On Some Regularity Criteria for Axisymmetric Navier–Stokes Equations

Joanna Renčławowicz and Wojciech M. Zajączkowski

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Abstract. We point out some criteria that imply regularity of axisymmetric solutions to Navier–Stokes equations. We show that boundedness of $\|v_r/\sqrt{r^3}\|_{L_2(\mathbb{R}^3 \times (0,T))}$ as well as boundedness of $\|\omega_\varphi/\sqrt{r}\|_{L_2(\mathbb{R}^3 \times (0,T))}$, where v_r is the radial component of velocity and ω_φ is the angular component of vorticity, imply regularity of weak solutions.

Keywords. Navier–Stokes equation, Regularity criteria, Regular solutions.

1. Introduction

We consider the Cauchy problem to the three-dimensional axisymmetric Navier–Stokes equations:

$$\begin{aligned} v_t + v \cdot \nabla v - \nu \Delta v + \nabla p &= 0 & (x, t) \in \mathbb{R}^3 \times \mathbb{R}_+, \\ \operatorname{div} v &= 0, \\ v|_{t=0} &= v(0), \end{aligned} \tag{1.1}$$

where $x = (x_1, x_2, x_3)$, v is the velocity of the fluid motion with

$$v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3,$$

$p = p(x, t) \in \mathbb{R}^1$ denotes the pressure, ν is the viscosity coefficient and v_0 is given initial velocity field.

The first papers concerning regularity of axially symmetric solutions to the Navier–Stokes equations were independently proved by Ladyzhenskaya [1] and Yudovich–Ukhovskij [2] in 1968. In these papers axisymmetric solutions without swirl were considered. In the period 1999–2002 arised many papers concerning sufficient conditions on regularity of axisymmetric solutions [3–6]. Especially, conditions on one coordinate of velocity were considered. Recently there are many papers dealing with new sufficient conditions (see references of Lei and Zhang [7]).

Our aim is to derive some criteria guaranteeing regularity of solutions to the axisymmetric Navier–Stokes equations. By the regular solutions we mean smooth weak solutions obtained by the standard increasing regularity technique for smooth initial data. There is a lot of criteria for regularity of axisymmetric solutions (see [6–12] and the literature cited in these papers). In Sect. 2 we recall only such criteria that are useful for our analysis.

Since we are restricted to the axisymmetric solutions we introduce the cylindrical coordinates (r, φ, z) by the relations

$$x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z,$$

and corresponding unit vectors:

$$\bar{e}_r = (\cos \varphi, \sin \varphi, 0), \quad \bar{e}_\varphi = (-\sin \varphi, \cos \varphi, 0), \quad \bar{e}_z = (0, 0, 1).$$

Then the cylindrical components of velocity and vorticity ($\omega = \operatorname{rot} v$) for axisymmetric solutions (therefore, solutions independent of φ) are represented as

$$v = v_r(r, z, t)\bar{e}_r + v_\varphi(r, z, t)\bar{e}_\varphi + v_z(r, z, t)\bar{e}_z$$

and

$$\begin{aligned}\omega &= \omega_r(r, z, t)\bar{e}_r + \omega_\varphi(r, z, t)\bar{e}_\varphi + \omega_z(r, z, t)\bar{e}_z \\ &= -v_{\varphi,z}\bar{e}_r + (v_{r,z} - v_{z,r})\bar{e}_\varphi + \left(v_{\varphi,r} + \frac{v_\varphi}{r}\right)\bar{e}_z,\end{aligned}$$

where v_r, v_φ, v_z are radial, angular and axial components of velocity.

The axisymmetric motion can be described by the three quantities: $v_\varphi, \omega_\varphi$ and the stream potential ψ which are solutions to the following equations:

$$\begin{aligned}v_{\varphi,t} + v \cdot \nabla v_\varphi - \nu \left(\Delta - \frac{1}{r^2} \right) v_\varphi + \frac{v_r}{r} v_\varphi &= 0, \\ v_\varphi|_{t=0} &= v_\varphi(0), \\ \omega_{\varphi,t} + v \cdot \nabla \omega_\varphi - \nu \left(\Delta - \frac{1}{r^2} \right) \omega_\varphi - \frac{v_r}{r} \omega_\varphi - \frac{2}{r} v_\varphi v_{\varphi,z} &= 0, \\ \omega_\varphi|_{t=0} &= \omega_\varphi(0), \\ - \left(\Delta \psi - \frac{1}{r^2} \psi \right) &= \omega_\varphi,\end{aligned}\tag{1.2}$$

where $v \cdot \nabla = v_r \partial_r + v_z \partial_z$, $\Delta = \partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r$ and

$$v_r = -\psi_{,z}, \quad v_z = \frac{1}{r}(r\psi)_{,r}.$$

It is very convenient to introduce quantities u_1, ω_1, ψ_1 by the relations

$$v_\varphi = ru_1, \quad \omega_\varphi = r\omega_1, \quad \psi = r\psi_1.$$

Then Eqs. (1.2) simplify to

$$\begin{aligned}u_{1,t} + v \cdot \nabla u_1 - \nu \left(\Delta u_1 + \frac{2}{r} u_{1,r} \right) &= 2u_1 \psi_{1,z}, \\ u_1|_{t=0} &= u_1(0),\end{aligned}\tag{1.3}$$

$$\begin{aligned}\omega_{1,t} + v \cdot \nabla \omega_1 - \nu \left(\Delta \omega_1 + \frac{2}{r} \omega_{1,r} \right) &= 2u_1 u_{1,z}, \\ \omega_1|_{t=0} &= \omega_1(0),\end{aligned}\tag{1.4}$$

$$- \left(\Delta \psi_1 + \frac{2}{r} \psi_{1,r} \right) = \omega_1,\tag{1.5}$$

and

$$v_r = -r\psi_{1,z}, \quad v_z = \frac{1}{r}(r^2\psi_1)_{,r}.\tag{1.6}$$

We prove the following regularity criteria (see (1.7), (1.8)) which are scaling invariant:

Theorem 1. 1. Let (v, p) be an axisymmetric solution to the Navier–Stokes Eqs. (1.1) with the axisymmetric initial data and additionally $\operatorname{div} v(0) = 0$.

2. Assume that $\frac{v_\varphi^2(0)}{r}, \frac{\omega_\varphi(0)}{r}, \frac{\omega_r(0)}{r}, \omega_z(0)$ belong to $L_2(\mathbb{R}^3)$, and with the notation $u = rv_\varphi, u(0) \in L_\infty(\mathbb{R}^3) \cap L_s(\mathbb{R}^3), s \geq 3$.
3. Assume that there exists constant c_1 such that

$$\int_0^T dt \int_{\mathbb{R}^3} \frac{v_r^2}{r^3} dx \leq c_1 < \infty.\tag{1.7}$$

Then $v \in L_\infty(0, T; H^1(\mathbb{R}_{r_0}^3))$, where $\mathbb{R}_{r_0}^3 = \{x \in \mathbb{R}^3, r < r_0\}$ and $r_0 > 0$ is given. Assume additionally that $v(0) \in B_{\sigma,r}^{2-2/r}(\mathbb{R}_{r_0}^3)$ - Besov space. Then $v \in W_{\sigma,r}^{2,1}(\mathbb{R}_{r_0}^3 \times (0, T))$.

Remark 1.1. For $\sigma > 3, r = 2$ we have that $v \in L_\infty(\mathbb{R}_{r_0}^3 \times (0, T))$ so in view of [13] there is no singular points. In $\mathbb{R}_{r_0}^3 = \{x \in \mathbb{R}^3, r > r_0\}$ the axisymmetric problem (1.1) is two-dimensional so local regularity of v is evident.

Theorem 2. *Let the assumptions 1, 2 of Theorem 1 hold. If*

$$\int_0^T dt \int_{\mathbb{R}^3} \frac{\omega_\varphi^2}{r} dx \leq c_2 < \infty \quad (1.8)$$

then there exists a constant c_3 such that

$$\int_0^T dt \int_{\mathbb{R}^3} \frac{v_r^2}{r^3} dx \leq c_3 \int_0^T dt \int_{\mathbb{R}^3} \frac{\omega_\varphi^2}{r} dx \leq c_3 c_2. \quad (1.9)$$

2. Notation and Auxiliary Results

By $L_p(\mathbb{R}^3), p \in [1, \infty]$, we denote the Lebesgue space of integrable functions. By $L_{p,q}(\mathbb{R}^3 \times (0, T))$ we denote the anisotropic Lebesgue space with the following finite norm

$$\|u\|_{L_{p,q}(\mathbb{R}^3 \times (0, T))} = \left(\int_0^T \int_{\mathbb{R}^3} (|u(x, t)|^p dx)^{q/p} dt \right)^{1/q},$$

where $p, q \in [1, \infty]$.

We define Sobolev spaces $W_p^{2,1}(\mathbb{R}^3 \times (0, T))$ and $W_p^{2-2/p}(\mathbb{R}^3 \times (0, T))$ by

$$\begin{aligned} \|u\|_{W_p^{2,1}(\mathbb{R}^3 \times (0, T))} &= \left(\int_0^T \int_{\mathbb{R}^3} (|\nabla_x^2 u|^p + |u_t|^p + |u|^p) dx dt \right)^{1/p} < \infty, \\ \|u\|_{W_p^{2-2/p}(\mathbb{R}^3)} &= \left(\sum_{i \leq [2-2/p]} \int_{\mathbb{R}^3} |\nabla^i u|^p dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\nabla_x^{[2-2/p]} u(x) - \nabla_y^{[2-2/p]} u(y)|^p}{|x - y|^{3+p(2-2/p-[2-2/p])}} dx dy \right)^{1/p} \\ &< \infty, \end{aligned}$$

where $[l]$ is the integer part of l .

By $H^s(\mathbb{R}^3), s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we denote the Sobolev space $W_2^s(\mathbb{R}^3)$.

To find the form of $|\nabla v|$ we recall that

$$|\nabla v|^2 = |v_{,r}|^2 + |v_{,z}|^2.$$

Using that $v = v_r \bar{e}_r + v_\varphi \bar{e}_\varphi + v_z \bar{e}_z$, we obtain

$$|\nabla v|^2 = |v_{r,r}|^2 + |v_{r,z}|^2 + |v_{\varphi,r}|^2 + |v_{\varphi,z}|^2 + |v_{z,r}|^2 + |v_{z,z}|^2,$$

where we used the fact that vectors $\bar{e}_r, \bar{e}_\varphi, \bar{e}_z$ are orthonormal.

Lemma 2.1. *There exists a weak solution to problem (1.1) such that $v \in L_\infty(0, T; L_2(\mathbb{R}^3)) \cap L_2(0, T; H^1(\mathbb{R}^3))$ and the following estimate holds*

$$\int_{\mathbb{R}^3} |v(t)|^2 dx + \nu \int_0^t dt' \int_{\mathbb{R}^3} |\nabla v|^2 dx \leq c \int_{\mathbb{R}^3} |v(0)|^2 dx. \quad (2.1)$$

In the case of axisymmetric solutions the energy inequality (2.1) takes the form

$$\begin{aligned} \int_{\mathbb{R}^3} |v(t)|^2 dx + \nu \int_0^t dt' \int_{\mathbb{R}^3} \left(|\nabla v|^2 + \left| \frac{v_r}{r} \right|^2 + \left| \frac{v_\varphi}{r} \right|^2 \right) dx \\ \leq c \int_{\mathbb{R}^3} |v(0)|^2 dx. \end{aligned} \quad (2.2)$$

Proof. Equations (1.1)_{1,2} for the axially symmetric solutions assume the form

$$v_{r,t} + v \cdot \nabla v_r - \frac{v_\varphi^2}{r} - \nu \Delta v_r + \nu \frac{v_r}{r^2} = -p_{,r}, \quad (2.3)$$

$$v_{\varphi,t} + v \cdot \nabla v_\varphi + \frac{v_r}{r} v_\varphi - \nu \Delta v_\varphi + \nu \frac{v_\varphi}{r^2} = 0, \quad (2.4)$$

$$v_{z,t} + v \cdot \nabla v_z - \nu \Delta v_z = -p_{,z}, \quad (2.5)$$

$$v_{r,r} + v_{z,z} = -\frac{v_r}{r}, \quad (2.6)$$

where $v \cdot \nabla = v_r \partial_r + v_z \partial_z$, $\Delta u = \frac{1}{r}(ru_{,r})_{,r} + u_{,zz}$.

Let $I = \{(\varphi, r, z) : r = 0\}$ denote the axis of symmetry. Any space of functions which vanish on I considered in this paper is a closure of $C_0^\infty(\mathbb{R}^3 \setminus I)$ in the corresponding norm. Then, we are looking for a priori estimate for functions v having such property.

Multiplying (2.4) by v_φ and integrating over $\mathbb{R}^3$ yields

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} v_\varphi^2 dx + \int_{\mathbb{R}^3} \frac{v_r}{r} v_\varphi^2 dx + \nu \int_{\mathbb{R}^3} (v_{\varphi,r}^2 + v_{\varphi,z}^2) dx + \nu \int_{\mathbb{R}^3} \frac{v_\varphi^2}{r^2} dx = 0.$$

Multiplying (2.3) by v_r , integrating over $\mathbb{R}^3$ implies

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} v_r^2 dx - \int_{\mathbb{R}^3} \frac{v_\varphi^2}{r} v_r dx + \nu \int_{\mathbb{R}^3} (v_{r,r}^2 + v_{r,z}^2) dx + \nu \int_{\mathbb{R}^3} \frac{v_r^2}{r^2} dx = - \int_{\mathbb{R}^3} p_{,r} v_r dx.$$

Multiplying (2.5) by v_z and integrating over $\mathbb{R}^3$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} v_z^2 dx + \nu \int_{\mathbb{R}^3} (v_{z,r}^2 + v_{z,z}^2) dx = - \int_{\mathbb{R}^3} p_{,z} v_z dx.$$

Adding the above equations and using (2.6) we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (v_r^2 + v_\varphi^2 + v_z^2) dx \\ & + \nu \int_{\mathbb{R}^3} (v_{r,r}^2 + v_{r,z}^2 + v_{\varphi,r}^2 + v_{\varphi,z}^2 + v_{z,r}^2 + v_{z,z}^2) dx \\ & + \nu \int_{\mathbb{R}^3} \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx = 0. \end{aligned} \quad (2.7)$$

Integrating (2.7) with respect to time from 0 to t , $t \leq T$, yields

$$\begin{aligned} & \nu \|v(t)\|_{L_2(\mathbb{R}^3)}^2 + \nu \int_0^t \int_{\mathbb{R}^3} (|v_{,r}|^2 + |v_{,z}|^2) dx dt' \\ & + \nu \int_0^t \int_{\mathbb{R}^3} \left(\frac{v_r^2}{r^2} + \frac{v_\varphi^2}{r^2} \right) dx dt' \leq \frac{1}{2} \|v(0)\|_{L_2(\mathbb{R}^3)}^2. \end{aligned}$$

This ends the proof. $\square$

To derive energy estimates in the proof of Lemma 2.1 we use the ideas from the proof of Theorem 3.1 from [14, Ch.3]. The notion of a suitable weak solution was introduced by Caffarelli et al. [13] in the famous paper. Our aim is to show that either (1.7) or (1.8) implies that a suitable weak solution to problem (1.1) does not contain singular points. This means that (v, p) is a regular solution to (1.1). In other words it means that if $v(0) \in W_p^{2-2/p}(\mathbb{R}^3)$ then $v \in W_p^{2,1}(\mathbb{R}^3 \times R_+)$ for any $p \in (1, \infty)$. Hence for $p > \frac{5}{2}$ we have that $v \in L_\infty(\mathbb{R}^3 \times R_+)$ so it is also bounded locally. Therefore v has no singular points (see [13]). To show this we use results of J. Neustupa, M. Pokorný and O. Kreml (see [5, 6, 10]). To clarify presentation we recall the results.

From [5] it follows that

Lemma 2.2 [5]. *Let v be an axisymmetric suitable weak solution to problem (1.1). Suppose that there exists a subdomain $D \subset \mathbb{R}^3 \times \mathbb{R}_+$ such that the angular component v_φ of v belongs to $L_{s,r}(D)$ where*

1. *Either $s \in [6, \infty]$, $r \in [20/7, \infty]$ and $2/r + 3/s \leq 7/10$.*
2. *or $s \in [24/5, 6]$, $r \in [10, \infty]$ and $2/r + 3/s \leq 1 - 9/(5s)$.*

Then v has no singular points in D .

Lemma 2.3 [10]. *Let v be an axisymmetric suitable weak solution to problem (1.1). Suppose that there exists a subdomain $D \subset \mathbb{R}^3 \times \mathbb{R}_+$ such that the angular component v_φ of v belongs to $L_{s,r}(D)$ where*

$$s \in \left(\frac{24}{7}, 4 \right], \quad r \in \left(\frac{8s}{7s-24}, \infty \right], \quad \frac{3}{s} + \frac{2}{r} < \frac{7}{4} - \frac{3}{s}.$$

Then v has no singular points in D .

By swirl we denote

$$u = rv_\varphi.$$

From (1.2)₁ it follows that u satisfies the equation

$$u_{,t} + v \cdot \nabla u - \nu \Delta u + \frac{2\nu}{r} u_{,r} = 0.$$

Lemma 2.4 (See [3]). *Let $u(0) = rv_\varphi(0) \in L_\infty(\mathbb{R}^3)$. Then*

$$\|u\|_{L_\infty(0,T;L_\infty(\mathbb{R}^3))} \leq \|u(0)\|_{L_\infty(\mathbb{R}^3)} \quad \text{for any } T. \quad (2.8)$$

Remark 2.5. From (2.2) and (2.8) we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} |v_\varphi|^4 dx dt &= \int_0^t \int_{\mathbb{R}^3} r^2 v_\varphi^2 \frac{v_\varphi^2}{r^2} dx dt \\ &\leq \|rv_\varphi\|_{L_\infty(\mathbb{R}^3 \times (0,T))}^2 \int_0^t \int_{\mathbb{R}^3} \frac{v_\varphi^2}{r^2} dx dt \leq c \|u(0)\|_{L_\infty(\mathbb{R}^3)}^2 \|v(0)\|_{L_2(\mathbb{R}^3)}^2. \end{aligned} \quad (2.9)$$

We recall also Lemma 3.1 from [9]

Lemma 2.6 (See [9]). *Assume that (v, p) is regular axisymmetric solution to the Navier–Stokes Eqs. (1.1). Assume that $\frac{v_\varphi(0)}{\sqrt{r}} \in L_4(\mathbb{R}^3)$, $\tilde{\nabla} \frac{v_r}{r} \in L_{4/3}(0, T; L_2(\mathbb{R}^3))$ when $\tilde{\nabla} = (\partial_r, \partial_z)$. Then the following estimate holds*

$$\begin{aligned} &\frac{1}{2} \left\| \frac{v_\varphi^2(t)}{r} \right\|_{L_2(\mathbb{R}^3)}^2 + \frac{1}{4} \int_0^t \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}^3)}^2 dt' + \frac{3}{4} \int_0^t \left\| \frac{v_\varphi}{r} \right\|_{L_4(\mathbb{R}^3)}^4 dt' \\ &\leq \frac{1}{4} \exp \left[c \int_0^t \left\| \tilde{\nabla} \frac{v_r}{r} \right\|_{L_2(\mathbb{R}^3)}^{4/3} dt' \right] \left\| \frac{v_\varphi^2(0)}{r} \right\|_{L_2(\mathbb{R}^3)}^2. \end{aligned} \quad (2.10)$$

Proof. Consider the following problem for v_φ in $\mathbb{R}_\varepsilon^3 = \{x \in \mathbb{R}^3 : r > \varepsilon\}$, $\varepsilon > 0$,

$$\begin{aligned} v_{\varphi,t} + v \cdot \nabla v_\varphi - \nu \Delta v_\varphi + \nu \frac{v_\varphi}{r^2} + \frac{v_r v_\varphi}{r} &= 0, \\ v_\varphi|_{r=\varepsilon} &= 0, \quad v_\varphi|_{r \rightarrow \infty} = 0. \end{aligned} \quad (2.11)$$

Multiplying (2.11) by $\frac{v_\varphi^3}{r^2}$, integrating over $\mathbb{R}_\varepsilon^3$ and using boundary conditions yields

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \frac{3}{4} \nu \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \frac{3}{4} \nu \left\| \frac{v_\varphi}{r} \right\|_{L_4(\mathbb{R}_\varepsilon^3)}^4 &= -\frac{3}{2} \int_{\mathbb{R}_\varepsilon^3} \frac{v_r}{r} \frac{v_\varphi^2}{r} \frac{v_\varphi^2}{r} dx \\ &\leq \frac{3}{2} \left\| \frac{v_r}{r} \right\|_{L_6(\mathbb{R}_\varepsilon^3)} \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)} \left\| \frac{v_\varphi^2}{r} \right\|_{L_3(\mathbb{R}_\varepsilon^3)} \leq c \left\| \tilde{\nabla} \frac{v_r}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)} \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^{3/2} \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^{1/2} \\ &\leq \frac{\nu}{2} \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + c \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 \left\| \tilde{\nabla} \frac{v_r}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^{4/3}. \end{aligned}$$

Simplifying we have

$$\frac{1}{4} \frac{d}{dt} \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \frac{\nu}{4} \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \frac{3}{4} \nu \left\| \frac{v_\varphi}{r} \right\|_{L_4(\mathbb{R}_\varepsilon^3)}^4 \leq c \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 \left\| \tilde{\nabla} \frac{v_r}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^{4/3}.$$

By the Gronwall lemma we have

$$\begin{aligned} \frac{1}{4} \left\| \frac{v_\varphi^2(t)}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \frac{\nu}{4} \int_0^t \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 dt' + \frac{3}{4} \nu \int_0^t \left\| \frac{v_\varphi}{r} \right\|_{L_4(\mathbb{R}_\varepsilon^3)}^4 dt' \\ \leq \frac{1}{4} \exp \left[c \int_0^t \left\| \tilde{\nabla} \frac{v_r}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^{4/3} dt' \right] \left\| \frac{v_\varphi^2(0)}{r} \right\|_{L_2(\mathbb{R}_\varepsilon^3)}^2. \end{aligned}$$

Passing with $\varepsilon \rightarrow 0$ we derive (2.10) and conclude the proof. $\square$

Remark 2.7. Formula (2.4) in [9] has the form

$$\left\| \tilde{\nabla} \frac{v_r}{r} \right\|_{L_q(\mathbb{R}^3)} \leq c(q) \left\| \frac{\omega_\varphi}{r} \right\|_{L_q(\mathbb{R}^3)}, \quad 1 < q < \infty. \quad (2.12)$$

Consider problem (1.3).

Lemma 2.8. *Let the assumptions of Lemma 2.6 be satisfied. Assume additionally that $w_1(0) \in L_2(\mathbb{R}^3)$, $u_1(0) \in L_4(0, t; L_4(\mathbb{R}^3))$. Then the following estimate holds*

$$\begin{aligned} \frac{1}{2} \|\omega_1(t)\|_{L_2(\mathbb{R}^3)}^2 + \frac{\nu}{2} \int_0^t \|\nabla \omega_1(t')\|_{L_2(\mathbb{R}^3)}^2 dt' \\ \leq \frac{2}{\nu} \int_0^t \|u_1(t')\|_{L_4(\mathbb{R}^3)}^4 dt' + \frac{1}{2} \|\omega_1(0)\|_{L_2(\mathbb{R}^3)}^2, \quad t \leq T. \end{aligned} \quad (2.13)$$

Proof. Consider the problem in $\mathbb{R}_\varepsilon^3$

$$\begin{aligned} \omega_{1,t} + v \cdot \nabla \omega_1 - \nu \left(\Delta \omega_1 + \frac{2}{\nu} \omega_{1,r} \right) &= 2u_1 u_{1,z}, \\ \omega_1|_{r=\varepsilon} &= 0, \quad \omega_1|_{r \rightarrow \infty} = 0. \end{aligned} \quad (2.14)$$

Multiplying (2.14) by ω_1 , integrating over $\mathbb{R}_\varepsilon^3$, using the boundary conditions yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega_1(t)\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \frac{\nu}{2} \|\nabla \omega_1\|_{L_2(\mathbb{R}_\varepsilon^3)}^2 + \nu \int_{\mathbb{R}} \omega_1^2|_{r=\varepsilon} dz \\ \leq \frac{2}{\nu} \int_{\mathbb{R}^3} |u_1|^4 dx. \end{aligned}$$

Integrating with respect to time and passing with $\varepsilon \rightarrow 0$ gives (2.13). This concludes the proof. $\square$

Introduce the quantities

$$(\Phi, \Gamma) = \left(\frac{\omega_r}{r}, \frac{\omega_\varphi}{r} \right)$$

which are solutions to the equations

$$\begin{aligned} \partial_t \Phi + v \cdot \nabla \Phi - \nu \left(\Delta + \frac{2}{r} \right) \Phi - (\omega_r \partial_r + \omega_z \partial_z) \frac{v_r}{r} &= 0, \\ \partial_t \Gamma + v \cdot \nabla \Gamma - \nu \left(\Delta + \frac{2}{r} \right) \Gamma + 2 \frac{v_r}{r} \Phi &= 0. \end{aligned}$$

Remark 2.9. In the proof of Theorem 1.1, Case 1 in [9] there is derived the formula (3.8) in [9] in the form

$$\begin{aligned} & \|\Phi(t)\|_{L_2(\mathbb{R}^3)}^2 + \|\Gamma(t)\|_{L_2(\mathbb{R}^3)}^2 \\ & + \nu \int_0^t (\|\tilde{\nabla} \Phi(t')\|_{L_2(\mathbb{R}^3)}^2 + \|\tilde{\nabla} \Gamma(t')\|_{L_2(\mathbb{R}^3)}^2) dt' \\ & \leq \exp[c(1 + \|r^d v_\varphi\|_{L_{p,q}(\mathbb{R}^3 \times (0,t))})] (\|\Phi(0)\|_{L_2(\mathbb{R}^3)}^2 + \|\Gamma(0)\|_{L_2(\mathbb{R}^3)}^2), \end{aligned} \quad (2.15)$$

where

$$\begin{aligned} 3/p + 2/q &\leq 1 - d, \quad 0 \leq d < 1, \\ \frac{3}{1-d} &< p \leq \infty, \quad \frac{3}{1-d} \leq q < \infty. \end{aligned}$$

Let us recall some properties of weak solutions to (1.1).

Lemma 2.10 (See [15, Ch.2, Sect. 3]). *For arbitrary $v \in L_\infty(0, T; L_2(\Omega)) \cap L_2(0, T; H^1(\Omega))$ the inequality holds:*

$$\|v\|_{L_{p,q}(\mathbb{R}^3 \times (0,T))}^2 \leq c \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |v|^2 dx + c \int_0^T dt \int_{\mathbb{R}^3} |\nabla v|^2 dx, \quad (2.16)$$

where

$$\frac{3}{p} + \frac{2}{q} \geq \frac{3}{2}.$$

3. Sufficient Conditions for Regularity

Let

$$u_\alpha = \frac{u}{r^\alpha}, \quad \alpha \in (0, 1).$$

Then u_α satisfies

$$\begin{aligned} u_{\alpha,t} + v \cdot \nabla u_\alpha + \alpha \frac{v_r}{r} u_\alpha - \nu \Delta u_\alpha + \frac{2\nu(1-\alpha)}{r} u_{\alpha,r} + \frac{\nu\alpha(2-\alpha)}{r^2} u_\alpha &= 0, \\ u_\alpha|_{t=0} &= u_\alpha(0). \end{aligned} \quad (3.1)$$

Lemma 3.1. *Let $u(0) \in L_\infty(\mathbb{R}^3)$. Assume that c_1 is a constant and*

$$\int_0^T \int_{\mathbb{R}^3} \frac{v_r^2}{r^3} dx dt \leq c_1^2,$$

Let

$$\begin{aligned} c_2 &= \frac{1}{4\nu\alpha(2-\alpha)} \|u(0)\|_{L_\infty(\mathbb{R}^3)}^s c_1^2 + \frac{1}{s} \|u(0)\|_{L_s(\mathbb{R}^3)}^s, \\ \text{where } \alpha &= \frac{3}{s}, \quad u_\alpha = v_\varphi r^{1-\alpha}. \end{aligned}$$

Then

$$\begin{aligned} r^d v_\varphi &\in L_{p,q}(\mathbb{R}^3 \times (0, T)), \quad \frac{3}{p} + \frac{2}{q} \geq \frac{3}{s} = \alpha, \\ d &= 1 - \alpha \quad \text{and} \quad \|r^d v_\varphi\|_{L_{p,q}(\mathbb{R}^3 \times (0, T))}^s \leq c_2. \end{aligned} \quad (3.2)$$

Proof. Multiplying (3.1) by $u_\alpha |u_\alpha|^{s-2}$, integrating the result over $\mathbb{R}^3$ and using that $u_\alpha \in C^\infty(\mathbb{R}^3 \setminus I)$, and u_α has compact support for large, finite $x \in \mathbb{R}^3$, we obtain

$$\begin{aligned} \frac{1}{s} \frac{d}{dt} \int_{\mathbb{R}^3} |u_\alpha|^s dx + \alpha \int_{\mathbb{R}^3} \frac{v_r}{r} |u_\alpha|^s dx + \nu \int_{\mathbb{R}^3} |\nabla |u_\alpha|^{s/2}|^2 dx \\ + \nu \alpha (2 - \alpha) \int_{\mathbb{R}^3} \frac{|u_\alpha|^s}{r^2} dx = 0. \end{aligned}$$

The second term in the l.h.s. of the above equality can be estimated by

$$\left| \int_{\mathbb{R}^3} v_r |u_\alpha|^{s/2} \frac{|u_\alpha|^{s/2}}{r} dx \right| \leq \frac{\varepsilon}{2} \int_{\mathbb{R}^3} \frac{|u_\alpha|^s}{r^2} dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}^3} v_r^2 |u_\alpha|^s dx \equiv I.$$

Using Lemma 2.4, the second integral in I is bounded by

$$\|u(0)\|_{L_\infty(\mathbb{R}^3)}^s \int_{\mathbb{R}^3} \frac{v_r^2}{r^{\alpha s}} dx.$$

Employing this estimate, with $\varepsilon = \nu \alpha (2 - \alpha)$, integrating the result with respect to time and using the density argument, yields

$$\begin{aligned} \frac{1}{s} \int_{\mathbb{R}^3} |u_\alpha(t)|^s dx + \nu \int_0^t dt' \int_{\mathbb{R}^3} |\nabla |u_\alpha|^{s/2}|^2 dx \\ + \frac{\nu \alpha (2 - \alpha)}{2} \int_0^t dt' \int_{\mathbb{R}^3} \frac{|u_\alpha|^s}{r^2} dx \\ \leq \frac{\|u(0)\|_{L_\infty(\mathbb{R}^3)}^s}{4\nu \alpha (2 - \alpha)} \int_0^t dt' \int_{\mathbb{R}^3} \frac{v_r^2}{r^{\alpha s}} dx + \frac{1}{s} \int_{\mathbb{R}^3} |u_\alpha(0)|^s dx, \quad t \leq T. \end{aligned} \quad (3.3)$$

In view of the assumptions of the lemma, the r.h.s. of (3.3) is bounded by a constant c_2 . Then the density argument and Lemma 2.10 imply

$$\|u_\alpha^{s/2}\|_{L_{p,q}(\mathbb{R}^3 \times (0, T))} \leq c_2^{1/2} \quad \text{with} \quad \frac{3}{p} + \frac{2}{q} \geq \frac{3}{2}.$$

Hence

$$\|u_\alpha\|_{L_{p',q'}(\mathbb{R}^3 \times (0, T))}^{s/2} = \|u_\alpha\|_{L_{sp/2, sq/2}(\mathbb{R}^3 \times (0, T))}^{s/2} \leq c_2^{1/2}$$

where

$$\frac{3}{p'} + \frac{2}{q'} \geq \frac{3}{2} \cdot \frac{2}{s} = \frac{3}{s}.$$

Comparing the above approach with (2.15) we have that $d = 1 - \alpha$ and the regularity criterion has the form

$$r^d v_\varphi \in L_{q'}(0, T; L_{p'}(\mathbb{R}^3)), \quad \frac{3}{p'} + \frac{2}{q'} \leq 1 - d, \quad 0 \leq d < 1.$$

Hence

$$\frac{3}{p'} + \frac{2}{q'} \geq \frac{3}{s} = \alpha.$$

Therefore $\alpha s = 3$. □

Corollary 3.2. *From (3.2), (2.15) and (2.10) we obtain*

$$\left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}^3)}^2 + \int_0^t \left\| \tilde{\nabla} \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}^3)}^2 dt' \leq c_3, \quad t \leq T,$$

where c_3 depends on the constants from the r.h.s. of (3.2), (2.15) and (2.10).

In view of (2.16) we have

$$\left\| \frac{v_\varphi^2}{r} \right\|_{L_{p,q}(\mathbb{R}^3 \times (0,T))} \leq cc_3, \quad (3.4)$$

where

$$\frac{3}{p} + \frac{2}{q} \geq \frac{3}{2}.$$

Let $\mathbb{R}_{r_0}^3 = \{x \in \mathbb{R}^3 : r \leq r_0\}$. Then (3.4) implies

$$\left\| v_\varphi^2 \right\|_{L_{p,q}(\mathbb{R}_{r_0}^3 \times (0,T))} \leq \left\| \frac{v_\varphi^2}{r} \right\|_{L_{p,q}(\mathbb{R}_{r_0}^3 \times (0,T))} \leq cc_3. \quad (3.5)$$

Hence (3.5) implies that $v_\varphi \in L_{p',q'}(\mathbb{R}_{r_0}^3 \times (0,T))$, where

$$\frac{3}{p'} + \frac{2}{q'} \geq \frac{3}{4}.$$

Consider Lemma 2.3. Let $s = 4$. Then $r = 8$, so

$$\frac{3}{4} + \frac{2}{8} = 1$$

Since $\frac{3}{4} < 1$, v_φ satisfies assumptions of Lemma 2.3. Hence v has no singular points in $\mathbb{R}_{r_0}^3 \times (0,T)$.

Next we show that axisymmetric solutions to problem (1.1) do not have singular points in the region located in a positive distance from the axis of symmetry.

In [13] is shown that singular points of v in the axisymmetric case may appear on the axis of symmetry only. Therefore in any region located in a positive distance from the axis of symmetry there is no singular points of v . However, we want to show that statement explicitly. Therefore, we proceed as follows.

Consider Eq. (1.2)₁. Let $\chi = \chi(r)$ be a smooth function such that

$$\chi(r) = \begin{cases} 1 & r \geq 2r_0, \\ 0 & r \leq r_0. \end{cases}$$

We multiply (1.2)₁ by χ and introduce the notation $\hat{v}_\varphi = v_\varphi \chi$. Then $\hat{v}_\varphi$ satisfies

$$\begin{aligned} \hat{v}_{\varphi,t} + v \cdot \nabla \hat{v}_\varphi - \nu \Delta \hat{v}_\varphi + \frac{v_r}{r} \hat{v}_\varphi + \nu \frac{\hat{v}_\varphi}{r^2} \\ = v \cdot \nabla \chi v_\varphi - 2\nu \nabla v_\varphi \nabla \chi - \nu v_\varphi \Delta \chi, \\ \hat{v}_\varphi|_{t=0} = \hat{v}_\varphi(0). \end{aligned} \quad (3.6)$$

Multiplying (3.6) by $\hat{v}_\varphi |\hat{v}_\varphi|^{s-2}$ and integrating over $\mathbb{R}^3 \times (0,t)$ yields

$$\begin{aligned} \frac{1}{s} \|\hat{v}_\varphi(t)\|_{L_s(\mathbb{R}^3)}^s + \nu \int_0^t \|\nabla |\hat{v}_\varphi|^{s/2}\|_{L_2(\mathbb{R}^3)}^2 dt' \\ + \nu \int_0^t \left\| \frac{|\hat{v}_\varphi|^{s/2}}{r} \right\|_{L_2(\mathbb{R}^3)}^2 dt' + \int_0^t \int_{\mathbb{R}^3} \frac{v_r}{r} |\hat{v}_\varphi|^s dx dt' \\ = \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla \chi v_\varphi - 2\nu \nabla v_\varphi \nabla \chi - \nu v_\varphi \Delta \chi) \hat{v}_\varphi |\hat{v}_\varphi|^{s-2} dx dt'. \end{aligned} \quad (3.7)$$

In view of Lemma 2.10 the first two terms on the l.h.s. of (3.7) are estimated from below by

$$\|\hat{v}_\varphi\|_{L^{\frac{5s}{3}}(\mathbb{R}^3 \times (0,t))}^s. \quad (3.8)$$

Next we estimate non-positive terms in (3.7). In view of the energy estimate (2.1) the last term on the l.h.s. of (3.7) is bounded by

$$\|v_r\|_{L^{\frac{10}{3}}(\mathbb{R}^3 \times (0,t))} \|\hat{v}_\varphi\|_{L^{\frac{20}{7s}}(\mathbb{R}^3 \times (0,t))}^s,$$

where the second factor can be always absorbed by (3.8) because $5/3 > 10/7$. In view of (2.9) and (2.1) the first integral on the r.h.s. of (3.7) is bounded by

$$\|v\|_{L^{\frac{10}{3}}(\mathbb{R}^3 \times (0,t))} \|v_\varphi\|_{L^4(\mathbb{R}^3 \times (0,t))} \|\hat{v}_\varphi\|_{L^{\frac{20}{9(s-1)}}(\mathbb{R}^3 \times (0,t))}^{s-1}. \quad (3.9)$$

The last factor in (3.9) can be absorbed by (3.8) for $\frac{20}{9}(s-1) \leq \frac{5}{3}s$ which holds for $s \leq 4$. Then (3.8) yields estimate for

$$\|\hat{v}_\varphi\|_{L^{\frac{20}{3}}(\mathbb{R}^3 \times (0,t))}.$$

Finally we estimate the last two integrals on the r.h.s. of (3.7). We write them in the form

$$\begin{aligned} & -\nu \int_0^t \int_{\mathbb{R}^3} (2\nabla v_\varphi \nabla \chi + v_\varphi \Delta \chi) \hat{v}_\varphi |\hat{v}_\varphi|^{s-2} dx dt' \\ & = -\nu \int_0^t \int_{\mathbb{R}^3} (2\nabla v_\varphi \nabla \chi v_\varphi \chi |\hat{v}_\varphi|^{s-2} + v_\varphi \Delta \chi \hat{v}_\varphi |\hat{v}_\varphi|^{s-2}) dx dt' \\ & = -\nu \int_0^t \int_{\mathbb{R}^3} (\nabla v_\varphi^2 \nabla \chi \chi |\hat{v}_\varphi|^{s-2} + v_\varphi \Delta \chi \hat{v}_\varphi |\hat{v}_\varphi|^{s-2}) dx dt' \\ & = -\nu \int_0^t \int_{\mathbb{R}^3} (-v_\varphi^2 \Delta \chi \chi |\hat{v}_\varphi|^{s-2} - v_\varphi^2 |\nabla \chi|^2 |\hat{v}_\varphi|^{s-2} \\ & \quad - v_\varphi^2 \nabla \chi \chi \nabla |\hat{v}_\varphi|^{s-2} + v_\varphi \Delta \chi \hat{v}_\varphi |\hat{v}_\varphi|^{s-2}) dx dt' \\ & = \nu \int_0^t \int_{\mathbb{R}^3} (v_\varphi^2 |\nabla \chi|^2 |\hat{v}_\varphi|^{s-2} + v_\varphi^2 \nabla \chi \chi \nabla |\hat{v}_\varphi|^{s-2}) dx dt' = I_1 + I_2, \end{aligned}$$

where

$$|I_1| \leq c \|v_\varphi\|_{L^4(\mathbb{R}^3 \times (0,t))} \|\hat{v}_\varphi\|_{L^{2(s-2)}(\mathbb{R}^3 \times (0,t))}^{s-2}$$

and the first factor is bounded in view of Remark 2.5 and the second factor is absorbed by (3.8) for $2(s-2) \leq \frac{5}{2}s$ which holds for $s \leq 12$. Finally

$$\begin{aligned} I_2 & = \nu(s-2) \int_0^t \int_{\mathbb{R}^3} v_\varphi^2 \nabla \chi \chi |\hat{v}_\varphi|^{s-3} \nabla |\hat{v}_\varphi| dx dt' \\ & \leq \frac{\nu(s-2)}{s} \int_0^t \int_{\mathbb{R}^3} v_\varphi^2 \nabla \chi \chi |\hat{v}_\varphi|^{s/2-2} \nabla |\hat{v}_\varphi|^{s/2} dx dt' \\ & \leq \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla |\hat{v}_\varphi|^{s/2}|^2 dx dt' + \frac{c}{\varepsilon} \int_0^t \int_{\mathbb{R}^3} |v_\varphi|^2 |\hat{v}_\varphi|^{s-2} dx dt', \end{aligned}$$

where the first integral is absorbed by the second term on the l.h.s. of (3.7) and the second is bounded by

$$\left(\int_0^t \int_{\mathbb{R}^3} |v_\varphi|^4 dx dt' \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^3} |\hat{v}_\varphi|^{2(s-2)} dx dt' \right)^{\frac{1}{2}} = I_3.$$

The first factor in I_3 is bounded in virtue of Remark 2.5 and the second factor is absorbed by (3.8) if $2(s-2) \leq \frac{5}{3}s$ so $s \leq 12$. Summarizing, we obtain the estimate

$$\|\hat{v}_\varphi\|_{L^{\frac{20}{3}}(\mathbb{R}^3 \times (0,t))} \leq c. \quad (3.10)$$

We observe that the estimate (3.10) above is not strong enough to apply Lemma 2.2(1) because for $s = r$ it is required that $s \geq \frac{50}{7}$.

To increase regularity we introduce a new smooth cut-off function

$$\check{\chi}(r) = \begin{cases} 1 & r \geq 3r_0, \\ 0 & r \leq 2r_0 \end{cases}$$

Introducing notation $\check{v}_\varphi = v_\varphi \check{\chi}$ we replace (3.6) by

$$\begin{aligned} \check{v}_{\varphi,t} + v \cdot \nabla \check{v}_\varphi - \nu \Delta \check{v}_\varphi + \frac{v_r}{r} \check{v}_\varphi + \nu \frac{\check{v}_\varphi}{r^2} \\ = v \cdot \nabla \check{\chi} \hat{v}_\varphi - 2\nu \nabla \hat{v}_\varphi \nabla \check{\chi} - \nu \hat{v}_\varphi \Delta \check{\chi}, \\ \check{v}_\varphi|_{t=0} = \check{v}_\varphi(0). \end{aligned}$$

where we can use (3.10). Hence the above problem increases regularity of (3.10), so we can meet assumptions of Lemma 2.2(1). This concludes the proof.

Lemma 3.3. *Assume that*

$$\int_0^T dt \int_{\mathbb{R}^3} \frac{\omega_\varphi^2}{r} dx < \infty.$$

Then there exists a constant c such that

$$\int_0^T dt \int_{\mathbb{R}^3} \frac{v_r^2}{r^3} dx \leq c \int_0^T dt \int_{\mathbb{R}^3} \frac{\omega_\varphi^2}{r} dx. \quad (3.11)$$

Proof. From (1.2)₃ we have

$$-\Delta \psi_{,z} + \frac{1}{r^2} \psi_{,z} = \omega_{\varphi,z}, \quad (3.12)$$

where $v_r = -\psi_{,z}$.

To simplify further considerations we introduce the notation

$$u = \psi_{,z}, \quad f = \omega_\varphi.$$

Then (3.12) takes the form

$$-\Delta u + \frac{1}{r^2} u = f_{,z}. \quad (3.13)$$

Recall that I is the axis of symmetry. Let $C_0^\infty(\mathbb{R}^3 \setminus I)$ be the set of smooth functions vanishing near I and outside a compact set.

Let $H_0^1(\mathbb{R}^3)$ be the closure of $C_0^\infty(\mathbb{R}^3 \setminus I)$ in the norm

$$\|u\|_{H_0^1(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + \frac{1}{r^2} |u|^2) dx \right)^{\frac{1}{2}}.$$

Recall that functions from $H_0^1(\mathbb{R}^3)$ vanish on I . By the weak solution to (3.13) we mean a function $u \in H_0^1(\mathbb{R}^3)$ satisfying the integral identity

$$\int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla \chi + \frac{1}{r^2} u \chi \right) dx = - \int_{\mathbb{R}^3} f \chi_{,z} dx, \quad (3.14)$$

which holds for any smooth function χ belonging to $H_0^1(\mathbb{R}^3)$. Introducing the scalar product

$$(u, v)_{H_0^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + \frac{1}{r^2} uv \right) dx, \quad (3.15)$$

we can write (3.14) in the following short form

$$(u, \chi)_{H_0^1(\mathbb{R}^3)} = -(f, \chi, z)_{L_2(\mathbb{R}^3)},$$

where

$$(u, v)_{L_2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} uv dx.$$

For $f \in L_2(\mathbb{R}^3)$, $(f, \chi, z)_{L_2(\mathbb{R}^3)}$ is a linear functional on $H_0^1(\mathbb{R}^3)$. Hence we have

$$|(f, \chi, z)_{L_2(\mathbb{R}^3)}| \leq \|f\|_{L_2(\mathbb{R}^3)} \|\chi, z\|_{L_2(\mathbb{R}^3)} \leq \|f\|_{L_2(\mathbb{R}^3)} \|\chi\|_{H_0^1(\mathbb{R}^3)}.$$

Hence by the Riesz Theorem there exists $F \in H_0^1(\mathbb{R}^3)$ such that

$$-(f, \chi, z)_{L_2(\mathbb{R}^3)} = (F, \chi)_{H_0^1(\mathbb{R}^3)}.$$

Therefore there exists a solution to the integral identity (3.15) such that $u = F \in H_0^1(\mathbb{R}^3)$ and the estimate holds

$$\|u\|_{H_0^1(\mathbb{R}^3)} \leq c \|f\|_{L_2(\mathbb{R}^3)}.$$

It is clear that the solution is unique.

Since u vanishes for $r = 0$, $u = \psi, z$ so $\psi|_{r=0} = 0$ also. Therefore we can look for approximate weak solution satisfying the integral identity

$$\int_{\mathbb{R}_\varepsilon^3} \left(\nabla u \cdot \nabla \chi + \frac{1}{r^2} u \chi \right) dx = - \int_{\mathbb{R}_\varepsilon^3} f \chi, z, \quad (3.16)$$

where $\mathbb{R}_\varepsilon^3 = \{x \in \mathbb{R}^3 : r > \varepsilon\}$, $\varepsilon > 0$. Let $\chi = \frac{\psi, z}{r^\alpha}$. Then recalling notation $u = \psi, z$, $f = \omega_\varphi$ identity (3.16) takes the form

$$\int_{\mathbb{R}_\varepsilon^3} \left(\nabla \psi, z \frac{\nabla \psi, z}{r^\alpha} + \frac{1}{r^{2+\alpha}} |\psi, z|^2 \right) dx = - \int_{\mathbb{R}_\varepsilon^3} \frac{\omega_\varphi}{r^\alpha} \psi, z z dx.$$

Performing differentiation we have

$$\begin{aligned} \int_{\mathbb{R}_\varepsilon^3} \frac{1}{r^\alpha} |\nabla \psi, z|^2 dx - \alpha \int_{\mathbb{R}_\varepsilon^3} \nabla \psi, z \psi, z \nabla r r^{-\alpha-1} dx \\ + \int_{\mathbb{R}_\varepsilon^3} \frac{\psi, z^2}{r^{2+\alpha}} dx = - \int_{\mathbb{R}_\varepsilon^3} \frac{\omega_\varphi}{r^\alpha} \psi, z z dx. \end{aligned} \quad (3.17)$$

The second integral on the l.h.s. of (3.17) takes the form

$$\begin{aligned} -\alpha \int_{\mathbb{R}_\varepsilon^3} \partial_r \psi, z \psi, z r^{-\alpha-1} r dr dz &= -\frac{\alpha}{2} \int_{\mathbb{R}_\varepsilon^3} (\psi, z^2), r r^{-\alpha} dr dz \\ &= -\frac{\alpha}{2} \int_{\mathbb{R}_\varepsilon^3} \partial_r (\psi, z^2 r^{-\alpha}) dr dz - \frac{\alpha^2}{2} \int_{\mathbb{R}_\varepsilon^3} \psi, z^2 r^{-\alpha-1} dr dz \\ &= \frac{\alpha}{2} \int_{\mathbb{R}_\varepsilon^3} \psi, z^2 r^{-\alpha} |_{r=\varepsilon} dz - \frac{\alpha^2}{2} \int_{\mathbb{R}_\varepsilon^3} \psi, z^2 r^{-\alpha-2} dx. \end{aligned}$$

Using this in (3.17) and applying the Hölder and Young inequalities to the r.h.s. term yields

$$\int_{\mathbb{R}_\varepsilon^3} \frac{1}{r^\alpha} |\nabla \psi, z|^2 dx + \left(1 - \frac{\alpha^2}{2}\right) \int_{\mathbb{R}_\varepsilon^3} \frac{\psi, z^2}{r^{2+\alpha}} dx \leq c \int_{\mathbb{R}_\varepsilon^3} \frac{\omega_\varphi^2}{r^\alpha} dx. \quad (3.18)$$

We have to emphasize that ψ in (3.18) is an approximate function. This should be denoted with ψ^ε but we omitted it for simplicity.

Passing with $\varepsilon \rightarrow 0$, setting $\alpha = 1$ and integrating this inequality with respect to time implies (3.11). This concludes the proof. $\square$

Proof of Theorem 1

Proof. From (2.10) and (2.12) we have

$$\begin{aligned} & \left\| \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}^3)}^2 + \int_0^t \left\| \nabla \frac{v_\varphi^2}{r} \right\|_{L_2(\mathbb{R}^3)}^2 dt' \\ & \leq \exp \left(c \int_0^t \left\| \frac{\omega_\varphi}{r} \right\|_{L_2(\mathbb{R}^3)}^{4/3} dt' \right) \cdot \left\| \nabla \frac{v_\varphi^2(0)}{r} \right\|_{L_2(\mathbb{R}^3)}^2, \quad t \leq T. \end{aligned}$$

Next (2.15) implies

$$\begin{aligned} & \left\| \frac{\omega_r}{r} \right\|_{L_2(\mathbb{R}^3)}^2 + \left\| \frac{\omega_\varphi}{r} \right\|_{L_2(\mathbb{R}^3)}^2 \\ & \leq c \exp(\|u_\alpha\|_{L_{p,q}(\mathbb{R}^3 \times (0,t))}) \left(\left\| \frac{\omega_r(0)}{r} \right\|_{L_2(\mathbb{R}^3)}^2 + \left\| \frac{\omega_\varphi(0)}{r} \right\|_{L_2(\mathbb{R}^3)}^2 \right), \\ & \frac{3}{p} + \frac{2}{q} = \alpha, \quad \alpha \leq 1, \quad t \leq T. \end{aligned} \tag{3.19}$$

Finally, in view of (1.7) and Lemma 3.1 we get

$$\|u_\alpha\|_{L_{p,q}(\mathbb{R}^3 \times (0,t))} \leq c(\|u(0)\|_{L_\infty(\mathbb{R}^3)} + \|u(0)\|_{L_s(\mathbb{R}^3)}),$$

where $\alpha = \frac{3}{s}$, $s \geq 3$, $t \leq T$. These estimates imply that

$$\begin{aligned} & \|v_\varphi\|_{L_{p',q'}(\mathbb{R}_{r_0}^3 \times (0,t))} \\ & \leq h \left(\left\| \frac{v_\varphi^2(0)}{r} \right\|_{L_2(\mathbb{R}^3)}, \left\| \frac{\omega_\varphi(0)}{r} \right\|_{L_2(\mathbb{R}^3)}, \left\| \frac{\omega_r(0)}{r} \right\|_{L_2(\mathbb{R}^3)}, \|u(0)\|_{L_\infty(\mathbb{R}^3)}, \|u(0)\|_{L_s(\mathbb{R}^3)} \right), \\ & \frac{3}{p'} + \frac{2}{q'} \geq \frac{3}{4}, \end{aligned}$$

where h is some positive increasing function of its arguments. Hence Lemma 2.3 implies local regularity.

To make statement more explicit we obtain from (3.19) for $r < r_0$ and from Step 5 of the proof of Theorem 1 from [5] that

$$\|\omega\|_{L_\infty(0,T;L_2(\mathbb{R}_{r_0}^3))} \leq C(data),$$

where $data$ are data from the assumptions of the Step 1 of the proof of Theorem 1 from [5].

Considering the problem

$$\operatorname{rot} v = \omega,$$

$$\operatorname{div} v = 0$$

and the local technique from [15, Ch.4, Sect.10], we have

$$\|v\|_{L_\infty(0,T;L_6(\mathbb{R}_{r_0}^3))} \leq C(data).$$

Consider the problem

$$v_t - \nu \Delta v + \nabla p = -v \cdot \nabla v,$$

$$\operatorname{div} v = 0,$$

$$v|_{t=0} = v(0).$$

Employing the result of Solonnikov, estimate for v above and some interpolation we get

$$\|v\|_{W_{\sigma,r}^{2,1}(\mathbb{R}_0^3 \times (0,T))} \leq c(data) + c\|v\|_{B_{\sigma,r}^{2-2/r}(\mathbb{R}_0^3 \times (0,T))},$$

where $\sigma < 6$, r arbitrary. This proves the second part of Theorem 1. $\square$

Proof of Theorem 2 follows from Theorem 1 and Lemma 3.3.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest

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Joanna Renclawowicz and Wojciech M. Zajączkowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warsaw
Poland
e-mail: jr@impan.pl

Wojciech M. Zajączkowski
Institute of Mathematics and Cryptology
Military University of Technology
Kaliskiego 2
00-908 Warsaw
Poland

Wojciech M. Zajączkowski
e-mail: wz@impan.pl

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