nabla Q_n - \nabla q_i\|_{L^2(\Omega' \setminus \overline{\Omega})} \end{aligned}$$

and

$$\begin{aligned} &\|\nabla Q_n - \nabla q_i\|_{L^2(\Omega' \setminus \overline{\Omega})} \\ &= \left\| \sum_{i=1}^{N_n} \alpha_i^n \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} \phi_i^\Omega|_{\partial\Omega} - c_i & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} \right. \\ &\quad \left. - \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} 0 & \text{on } \partial\Omega \\ |\Omega'|^{\frac{1}{2}} (f \tilde{q}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i) |\Omega'|^{-\frac{1}{2}} & \text{on } \partial\Omega' \end{cases} \right. \\ &\quad \left. - \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} u_i^\Omega|_{\partial\Omega} & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} \right\|_{L^2(\Omega' \setminus \overline{\Omega})^d} \\ &= \left\| \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} \sum_{i=1}^{N_n} \alpha_i^n (\phi_i^\Omega|_{\partial\Omega} - c_i) - \overbrace{u_i^\Omega|_{\partial\Omega}}^{=\tilde{q}_i|_{\partial\Omega}} & \text{on } \partial\Omega \\ -(f_\Omega \tilde{q}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i) & \text{on } \partial\Omega' \end{cases} \right. \\ &\quad \left. + \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \overbrace{\left(\oint_{\Omega} \tilde{q}_i \, dx + \sum_{i=1}^{N_n} \alpha_i^n c_i \right)}^{=0} \right\|_{L^2(\Omega' \setminus \overline{\Omega})^d} \\ &= \left\| \nabla L_{\Omega' \setminus \overline{\Omega}}^{-1} \begin{cases} \sum_{i=1}^{N_n} \alpha_i^n \phi_i^\Omega|_{\partial\Omega} - (\tilde{q}_i|_{\partial\Omega} - f_\Omega \tilde{q}_i \, dx) & \text{on } \partial\Omega \\ 0 & \text{on } \partial\Omega' \end{cases} \right\|_{L^2(\Omega' \setminus \overline{\Omega})^d} \\ &\stackrel{(3.2.4)}{\leq} C \left\{ \left\| \sum_{i=1}^{N_n} \alpha_i^n \phi_i^\Omega|_{\partial\Omega} - (\tilde{q}_i - \oint_{\Omega} \tilde{q}_i \, dx)|_{\partial\Omega} \right\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|0\|_{H^{\frac{1}{2}}(\partial\Omega')} \right\} \\ &\leq \overline{C} \left\| \sum_{i=1}^{N_n} \alpha_i^n \phi_i^\Omega - (\tilde{q}_i - \oint_{\Omega} \tilde{q}_i \, dx) \right\|_{H^1(\Omega)} \leq \tilde{C} \|\tilde{\mathbf{h}}_n - \nabla(\tilde{q}_i - \oint_{\Omega} \tilde{q}_i \, dx)\|_{L^2(\Omega)^d} \xrightarrow{(A.3.4)} 0. \end{aligned}$$

Hence, $q_i \in \mathcal{R}_{temp}$. □

A.4. Miscellaneous

Lemma A.4.1. *Let X be a Hilbert space and $(X_n)_{n \in \mathbb{N}}$ a sequence of closed subspaces of X . Let Π_{X_n} be a projector onto X_n such that*

- $\Pi_{X_n} F \xrightarrow{n \rightarrow \infty} F$ in X for all $F \in X$,
- $\|\Pi_{X_n} F\|_X \leq C \|F\|_X \quad \forall F \in X$ independently of $n \in \mathbb{N}$.

If $F \in L^p(I; X)$, $p \in [1, \infty)$, then we have the convergence $\|\Pi_{X_n} F(t) - F(t)\|_X \rightarrow 0$ pointwise for almost all $t \in I$ and in $L^p(I; X)$.

A.5. Notation

In this section we collect our notation and provide information on its usage. We start with general notation.

Bold face characters always denote vector valued quantities in $\mathbb{R}^d$, where $d \in \{2, 3\}$. The term $\mathbf{Du}$ denotes the $(d \times d)$ -tensor field of the symmetric gradient of the vector field $\mathbf{u}$. Plain characters are scalars or elements of abstract vector spaces.

General notation.

Ω, Ω'	Spatial domains, where $\Omega \subset \Omega'$. See (H1) for further assumptions.
T, I	Final time $T > 0$ or time interval $(0, T)$, respectively.
d	Spatial dimension, $d \in \{2, 3\}$. Will only be specified, if necessary.
$\boldsymbol{\nu}$	Outer unit normal vector.
$(\cdot)_t, \partial_t$	Differentiation w.r.t. time.
$\mathbf{D}$	Symmetric gradient of vector fields.
$\mathcal{L}_{(\cdot)}^{(\cdot)}, \mathcal{R}_{(\cdot)}^{(\cdot)}$	Terms of the left-hand side or right-hand side—respectively—of the equations in the system of ordinary differential equations (5.16), where lower indices denote the ordinal number from left to right and upper indices denote the correspondence to the individual equation. For instance, $\mathcal{L}_1^R$ denotes the the first term on the left-hand side of the equation associated to the unknown R , which is the magnetostatic equation (5.16c).

Next, we list the quantities and parameters that occur in our model.

Quantities and parameters.

$\mathbf{u}, p$	Velocity field and pressure, see (1.1a), (1.1b).
c	Particle number density, see (1.1c).
$\mathbf{h}, R$	Magnetic field $\mathbf{h} := \nabla R$ and its potential R , see (1.1e), (1.1g).
$\mathbf{h}_a$	Given external magnetic field, satisfying (1.4).
$\mathbf{m}$	Magnetization, see (1.1f).
$\mathbf{V}_{\text{part}}$	Convective velocity of particle density, see (1.1d).
χ	Susceptibility $\chi = \chi(c, \mathbf{h})$, see (2.1), assumed to be bounded, see e.g. (H4).
f_2, f_2^n	Nonlinear mobility function for the particle density, chosen as $f_2(c) = c^m$, $m \in [0, 2]$. Existence of solutions established for $m = 2$. $(f_2^n)_{n \in \mathbb{N}}$ is an approximating sequence, see (8.1).
g_s^L	Regularized entropic function, see (5.1). For convenience, we assume $0 < s < e < L$, where e is Euler's number.
σ	Regularization parameter in the magnetization equation (1.1f).
σ_c	Regularization parameter used in the particle density equation (5.7b) of the weak formulation.
$\alpha_0, \dots, \alpha_2, \beta$	Generic parameters defining the energy of the system (1.1), see [17].
K, D	Mobility and diffusion parameters in the evolution equation of the particle density c .
ρ_0, η	Density and dynamic viscosity of the carrier fluid.
μ_0, τ_{rel}	Vacuum permeability/magnetic constant and relaxation time for rearrangement of magnetic spins in alignment with the magnetic field.
$\mathbf{u}_n, c_n, R_n, \mathbf{m}_n, \mathbf{h}_n$	Members of a sequence of solutions either to the Galerkin scheme (5.16) or to (5.7) in the regularized case, see Theorem 7.2. There, $\mathbf{h}_n := \nabla R_n$.
$(\mathbf{V}_{\text{part}})_n$	Members of a sequence of convective velocities, defined by approximate solutions, see (6.2) on the discrete level and (8.4) on the level of solutions to the TMR-model.

Operators.

$\text{curl}, \times$	The curl and $\times$ operators defined in three dimensions as usual and defined according to Definition 2.1 in two dimensions.
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$(\cdot)_s, (\cdot)^L$	Cut-off operator from below at $s > 0$, cf. (5.4) or cut-off from above at $L > 0$, cf. (8.2), respectively. If both are used at once, we write $(\cdot)_s^L$.
$\Pi_{\mathcal{C}_n}, \Pi_{\mathcal{U}_n}, \Pi_{\mathcal{M}_n}$	L^2 -orthogonal projection operators onto $\mathcal{C}_n, \mathcal{U}_n, \mathcal{M}_n$, see (5.9), based on the basis representations (4.7), (4.4) and (3.1.13). Those are stable in the L^2 -norm—due to orthogonality—and other norms, cf. (5.11).
$\Pi_{\mathcal{R}_n}$	Projection operator, cf. (5.9), based on the basis (3.2.12). No stability result needed for this operator.
$\Pi_{\mathcal{H}_n}$	L^2 -orthogonal projection operator based on the basis (3.2.12), which is related to the basis of $\mathcal{H}$, according to (3.2.15). See its definition in (5.8). This projector is also stable, (5.13), in the L^2 -norm as well as the norm of the sum $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\circ$, cf. (3.1.2), due to orthogonality of the bases of $\mathcal{S}, \mathcal{S}^\circ$, cf. (3.1.10).

Function spaces.

$\mathcal{M}, \tilde{\mathcal{M}}$	The space $\mathcal{M}$ is used as ansatz space for the magnetization equation (5.7d), see (3.3). The space $\tilde{\mathcal{M}}$ is identical to $\mathcal{M}$ but the latter is equipped with the $H^3(\Omega)^d$ -norm and the former with the norm of the L^2 -orthogonal and direct sum $\mathcal{M} = \mathcal{S} \oplus \mathcal{S}^\circ \oplus \mathcal{V}$, where each subspace is equipped with the $H^3(\Omega)^d$ -norm. Concerning the direct sum see (3.1.12), (3.1.2).
$\mathcal{H}, \mathcal{S}, \mathcal{S}^\circ$	Special subsets of $\mathcal{M}$. The gradient fields in $\mathcal{M}$ are denoted by $\mathcal{H}$, cf. (3.1.1), while those gradient fields that have constant trace on $\partial\Omega$ are denoted by $\mathcal{S}$, cf. (3.1.2). The sum $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\circ$ is direct and all spaces are equipped with the $H^3(\Omega)^d$ -norm.
$\mathcal{V}$	Vector fields of $\mathcal{M}$ that are no gradient fields. The complement $\mathcal{V}$ is L^2 -orthogonal to the gradient fields $\mathcal{H}$ of $\mathcal{M}$, see (3.1.12).
$\mathcal{R}$	Ansatz space for the magnetostatic equation (5.7c), see (3.4).
$\mathcal{U}$	Ansatz space for the Navier-Stokes equations (5.7a), see (4.1).
$\mathcal{U}_n, \mathcal{C}_n, \mathcal{R}_n, \mathcal{M}_n, \mathcal{H}_n$	Finite dimensional subspaces of $\mathcal{U}, \mathcal{C}, \mathcal{R}, \mathcal{M}, \mathcal{H}$ used for a Galerkin approximation argument, see Definition 5.3.
$\mathcal{H}_{\text{special}}, H(V)$	Special subsets of $\nabla[\mathcal{R}]$, defined in (6.4), (6.5) and used for time-compactness estimates in the Galerkin approximation setting.
$L^p(I; L_{\text{loc}}^q(\Omega))$	Space of functions f that satisfy $f \in L^p(I; L^q(\hat{\Omega}))$ for all $\hat{\Omega} \subset\subset \Omega$. In addition, whenever the notion $L_{\text{loc}}^{(\cdot)}(\cdot)$ inside a logical statement A occurs, it is an abbreviation of $\forall \hat{\Omega} \subset\subset \Omega : A$ holds.
$L^p(I; W_{\text{loc}}^{k,q}(\Omega))$	Analogously defined as $L^p(I; L_{\text{loc}}^q(\Omega))$.
$L^{p-}(\cdot), L^{p+}(\cdot)$	In a logical statement A , by the notion $L^{p-}(\cdot)$ we mean $\forall r \in [1, p) : A$ holds. By $L^{p+}(\cdot)$ we mean $\exists r > p : A$ holds.
$H(\text{div})(\cdot), H(\text{div}_0)(\cdot)$	Space of $L^2(\cdot)^d$ -functions with distributional div-operator in $L^2(\cdot)$ in the former case. In the latter case, the divergence vanishes, additionally.
$H(\text{curl})(\cdot), H(\text{curl}_0)(\cdot)$	Analogously defined as the $H(\text{div})$ -spaces but with distributional curl instead of div.
$H_{n0}(\text{div})(\cdot)$	Space of $H(\text{div})(\cdot)$ -functions with vanishing distributional normal trace.
$H_{t0}(\text{curl})(\cdot)$	Space of $H(\text{curl})(\cdot)$ -functions with vanishing distributional tangential trace.
$H(\text{div}, \text{curl})(\cdot)$	Intersection $H(\text{div})(\cdot) \cap H(\text{curl})(\cdot)$.
$H(\text{div}_0, \text{curl}_0)(\cdot)$	Intersection $H(\text{div}_0)(\cdot) \cap H(\text{curl}_0)(\cdot)$.
$H_*^2(\Omega)$	Space of H^2 -functions with homogeneous Neumann boundary data, see (5.5).

Basis functions and their auxiliary functions.

$\mathbf{s}_i, \mathbf{s}_i^\perp$	Basis functions of $\mathcal{S}$ or $\mathcal{S}^\circ$, respectively, see (3.1.10).
$\mathbf{h}_i$	Basis functions of $\mathcal{H}$, combining $\mathbf{s}_j$ and $\mathbf{s}_j^\perp$, cf. (3.1.11).
$\mathbf{m}_i$	Basis functions of $\mathcal{V}$.
$\Psi_i^{\mathbf{m}}$	Basis functions of $\mathcal{M}$ combining $\mathbf{m}_j, \mathbf{h}_j$, see (3.1.13).

ϕ_i^Ω	Potentials (on Ω) of basis functions $\mathfrak{h}_i$, mean value free on Ω , cf. (3.2.1).
$\tilde{R}_i, \mathbf{R}_i$	Potentials—partly based on ϕ_i^Ω —that are used to construct a basis of $\mathcal{R}$, see (3.2.5). Potentials $\mathbf{R}_i$ are mean value free on Ω' , cf. (3.2.6).
$\psi_i^{\text{dir}}, \tilde{p}_i$	Eigenfunctions ψ_i^{dir} of the homogeneous Dirichlet Laplace problem on Ω , cf. (3.2.9), and their constant extensions $\tilde{p}_i$ onto $\Omega' \setminus \bar{\Omega}$, see (3.2.10).
u_i^V	Eigenfunctions of the homogeneous Neumann Laplace operator on V . In this paper, $V = \Omega$ or $V = \Omega'$.
$\tilde{q}_i$	Special testfunctions used in Section 3, cf. (3.2.11), constructed with the help of the functions u_i^Ω .
p_i, q_i	Mean value free—cf. (3.2.10), (3.2.11)—variants of $\tilde{p}_i, \tilde{q}_i$ used as testfunctions together with R_{2i+1} to prove the completeness of the functions R_i in $\mathcal{R}$.
ψ_i^R	Basis functions of $\mathcal{R}$ combining the functions R_{2i} and re-orthogonalized functions based on R_{2i-1} , see (3.2.13) and (3.2.14).
$\Psi_i^{\mathfrak{U}}$	Basis functions of $\mathcal{U}$, see (4.4).
ψ_i^c	Basis functions of $L^2(\Omega)$ which are used for the particle density equation (5.7b), see (4.7).

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