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Existence and Uniqueness of Weak Solutions to the Two-Dimensional Stationary Navier–Stokes Exterior Problem

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Dedicated to Professor Reinhard Farwig on the occasion of his sixtieth birthday.

Abstract. This paper is concerned with the stationary Navier–Stokes equation in two-dimensional exterior domains with external forces and inhomogeneous boundary conditions, and shows the existence of weak solutions. This solution enjoys a new energy inequality, provided the total flux is bounded by an absolute constant. It is also shown that, under the symmetry condition, the weak solutions tend to 0 at infinity. This paper also provides two criteria for the uniqueness of weak solutions under the assumption on the existence of one small solution which vanishes at infinity. In these criteria the aforementioned energy inequality plays a crucial role.

Mathematics Subject Classification. 35Q30.

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1. Introduction

Let Ω be an exterior domain in the plane $\mathbb{R}^2$ with $C^{2+\gamma}$-boundary Γ with some $\gamma \in (0,1)$. We are concerned with the following stationary Navier–Stokes equation in Ω:

$$-\Delta w(x) + \big(w(x)\cdot\nabla\big)w(x) + \nabla\pi(x) = f(x) \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot w(x) = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$w(x) = a(x) \qquad \text{on } \Gamma, \tag{1.3}$$

where the vector-valued unknown function $w(x)$, the scalar-valued unknown function $\pi(x)$ and the vector-valued given function $f(x)$ stand for the velocity, the pressure and the external force respectively. As is known in Russo and Simader [30], the solution does not satisfy the boundary condition

$$w(x) \to 0 \text{ as } |x| \to \infty \tag{1.4}$$

in general. Throughout this paper we assume that the external force $f(x)$ is given by the formula $f(x) = \nabla\cdot F(x) = \left(\sum_{j=1}^{2}\frac{\partial F_{jk}}{\partial x_j}(x)\right)_{k=1}^{2}$ with a 2×2 matrix $\big(F_{jk}(x)\big)_{j,k=1}^{2} \in \big(L^2(\Omega)\big)^4$.

In the pioneering work by Finn and Smith [9], the study on the stationary Navier–Stokes equation in two-dimensional exterior domains started, first under the assumption that $w(x)$ is close to a definite vector

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$w_\infty \neq 0$ at infinity with no external force. This problem was considered by Gilbarg and Weinberger [15,16] and Amick [2,3]. In particular, [2,3] considered the case

$$\Omega \text{ is invariant under the mapping } \pi_1 : (x_1, x_2) \mapsto (-x_1, x_2) \tag{R2I}$$

and

$$w_1(-x_1, x_2) = -w_1(x_1, x_2), \quad w_2(-x_1, x_2) = w_2(x_1, x_2), \tag{R2E}$$

where $w_\infty = (0, c)$ with some $c \neq 0$. This result is improved by Russo [27]. Then Galdi and Simader [12] considered the problem for external force $f(x)$ with little regularity, and Galdi and Sohr [13], Sazonov [31] and Russo [26] obtained precise asymptotic behavior of $w(x)$ and $\pi(x)$. (For more complete references, see Galdi [11].)

For the case $w_\infty = 0$ with external force, Russo [25] obtained the existence of weak solutions under the assumption that $|\alpha|$ is sufficiently small. Here the total flux α is defined by $\alpha = \int_\Gamma a(x) \cdot n(x)\, ds(x)$, where $n(x)$ denotes the unit normal vector of Γ at x outward of Ω.

Furthermore, Galdi [10, Section 3] and Pileckas and Russo [24], as well as [25], posed the condition

$$\begin{aligned} &\text{The set } \Omega \text{ is invariant under the mappings} \\ &\pi_1 : (x_1, x_2) \mapsto (-x_1, x_2), \quad \pi_2 : (x_1, x_2) \mapsto (x_1, -x_2) \end{aligned} \tag{D4I}$$

with a specific coordinate variables (x_1, x_2) on Ω, and the condition

$$\begin{cases} f_1(-x_1, x_2) = -f_1(x_1, x_2), & f_2(-x_1, x_2) = f_2(x_1, x_2), \\ f_1(x_1, -x_2) = f_1(x_1, x_2), & f_2(x_1, -x_2) = -f_2(x_1, x_2) \end{cases} \tag{D4E}$$

on the external force $f(x) = \big(f_1(x), f_2(x)\big)$, and (D4E) on the boundary value $a(x)$, and showed the existence of a weak solution $w(x) = \big(w_1(x), w_2(x)\big)$ satisfying (D4E), which is exactly the same as [10, (3.18)]. Furthermore, [25] showed that $w(x)$ tends to 0 in the average, [Precise definition is given in (2.9).] and that $w(x)$ tends to 0 pointwise if the external force has compact support. We here regard this condition above from the viewpoint of transformation groups, as is in Brandolese [6] which obtained sharp decay order of solutions to the nonstationary Navier–Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$. The condition (D4I) can be described as $\pi_j(\Omega) = \Omega$ for $j = 1, 2$, and the condition (D4E) can be described as $f\big(\pi_j(x)\big) = \pi_j\big(f(x)\big)$. In other words, the domain Ω is invariant, and the vector field $f(x)$ is equivariant, with respect to the action of the group $D_4 = \{id, \pi_1, \pi_2, \pi_1\pi_2\}$.

Later, Russo [28] relaxed the assumptions (D4I) and (D4E) to

$$\begin{aligned} &\text{The set } \Omega \text{ is invariant under the mapping} \\ &\pi_1\pi_2 : x = (x_1, x_2) \mapsto -x = (-x_1, -x_2), \end{aligned} \tag{C2I}$$

and

$$f_j(-x) = -f_j(x) \text{ for } j = 1, 2 \text{ and every } x \in \Omega, \tag{C2E}$$

and showed the same property satisfying $w(x)$ satisfying (C2E). Russo [29] studied the decay order of the solutions at infinity in detail. In other words, Ω is invariant, and $f(x)$ is equivariant, under the group C_2 generated by $x \mapsto -x$. Most of these works treated the solution $\big(w(x), \pi(x)\big)$ satisfying $\nabla w \in \big(L^2(\Omega)\big)^4$.

Recently, Korobkov et al. [20] proved, among others, the existence of the solution with no assumptions on the flux of each boundary components for multiply connected exterior domains under the assumptions (C2I) and (C2E) and some assumptions on the obstacles.

However, in the author's knowledge, there are few results on the solutions decaying sufficiently so that its stability for the nonstationary problem under initial perturbation is assumed, and the uniqueness in these classes is obtained in this class for exterior domains is obtained only by [28] without the exterior force.

For these problems, the author [33] showed the existence, together with the uniqueness in the small, of the solution of the stationary Navier–Stokes equation on the whole plane under the assumption that the small external force $f(x) = \nabla \cdot F(x)$ decays like $|x|^{-2}$ as $|x| \to \infty$ and satisfies the condition

$$f(x^\perp) = \big(f(x)\big)^\perp; \text{ namely, } f_1(x^\perp) = -f_2(x), f_2(x^\perp) = f_1(x), \tag{C4E}$$

where $x^\perp = (-x_2, x_1)$, as well as (D4E). The solutions decays like $|x|^{-1}$ as $x \to \infty$; in other words, decays like the derivatives of the fundamental solution of the Laplacian. This class is invariant under the scaling $w_\rho(x) = \rho w(\rho x)$ keeping (1.1) invariant with suitable scaling for $p(x)$ and $f(x)$. We call these solutions critically decaying. (Its definition in somewhat generalized form, which does not imply pointwise estimate, is given in Remark 2.13.) In the terminology of function spaces, critically decaying solutions belong to the weak-L^2 space. It is also shown that, if $F(x)$ decays more rapidly, then $w(x)$ decays more rapidly (up to $|x|^{-2}$). This result is also obtained by Guillod [17, Section 3] without the assumption that $f(x)$ is of the form $f(x) = \nabla \cdot F(x)$. Very recently, Decaster and Iftimie [8] obtained sharp decay and asymptotic profiles of the solutions.

Then [34] showed that the same result as [33] holds for the exterior problem provided the domain Ω satisfies the symmetry condition

$$\text{The set } \Omega \text{ is invariant under the mapping } \sigma : x \mapsto x^\perp \tag{C4I}$$

as well as (D4I), and the external force $f(x)$ and the boundary value $a(x)$ satisfies (D4E) and (C4E) with $\alpha = 0$. Recently, Guillod [17] obtained sharp asymptotic behavior at infinity of the solution above. Notice that (D4I) and (C4I) imply the invariance of Ω for the square dihedral group D_4 , and (D4E) and (C4E) imply the equivariance for D_4 .

Then Nakatsuka [22] proved the weak-strong uniqueness; namely, he showed that, if the exterior domain satisfies (D4I) and if there exists a sufficiently small critically decaying solution of (1.1)–(1.4) with $a(x) \equiv 0$ satisfying the condition (D4E), then every weak solution of (1.1)–(1.4) satisfying the energy inequality and the same symmetry property coincides with the critically decaying solution. He also showed that, if there exists a sufficiently small supercritically decaying solution of (1.1)–(1.4) with $a(x) \equiv 0$, every weak solution satisfying the energy inequality coincides with the supercritically decaying solution. For the proof he first showed that the critically decaying solutions satisfy the energy identity.

Further, Galdi and Yamazaki [14] showed that the solutions above are stable under initial L^2-perturbation with the symmetry property (D4E) with no restriction on the size, and the author [35] gave convergence rate in various function spaces. The author [34] also showed that, if $f(x)$ decays more rapidly, then the solution decays faster than the derivatives of the fundamental solution. We call these solutions supercritically decaying. In this case the stability above holds true for initial L^2-perturbation without symmetry or size restriction. Very recently, Guillod [18] showed that the critically decaying solution is stable under general initial L^2-perturbation. (The precise definition of supercritically decaying solutions, which does not imply pointwise estimate, is given in Remark 2.10 in somewhat generalized form.) Note that the divergence theorem implies that the outflow condition $\alpha = 0$ is necessary for the existence of supercritically decaying solutions.

On the other handy, the author [36] proved, assuming only (C4I) on the domain Ω, and assuming only (C4E) on $a(x)$ and $f(x) = \nabla F(x)$, where $a(x)$ and $F(x)$ are small in appropriate function spaces, the unique existence of small critically decreasing solutions satisfying the symmetry condition (C4E). In this work we imposed the invariance and equivariance under the cyclic group C_4 .

For the existence of critically decreasing solutions, the condition $\alpha = 0$ is not necessary. The author [36] also showed that the solutions above become supercritically decreasing if $F(x)$ satisfies a sharper decay condition and if, as well as the outflow condition $\alpha = 0$, the data $a(x)$ and $F(x)$ are sufficiently small. We observe that the conditions (C4I) and (C4E) are independent of the choice of the coordinate axes, and includes rotationally symmetric functions naturally. However, the results above implies only the uniqueness of small solutions. The main difficulty in the two-dimensional case is that the function with finite Dirichlet integral is not bounded in general, and we must modify Hardy's inequality (see Propositions 3.5 and 7.1) in general. (see Adimurthi et al. [1].)

The purpose of this paper is twofold. The first one is a slight generalization of constructing weak solutions with external force $f(x) = \nabla \cdot F(x)$ with neither any sort of symmetry conditions nor the smallness conditions. In order to treat large $a(x)$, we construct a corrector potential $G(x)$ so that there exists a solution $v(x)$ of the Stokes equation with given boundary condition and the external force $\nabla G(x)$ is critically decreasing, and consider the equation for $u(x) = w(x) - v(x)$ with homogeneous boundary condition. The correction potential $G(x)$ is a compactly supported function which cancels the angular momentum of the boundary value, and the function $v(x)$ is critically decreasing function whose profile is determined by α, However, these functions do not cancel the momentum, and hence $v(x)$ does not describe the asymptotic profile of $w(x)$. Indeed, for the steady flow $w(x) \equiv w_0 \neq 0$ outside a disk with $F(x) \equiv 0$ and $w(x) = w_0$ on the boundary, we construct $G(x)$ so that $v(x)$ is supercritically decreasing, and the asymptotic profile $w(x) = w_0$ at infinity reappears as the homogeneous boundary value problem with external force $-\nabla G(x)$. This fact also implies that, even if the boundary value vanishes and the external force is compactly supported, the asymptotic behavior of the solution may not be trivial. Hence our existence theorem covers the case where $w(x)$ is not decreasing as $|x| \to \infty$. We then apply the fixed point theorem on disks, whose centers converge to the image of $v(x)$ by the orthogonal projection to $H^1_{0,\sigma}(\Omega)$, which is denoted by $\overline{v}$. In this case the solution does not necessarily satisfy (1.4).

If the domain Ω satisfies the symmetry condition (C2I) and the external force and the boundary conditions satisfies (C2E), then we can construct a solution $w(x)$ satisfying (C2E). Since C_2 is a subgroup of C_4, these conditions are also independent of the choice of coordinate axes. In this case the solution satisfies (1.4) in the sense that $w(x)$ tends to 0 in the average.

Moreover, the solution satisfies an energy inequality provided $|\alpha|$ is bounded by an absolute constant. This inequality is a generalization of the one employed in [22], where the case $a(x) = 0$. This inequality in this paper seems to be new since it is applicable to solutions which fail to satisfy (1.4), and plays a crucial role in the study of weak-strong uniqueness. In order to treat the case where the angular momentum of the solution is not zero, we use the result of Coifman et al. [7].

The second purpose is to give a condition for the weak-strong uniqueness under non-homogeneous boundary conditions and with external force under the assumption that $a(x)$ is small and that $w(x)$ decays sufficiently fast. Namely, suppose that $w(x)$ is a weak solution such that $u(x) = w(x) - v(x)$ is small and supercritically decaying. (This implies the smallness of $a(x)$ and $f(x)$.) Then every weak solution $w'(x)$ such that $u'(x) = w'(x) - v(x)$ satisfies the energy inequality but not necessarily satisfies (1.4), however large it may be, coincides with $w(x)$. In particular, the weak solution constructed in the previous results coincides with $w(x)$. Under the symmetry condition the assumption on decay property can be relaxed. Namely, suppose that Ω satisfies (C2I), and that ∇F and $a(x)$ satisfies (C2E). Moreover, suppose that $w(x)$ is a weak solution such that $u(x) = w(x) - v(x)$ is a small critically decaying function satisfying (C2E). Then every weak solution $w'(x)$ satisfying (C2E) such that $u'(x) = w'(x) - v(x)$ satisfies (C2E) and the energy inequality, must coincide with $w(x)$. This result implies the uniqueness of the solutions obtained in [36] as well as those in [33,34]. In particular, if $w'(x)$ is the weak solution constructed in the previous result such that $u'(x) = w'(x) - v(x)$ satisfies (C2E), then $w'(x)$ coincides with $w(x)$. With (C2I) and (C2E) we have Hardy's inequality. Notice that $u(x)$ need not satisfy an assumption of pointwise estimate in either case.

In the case Ω satisfies (D4I), $f(x)$ satisfies (D4E) and $a(x) = 0$, our assumption on the smallness of the weighted L^p-norm is a slight generalization of the assumption on the smallness of the pointwise estimate in [22]. Moreover, we prove the energy identity under weaker assumption in which no pointwise estimate is necessary. To this end we prove a sharp version of Hardy's inequality.

In addition to the property above on the uniqueness, the results in [14,35] on the stability under initial L^2-perturbation with no restriction on the size holds, and we can replace the symmetry condition (D4E) by (C2E) by applying the improved Hardy's inequality. In other words, solutions in [33,34,36] have similar property on uniqueness and stability as physically reasonable solutions in the three-dimensional setting.

This paper is organized as follows. In Sect. 2 the notation is fixed and main results are stated. In Sect. 3 we list up some facts necessary in the proof. Proof of some lemmata are given in the Appendix. Then corrector potentials and weak solutions are constructed in Sects. 4 and 5 respectively. The uniqueness is proved in Sect. 6. Finally in Sect. 7 we state the improvement in symmetric cases.

2. Notations and Main Results

We first introduce some function spaces. For a domain $U \subset \mathbb{R}^2$, let $C_0^\infty(U)$ denote the set of infinitely differentiable functions on U supported by a compact subset of U, and let $C_{0,\sigma}^\infty(U)$ denote the set of vector-valued functions $\varphi(x) = \big(\varphi_1(x), \varphi_2(x)\big) \in \big(C_0^\infty(U)\big)^2$ such that $\nabla \cdot \varphi \equiv 0$. Next, for a domain U in $\mathbb{R}^2$ and $q \in [1,\infty]$, let $L^q(U)$ denote the standard Lebesgue spaces. The norm of $\big(L^q(U)\big)^m$ with $m \in \mathbb{N}$ is denoted by $\|\cdot\|_q$. For $p \in (1,\infty)$ and $r \in [1,\infty]$, let $L^{q,r}(U)$ denote the set of Lorentz space. The norm of $\big(L^{q,r}(U)\big)^m$ for $m \in \mathbb{N}$ is denoted by $\|\cdot\|_{q,r}$.

Then we have the following properties. (see Bergh and Löfström [4] or Triebel [32] for example.) First, there exist a inclusion relation $L^{q,r}(U) \subset L^{q,s}(U)$ provided $r < s$. Second, the space $L^{q,q}(U)$ coincides with the Lebesgue space $L^q(U)$, and the space $L^{q,\infty}(U)$ coincides with the weak-L^q space on U. Third, for $1 \le r < \infty$, the space $C_0^\infty(U)$ is dense in $L^{q,r}(U)$, while this property fails if $r = \infty$. Let $L^{q,\infty-}(U)$ denote the closure of $C_0^\infty(U)$ in $L^{q,\infty}(U)$. Fourth, if $1 \le r < \infty$, we have the duality property $\big(L^{q,r}(U)\big)' = L^{q/(q-1),r/(r-1)}(U)$, while $\big(L^{q,\infty-}(U)\big)' = L^{q/(q-1),1}(U)$. Fifth, if $1 \le q_0 < q_1 \le \infty$, $1 \le r \le \infty$ and $0 < \theta < 1$, the real interpolation space $\big(L^{q_0}(U), L^{q_1}(U)\big)_{\theta,r}$ coincides with $L^{q,r}(U)$ up to equivalence of the norms, where $1/q = (1-\theta)/q_0 + \theta/q_1$.

Suppose that U is either a whole plane $\mathbb{R}^2$, a bounded domain or an exterior domain with $C^{2+\gamma}$ boundary, and let Γ denote the boundary of U. Then, for every $q \in (1,\infty)$, there exists a direct sum decomposition $\big(L^q(U)\big)^2 = L^q_\sigma(U) \oplus G_q(U)$, where

$$L^q_\sigma(U) = \left\{ u \in \big(L^q(U)\big)^2 \;\middle|\; \nabla \cdot u = 0 \text{ in } U, n(x)\cdot u(x) = 0 \text{ on } \Gamma \right\}$$

and

$$G_q(U) = \left\{ \nabla f \in \big(L^q(U)\big)^2 \;\middle|\; f \in L^q_{\mathrm{loc}}(U) \right\}.$$

Let P_q denote the projection on $\big(L^q(U)\big)^2$ onto $L^q_\sigma(U)$ associated with the decomposition above. Then we have $P_{q_0} = P_{q_1}$ on $\big(L^{q_o}(U) \cap L^{q_1}(U)\big)^2$. Hence we can define the projection $P_{q,r}$ on $\big(L^{q,r}(\Omega)\big)^2$ for every $q \in (1,\infty)$ and $r \in [1,\infty]$ by real interpolation. Let $L^{q,r}_\sigma(U)$ denote the range of $P_{q,r}$.

For $s \in \mathbb{R}$, let $H^s(\mathbb{R}^2)$ and $\dot H^s(\mathbb{R}^2)$ denote the Sobolev space and the homogeneous Sobolev space, equipped with the norms

$$\|U\|_{H^s(\mathbb{R}^2)} = \left\| \mathcal{F}^{-1}\left[\left(1+|\xi|^2\right)^{s/2} \mathcal{F}[u](\xi) \right] \right\|_2 < \infty,$$

and

$$\|u\|_{\dot H^s(\mathbb{R}^2)} = \left\| \mathcal{F}^{-1}\left[|\xi|^s \mathcal{F}[u](\xi) \right] \right\|_2 < \infty$$

respectively, and for $q, r \in [1,\infty]$ and $s \in \mathbb{R}$, let $\dot B^s_{q,r}$ denote the homogeneous Besov space. (see [4] or [32] for the definition.) In this paper we only use the inclusion relation $\dot B^s_{q,r} \subset \dot B^s_{q,\rho}$ if $r < \rho$, $\dot B^s_{p,p} \subset \dot H^s_p \subset \dot B^s_{p,2}$ for $1 < p \le 2$ and $\dot B^s_{p,2} \subset \dot H^s_p \subset \dot B^s_{p,p}$ for $2 \le p < \infty$, the embedding theorem $\dot B^s_{p,r} \subset \dot B^{s-n/p+n/q}_{q,r}$ for $q \ge p$, the characterization of homogeneous Besov spaces by the norm of differences, and the real interpolation property $\dot B^s_{p,q} = \left(\dot H^{s_0}_p, \dot H^{s_1}_p\right)_{\theta,q}$, where $s_0 \ne s_1$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. In general homogeneous spaces are defined only modulo polynomials, but if either $s < n/p$, or $s = n/p$ and $q = 1$ for the Besov space, the modulo classes in $\dot H^s_p$ and $\dot B^s_{p,q}$ have a canonical representative which decays as $|x| \to \infty$, and hence these spaces can be considered as function spaces on $\mathbb{R}^n$. For a domain U

in $\mathbb{R}^2$ and $k \in \mathbb{N}$, let $H^k(U)$ and $\dot{H}^k(U)$ denote the set of the restrictions of the elements of $H^k(\mathbb{R}^2)$ and $\dot{H}^k(\mathbb{R}^2)$ on U equipped with the norms

$$\|u\|_{H^k(U)} = \inf\left\{ \|\tilde{u}\|_{H^k(\mathbb{R}^2)} \,\middle|\, \tilde{u}|_U = u \right\}$$

and

$$\|u\|_{\dot{H}^k(U)} = \inf\left\{ \|\tilde{u}\|_{\dot{H}^k(\mathbb{R}^2)} \,\middle|\, \tilde{u}|_U = u \right\}$$

respectively, and let $\dot{H}^k_0(U)$ denote the closure of $C_0^\infty(U)$ in $\dot{H}^k(U)$. In particular, if U is bounded, the space $\dot{H}^k_0(U)$ is defined as a set of functions even if $k \geq 1$. Let $H^k_0(U)$ denote this space. Furthermore, let $\dot{H}^k_{0,\sigma}(U)$ denote the closure of $C^\infty_{0,\sigma}(\Omega)$ in $\left(\dot{H}^k(U)\right)^2$, and we write it $H^k_{0,\sigma}(U)$ if U is bounded.

Remark 2.1. The functions $u \in \left(\dot{H}^1_0(\Omega)\right)^2$ satisfying $\nabla \cdot u \equiv 0$ on Ω belongs to $\dot{H}^1_{0,\sigma}(\Omega)$, as is shown in Heywood [19, Section 2] for example.

For a domain U and $s > 0$ such that $s \notin \mathbb{N}$, Let $C^s(\overline{U})$ denote the Hölder space on $\overline{U}$. For a closed curve Γ of $C^{2+\gamma}$ class and $s > 0$, let $H^s(\Gamma)$ denote the Sobolev space on Γ.

Finally, for a scalar-valued function $f(x)$ we write

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x)\right), \quad \nabla^\perp f(x) = \left(-\frac{\partial f}{\partial x_2}(x), \frac{\partial f}{\partial x_1}(x)\right)$$

and a vector-valued function $u(x) = \big(u_1(x), u_2(x)\big)$ we write

$$\nabla \cdot u(x) = \frac{\partial u_1}{\partial x_1}(x) + \frac{\partial u_2}{\partial x_2}(x), \quad \nabla \times u(x) = \frac{\partial u_2}{\partial x_1}(x) - \frac{\partial u_1}{\partial x_2}(x).$$

Then we have $\Delta u(x) = \nabla\big(\nabla \cdot u(x)\big) + \nabla^\perp\big(\nabla \times u(x)\big)$ for a vector-valued function $u(x)$.

In the sequel let Ω be a fixed exterior domain with $C^{2+\gamma}$-boundary Γ. We introduce the notion of weak solutions.

Definition 2.2. We say that $w(x) \in \left(\dot{H}^1(\Omega)\right)^2$ satisfying (1.2)–(1.3) is a weak solution of (1.1)–(1.3) if it satisfies the identity

$$(\nabla w, \nabla \varphi) + (F - w \otimes w, \nabla \varphi) = 0 \tag{2.1}$$

for every $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$. Here $a \otimes b$ denotes the 2×2 matrix $\big(a_j b_k\big)_{j,k=1}^2$, and $(F, \nabla w)$ denotes the sum $\displaystyle\int_\Omega \sum_{j=1}^2 \sum_{k=1}^2 F_{j,k}(x) \frac{\partial w_k}{\partial x_j}(x)\, dx.$

We construct weak solutions of (1.1)–(1.3) by writing $w(x) = v(x) + u(x)$, where $v(x) \in \big(\dot{H}^1(\Omega)\big)^2$ is a function satisfying the linear system

$$-\Delta v(x) = \nabla \cdot G(x) \qquad \text{in } \Omega, \tag{2.2}$$

$$\nabla \cdot v(x) = 0 \qquad \text{in } \Omega, \tag{2.3}$$

$$v(x) = a(x) \qquad \text{on } \Gamma \tag{2.4}$$

with a certain function $G(x) \in \big(L^2(\Omega)\big)^4$ with a bounded support of satisfying the estimate $\|G\|_2 \leq C\|a\|_{H^{1/2}(\Gamma)}$ with some positive constant C. This function is the corrector potential.

Fix a positive integer J such that

$$B(0, 2^{J-1}) \cup \Omega = \mathbb{R}^2, \tag{2.5}$$

where $B(a, r)$ denotes the open ball with center a and radius r, and put $\tilde{\Omega} = \{x \in \Omega \mid |x| < 2^{J+1}\}$ and $D = \{x \mid 2^J \leq |x| \leq 2^{J+1}\}$. We also fix a smooth function $g(r)$ on $(0, \infty)$ such that $g(r) = 0$ for $r \leq 2^{J-1}$

and $g(r) = \log r$ for $r \geq 2^J$. Furthermore, for q such that $2 < q \leq \infty$, we define a function $\lambda_q(t)$ on $[0, \infty)$ by $\lambda_q(t) = t^{1-2/q}\big(\log(t+e)\big)^{1-1/q}$.

We now state the existence the corrector functions satisfying (2.2)–(2.4) with suitable $G(x)$.

Proposition 2.3. *Suppose that the exterior domain Ω satisfies (2.5). Then there exist a positive constant C and a bounded linear mapping I_1 from $\big(H^{1/2}(\Gamma)\big)^2$ to $\Big(\dot{H}^1(\Omega)\Big)^2$ and I_2 from $\big(H^{1/2}(\Gamma)\big)^2$ to $\big(H^1(\Omega)\big)^2$ such that, for $a(x) \in \big(H^{1/2}(\Gamma)\big)^2$, the function $v(x) = I_1[a](x) + I_2[a](x)$ satisfies (2.2)–(2.4) with $G(x) \in \big(L^2(D)\big)^4$ such that $\|G\|_2 \leq C\|a\|_{H^{1/2}(\Gamma)}$. Furthermore, for every $q \in (2, \infty)$, there exists a positive constant C_q such that the estimate $\|I[a](x)\lambda_q(|x|)\|_q \leq C_q\|a\|_{H^{1/2}(\Gamma)}$ holds.*

Moreover, we can write $I_1[a] = \dfrac{\alpha(x-c)}{2\pi|x-c|^2} + I_3[a]$, where $c \in \mathbb{R}^2\backslash\overline{\Omega}$ and there exists a positive number C such that we have $|I_3(x)| \leq \dfrac{C}{|x-c|^2}$.

Furthermore, if Ω satisfies (C2I) *and $a(x)$ satisfies* (C2E)*, then $I[a](x)$ also satisfies* (C2E)*, and $G(x)$ satisfies*

$$G(-x) = G(x) \text{ for every } x \in \Omega. \tag{C2AE}$$

In the sequel we write $v^{(j)} = I_j[a]$ for $j = 1, 2, 3$ and $\tilde{v} = v^{(2)} + v^{(3)}$.

Remark 2.4. Note that, as is stated in [30], the solution of the equation (2.2) does not decay as $|x| \to \infty$, but here we *construct* $G(x)$ so that there exists a solution $v(x)$ which enjoys (2.3)–(2.4) as well. This behavior is independent of the asymptotic profile of $w(x)$ in general.

Then $w(x) \in \Big(\dot{H}^1(\Omega)\Big)^2$ satisfying $\nabla \cdot w(x) = 0$ is a weak solution of (1.1)–(1.3) if and only if $u(x) = w(x) - v(x)$ satisfies $u(x) \in \Big(\dot{H}^1_0(\Omega)\Big)^2$, $\nabla \cdot u(x) = 0$ and the identity

$$(\nabla u, \nabla\varphi) - \big((u+v)\otimes(u+v), \nabla\varphi\big) + \big(H, \nabla\varphi\big) = 0 \tag{2.6}$$

for every $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$, where $H(x) = F(x) - G(x)$. Indeed, subtracting the equality $(\nabla v, \nabla\varphi) = (-\Delta v, \varphi) = (\nabla \cdot G, \varphi) = (-G, \nabla\varphi)$ from (2.1) we obtain (2.6).

Then our result concerning the existence and the energy inequality is the following theorem.

Theorem 2.5. *Suppose that Ω satisfies (2.5). Then, for every $a(x) \in \big(H^{1/2}(\Gamma)\big)^2$ and every $F(x) \in \big(L^2(\Omega)\big)^4$, there exists a weak solution $w(x)$ of (1.1)–(1.3). Furthermore, if $|\alpha| \leq C_0$, where C_0 is an absolute constant, then $u(x) = w(x) - v(x)$ satisfies the energy inequality*

$$\|\nabla u\|_2{}^2 + (-u\otimes\tilde{v} - v\otimes v + H, \nabla u) + \alpha\Phi(u, u) \leq 0, \tag{2.7}$$

where $\Phi(u, v) = \dfrac{1}{2\pi}\log(|x-c|)\sum_{h=1}^2\sum_{m=1}^2 \dfrac{\partial\varphi_m}{\partial x_h}\dfrac{\partial\psi_h}{\partial x_m}$ for φ, $\psi \in H^1_{0,\sigma}(\Omega)$, and the estimate $\|\nabla(u - \overline{v})\|_2 \leq R_{v,H}$, where $\overline{v}$ and $R_{v,H}$ will be defined by (5.1) and (5.2) respectively.

Note that $\alpha = 0$ is necessary in general, at least formally, so that the term $(u \otimes v, \nabla u)$ to be well-defined. Thanks to the estimate (5.11) the term $\Phi(u, u)$ is well defined for $u \in H^1_{0,\sigma}(\Omega)$. Although $v(x)$ is critically decreasing, the solution $u(x)$ does not decay as $|x| \to \infty$ in general. Hence $v(x)$ does not describe the asymptotic behavior of $w(x)$ at infinity, but it plays a crucial role in the proof of the existence.

Although the choice of the pair $\big(G(x), v(x)\big)$ is not unique, the validity of the energy inequality (2.7) is independent of the choice of the pair. In fact, we have the following proposition.

Proposition 2.6. *Suppose that Ω satisfies (2.5) and that $w(x)$ is a weak solution of (1.1)–(1.3) such that $u(x) = w(x) - v(x)$ satisfies (2.7). If $\big(v'(x), G'(x)\big)$ is another pair satisfying the conclusion of Proposition 2.3, then $u'(x) = w(x) - v'(x)$ also satisfies (2.7) with $u(x)$ and $v(x)$ replaced by $u'(x)$ and $v'(x)$ respectively.*

We next give a condition sufficient for the equality in (2.7).

Theorem 2.7. *Suppose that $w(x)$ is a weak solution of the system (1.1)–(1.3) satisfying $\nabla w \in \big(L^2(\Omega)\big)^4$ and $w \in \big(L^4(\Omega)\big)^2$. Then $u(x) = w(x) - v(x)$ enjoys (2.7) with equality, and in this case the left-hand side of (2.7) can be written as*

$$\|\nabla u\|_2{}^2 - \big((u+v)\otimes v, \nabla u\big) + (H, \nabla u). \tag{2.8}$$

Remark 2.8. If w satisfies $\nabla w \in \big(L^2(\Omega)\big)^4$ and $w \in \big(L^{2,\infty}(\Omega)\big)^2$, then we have $u \in \dot{H}^1_{0,\sigma}(\Omega) \cap L^{2,\infty}_\sigma(\Omega)$. Hence the sharp Gagliardo–Nirenberg theorem (Lemma 3.2) implies u, $w \in \big(L^4(\Omega)\big)^2$. In this sense $w(x)$ enjoys (1.4).

Next we state our results on the uniqueness of the solutions for domains which do not necessarily satisfy (C2I) or (C2E).

Theorem 2.9. *Let Ω be an exterior domain satisfying (2.5). Then there exists a positive constant C_Ω such that, for q such that $2 < q \le \infty$, there exists a positive constant $\delta_{q,\Omega}$ such that the following assertion holds. Suppose that $a(x) \in \big(H^{1/2}(\Gamma)\big)^2$ satisfying the estimate $\|a\|_{H^{1/2}(\Gamma)} < C_\Omega$, and that $w(x)$ is a weak solution of (1.1)–(1.3) with $v(x)$ above satisfying (1.4) in the sense that $u(x) = w(x) - v(x)$ satisfies $\|u(x)\lambda_q(|x|)\|_q < \delta_{q,\Omega}$ and that $w'(x)$ is another weak solution of (1.1)–(1.3) such that $u'(x) = w'(x) - v(x)$ satisfies the energy inequality (2.7). Then we have $w'(x) \equiv w(x)$. In particular, if $w'(x)$ is the weak solution given in Theorem 2.5, then we have $w'(x) = w(x)$.*

Here the smallness of a implies the smallness of $|\alpha|$. Observe that there exists no restriction on the size of $w'(x)$.

Remark 2.10. The condition

$$\|u(x)\lambda_q(|x|)\|_q < \infty \tag{SC_q}$$

with some $q \in (2, \infty]$ is the precise definition of supercritically decaying functions. If $q_0 \neq q_1$, the condition (SC_{q_0}) is neither necessary nor sufficient for (SC_{q_1}). Indeed, suppose that $2 \le q_0 < q_1 < \infty$. Then the function $u(x)$ such that $|u(x)| \sim \big(|x|\log(e+|x|)\big)^{-1}\big(\log\big(e+\log(e+|x|)\big)\big)^{-q_0}$ satisfies (SC_q) for every $q \in (q_0, \infty]$ but not (SC_{q_0}). On the other hand, if $\varphi(x) \in C_0^\infty(\Omega)$, $c \in \Omega\backslash\{0\}$ and $\varphi(c) \neq 0$, the function $u(x)$ such that $|u(x)| \sim |x-c|^{-\beta}\varphi(x)$ with some $\beta \in (0,1)$ satisfies (SC_q) for every $q \in (2, 2/\alpha)$ but not $(\mathrm{SC}_{2/\beta})$.

We next consider the results in the class of functions satisfying (C2E) under the assumption that Ω satisfies (C2I). In this case we modify the choice of I_1 as follows: If $0 \in \mathbb{R}^2\backslash\overline{\Omega}$, then we take $c = 0$. Otherwise, if $c \in \mathbb{R}^2\backslash\overline{\Omega}$. In this case we take $I_1[a] = \dfrac{\alpha(x-c)}{4\pi|x-c|^2} + \dfrac{\alpha(x+c)}{4\pi|x+c|^2} + I_3[x]$. Then $v^{(j)}(x)$ satisfies (C2E). In this case we have the following existence theorem, which states that our solution decays in the average as $|x| \to \infty$.

Theorem 2.11. *In addition to the assumption of Theorem 2.5, we assume that Ω satisfies* (C2I), *$a(x)$ satisfies* (C2E) *and $F(x)$ satisfies* (C2AE). *Then there exists a weak solution $w(x)$ of (1.1)–(1.3) satisfying the condition* (C2E) *and* (1.4) *in the sense that*

$$\int_0^{2\pi} |w(r\cos\theta, r\sin\theta)|^2\, d\theta \to 0 \text{ as } r \to \infty. \tag{2.9}$$

holds. Moreover, if $\nabla \cdot F(x)$ is of bounded support, then we have $w(r\cos\theta, r\sin\theta) \to 0$ uniformly in $\theta \in [0, 2\pi]$ as $r \to \infty$. If $|\alpha| < C_0$, Then $u(x) = w(x) - v(x)$ satisfies (2.7).

Note that the expression (2.8) makes sense in this case. For the uniqueness of solutions satisfying (C2E), we have the following theorem. Note that, as in Theorem 2.9, there is no restriction on the size of $w'(x)$.

Theorem 2.12. *There exists a positive constant C_Ω such that, for every $2 < q \le \infty$, there exists a positive constant δ'_q such that the following assertion holds. Suppose that $\|a\|_{H^{1/2}(\Omega)} < C_\Omega$ and that Ω satisfies* (C2I) *and* (2.5)*, and that $a(x) \in \left(H^{1/2}(\Gamma)\right)^2$. Suppose moreover that $w(x)$ is a weak solution of* (1.1)–(1.3) *with $v(x)$ above satisfying* (1.4) *in the sense that $u(x) = w(x) - v(x)$ satisfies* (C2E) *and $\left\||x|^{1-2/q}u(x)\right\|_{q,\infty} < \delta'_q$, and that $w'(x)$ is another weak solution of* (1.1)–(1.3) *such that $u'(x) = w'(x) - v(x)$ satisfying* (C2E) *and the energy inequality* (2.7)*. Then we have $w'(x) \equiv w(x)$. In particular, if $a(x)$ satisfies* (C2E) *and if $w'(x)$ is the weak solution given in Theorem* 2.11*, then we have $w'(x) = w(x)$.*

Remark 2.13. The condition

$$\left\||x|^{1-2/q}u(x)\right\|_{q,\infty} < \infty \tag{C_q}$$

with some $q \in (2, \infty]$ is the precise definition of critically decaying functions in the symmetric setting. If (C_q) holds with some $q_0 \in (2, \infty]$, then (C_{q_1}) holds for every $q_1 \in (2, q)$, as we see from the estimate

$$\left\||z|^{1-2/q_1}u(x)\right\|_{q_1,\infty} \le C\left\||x|^{-2(q_0-q_1)/q_0q_1}\right\|_{q_0q_1/(q_0-q_1),\infty}\left\||x|^{1-2/q_0}u(x)\right\|_{q_0,\infty}$$

by way of Lemma 3.1. Hence we see that (C_{q_0}) is strictly more restrictive than (C_{q_1}). In particular, if $|x|u(x)$ is bounded, then $u(x)$ satisfies (C_q) for every $q \in (2, \infty)$.

Remark 2.14. We cannot take $q = 2$ in Theorems 2.9 and 2.12, since the estimate (6.3) fails in this case.

3. Preparatory Lemmata

We first give the sharp Hölder estimate, which is stated by O'Neil [23] without rigorous proof. We give a simple proof in Appendix A.

Lemma 3.1. *Suppose that p, q, $r \in (1, \infty)$ satisfy $1/r = 1/p + 1/q$, and that α, β, $\gamma \in [1, \infty]$ satisfy $1/\gamma = 1/\alpha + 1/\beta$. Then there exists a positive constant C such that, if $f(x) \in L^{p,\alpha}$ and $g(x) \in L^{q,\beta}$, then $f(x)g(x) \in L^{r,\gamma}$, and we have the estimate $\|fg\|_{r,\gamma} \le C\|f\|_{p,\alpha}\|g\|_{q,\beta}$.*

We also have the generalized Gagliardo–Nirenberg inequality.

Lemma 3.2. *Let U be a domain in $\mathbb{R}^2$ with C^2 boundary, and suppose that p, $q \in (1, \infty)$ and that $\max\{p, q\} < r \le \infty$. We also assume that $2/q - 1 < 2/r$. Then there exists a positive constant C such that, for every $u \in L^{p,\infty}(U)$ satisfying $\nabla u \in \left(L^{q,\infty}(U)\right)^2$, we have $u \in L^{r,1}(U)$ in the case $r < \infty$, and $u \in L^\infty(U)$ in the case $r = \infty$. Furthermore, if we put $\theta = \dfrac{2/p - 2/r}{1 + 2/p - 2/q}$, we have the estimate*

$$\|u\|_{r,1} \le C\|u\|_{p,\infty}{}^{1-\theta}\|\nabla u\|_{q,\infty}{}^{\theta} \qquad \textit{for } r < \infty,$$
$$\|u\|_{\infty} \le C\|u\|_{p,\infty}{}^{1-\theta}\|\nabla u\|_{q,\infty}{}^{\theta} \qquad \textit{for } r = \infty.$$

Proof. First, choose s such that $\max\{p, q\} < s < r$. There exists a function $\tilde{u}$ on $\mathbb{R}^2$ such that $\tilde{u}|_U = u$ and that the inequalities $\|\tilde{u}\|_{p,\infty} \le C\|u\|_{p,\infty}$ and $\|\nabla\tilde{u}\|_{q,\infty} \le C\|\nabla u\|_{q,\infty}$ hold with a positive constant C independent of u. Hence, by the Sobolev embedding theorem, we have $\tilde{u} \in \dot{B}^{2/s-2/p}_{s,\infty}$ with the estimate

$$\|\tilde{u}\|_{\dot{B}^{2/s-2/p}_{s,\infty}} \le C\|\tilde{u}\|_{p,\infty} \le C\|u\|_{p,\infty} \tag{3.1}$$

with a positive constant C independent of u, and $\tilde{u} \in \dot{B}^{1+2/s-2/q}_{s,\infty}$ with the estimate

$$\|\tilde{u}\|_{\dot{B}^{1+2/s-2/q}_{s,\infty}} \le C\|\nabla\tilde{u}\|_{q,\infty} \le C\|\nabla u\|_{q,\infty}. \tag{3.2}$$

In view of the equality

$$\theta\left\{\left(1 + \frac{2}{s} - \frac{2}{q}\right) - \left(\frac{2}{s} - \frac{2}{p}\right)\right\} + \frac{2}{s} - \frac{2}{p} = \frac{2}{p} - \frac{2}{r} + \frac{2}{s} - \frac{2}{p} = \frac{2}{s} - \frac{2}{r},$$

we can apply real interpolation to obtain

$$\|\tilde{u}\|_{\dot{B}^{2/s-2/r}_{s,1}} \le C\|u\|_{p,\infty}{}^{1-\theta}\|\nabla u\|_{q,\infty}{}^{\theta} \tag{3.3}$$

from (3.1) and (3.2).

If $r=\infty$, we obtain the conclusion from the estimate above and the inclusion relation $\dot{B}^{2/s}_{s,1} \subset L^\infty(\mathbb{R}^2)$.

If $r<\infty$, put $\delta = \min\{1/r, 1/s-1/r\}$. Then we have

$$\dot{B}^{2/s-2/r}_{s,1} = \left(\dot{B}^{2/s-2/r-\delta}_{s,1}, \dot{B}^{2/s-2/r+\delta}_{s,1}\right)_{1/2,1}. \tag{3.4}$$

From the choice of δ, we have $2/s-2/r-\delta>0$ and $2/s-2/r+\delta<2/s$. Hence, putting $r_0 = 2r/(2+\delta r)$ and $r_1 = 2r/(2-\delta r)$, we have

$$s < \frac{2rs}{r+s} \le r_0 < r < r_1 \le 2r < \infty,$$

and the Sobolev embedding theorem implies

$$\dot{B}^{2/s-2/r-\delta}_{s,1} \subset \dot{H}^{2/s-2/r-\delta}_{s} \subset L^{r_0}(\mathbb{R}^2), \quad \dot{B}^{2/s-2/r+\delta}_{s,1} \subset \dot{H}^{2/s-2/r+\delta}_{s} \subset L^{r_1}(\mathbb{R}^2).$$

It follows from these facts and (3.4) that

$$\dot{B}^{2/s-2/r}_{s,1} \subset \left(L^{r_0}(\mathbb{R}^2), L^{r_1}(\mathbb{R}^2)\right)_{1/2,1} = L^{r,1}(\mathbb{R}^2).$$

The conclusion follows from this inclusion relation and (3.3). □

We next recall the Bogovskii lemma, which is shown by Bogovskii [5].

Lemma 3.3. *Suppose that D is a bounded domain with $C^{2+\gamma}$ boundary for some $\gamma>0$. Then there exists a constant C such that, if $f(x)\in L^2(D)$ satisfies the condition $\int_D f(x)\,dx = 0$, then there exists a vector-valued function $u(x)\in \left(H^1_0(D)\right)^2$ satisfying the estimate $\|\nabla u\|_2 \le C\|f\|_2$ and the equality $\nabla\cdot u(x) = f(x)$ holds on D. Moreover, the positive constant C is invariant under conformal transformations of D. Furthermore, if $f(x)\in H^1(D)$, then we have $u(x) \in \left(H^1_0(D)\cap H^2(D)\right)^2$, and the estimate $\left\|\nabla^2 u\right\|_2 \le C\|\nabla f\|_2$ holds.*

From this lemma we have the following decomposition. Put $U = \{x\in\mathbb{R}^2 \mid |x| > 2^{J+2}/3\}$. Then we have the following lemma.

Lemma 3.4. *There exists a positive constant C such that the following assertion holds.*

- (i) *For every $u(x)\in \dot{H}^1_{0,\sigma}(\Omega)$, we have $u_1(x)\in H^1_{0,\sigma}(\tilde{\Omega})$ and $u_2(x)\in \dot{H}^1_{0,\sigma}(U)$ such that $u(x) = u_1(x)+u_2(x)$ on $\left\{x\in\mathbb{R}^2 \,\middle|\, 2^{J-1}3 < |x| < 2^{J+1}\right\}$, and we have the estimates $\|\nabla u_j\|_2 \le C\|\nabla u\|_2$ for $j=1,2$ and $\|u_1\|_2 \le C\|\nabla u\|_2$.*
- (ii) *For every $q\in[2,\infty)$ there exists a positive constant C_q such that, if $u(x)\in L^q_\sigma(\Omega)$ as well as $u(x)\in H^1_{0,\sigma}(\Omega)$, then the function $u_2(x)$ in Assertion* (i) *satisfies $u_2(x)\in L^q_\sigma(\Omega)$ with the estimate $\|u_2\|_q \le \|u\|_q + C_q\|\nabla u\|_2$.*
- (iii) *For $q\in(2,\infty)$ and $r\in[1,\infty]$, there exists a positive constant $C_{q,r}$ such that, if $u(x)\in L^{q,r}_\sigma(\Omega)$ as well as $u(x)\in H^1_{0,\sigma}(\Omega)$, then the function $u_2(x)$ in Assertion* (i) *satisfies $u_2(x)\in L^{q,r}_\sigma(\Omega)$ with the estimate $\|u_2\|_{q,r} \le \|u\|_{q,r} + C_{q,r}\|\nabla u\|_2$.*

Proof. We first prove Assertion (i). Let $\chi(t)$ be a monotone-decreasing C^∞-function on $\mathbb{R}$ such that $\chi(t)\equiv 1$ for $t\le 5/3$ and that $\chi(t)\equiv 0$ for $t\ge 11/6$. Then we have

$$\nabla\cdot\left(\chi\left(2^{-J}|x|\right)u(x)\right) = 2^{-J}\nabla\chi\left(2^{-J}|x|\right)\cdot u(x) = \chi'\left(2^{-J}|x|\right)2^{-J}\frac{x}{|x|}\cdot u(x).$$

In particular, $\nabla \cdot \left(\chi\left(2^{-J}|x|\right) u(x)\right) \neq 0$ implies $2^{J-1}3 < |x| < 2^{J+1}$. Since $\chi\left(2^{-J}|x|\right) \equiv 0$ on $|x| = 2^{J+1}$ and $\chi\left(2^{-J}|x|\right) \equiv 1$ on $|x| = 2^{J-1}3$, we have

$$\begin{aligned}\int_{2^{J-1}3\leq|x|\leq 2^{J+1}} \nabla \cdot \left(\chi\left(2^{-J}|x|\right) u(x)\right) \, dx &= \oint_{|x|=2^{J-1}3} \big(-n(x)\big) \cdot u(x)\, ds(x) \\ &= -\int_{x\in\Omega, |x|\leq 2^{J-1}3} \big(-\nabla \cdot u(x)\big)\, dx + \oint_{\Gamma} n(x) \cdot u(x)\, ds(x) = 0\end{aligned}$$

in view of the fact $u(x)|_\Gamma \equiv 0$. It follows that there exists a function

$$v(x) \in \left(H_0^1\left(\left\{x \in \mathbb{R}^2 \;\middle|\; 2^{J-1}3 < |x| < 2^{J+1}\right\}\right)\right)^2$$

such that $\nabla \cdot v(x) = \nabla \cdot \left(\chi\left(2^{-J}|x|\right) u(x)\right)$. Now put

$$u_1(x) = \chi\left(2^{-J}|x|\right) u(x) - v(x) \text{ and } u_2(x) = \left(1 - \chi\left(2^{-J}|x|\right)\right) u(x) + v(x).$$

Then we have $u_1(x) \in H^1_{0,\sigma}(\tilde{\Omega})$, $u_2(x) \in H^1_{0,\sigma}(U)$ and $u_1(x) + u_2(x) \equiv u(x)$. We also have

$$\|\nabla v\|_2 \leq C \left\| \chi'\left(2^{-J}|x|\right) 2^{-J} \frac{x}{|x|} \cdot u \right\|_2 \leq C\|u\|_{L^2(\tilde{\Omega})}$$

and

$$\left\|\nabla\left(\chi\left(2^{-J}|x|\right) u(x)\right)\right\|_2 \leq \sup_{t\in\mathbb{R}} |\chi'(t)|\|u\|_{L^2(\tilde{\Omega})} + \|\nabla u\|_2.$$

From these estimates we have

$$\|\nabla u_1\|_2 \leq \|\nabla u\|_2 + C\|u\|_{L^2(\tilde{\Omega})}. \tag{3.5}$$

Since $u|_\Gamma = 0$, we can apply the Poincaré inequality to $u(x)$ on $\tilde{\Omega}$ to conclude that there exists a positive integer C such that the estimate $\|u\|_{L^2(\tilde{\Omega})} \leq C\|\nabla u\|_2$ holds. Substituting this estimate into (3.5) we obtain $\|\nabla u_1\|_2 \leq C\|\nabla u\|_2$. This inequality yields $\|\nabla u_2\|_2 \leq C\|\nabla u\|_2$ since $u_2(x) = u(x) - u_1(x)$.

We turn to the proof of Assertion (ii). Applying the Poincaré inequality and the Sobolev embedding theorem to $u_1(x)$, we obtain $\|u_1\|_q \leq C\|\nabla u\|_2$ with a positive constant C depending on q. If $u(x) \in L^q_\sigma(\Omega)$ as well, it follows that $\|u_2\|_q \leq \|u\|_q + \|u_1\|_q \leq \|u\|_q + C\|\nabla u\|_2$.

Assertion (iii) follows from Assertion (ii) and real interpolation. □

At the end of this section we give a refined version of Hardy's inequality, whose proof is given in Appendix B.

Proposition 3.5. *Suppose that $2 \leq \rho < \infty$. Then, for every $\varepsilon_0 > 0$, there exists a positive constant C_{ρ,ε_0} such that, if Ω is an exterior domain and if $u(x) \in \dot{H}^1_0(\Omega)$, then we have $u(x)/\lambda_{2\rho/(\rho-2)}(|x|) \in L^\rho(\Omega)$ with the estimate $\left\|u(x)/\lambda_{2\rho/(\rho-2)}(|x|)\right\|_\rho \leq C_{\rho,\varepsilon_0}\|\nabla u\|_2$.*

4. Construction of Corrector Functions

Let $U_1, \ldots, U_L$ denote the connected components of $\mathbb{R}^2\backslash\overline{\Omega}$. Since Ω is connected, every U_ℓ is simply connected and arcwise connected. Let Γ_ℓ be the boundary of U_ℓ for $\ell = 1, \ldots, L$. Next, for $\ell = 1, \ldots, L$, we put

$$\alpha_\ell = \oint_{\Gamma_\ell} a(x) \cdot n(x)\, dx.$$

Then we have $\alpha = \alpha_1 + \cdots + \alpha_L$. Then Proposition 2.3 follows from the following one.

Proposition 4.1. *Choose $c_\ell \in U_\ell$ for every $\ell = 1, \ldots, L$. Then we introduce a linear operator $I_1[a]$ on $\left(H^{1/2}(\Gamma)\right)^2$ as follows: Put $v_{0,\ell}(x) = \dfrac{\alpha_\ell(x - c_\ell)}{2\pi|x - c_\ell|^2}$ for every $\ell = 1, \ldots, L$, and put $I_1[a] = v^{(1)}(x) = \sum_{\ell=1}^{L} v_{0,\ell}(x)$. Then we have the following assertions.*

- (i) *The mapping I_1 is bounded from $\left(H^{1/2}(\Gamma)\right)^2$ to $\left(\dot{H}^1(\Omega)\right)^2$, and the function $v_1(x)$ satisfies the equality $\Delta v_1(x) = 0$ on Ω and the estimate $\left\|\nabla v^{(1)}(z)\right\|_2 \leq C \sum_{\ell=1}^{L} |\alpha_\ell|$. Furthermore, for every $r \in (2, \infty]$ there exists a positive constant C_r such that $v^{(1)} \in \left(L^r(\Omega)\right)^2$ and that $\left\|v^{(1)}(z)\right\|_r \leq C_r \sum_{\ell=1}^{L} |\alpha_\ell|$.*
- (ii) *There exists another bounded mapping I_2 from $\left(H^{1/2}(\Gamma)\right)^2$ to $\left(H^1(\Omega)\right)^2$, and $v^{(2)}(x) = I_2[a]$ satisfies the equalities $\nabla \cdot v^{(2)}(x) = 0$ in $\tilde{\Omega}$, $v^{(2)}(x) = 0$ on $\{x \in \Omega \mid |x| \geq 2^{J+1}\}$ and $v^{(2)}(x) = a(x) - v^{(1)}(x)$ on Γ. Moreover, the function $v(x) = v^{(1)}(x) + v^{(2)}(x)$ satisfies (2.2)–(2.4) with $G(x) \in \left(L^2(D)\right)^4$ such that $\|G\|_2 \leq C\|a\|_{H^{1/2}(\Gamma)}$.*
- (iii) *For every $c \in \mathbb{R}^2 \backslash \overline{\Omega}$, the mapping $I_3[a] = \dfrac{\alpha}{2\pi} \dfrac{x - c}{|x - c|^2} - I_1[a]$ satisfies $\lambda_q(x) I_3[a](x) \in L^2(\Omega) \cap L^\infty(\Omega)$ for every $q \in (2, \infty)$.*

Proof. It follows from the constructions that $\nabla \cdot v_{0,\ell}(x) = 0$ and that $\nabla \times v_{0,\ell}(x) = 0$ outside U_ℓ. Moreover, we have $\oint_{\Gamma_\ell} n(x) \cdot v_{0,m}(x)\, ds(x) = \delta_{\ell m} \alpha_\ell$. This implies that $v^{(1)}(x) = \sum_{\ell=1}^{L} v_{0,\ell}(x)$ satisfies $\nabla \cdot v^{(1)}(x) = \nabla \times v^{(1)}(x) = 0$, and hence $\Delta v^{(1)}(x) = 0$, in Ω. Furthermore, $\nabla v^{(1)}(x) \in \left(L^2(\Omega)\right)^4$ holds, and for every $r \in (2, \infty]$ there exists a positive constant C such that $v^{(1)}(x)$ satisfies the inequality $\left\|\nabla v^{(1)}\right\|_{L^2(\Omega)} + \left\|v^{(1)}\right\|_{L^r(\Omega)} \leq C \sum_{\ell=1}^{L} |\alpha_\ell|$. This completes the proof of Assertion (i).

We next prove Assertion (ii). Put $b(x) = a(x) - v^{(1)}(x)|_\Gamma$. Then we have $b \in H^{1/2}(\Gamma)$ satisfying with the estimate $\|b\|_{H^{1/2}(\Gamma)} \leq C\|a\|_{H^{1/2}(\Gamma)}$ with some positive constant C. Moreover, by construction, we see that $b(x)$ satisfies $\oint_{\Gamma_\ell} n(x) \cdot b(x)\, ds(x) = 0$ for every $\ell = 1, \ldots, L$. It follows that the system

$$\begin{aligned}
-\Delta w(x) + \nabla \pi(x) &= 0 && \text{in } \tilde{\Omega},\\
\nabla \cdot w(x) &= 0 && \text{in } \tilde{\Omega},\\
w(x) &= 0 && \text{on } |x| = 2^{J+1},\\
w(x) &= b(x) && \text{in } x \in \Gamma
\end{aligned}$$

admits a solution $\left(w(x), \pi(x)\right) \in H^1_\sigma(\tilde{\Omega}) \times L^2(\tilde{\Omega})$, which is unique up to constant in $\pi(x)$, and there exists a positive constant C such that we have the estimate $\|\nabla w\|_2 \leq C\|b\|_{H^{1/2}(\Gamma)} \leq C\|a\|_{H^{1/2}(\Gamma)}$. We now put $\tilde{w}(x) = \chi\left(2^{-J}|x|\right) w(x)$. Then we have

$$\begin{aligned}
\nabla \cdot \tilde{w}(x) &= g(x) && \text{in } \tilde{\Omega},\\
\tilde{w}(x) &= 0 && \text{on } |x| = 2^{J+1},\\
\tilde{w}(x) &= b(x) && \text{on } \Gamma,
\end{aligned}$$

where $g(x) = \dfrac{2^{-J}\chi'\left(2^{-J}|x|\right) x \cdot w(x)}{|x|} \in H^1_0(D)$ satisfies the estimate

$$\|\nabla \tilde{w}\|_2 + \|\nabla g\|_2 \leq C\|\nabla w\|_2 \leq C\|a\|_{H^{1/2}(\Gamma)}.$$

Since $\tilde{w}(x) = 0$ on $\{x \mid |x| = 2^{J+1}\}$ and $\nabla w(x) = 0$ on $\{x \in \Omega \mid |x| \le 2^J\}$, we have

$$\begin{aligned}\int_D g(x)\,dx &= \int_{|x|=2^{J+1}} n(x)\cdot\tilde{w}(x)\,ds(x) - \int_{|x|=2^J} n(x)\cdot\tilde{w}(x)\,ds(x)\\ &= -\int_{|x|=2^J} n(x)\cdot w(x)\,ds(x) + \int_\Gamma n(x)\cdot b(x)\,ds(x)\\ &= -\int_{\{x\in\Omega \mid |x|\le 2^J\}} \nabla\cdot w(x)\,dx = 0.\end{aligned}$$

Hence Lemma 3.3 implies that there exists $\varphi \in \left(H_0^1(D)\cap H^2(D)\right)^2$ satisfying the equality $\nabla\cdot\varphi(x) = g(x)$ and the estimate $\left\|\nabla^2\varphi\right\|_2 \le C\|\nabla g\|_2 \le C\|a\|_{H^{1/2}(\Gamma)}$. Moreover, since $\varphi(x) = 0$ holds on $|x| = 2^J$ and $|x| = 2^{J+1}$, we have $\nabla\times\varphi(x) = 0$ on $|x| = 2^J$ and $|x| = 2^{J+1}$. This implies that $\nabla\times\varphi(x) \in H_0^1(D)$ with the estimate $\|\nabla(\nabla\times\varphi)\|_2 \le C\|a\|_{H^{1/2}(\Gamma)}$. Putting $v^{(2)}(x) = \tilde{w}(x) - \varphi(x)$, we see that $v^{(2)}(x)$ satisfies the estimate

$$\left\|\nabla v^{(2)}\right\|_2 + \left\|v^{(2)}\right\|_r \le \|\nabla\tilde{w}\|_2 + \|\tilde{w}\|_r + \|\nabla\varphi\|_2 + \|\varphi\|_r \le C\|a\|_{H^{1/2}(\Gamma)}$$

and the system

$$\begin{aligned} -\Delta v^{(2)}(x) &= -\nabla^\perp\big(\nabla\times\tilde{w}(x) - \nabla\times\varphi(x)\big) - \nabla\big(\nabla\cdot v^{(2)}(x)\big)\\ &= \nabla^\perp\big(\nabla\times\varphi(x) - h(x)\big) && \text{in } \Omega,\\ \nabla\cdot v^{(2)}(x) &= 0 && \text{in } \Omega,\\ v^{(2)}(x) &= 0 && \text{on } |x| = 2^{J+1},\\ v^{(2)}(x) &= b(x) && \text{on } \Gamma,\end{aligned}$$

where $h(x) \in L^2(D)$ with the estimate $\|h\|_2 \le C\|\nabla w\|_2 \le C\|a\|_{H^{1/2}(\Gamma)}$. Hence we have the equality $-\Delta v^{(2)}(x) = \nabla\cdot G(x)$ with a matrix

$$G(x) = \begin{pmatrix} 0 & \nabla\times\varphi(x) - h(x)\\ h(x) - \nabla\times\varphi(x) & 0\end{pmatrix} \in \left(L^2(D)\right)^4$$

with the estimate $\|G\|_2 \le C\|a\|_{H^{1/2}(\Gamma)}$, and we see that $v(x) = v^{(1)}(x) + v^{(2)}(x)$ satisfies the system (2.2)–(2.4) with the estimate $\|\nabla v\|_2 + \|G\|_2 \le C\|a\|_{H^{1/2}(\Gamma)}$. This completes the proof of Assertion (ii).

Further, we have the expression $I_3[a] = \sum_{\ell=1}^L \frac{\alpha_\ell}{2\pi}\left(\frac{x-c}{|x-c|^2} - \frac{x-c_\ell}{|x-c_\ell|^2}\right)$. It follows that there exists a positive constant C such that $|I_3[a]| \le \dfrac{C}{|x-c|^2}$ on Ω, which implies Assertion (iii). □

5. Construction of Weak Solutions

We now prove Theorems 2.5 and 2.11. Put $\Omega_j = \left\{x \in \Omega \,\middle|\, |x| < 2^j\right\}$ for every $j \ge J+1$, and let $v^{(\ell)} = I_\ell[a]$ for $\ell = 1, 2$, where the operators I_ℓ are defined in Proposition 4.1. Next we define $\overline{v}_j \in H^1_{0,\sigma}(\Omega_j)$ and $R_{v,H}$ as follows:

Let $\overline{v}$ and $\overline{v}_j$ denote the image of $v^{(2)}$ by the orthogonal projection with respect to the norm$\|\nabla u\|_2$ onto the closed subspace $H^1_{0,\sigma}(\Omega)$ and onto $H^1_{0,\sigma}(\Omega_j)$ respectively. (5.1)

$$R_{v,H} = 2\|v\|_4\sqrt{\|\nabla v\|_2} + 2\sqrt{\|H\|_2} + 3\left\|\nabla v^{(2)}\right\|_2 + 2\left\|\nabla v^{(1)}\right\|_2. \tag{5.2}$$

Then we have the following proposition.

Proposition 5.1. *Suppose that $H(x) \in \big(L^2(\Omega)\big)^4$ and that $a(x) \in \big(H^{1/2}(\Gamma)\big)^2$. Then, for every $j \geq J+1$, there exists a function $u^{(j)} \in H_{0,\sigma}(\Omega_j)$ satisfying the estimate*

$$\left\|\nabla\left(u^{(j)}+\overline{v}_j\right)\right\|_2 \leq R_{v,H} \tag{5.3}$$

such that, for every $\varphi \in C^\infty_{0,\sigma}(\Omega_j)$, we have the identity

$$\left(\nabla u^{(j)}, \nabla\varphi\right) = \left(\left(u^{(j)}+v\right) \otimes \left(u^{(j)}+v\right) - H, \nabla\varphi\right). \tag{5.4}$$

Furthermore, if Ω satisfies (C2I), *$H(x)$ satisfies* (C2AE) *and $a(x)$, $v^{(1)}(x)$ and $v^{(2)}(z)$ satisfies* (C2E), *then we can take $u^{(j)}$ satisfying* (C2E).

We now prove the above proposition. Since $u \in H^1_{0,\sigma}(\Omega)$ with compact support, we obtain

$$\begin{aligned}\big((u+v)\otimes u, \nabla u\big) &= 0,\\ \big((u+v)\otimes u, \nabla v\big) + \big((u+v)\otimes v, \nabla u\big) &= 0,\\ \big(u \otimes v, \nabla v\big) &= 0\end{aligned}$$

by integrating by parts. Hence

$$\big|\big((u+v)\otimes(u+v), \nabla(u+v)\big)\big| = \big|(v\otimes v, \nabla v)\big| \leq \|v\|_4{}^2\|\nabla v\|_2.$$

It follows that

$$\begin{aligned}\big|\big((u+v)\otimes(u+v) - H, \nabla(u+v)\big)\big| &\leq \|v\|_4{}^2\|\nabla v\|_2 + \|H\|_2\|\nabla(u+v)\|_2\\ &\leq \|v\|_4{}^2\|\nabla v\|_2 + \|H\|_2{}^2 + \frac{\|\nabla(u+v)\|_2{}^2}{4}.\end{aligned} \tag{5.5}$$

We next observe that, for every $j \geq J+1$ and every $G \in \big(L^2(\Omega_j)\big)^4$, the functional on $\dot{H}^1_\sigma(\Omega_j)$ defined by $\varphi \mapsto (G, \nabla\varphi)$ is bounded. Hence we can define a bounded linear operator S_j from $\big(L^2(\Omega_j)\big)^4$ to $H^1_{0,\sigma}(\Omega_j)$ defined by the equality $(\nabla S_j[G], \nabla\varphi) = (G, \nabla\varphi)$ for every $\varphi \in H^1_{0,\sigma}(\Omega_j)$. We now introduce the mapping T_j from the space $X_j = L^4_\sigma(\Omega_j)$ to $H^1_{0,\sigma}(\Omega_j)$ by the equality $T_j[u] = S_j\big[(u+v)\otimes(u+v) - H\big]$. Then T_j is a continuous mapping into the space $Y_j = X_j \cap H^1_{0,\sigma}(\Omega_j)$. Since the inclusion $Y_j \to X_j$ is a compact operator, the operator T_j restricted on Y_j is a compact mapping into itself.

Then we have the following lemma.

Lemma 5.2. *There exists at least one fixed point of the mapping T_j in the set $D_{j,R_{v,H}} = \big\{u \in Y_j \;\big|\; \|\nabla(u+\overline{v}_j)\|_2 \leq R_{v,H}\big\}$.*

Proof. Consider the mapping U defined by

$$U[u] = \begin{cases} T_j[u] & \text{if } \big\|\nabla\big(T_j[u]+\overline{v}_j\big)\big\|_2 \leq R_{v,H},\\ \dfrac{R_{v,H}\big(T_j[u]+\overline{v}_j\big)}{\big\|\nabla\big(T_j[u]+\overline{v}_j\big)\big\|_2} - \overline{v}_j & \text{if } \big\|\nabla\big(T_j[u]+\overline{v}_j\big)\big\|_2 > R_{v,H}.\end{cases}$$

Since $D_{j,R_{v,H}}$ is a convex closed set and U is a compact mapping from Y_j into $D_{j,R_{v,H}}$, Schauder's theorem implies that there exists at least one fixed point of U in $D_{j,R_{vH}}$. If a fixed point $u \in D_{j,R_{v,H}}$ of U satisfies the inequality $\big\|\nabla\big(T_j[u]+\overline{v}_j\big)\big\|_2 > R_{v,H}$, then we have

$$\big\|\nabla\big(u+\overline{v}_j\big)\big\|_2 = \big\|\nabla\big(U[u]+\overline{v}_j\big)\big\|_2 = R_{v,H}. \tag{5.6}$$

On the other hand, we have $\overline{v}_j \in H^1_{0,\sigma}(\Omega)$. Hence, for every $\varphi \in H^1_{0,\sigma}(\Omega_j)$, we obtain $\big(\nabla\big(\varphi+\overline{v}_j\big), \nabla v^{(1)}\big) = \big(\varphi+\overline{v}_j, -\Delta v^{(1)}\big) = 0$ by integrating by parts, and $\big(\nabla\big(\varphi+\overline{v}_j\big), \nabla\left(v^{(2)}-\overline{v}_j\right)\big) = 0$. It follows that

$$\big(\nabla\big(\varphi+\overline{v}_j\big), \nabla(u+v)\big) = \big(\nabla\big(\varphi+\overline{v}_j\big), \nabla\big(u+\overline{v}_j\big)\big) \tag{5.7}$$

and

$$\|\nabla(\varphi+v)\|_2{}^2 = \left\|\nabla\big(\varphi+\overline{v}_j\big)\right\|_2{}^2 + \left\|\nabla\left(v^{(2)}-\overline{v}_j+v^{(1)}\right)\right\|_2{}^2$$

for every $\varphi \in H^1_{0,\sigma}(\Omega_j)$. Substituting (5.5) and applying (5.7) once again, we obtain

$$\begin{aligned}
\|\nabla(u+v)\|_2{}^2 &= \big(\nabla\big(U[u]+\overline{v}_j\big), \nabla\big(u+\overline{v}_j\big)\big) + \left\|\nabla\left(v^{(2)}-\overline{v}_j+v^{(1)}\right)\right\|_2{}^2 \\
&\le \frac{R_{v,H}}{\left\|\nabla\big(T_j[u]+\overline{v}_j\big)\right\|_2}\big(\nabla\big(T_j[u]+\overline{v}_j\big), \nabla(u+v)\big) \\
&\qquad + 2\left\|\nabla\left(v^{(2)}-\overline{v}_j\right)\right\|_2{}^2 + 2\left\|\nabla v^{(1)}\right\|_2{}^2 \\
&= \frac{R_{v,H}}{\left\|\nabla\big(T_j[u]+\overline{v}\big)\right\|_2}\big(\nabla S_j[(u+v)\otimes(u+v)-H], \nabla(u+v)\big) \\
&\qquad + \big(\nabla\overline{v}_j, \nabla(u+v)\big) + 2\left\|\nabla v^{(2)}\right\|_2{}^2 + 2\left\|\nabla v^{(1)}\right\|_2{}^2 \\
&= \frac{R_{v,H}}{\left\|\nabla\big(T_j[u]+\overline{v}\big)\right\|_2}\left(\|v\|_4{}^2\|\nabla v\|_2 + \|H\|_2{}^2 + \frac{\|\nabla(u+v)\|_2{}^2}{4}\right) \\
&\qquad + \frac{\left\|\nabla v^{(2)}\right\|_2{}^2}{2} + \frac{\|\nabla(u+v)\|_2{}^2}{2} + 2\left\|\nabla v^{(2)}\right\|_2{}^2 + 2\left\|\nabla v^{(1)}\right\|_2{}^2.
\end{aligned}$$

If $\|\nabla(u+v)\|_2 = 0$, then $\left\|\nabla\big(u+\overline{v}_j\big)\right\|_2 = 0 < R_{v,H}$. Otherwise, the inequality $R_{v,H} < \left\|\nabla\big(T_j[u]+\overline{v}_j\big)\right\|_2$ implies

$$\begin{aligned}
\|\nabla(u+v)\|_2{}^2 &< \|v\|_4{}^2\|\nabla v\|_2 + \|H\|_2{}^2 + \frac{\|\nabla(u+v)\|_2{}^2}{4} + \frac{\left\|\nabla v^{(2)}\right\|_2{}^2}{2} \\
&\qquad + \frac{\|\nabla(u+v)\|_2{}^2}{2} + 2\left\|\nabla v^{(2)}\right\|_2{}^2 + 2\left\|\nabla v^{(1)}\right\|_2{}^2 \\
&= \frac{3\|\nabla(u+v)\|_2{}^2}{4} + \|v\|_4{}^2\|\nabla v\|_2 + \|H\|_2 + \frac{5\left\|\nabla v^2\right\|_2{}^2}{2} + 2\left\|\nabla v^{(1)}\right\|_2{}^2.
\end{aligned}$$

This implies

$$\begin{aligned}
\left\|\nabla\big(u+\overline{v}_j\big)\right\|_2{}^2 &\le \|\nabla(u+v)\|_2{}^2 \\
&< 4\|v\|_4{}^2\|\nabla v\|_2 + 4\|H\|_2 + 5\left\|\nabla v^{(2)}\right\|_2{}^2 + 4\left\|\nabla v^{(1)}\right\|_2{}^2 \le R_{v,H}^2.
\end{aligned}$$

Hence we have $\left\|\nabla\big(u+\overline{v}_j\big)\right\|_2 < R_{v,H}$ in either case. This contradicts (5.6). It follows that every fixed point u of U satisfies $\|\nabla\left(T_j[u]+\overline{v}_j\right)\|_2 \le R_{v,H}$, in which case $T_j[u] = U[u] = u$ and hence $\|\nabla\left(u+\overline{v}_j\right)\|_2 \le R_{v,H}$. □

We return to the proof of Proposition 5.1. Let $u^{(j)}$ denote a fixed point of T given by Lemma 5.2. Then Lemma 5.2 implies (5.3). Next, for every $\varphi \in H^1_{0,\sigma}(\Omega_j)$, we have

$$\begin{aligned}
\left(\nabla u^{(j)}, \nabla\varphi\right) &= \left(\nabla T_j\left[u^{(j)}\right], \nabla\varphi\right) \\
&= \left(\nabla S_j\left[\left(u^{(j)}+v\right)\otimes\left(u^{(j)}+v\right)-H\right], \nabla\varphi\right) \\
&= \left(\left(u^{(j)}+v\right)\otimes\left(u^{(j)}+v\right)-H, \nabla\varphi\right).
\end{aligned}$$

This implies (5.4). □

Proof of Theorem 2.5. We first observe that $\overline{v}_j$ converges strongly to $\overline{v}$ in $\dot{H}^1_{0,\sigma}(\Omega)$. Next, since $\|\nabla u^{(j)}\|_2 \leq R_{v.H} + \|\nabla \overline{v}_j\|_2 \leq R_{v.H} + \|\nabla v\|_2$. there exists a subsequence $\left\{u^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$ converging weakly in $\dot{H}^1_{0,\sigma}(\Omega)$ such that $\left\|\nabla u^{\left(j(k)\right)}\right\|_2$ converges. Let u denote the weak limit of $\left\{u^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$. Then we have $\|\nabla(u+\overline{v})\|_2 \leq \lim_{k\to\infty}\left\|\nabla\left(u^{\left(j(k)\right)} - \overline{v}_{j(k)}\right)\right\|_2 \leq R_{v,H}$. Hence $u(x)$ satisfies (5.3). We also see that the sequence $\left\{u^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$ converges strongly to $u(x)$ in $\left(L^4(\Omega_j)\right)^2$. Indeed, putting $\tilde{u}^{\left(j(k)\right)}(x) = \chi(2^{-j}|x|)u^{\left(j(k)\right)}(x)$, we have $\tilde{u}^{\left(j(k)\right)}(x) \in \left(H^1_0(\Omega_{j+1})\right)^2$ and $\tilde{u}^{\left(j(k)\right)}(x) \equiv u^{\left(j(k)\right)}(x)$ on Ω_j. Then we have

$$\begin{aligned}\left\|\nabla\tilde{u}^{\left(j(k)\right)}\right\|_2 &\leq \left\|\chi\left(2^{-j}|\cdot|\right)\nabla u^{\left(j(k)\right)}\right\|_2 + 2^{-j}\left\|\chi'\left(2^{-j}|x|\right)\frac{x}{|x|}u^{\left(j(k)\right)}(x)\right\|_2\\ &\leq \left\|\nabla u^{\left(j(k)\right)}\right\|_2 + C2^{-j}\left\|u^{\left(j(k)\right)}\right\|_{L^2(\{x\,|\,|x|\leq 2^{j+1}\})}\end{aligned} \tag{5.8}$$

with a constant C independent of k, where $\tilde{u}^{\left(j(k)\right)}(x)$ is identified with its zero extension to $\mathbb{R}^2\backslash\Omega$. Since $\tilde{u}^{\left(j(k)\right)}(0) = 0$ outside Ω_{j+1} , Poincaré inequality yields

$$\left\|\tilde{u}^{\left(j(k)\right)}\right\|_{L^2(\{x\,|\,|x|\leq 2^{j+1}\})} \leq C2^j\left\|\nabla\tilde{u}^{\left(j(k)\right)}\right\|_{L^2(\{x\,|\,|x|\leq 2^{j+1}\})} \tag{5.9}$$

with a constant C independent of k. Substituting this estimate into (5.8), we see that $\left\{\nabla\tilde{u}^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$ is bounded in $\left(L^2(\Omega_{j+1})\right)^4$. Then the sequence $\left\{\tilde{u}^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$ converges weakly in $\left(H^1_0(\Omega_{j+1})\right)^2$, and hence strongly in $\left(L^4(\Omega_{j+1})\right)^2$, to $\chi\left(2^{-j}|x|\right)u(x)$. It follows that $\left\{u^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$ converges to u strongly in $\left(L^4(\Omega_j)\right)^2$.

We now show that this $u(x)$ is a weak solution. Suppose that $\varphi(x) \in C^\infty_{0,\sigma}(\Omega)$. Then there exists an integer $j \geq J+1$ such that $\operatorname{supp}\varphi \subset \Omega_j$. Since $v(x) \in L^4(\Omega)$ and $\left\{u^{\left(j(k)\right)}\right\}_{k=1}^{\infty}$ converges to $u(x)$ strongly in $\left(L^4(\Omega_j)\right)^2$, it follows that the sequence $\left\{\left(u^{\left(j(k)\right)}+v\right)\otimes\left(u^{\left(j(k)\right)}+v\right)\right\}_{k=1}^{\infty}$ converges strongly in $\left(L^2(\Omega_j)\right)^4$ to $(u+v)\otimes(u+v)$, and hence $\lim_{k\to\infty}\left(\nabla u^{\left(j(k)\right)}, \nabla\varphi\right) = (\nabla u, \nabla\varphi)$ and

$$\lim_{k\to\infty}\left(\left(u^{\left(j(k)\right)}+v\right)\otimes\left(u^{\left(j(k)\right)}+v\right), \nabla\varphi\right) = \left((u+v)\otimes(u+v), \nabla\varphi\right).$$

From these facts and the equality

$$\left(\nabla u^{\left(j(k)\right)}, \nabla\varphi\right) - \left(\left(u^{\left(j(k)\right)}+v\right)\otimes\left(u^{\left(j(k)\right)}+v\right), \nabla\varphi\right) + (H, \nabla\varphi) = 0 \tag{5.10}$$

for every k such that $j(k) \geq j$, we conclude (2.6). Since $\varphi \in C^\infty_{0,\sigma}(\Omega)$ is arbitrary, the function $u(x)$ is a weak solution.

It remains only to show (2.7). Integrating by parts and substituting $\varphi = u^{(j(k))}$ in (5.10), we obtain

$$\left\|\nabla u^{(j(k))}\right\|_2^2 = \left(v\otimes v - H, u^{(j(k))}\right) + \left(u^{(j(k))}\otimes \tilde{v}, u^{(j(k))}\right) - \alpha\Phi\left(u^{(j(k))}, u^{(j(k))}\right)$$
$$= I_1^{(k)} + I_2^{(k)} - \alpha\Phi\left(u^{(j(k))}, u^{(j(k))}\right),$$

where $I_1^{(k)} = \left(v\otimes v - H, \nabla u^{(j(k))}\right)$, $I_2^{(k)} = \left(u^{(j(k))}\otimes\tilde{v}, \nabla u^{(j(k))}\right)$.

We easily see that $I_1^{(k)} \to (v\otimes v - H, \nabla u)$ as $k\to\infty$.

We next verify that $u^{(j(k))}\otimes\tilde{v}$ converges strongly to $u\otimes\tilde{v}$ in $\left(L^2(\Omega)\right)^4$. We first observe that Proposition 3.5 implies that the set $\left\{ \frac{u^{(j(k))}(x)}{\lambda_4(x)} \,\middle|\, j\in\mathbb{N}\right\} \cup \left\{\frac{u(x)}{\lambda_4(x)}\right\}$ is bounded in $\left(L^4(\Omega)\right)^2$, and the estimate $\lambda_4(x)|\tilde{v}(x)| \le C|x|^{-5/3}$ holds for $|x|\le 2^J$. Hence, for every $R\ge 2^J$, we have

$$\left\|\left(u^{(j(k))}(x) - u(x)\right)|\tilde{v}(x)|\right\|_{L^2(\{x||x|\ge R\})}$$
$$\le C\left\|\frac{u^{(j(k))}(x)-u(x)}{\lambda_4(x)}\right\|_4 \left\||x|^{-5/3}\right\|_{L^4(\{x||x|\ge R\})} \le CR^{-2/3}.$$

This implies that, for every $\varepsilon > 0$, we can take $R > 0$ so large that $\left\|\left(u^{(j(k))}(x) - u(x)\right)|\tilde{v}(x)|\right\|_{L^2(\{x||x|\ge R\}} < \varepsilon$ holds for every k. On the other hand, $u^{(j(k))}$ converges strongly in $\left(L^4(\Omega\cap\{x \mid |x|\le R\}\right)^2$. Hence we have

$$\left\|\left(u^{(j(k))}(x) - u(x)\right)|\tilde{v}(x)|\right\|_{L^2(\{x||x|\le R\})}$$
$$\le \left\|\left(u^{(j(k))}(x) - u(x)\right)\right\|_{L^4(\{x||x|\le R\})} \|\tilde{v}(x)\|_4 \to 0$$

as $k\to\infty$. This implies $\limsup_{k\to\infty}\left\|\left(u^{(j(k))}(x) - u(x)\right) I_2^{(k)}(x)|\right\|_2 \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $u^{(j(k))}\otimes\tilde{v}$ converges strongly to $u\otimes\tilde{v}$ in $\left(L^2(\Omega)\right)^4$. This implies that we see that $I_2^{(k)} \to (u\otimes\tilde{v}, \nabla u)$ as $k\to\infty$.

We finally consider $-\alpha\Phi\left(u^{(j(k))}, u^{(j(k))}\right)$. Applying Fefferman-Stein duality and the result of [7], we obtain the estimate

$$|\Phi(\varphi,\psi)| \le C_1\|\log|x-c|\|_{BMO}\left\|\frac{\partial\varphi_m}{\partial x_h}\frac{\partial\psi_h}{\partial x_m}\right\|_{\mathcal{H}^1} \le C_2\|\nabla\varphi\|_2\|\nabla\psi\|_2 \tag{5.11}$$

with absolute constants C_1 and C_2, where φ and ψ are identified with their zero extensions to $\mathbb{R}^2$. In the sequel we assume that $|\alpha| < 1/C_2$.

If $\lim_{k\to\infty}\left\|\nabla u^{(j(k))}\right\|_2 = \|\nabla u\|_2$ holds, then $\left\{u^{(j(k))}\right\}_{k=1}^{\infty}$ converges strongly to u in $H^1_{0,\sigma}(\Omega)$, which implies that $\left\{-\Phi\left(u^{(j(k))}, u^{(j(k))}\right)\right\}_{k=1}^{\infty}$ converges to $-\Phi(u,u)$. Hence

$$\|\nabla u\|_2{}^2 \le (u\otimes\tilde{v} + v\otimes v - H, \nabla u) - \alpha\Phi(u,u) \tag{5.12}$$

holds with equality.

Suppose that $\lim_{k\to\infty} \left\|\nabla u^{\left(j(k)\right)}\right\|_2 > \|\nabla u\|_2$. Then there exist $K \in \mathbb{N}$ and $\varepsilon > 0$ such that $k \geq K$ implies $\left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 > \|\nabla u\|_2^2 + \varepsilon$. We next have

$$\begin{aligned}
&\left|\Phi\left(u^{\left(j(k)\right)}, u^{\left(j(k)\right)}\right) - 2\Phi\left(u^{\left(j(k)\right)} u, u\right) + \Phi(u,u)\right| \\
&\quad = \left|\Phi\left(u^{\left(j(k)\right)} - u, u^{\left(j(k)\right)} - u\right)\right| \leq C_2 \left\|\nabla\left(u^{\left(j(k)\right)} - u\right)\right\|_2^2 \\
&\quad = C_2 \left\{\left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 - 2\left(\nabla u^{\left(j(k)\right)}, \nabla u\right) + \|\nabla u\|_2^2\right\}.
\end{aligned}$$

Since the mapping $\varphi \mapsto \Phi(\varphi, u)$ is a bounded linear functional on $H_{0,\sigma^1}(\Omega)$, we have $\Phi\left(u^{\left(j(k)\right)}, u\right) \to \Phi(u,u)$ as $k \to \infty$. In the same way we have $\left(\nabla u^{\left(j(k)\right)}, \nabla u\right) \to \|\nabla u\|_2^2$ as $k \to \infty$. By taking K larger if necessary,, we may assume that $k \geq K$ implies

$$\begin{aligned}
&\left|\alpha\Phi\left(u^{\left(j(k)\right)}, u^{\left(j(k)\right)}\right) - \alpha\Phi(u,u)\right| \\
&\quad < |\alpha| C_2 \left(\left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 - \|\nabla u\|_2^2\right) + \frac{(1-|\alpha|C_2)\varepsilon}{3}
\end{aligned}$$

for $k \geq K$.

By taking K larger if necessary, we may assume that $k \geq K$ implies

$$\begin{aligned}
\left|I_1^{(k)} - (v \otimes v - H, \nabla u)\right| &< \frac{(1-|\alpha|C_2)\varepsilon}{3}, \\
\left|I_2^{(k)} - (u \otimes \tilde{v}, \nabla u)\right| &< \frac{(1-|\alpha|C_2)\varepsilon}{3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
0 &= \left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 - I_1^{(k)} - I_2^{(k)} + \alpha\Phi\left(u^{\left(j(k)\right)}, u^{\left(j(k)\right)}\right) \\
&> \left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 - (v \otimes v - H + u \otimes \tilde{v}, \nabla u) - \frac{2(1-|\alpha|C_2)\varepsilon}{3} \\
&\quad + \alpha\Phi(u,u) - |\alpha| C_2 \left(\left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 - \|\nabla u\|_2^2\right) - \frac{(1-|\alpha|C_2)\varepsilon}{3} \\
&= (1-|\alpha|C_2)\left\|\nabla u^{\left(j(k)\right)}\right\|_2^2 + |\alpha| C_2 \|\nabla u\|_2^2 - (v \otimes v - H + u \otimes \tilde{v}, \nabla u) \\
&\quad + \alpha\Phi(u,u) - (1-|\alpha|C_2)\varepsilon \\
&> \|\nabla u\|_2^2 - (v \otimes v + u \otimes \tilde{v} - H, \nabla u) + \alpha\Phi(u,u).
\end{aligned}$$

This implies (5.12). □

The existence in Theorem 2.11 can be proved in the same way, by observing the fact that $|x|^{1/2}v(x)\cdot\nabla u(x)\in\big(L^{4/3,1}(\Omega)\big)^2$ and replacing the weak convergence of $\nabla u^{(j(k))}(x)/\lambda_4(|x|)$ in $\big(L^4(\Omega)\big)^2$ to the weak-$*$ convergence of $\nabla u^{(j(k))}(x)/|x|^{1/2}$ in $\big(L^{4,\infty}(\Omega)\big)^2$. The estimate (5.9) follows from the fact $\displaystyle\int_{|x|\le 2^{j+1}} u^{(j(k))}(x)\,dx=0$.

6. Uniqueness of Weak Solutions

In this section we prove Theorems 2.7 and 2.9. We start with the following proposition, which provides a series of test functions approximating weak solutions. This proposition corresponds to the result in [22, Proposition 3.1], in which the existence of a sequence converging with respect to the weak-$*$ topology of L^∞ is proved.

Proposition 6.1. *Suppose that $2<q\le\infty$ and that $u\in H^1_{0,\sigma}(\Omega)$. Then we have the following assertions.*

(i) *Suppose in addition that $q\ne\infty$ and that $u(x)\in L^q_\sigma(\Omega)$. Then there exists a sequence $\{\varphi_j(x)\}_{j=1}^\infty$ such that $\nabla\varphi_j\to\nabla u$ in $\big(L^2(\Omega)\big)^4$ and $\varphi_j\to u$ in $L^q_\sigma(\Omega)$ as $j\to\infty$.*

(ii) *Suppose in addition that $\lambda_q(|x|))u(x)\in\big(L^q(\Omega)\big)^2$. Then there exists a sequence $\{\varphi_j(x)\}_{j=1}^\infty$ in $C^\infty_{0,\sigma}(\Omega)$ such that $\nabla\varphi_j\to\nabla u$ in $\big(L^2(\Omega)\big)^4$ and that $\lambda_q(|x|)\varphi_j(x)\to\lambda_q(|x|))u(x)$ as $j\to\infty$, weakly if $2<q<\infty$ and $*$-weakly if $q=\infty$.*

(iii) *Suppose in addition that $q\ne\infty$, Ω satisfies* (C2I), *and that $u(x)$ satisfies* (C2E) *and $|x|^{1-2/q}u(x)\in\big(L^{q,\infty}(\Omega)\big)^2$. Then there exists a sequence $\{\varphi_j(x)\}_{j=1}^\infty$ in $C^\infty_{0,\sigma}(\Omega)$ such that $\varphi_j(x)$ satisfies* (C2E) *for every j, $\nabla\varphi_j\to\nabla u$ in $\big(L^2(\Omega)\big)^4$ and that $|x|^{1-2/q}\varphi_j(x)\to|x|^{1-2/q}u(x)$ $*$-weakly in $\big(L^{q,\infty}(\Omega)\big)^2$ as $j\to\infty$.*

Proof. Assertion (i) is proved in Kozono and Sohr [21, Theorem 2].

We next prove Assertion (ii). We first decompose $u(x)=u_1(x)+u_2(x)$ as in Lemma 3.4. Then there exists a sequence $\{\varphi^{(1)}_j(x)\}_{j=0}^\infty$ in $C^\infty_{0,\sigma}(\tilde\Omega)$ such that $\nabla\varphi^{(1)}_j\to\nabla u_1$ in $\big(L^2(\tilde\Omega)\big)^4$ as $j\to\infty$. Then the Poincaré inequality yields $\varphi^{(1)}_j\to u_1$ in $L^q_\sigma(\tilde\Omega)$ as $j\to\infty$. Since $\lambda_q(|x|)$ is bounded on $\tilde\Omega$, we have $\lambda_q(|x|)\varphi^{(1)}_j(x)\to\lambda_q(|x|)u_1(x)$ in $\big(L^q(\Omega)\big)^2$ as $j\to\infty$ as well.

We next consider $u_2(x)$. For every $k\in\mathbb{N}$, put $\tilde u_{2,k}(x)=\chi\left(2^{-J}|x|\right)u_2\left(2^kx\right)$, where χ is the same function in the proof of Lemma 3.4. Then we have

$$\nabla\cdot\tilde u_{2,k}(x)=\big(\nabla\chi\left(2^{-J}|x|\right)\big)\cdot u_2(2^kx)=\chi'\left(2^{-J}|x|\right)2^{-J}\frac{x}{|x|}\cdot u_2(2^kx),$$

which is supported in $D=\left\{x\;\middle|\;2^{J-1}3<|x|<2^{J+1}\right\}$. Moreover we have $\displaystyle\int_{2^{J-1}3<|x|<2^{J+1}}\nabla\cdot\tilde u_{2,k}(x)\,dx=0$ in the same way as in the proof of Lemma 3.4. It follows that there exists a function $\tilde v_k(x)\in\big(H^1_0(D)\big)^2$ such that the identity $\nabla\cdot\tilde v_k(x)\equiv-\nabla\cdot\tilde u_{2,k}(x)$ holds, and that the estimate $\|\nabla\tilde v_k\|_2\le C\|\nabla\cdot\tilde u_{2,k}\|_2$ holds with a positive constant C independent of k. We also have $\|\nabla\cdot\tilde u_{2,k}\|_2\le C\left\|\frac{x}{|x|}\cdot u_2\left(2^kx\right)\right\|_2$. Now put

$$\begin{aligned}&u_{2,k}(x)=\tilde u_{2,k}\left(2^{-k}x\right)=\chi\left(2^{-k}|x|\right)u_2(x),\quad v_k(x)=\tilde v_k\left(2^{-k}x\right)\\&D_k=\left\{x\;\middle|\;2^{J+k-1}3<|x|<2^{J+k+1}\right\}.\end{aligned}$$

Then we have $\nabla\big(u_{2,k}(x)+v_k(x)\big)\equiv0$. Moreover, we have

$$\begin{aligned}\|\nabla(u_2-u_{2,k})\|_2&\le\|\nabla u_2\|_{L^2(\|x||x|\ge2^{k+J-1}3\})}+\left\|u_2(x)\otimes\nabla\left(\chi(2^{-k}|x|)\right)\right\|_2\\&\le\|\nabla u_2\|_{L^2(\|x\,|\,|x|\ge2^{k+J-1}3\})}+C2^{-k}\|u_2(x)\|_{L^2(D_k)}\end{aligned}\tag{6.1}$$

and

$$\|\nabla v_k\|_2 = \|\nabla \tilde{v}_k\|_2 \le C\left\|\chi'(|x|)\frac{x}{|x|}\cdot u_2(2^k x)\right\|_2 \le C\big\|u_2(2^k x)\big\|_{L^2(D)} = C2^{-k}\|u_2\|_{L^2(D_k)} \tag{6.2}$$

with a positive constant C independent of k.

For every $k \in \mathbb{N}$, put $\varphi_k(|x|) = \chi\left(2^{-k-1}|x|\right) - \chi\left(2^{1-k}|x|\right)$. Then we have $\varphi_k(|x|) \equiv 1$ on D_k. On the other hand, Hölder's inequality implies

$$\|u_2\|_{L^2(D_k)} \le C\left\|\frac{\varphi_k(x)}{\lambda_q(|x|)}\right\|_{2q/(q-2)}\|\lambda_q(|x|)u_2(x)\|_q. \tag{6.3}$$

Here Proposition 3.5 implies

$$\|\varphi_k(x)/\lambda_q(|x|)\|_{2q/(q-2)} \le C\|\nabla\varphi_k(x)\|_2 = C \tag{6.4}$$

with a positive constant C independent of k. Substituting this estimate into (6.3), we see that $\|u_2\|_{L^2(D_k)}$ is bounded uniformly in k. Substituting this into (6.1) and (6.2), we see that $\nabla(u_{2,k}+v_k)$ converges to ∇u_2 in $\left(L^2(\Omega)\right)^4$ as $k\to\infty$. We also have

$$\big\|\lambda_q(|x|)\big(u_2(x)-u_{2,k}(x)-v_k(x)\big)\big\|_q \le \|\lambda_q(|x|)u_2(x)\|_{L^q(\{x\,|\,|x|\ge 2^{k+J}3\})} + \|\lambda_q(|x|)u_{2,k}(x)\|_{L^q(D_k)} + \|\lambda_q(|x|)v_k(x)\|_{L^q(D_k)}. \tag{6.5}$$

Here we have

$$\begin{aligned}\left(2^{k+J-1}3\right)^{1-2/q}&\left(\log\left(2^{k+J-1}3+e\right)\right)^{1-1/q} \le \lambda_q(|x|)\\ &\le \left(2^{k+J+1}\right)^{1-2/q}\left(\log\left(2^{k+J+1}+\frac{4e}{3}\right)\right)^{1-1/q}\\ &= \left(2^{k+J+1}\right)^{1-2/q}\left(2\log 2-\log 3+\log\left(2^{k+J-1}3+e\right)\right)^{1-1/q}\\ &\le C\left(2^{k+J}3\right)^{1-2/q}\left(\log\left(2^{k+J-1}3+e\right)\right)^{1-1/q}\end{aligned} \tag{6.6}$$

for $x \in D_k$. On the other hand, since $v_k \in \left(H_0^1(D)\right)^2$, (6.2) implies

$$\begin{aligned}\|v_k\|_{L^q(D_k)} &\le C2^{2k/q}\|\nabla v_k\|_{L^2(D_k)} \le C2^{-k}\|u_2\|_{L^2(D_k)}\\ &\le C2^{-2k/q}\|u_2\|_{L^{q,r}(D_k)} \le C\frac{2^{-2k/q}\|\lambda_q(|x|)u_2\|_{L^q(\{x||x|\ge 2^{k+J+1}\})}}{\lambda_q(2^{J+k-1}3)}.\end{aligned}$$

From this estimate and (6.6) we conclude

$$\begin{aligned}\|\lambda_q(|x|)v_k\|_{L^q(D_k)} &\le \lambda_q\left(2^{k+J+1}\right)\|v_k\|_{L^q(D_k)} \le C\lambda_q\left(2^{k+J-1}3\right)\|v_k\|_{L^q(D_k)}\\ &\le C\|\lambda_q(|x|)u_2\|_{L^q(\{x||x|\ge 2^{k+J+1}\})}.\end{aligned}$$

Substituting this estimate into (6.5), we obtain

$$\big\|\lambda_q(|x|)\big(u_2(x)-u_{2,k}(x)-v_k(x)\big)\big\|_q \le C\|\lambda_q(|x|)u_2(x)\|_{L^q(\{x||x|\ge 2^{k+J+1}\})}.$$

Putting $f_k(x) = \lambda_q(|x|)\big(u_{2,k}(x)+v_k(x)\big)$, we see that the sequence $\{f_k\}_{k=1}^{\infty}$ is bounded in $L^q(\Omega)$ and that $f_k(x) \equiv \lambda_q(|x|)u_2(x)$ on the set $\left\{x\in\Omega \;\middle|\; |x| \le 2^{k+J+1}\right\}$. Put $M = \max\left\{\sup_{k=1,2,\dots}\|f_k\|_q, \|\lambda_q(|x|)u_2(x)\|_q\right\}$. Suppose that $g(x) \in \left(L^{q/(q-1)}(\Omega)\right)^2$ and $\varepsilon > 0$. (We put $q/(q-1) = 1$ if $q = \infty$.) Then there exists a positive integer k_0 such that the inequality $\big\|g(x)\big(1-\chi\left(2^{-k}|x|\right)\big)\big\|_{L^{q/(q-1)}} < \varepsilon/2M$ holds for every $k \ge k_0$.

Suppose that $k \geq k_0+1$. Then $\chi\left(2^{-k_0}|x|\right) \neq 0$ implies $|x| < 2^{k_0+J-1}3 < 2^{k+J}$, which yields the equality $f_k(x) \equiv \lambda_q(|x|)u_2(x)$. It follows that

$$\begin{aligned}
&\big|(g, f_k) - \big(g(x), \lambda_q(|x|)u_2(x)\big)\big| \\
&\quad \leq \big|\big(g(x) - \chi\left(2^{-k_0}|x|\right) g(x), f_k\big)\big| \\
&\qquad + \big|\big(g(x) - \chi\left(2^{-k_0}|x|\right) g(x), \lambda_q(|x|)u_2(x)\big)\big| \\
&\quad \leq \big\|g(x) - \chi\left(2^{-k_0}|x|\right) g(x)\big\|_{q/(q-1)} \big(\|f_k\|_q + \|\lambda_q(|x|)u_2(x)\|_q\big) \\
&\quad \leq 2M \big\|\big(1 - \chi(2^{-k_0}|x|)\big)g(x)\big\|_{q/(q-1)} < \varepsilon.
\end{aligned}$$

Since $g(x) \in \left(L^{q/(q-1)}(\Omega)\right)^2$ and $\varepsilon > 0$ are arbitrary, we see that $\{f_k(x)\}_{k=1}^{\infty}$ converges weakly ($*$-weakly if $q = \infty$) in $\left(L^q(\Omega)\right)^2$ to $\lambda_q(|x|)u_2(x)$ as $k \to \infty$.

Finally, let $\psi(x)$ be a function in $C_0^\infty(\mathbb{R}^2)$ such that $\psi(x) \geq 0$, $\operatorname{supp}\psi \subset \left\{x \in \mathbb{R}^2 \;\middle|\; |x| < 2^{J-1}\right\}$ and that $\int_{\mathbb{R}^2} \psi(x)\,dx = 1$. For $\delta \in (0,1)$, we put

$$u_{2,\delta}(x) = \int_{\mathbb{R}^2} \psi(y) u_2(x - \delta y)\,dy$$

and

$$\varphi_{k,\delta}(x) = \int_{\mathbb{R}^2} \psi(y)\big(u_{2,k}(x - \delta y) + v_k(x - \delta y)\big)\,dy.$$

Then we have

$$\nabla \cdot \varphi_{k,\delta}(x) = \int_{\mathbb{R}^2} \psi(y) \nabla \cdot \big(u_{2,k}(x - \delta y) + v_k(x - \delta y)\big)\,dy = 0$$

and $\|\nabla(u_2 - \varphi_{j,\delta})\|_2 \leq I_1 + I_2$, where

$$I_1 = \|\nabla(u_2 - u_{2,\delta})\|_2 = \left\|\int_{\mathbb{R}^2} \psi(y)\big\{\nabla u_2(\cdot) - \nabla u_2(\cdot - \delta y)\big\}\,dy\right\|_2$$

and

$$\begin{aligned}
I_2 &= \|\nabla(u_{2,\delta} - \varphi_{j,\delta})\|_2 \\
&= \left\|\int_{\mathbb{R}^2} \psi(y)\big\{\nabla u_2(\cdot - \delta y) - \nabla u_{2,k}(\cdot - \delta y) - \nabla v_k(\cdot - \delta y)\big\}\,dy\right\|_2.
\end{aligned}$$

Here we have

$$I_1 \leq \int_{\mathbb{R}^2} \psi(y)\|\nabla u_2(\cdot) - \nabla u_2(\cdot - \delta y)\|_2\,dy \leq \sup_{|y| \leq \delta} \|\nabla u_2(\cdot) - \nabla u_2(\cdot - y)\|_2.$$

Hence we can take a monotone-decreasing sequence of positive numbers $\{\delta_\ell\}_{\ell=1}^{\infty}$ such that the estimate $\|\nabla u_2(\cdot) - \nabla u_2(\cdot - y)\|_2 < 1/2\ell$ holds provided $|y| \leq \delta_\ell$. This implies $I_1 < 1/2\ell$ provided $\delta \leq \delta_\ell$. We also have

$$I_2 \leq \int_{\mathbb{R}^2} \psi(y)\|\nabla u_2 - \nabla u_{2,k} - \nabla v_k\|_2\,dx = \|\nabla u_2 - \nabla u_{2,k} - \nabla v_k\|_2.$$

It follows that we can choose a monotone-increasing sequence of positive integers $\{k_\ell\}_{\ell=1}^{\infty}$ such that $I_2 < 1/2\ell$ provided $k \geq k_\ell$. Hence, putting $\phi_\ell = \varphi_{k_\ell,\delta_\ell}$, we have $\|\nabla(\phi_\ell - u_2)\|_2 < 1/\ell$ and $\phi_\ell \in C_{0,\sigma}^\infty\left(\left\{x \;\middle|\; 2^{J-1}3 < |x| < 2^{k+J+1}\right\}\right)$. It follows that $\{\phi_\ell\}_{\ell=1}^{\infty}$ is a sequence of $C_{0,\sigma}^\infty(\Omega)$ converging to u_2 in $\dot{H}_{0,\sigma}^1(\Omega)$ as $\ell \to \infty$.

Suppose that the assumption of Assertion (ii) is satisfied. Let $g(x)$ be an element of $\left(L^{q/(q-1)}(\Omega)\right)^2$. Then we have

$$\begin{aligned}
&\big(g(x), \lambda_q(|x|)u_2(x) - \lambda_q(|x|)\varphi_{k,\delta}(x)\big)\\
&\quad= \int_{|x|\geq 2^J} g(x)\lambda_q(|x|)u_2(x)\,dx\\
&\qquad - \int_{|x|\geq 2^{J-2}3} g(x)\lambda_q(|x|)\int_{|z|\geq 2^J}\frac{1}{\delta^2}\psi\left(\frac{x-z}{\delta}\right)\{u_{2,k}(z)+v_k(z)\}\,dz\,dx\\
&\quad= \int_{|z|\geq 2^J} g(z)\left\{\lambda_q(|z|)\big(u_2(z)-u_{2,k}(z)-v_k(z)\big)\right\}\,dz\\
&\qquad + \int_{|z|\geq 2^J}\lambda_q(|z|)\{u_{2,k}(z)+v_k(z)\}\\
&\qquad\quad \left(g(z) - \frac{1}{\delta^2}\int_{|x|\geq 2^{J-2}3}\frac{\lambda_q(|x|)}{\lambda_q(|z|)}g(x)\psi\left(\frac{x-z}{\delta}\right)\,dx\right)\,dz.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\big|\big(g(x), \lambda_q(|x|)u_2(x) - \lambda_q(|x|)\varphi_{k,\delta}(x)\big)\big|\\
&\quad\leq \left|\int_{|z|\geq 2^J} g(z)\left\{\lambda_q(|z|)\big(u_2(z)-u_{2,k}(z)-v_k(z)\big)\right\}\,dz\right| + C\|f_k\|_q\times\\
&\qquad \left\|\int_{|y|\leq 2^{J-2}}\left(g(\cdot) - \left(\frac{|\cdot|}{|\cdot-\delta y|}\right)^{1-2/q} g(\cdot-\delta y)\right)\psi(y)\,dy\right\|_{q/(q-1)}.
\end{aligned}\tag{6.7}$$

Hence, for every $\varepsilon > 0$, we can choose L so large that $\ell \geq L$ implies

$$\left\|\int_{|y|\leq 2^{J-2}}\left(g(\cdot) - \left(\frac{|\cdot|}{|\cdot-\delta_\ell y|}\right)^{1-2/q} g(\cdot-\delta_\ell y)\right)\psi(y)\,dy\right\|_{q/(q-1)} < \frac{\varepsilon}{2MC}.\tag{6.8}$$

Furthermore, we can choose $L' \geq L$ so large that $\ell \geq L'$ implies

$$\left|\int_{|z|\geq 2^J} g(z)\left\{\lambda_q(|z|)\big(u_2(z)-u_{2,k_\ell}(z)-v_{k_\ell}(z)\big)\right\}\,dz\right| < \frac{\varepsilon}{2}.\tag{6.9}$$

For every $\ell \leq L'$, we substitute the estimates (6.8) and (6.9) into (6.7) to obtain $\big|\big(g(x), \lambda_q(|x|)u_2(x) - \lambda_q(|x|)\varphi_{k_\ell,\delta_\ell}(x)\big)\big| < \varepsilon$. This implies that the sequence $\{\lambda_q(|x|)\phi_\ell(x)\}_{\ell=1}^\infty$ converges to $\lambda_q(|x|)u_2(x)$ weakly ($*$-weakly if $q=\infty$) in $\big(L^q(\Omega)\big)^2$ as $\ell\to\infty$. This completes the proof of Assertion (ii).

We finally prove Assertion (iii). By the symmetry we can easily see that the sequence $\{-\varphi_j(-x)\}_{j=1}^\infty$ satisfies all the requirements for $-u(-x)$. Then, for every $j\in\mathbb{N}$, the function $\Phi_j(x) = \big(\varphi_j(x) - \varphi(-x)\big)/2$ satisfies (C2E). Since $u(x) = -u(-x)$, the sequence $\{\Phi_j(x)\}_{j=1}^\infty$ converges to u_2 in $\dot H^1_{0,\sigma}(\Omega)$ as $j\to\infty$. It suffices to prove the $*$-weak convergence of a subsequence of $\{|x|^{1-2/q}\Phi_j(x)\}_{j=1}^\infty$ to $|x|^{1-2/q}u(x)$ in $\big(L^{q,\infty}(\Omega)\big)^2$ We can do this in the same way as Assertion (ii), by replacing $\lambda_q(|x|)$ by $|x|^{1-2/q}$ and $g(x)\in\big(L^{q/(q-1)}(\Omega)\big)^2$ by $g(x)\in\big(L^{q/(q-1),1}(\Omega)\big)^2$. This completes proof of Assertion (iii). □

We now prove Theorem 2.7. From the assumption, we can apply Proposition 6.1, (i) with $q=4$ to obtain a sequence of functions $\{\varphi_j(x)\}_{j=1}^\infty$ in $C^\infty_{0,\sigma}(\Omega)$ such that $\varphi_j\to u$ in $L^4_\sigma(\Omega)$ and $\nabla\varphi_j\to\nabla u$ in $\big(L^2(\Omega)\big)^4$ as $j\to\infty$. By assumption we have

$$\big(\nabla u(x), \nabla\varphi_j(x)\big) - \Big(\big(u(x)+v(x)\big)\otimes\big(u(x)+v(x)\big), \varphi_j(x)\Big) + \big(F(x), \nabla\varphi_j(x)\big) = 0\tag{6.10}$$

for every $j \in \mathbb{N}$. Since the functions ∇u, $(u+v)\otimes(u+v)$ and F belong to $\left(L^2(\Omega)\right)^4$ and since $\nabla\varphi_j \to \nabla u$ in $\left(L^2(\Omega)\right)^4$ as $j \to \infty$, we have

$$\begin{aligned}&(\nabla u, \nabla\varphi_j) - \big((u+v)\otimes v, \nabla\varphi_j\big) + (F, \nabla\varphi_j)\\ &\quad\to \|\nabla u\|_2 - \big((u+v)\otimes v, \nabla u\big) + (F, \nabla u) \text{ as } j \to \infty.\end{aligned} \tag{6.11}$$

On the other hand, since $\nabla\cdot u(x) = \nabla\cdot v(x) = 0$, we can integrate by parts to obtain $\big((u+v)\otimes\varphi_j, \nabla\varphi_j\big) = 0$. Hence we see that

$$\begin{aligned}\big|\big((u+v)\otimes u, \nabla\varphi_j\big)\big| &= \big|\big((u+v)\otimes(u-\varphi_j), \nabla\varphi_j\big)\big|\\ &\le (\|u\|_4 + \|v\|_4)\, \|u-\varphi_j\|_4 \|\nabla\varphi_j\|_2.\end{aligned}$$

Since $\nabla\varphi_j \to \nabla u$ in $\left(L^2(\Omega)\right)^4$ as $j\to\infty$, the sequence $\left\{\|\nabla\varphi_j\|_2\right\}_{j=1}^{\infty}$ is bounded. Since $\varphi_j \to u$ in $L^4_\sigma(\Omega)$ as $j\to\infty$, we have $\big|\big((u+v)\otimes u, \nabla\varphi_j\big)\big| \to 0$ as $j\to\infty$. Substituting this formula and (6.11) into (6.10), we obtain (2.8). Integration by parts yields $-\left(u, \dfrac{\alpha(x-c)}{2\pi|x-c|^2}, \nabla u\right) = \Phi(u,u)$, from which we obtain the conclusion. □

We next verify Proposition 2.6. To this end we assume that the pairs $\big(v(x), G(x)\big)$ and $\big(v'(x), G'(x)\big)$ satisfy the conclusion of Proposition 4.1, and that $u(x) = w(x) - v(x)$ satisfies (2.7). Put $t(x) = u'(x) - u(x) = v(x) - v'(x) = \tilde{v}(x) - \tilde{v}'(x)$. Then Proposition 4.1 implies that $t(x) \in H^1_\sigma(\Omega) \cap \left(L^4(\Omega)\right)^2$. It follows from Proposition 6.1, (i) that there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ in $C^\infty_{0,\sigma}(\Omega)$ such that $\varphi_j \to t$ in $\left(L^4(\Omega)\right)^2$ and $\nabla\varphi_j \to \nabla t$ in $\left(L^2(\Omega)\right)^4$ as $j\to\infty$. Then, for every j, we put

$$\begin{aligned}I_j &= \big(-(u+\varphi_j)\otimes(\tilde{v}-\varphi_j) - (v-\varphi_j)\otimes(v-\varphi_j) + F - G', \nabla(u+\varphi_j)\big)\\ &\quad + \|\nabla(u+\varphi_j)\|_2{}^2 + \alpha\Phi(u+\varphi_j, u+\varphi_j)\\ &= \big(-u\otimes\tilde{v} - v\otimes v + F - G', \nabla(u+\varphi_j)\big)\\ &\quad + \big(\varphi_j\otimes(v-\tilde{v}) + (u+v)\otimes\varphi_j, \nabla(u+\varphi_j)\big) + \|\nabla u\|_2{}^2 + 2(\nabla u, \nabla\varphi_j)\\ &\quad + \|\nabla\varphi_j\|_2{}^2 + \alpha\Phi(u,u)@ + 2\alpha\Phi(u,\varphi_j) + \alpha\Phi(\varphi_j,\varphi_j)\\ &= \big(-u\otimes\tilde{v} - v\otimes v + F - G', \nabla(u+\varphi_j)\big)\\ &\quad + \big(-(u+v)\otimes u, \nabla\varphi_j\big) + \|\nabla u\|_2{}^2 + 2(\nabla u, \nabla\varphi_j)\\ &\quad + \|\nabla\varphi_j\|_2{}^2 + \alpha\Phi(u,u) + \alpha\Phi(u,\varphi_j)\\ &= \|\nabla u\|_2{}^2 + (-u\otimes\tilde{v} - v\otimes v + F - G, \nabla u) + \alpha\Phi(u,u) + I_{j,1} + I_{j,2} + I_{j,3},\end{aligned}$$

where

$$\begin{aligned}I_{j,1} &= \big(\nabla u - (u+v)\otimes(u+v) + F - G, \nabla\varphi_j\big),\\ I_{j,2} &= \big(G - G' + \nabla\varphi_j, \nabla(u+\varphi_j)\big),\\ I_{j,3} &= \big(u\otimes(v-\tilde{v}), \nabla\varphi_j\big) + \alpha\Phi(u,\varphi_j).\end{aligned}$$

Since $u+v$ is a weak solution, the equality (2.6) implies $I_{j,1} = 0$. Next we obtain $I_{j,3} = 0$ by integrating in parts.

We finally consider $I_{j,2}$. From the equality

$$\begin{aligned}I_{j,2} - \big(\nabla\varphi_j, \nabla(u+\varphi_j)\big) &= -\big(\nabla\cdot(G-G'), u+\varphi_j\big) = \big(\Delta(v-v'), u+\varphi_j\big)\\ &= \big(\nabla(v'-v), \nabla(u+\varphi_j)\big),\end{aligned}$$

Since $\varphi_j \to t$ in $H^1_{0,\sigma}(\Omega)$ as $j\to\infty$, we conclude that

$$\lim_{j\to\infty} I_{j,2} = \lim_{j\to\infty}\big(\nabla(-t+\varphi_j), \nabla(u+\varphi_j)\big) = 0.$$

From these facts we conclude that

$$\lim_{j\to\infty} I_j = \|\nabla u\|_2{}^2 + (-u\otimes\tilde{v} - v\otimes v + F - G, \nabla u) + \alpha\Phi(u,u).$$

On the other hand, since $\varphi_j \to t$ in $H^1_{0,\sigma}(\Omega)\cap\big(L^4(\Omega)\big)^2$, we obtain

$$\lim_{j\to\infty} I_j = \|\nabla u'\|_2{}^2 + (-u'\otimes\tilde{v}' - v'\otimes v' + F - G', \nabla u') + \alpha\Phi(u',u').$$

This completes the proof. □

We next prove Theorem 2.9. We first observe that the assumption and Proposition 3.5 imply $u(x)\in L^4(\Omega)$, in view of the estimate

$$\big\|u^2\big\|_2 \le \left\|\frac{u(x)}{\lambda_q(x)}\right\|_{2q/(q-2)} \|\lambda_q(x)u(x)\|_q \le C_{\rho,\varepsilon_n}\|\nabla u\|_2\delta_{q,\Omega} < \infty.$$

By Remark 2.1 we have a sequence $\{\varphi_j\}_{j=1}^\infty$ of the functions in $C^\infty_{0,\sigma}(\Omega)$ such that $\nabla\varphi_j\to\nabla u'$ in $\big(L^2(\Omega)\big)^4$. Since u is a weak solution, we have

$$-(\nabla u,\nabla\varphi_j) + \big((v\otimes v + v\otimes u + u\otimes v + u\otimes u - F),\nabla\varphi_j\big) = 0.$$

Integrating by parts, we obtain

$$\big((-\nabla u + v\otimes v + u\otimes\tilde{v} + v\otimes u + u\otimes u - F),\nabla\varphi_j\big) - \alpha\Phi(u,\varphi_j) = 0.$$

Since

$$-\nabla u + v\otimes v + u\otimes\tilde{v} + v\otimes u + u\otimes u - F \in \big(L^2(\Omega)\big)^4,$$

we can let $j\to\infty$ to obtain

$$\big((-\nabla u + v\otimes v + u\otimes v + v\otimes u + u\otimes u - F),\nabla u'\big) - \alpha\Phi(u,u') = 0. \tag{6.12}$$

Next, from the assumption we can apply Proposition 6.1, (ii) to obtain a sequence $\{\psi_k(x)\}_{k=1}^\infty$ of functions in $C^\infty_{0,\sigma}(\Omega)$ such that $\nabla\psi_k\to\nabla u$ in $\big(L^2(\Omega)\big)^4$ and that $\lambda_q(|x|)\psi_k(x)\to\lambda_q(|x|)u(x)$ converges weakly (∗-weakly if $q=\infty$) in $\big(L^q(\Omega)\big)^2$ as $k\to\infty$.

Since $u'(x)$ is a weak solution, we have

$$\big((-\nabla u' + v\otimes v + u'\otimes v + v\otimes u' + u'\otimes u' - F),\nabla\psi_k\big) = 0.$$

Integrating by parts, we have

$$\big((-\nabla u' + v\otimes v + u'\otimes\tilde{v} + v\otimes u' + u'\otimes u' - F),\nabla\psi_k\big) - \alpha\Phi(u',\psi_k) = 0. \tag{6.13}$$

Adding the formulae (6.12), (6.13), (2.7) and the equality

$$\|\nabla u\|_2{}^2 - (v\otimes v,\nabla u) - (u\otimes\tilde{v},\nabla u) + (F,\nabla u) + \alpha\Phi(u,u) = 0,$$

which follows from Theorem 2.7, we obtain

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \le 0, \tag{6.14}$$

where

$$\begin{aligned}
I_1 &= \|\nabla u\|_2{}^2 - (\nabla u,\nabla u') - (\nabla u',\nabla\psi_k) + \|\nabla u'\|_2,\\
I_2 &= (F - v\otimes v,\nabla u) - (F - v\otimes v,\nabla\psi_k),\\
I_3 &= (v\otimes u,\nabla u') + (v\otimes u',\nabla\psi_k),\\
I_4 &= -(u\otimes\tilde{v},\nabla u) + (u\otimes\tilde{v},\nabla u') + (u'\otimes\tilde{v},\nabla\psi_k) - (u'\otimes\tilde{v},\nabla u'),\\
I_5 &= (u\otimes u,\nabla u') + (u'\otimes u',\nabla\psi_k),\\
I_6 &= \alpha\big(\Phi(u',u') - \Phi(u,u') - \Phi(u',\psi_k) + \Phi(u,u)\big).
\end{aligned}$$

We calculate each terms. First we see that

$$I_1 = \|\nabla u - \nabla u'\|_2{}^2 + (\nabla u',\nabla u - \nabla\psi_k).$$

It follows that $|I_1 - \|\nabla u - \nabla u'\|_2| \le \|\nabla u'\|_2 \|\nabla(u-\psi_k)\|_2$. Since $\nabla\psi_k \to \nabla u$ in $\big(L^2(\Omega)\big)^4$ as $k\to\infty$, we have

$$\lim_{k\to\infty} I_1 = \|\nabla(u-u')\|_2{}^2. \tag{6.15}$$

Next we see that

$$|I_2| \le \big|(F - v\otimes v, \nabla u - \nabla\psi_k)\big| \le \Big(\|F\|_2 + \|v\|_4{}^2\Big)\, \|\nabla(u-\psi_k)\|_2.$$

Hence, in the same way as (6.15), we have

$$\lim_{k\to\infty} I_2 = 0. \tag{6.16}$$

Third, we have $I_3 = \big(v\otimes(u-\psi_k), \nabla u'\big)$ by integrating by parts. Since $v(x)/\lambda_q(|x|) \in \big(L^{2q/(q-2)}(\Omega)\big)^2$, $\nabla u' \in \big(L^2(\Omega)\big)^4$ and $\lambda_q(x)\psi_k(x)$ converges weakly ($*$-weakly if $q=\infty$) in $(L^q)^2$ as $k\to\infty$, we obtain

$$\lim_{k\to\infty} I_3 = 0. \tag{6.17}$$

Fourth, we have

$$\begin{aligned} I_4 &= \big(u\otimes\tilde v, \nabla(u'-u)\big) + \big(u'\otimes\tilde v, \nabla(\psi_k - u')\big) \\ &= \big(u'\otimes\tilde v, \nabla(\psi_k - u)\big) + \big((u-u')\otimes\tilde v, \nabla(u'-u)\big). \end{aligned}$$

Since $\lambda_q(x)\tilde v \in \big(L^q(\Omega)\big)^2$, the estimate (6.4) implies $u'\otimes\tilde v \in \big(L^2(\Omega)\big)^4$. Hence, letting $k\to\infty$, we obtain

$$\lim_{k\to\infty} I_4 = \big((u-u')\otimes v, \nabla(u'-u)\big). \tag{6.18}$$

Fifth, we calculate

$$\begin{aligned} I_5 &= \big(u\otimes(u-\psi_k), \nabla u'\big) + (u\otimes\psi_k, \nabla u') + (u'\otimes u', \nabla\psi_k) \\ &= \big(u\otimes(u-\psi_k), \nabla u'\big) + (u\otimes\psi_k, \nabla u') - (u'\otimes\psi_k, \nabla u') \\ &= \big(u\otimes(u-\psi_k), \nabla u'\big) + \big((u-u')\otimes\psi_k, \nabla(u'-u)\big) \\ &\qquad + \big((u-u')\otimes\psi_k, \nabla(u-\psi_k)\big) \\ &= \big(u\otimes(u-\psi_k), \nabla u'\big) + \big((u-u')\otimes u, \nabla(u'-u)\big) \\ &\qquad + \big((u-u')\otimes(\psi_k - u), \nabla(u'-u)\big) + \big((u-u')\otimes\psi_k, \nabla(u-\psi_k)\big). \end{aligned}$$

It follows that

$$I_5 - \big((u-u')\otimes u, \nabla(u'-u)\big) = J_1 + J_2 + J_3, \tag{6.19}$$

where

$$\begin{aligned} J_1 &= \left(\lambda_q(|x|)\big(u(x)-\psi_k(x)\big), \left(\frac{u(x)}{\lambda_q(|x|)}\cdot\nabla\right) u'(x)\right), \\ J_2 &= \left(\lambda_q(|x|)\big(\psi_k(x)-u(x)\big), \left(\frac{u(x)-u'(x)}{\lambda_q(|x|)}\cdot\nabla\right)\big(u'(x)-u(x)\big)\right) \end{aligned}$$

and

$$J_3 = \left(\lambda_q(|x|)\psi_k(x), \left(\frac{u(x)-u'(x)}{\lambda_q(|x|)}\cdot\nabla\right)\big(u(x)-\psi_k(x)\big)\right).$$

Then we have $J_\ell \to 0$ as $k\to 0$ for $\ell = 1,2,3$. Indeed, Lemma 3.1 and Proposition 3.5, (ii) imply

$$\begin{aligned} \left\|\left(\frac{u(x)}{\lambda_q(|x|)}\cdot\nabla\right) u'(x)\right\|_{q/(q-1)} &\le C\left\|\frac{u(x)}{\lambda_q(|x|)}\right\|_{2q/(q-2)} \|\nabla u'\|_2 \\ &\le C\|\nabla u\|_2\|\nabla u'\|_2 \end{aligned}$$

with a positive constant C. Since $\lambda_q(|x|)\psi_k(x) \to \lambda_q(|x|)u$ holds weakly ($*$-weakly if $q=\infty$) in $\big(L^q(\Omega)\big)^2$, we have $J_1 \to 0$ as $k \to \infty$. In the same way we can prove $J_2 \to 0$ as $k \to \infty$. Finally, in view of Proposition 3.5, (ii), we can estimate

$$\begin{aligned}|J_3| &\le C\|\lambda_q(|x|)\psi_k(x)\|_q \left\|\frac{u(x)-u'(x)}{\lambda_q(|x|)}\right\|_{2q/(q-2)} \|\nabla(u-\psi_k)\|_2 \\ &\le C\|\lambda_q(|x|)\psi_k(x)\|_q \|\nabla(u-u')\|_2 \|\nabla(u-\psi_k)\|_2.\end{aligned}$$

with a positive constant C. Since $\lambda_q(|x|)\psi_k(x) \to \lambda_q(|x|)u$ holds weakly in $\big(L^q(\Omega)\big)^2$, the sequence $\Big\{\|\lambda_q(|x|)\psi_k(x)\|_{q,r}\Big\}_{k=1}^{\infty}$ is bounded. Since $\nabla\psi_k \to \nabla u$ strongly in $\big(L^2(\Omega)\big)^4$, we have $J_3 \to 0$ as $k \to \infty$. These facts imply

$$\lim_{k\to\infty} I_5 = \big((u-u')\otimes u, \nabla(u'-u)\big). \tag{6.20}$$

Finally, we have $I_6 = \alpha\Phi(u'-u, u'-u) + \alpha\Phi(u', u-\psi_k)$. It follows that

$$\lim_{k\to\infty} I_6 = \alpha\Phi(u'-u, u'-u). \tag{6.21}$$

Substituting (6.15)–(6.18), (6.20) and (6.21) into (6.14), we obtain

$$\begin{aligned}0 \ge \|\nabla(u-u')\|_2{}^2 &+ \big((u-u')\otimes\tilde{v}, \nabla(u'-u)\big) \\ &+ \big((u-u')\otimes u, \nabla(u'-u)\big) + \Phi(u'-u, u'-u).\end{aligned}$$

We can take $C_\Omega > 0$ sufficiently small so that, if $a(x) \in \big(H^{1/2}(\Omega)\big)^2$ satisfies the estimate $\|a\|_{H^{1/2}(\Omega)} < C_\Omega$, then we have

$$\|(u-u')\otimes v\|_2{}^2 + \alpha\Phi(u'-u, u'-u) \le \frac{\|\nabla(u-u')\|_2{}^2}{4}.$$

It follows that

$$\frac{3}{4}\|\nabla(u-u')\|_2{}^2 \le \big((u'-u)\otimes u, \nabla(u'-u)\big). \tag{6.22}$$

We then apply Lemma 3.1 and Proposition 3.5, (ii) to obtain

$$\begin{aligned}&\big|\big((u'-u)\otimes u, \nabla(u'-u)\big)\big| \\ &\quad\le C\|\lambda_q(|x|)u(x)\|_q \left\|\frac{u'(x)-u(x)}{\lambda_q(|x|)}\right\|_{2q/(q-2)} \|\nabla(u'-u)\|_2 \\ &\quad\le C\|\lambda_q(|x|)u(x)\|_q \|\nabla(u'-u)\|_2{}^2,\end{aligned}$$

where C depends on q and Ω. Substituting this estimate into (6.22) we obtain

$$\|\nabla(u-u')\|_2{}^2\left(\frac{3}{4} - C\|\lambda_q(|x|)u(x)\|_q\right) \le 0.$$

Hence, if $\|\lambda_q(x)u(x)\|_q < C_{q,\Omega} = 3/4C$, we have $\|\nabla(u-u')\|_2 = 0$. Since $u, u' \in \dot{H}^1_{0,\sigma}(\Omega)$, it follows that $u(x) \equiv u'(x)$ on Ω. Hence we obtain Theorem 2.9 by putting $\delta_{q,\Omega} = 3/4C$, which depends only on q and Ω.

7. Results on Symmetric Solutions

In this section we prove the results on the situation when Ω satisfies (C2I) and $a(x)$ satisfies (C2E). Since the argument is almost the same, we shall give main differences.

We first introduce Hardy's inequality with symmetry, whose proof is given in Appendix C.

Proposition 7.1. *Suppose that $2 \le \rho < \infty$. Then there exists a positive constant C_ρ depending only on ρ such that, if $u(x) \in \dot{H}^1_0(\Omega)$ satisfies*

$$\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\, d\theta = 0 \text{ for almost every } r > 0, \tag{7.1}$$

then we have $u(x)/|x|^{2/\rho} \in L^{\rho,2}(\Omega)$ with the estimate $\left\|u(x)/|x|^{2/\rho}\right\|_{\rho,2} \le C_\rho \|\nabla u\|_2$.

We next make a comment on the corrector function.

Proposition 7.2. *Suppose that Ω satisfies* (C2I) *and $a(x)$ satisfies* (C2E)*. Then we can construct $I_j[a]$ so that they satisfy* (C2E)*.*

Proof. For each ℓ, either U_ℓ is different from $-U_\ell$ or $U_\ell = -U_\ell$. Moreover, if $U_\ell = -U_\ell$, then $0 \in U_\ell$. Indeed, if C is a curve contained in U_ℓ which connects $\mathrm{P} \in U_\ell$ to $-\mathrm{P} \in -U_\ell = U_\ell$, then $C \cup -C$ is a closed curve in U_ℓ surrounding 0. Since U_ℓ is simply connected, it follows that $0 \in U_\ell$. In this case choose c_ℓ as follows: If $U_m = -U_\ell$ holds for some ℓ and m such that $\ell \ne m$, choose $c_m = -c_\ell$, and if $0 \in U_\ell$, choose $c_\ell = 0$. Moreover, if $a(x)$ satisfies (C2E) as well, then the equality $\alpha_\ell = \alpha_m$ holds for ℓ, m such that $U_m = -U_\ell$. In this case we have $\alpha_\ell = \alpha_m$ Hence $v^{(1)}(x)$ satisfies (C2E) in this case. Furthermore, in this case $b(x)$ satisfies (C2E), and so does $w(x)$. It follows that $g(x)$ satisfies (C2AE). Hence we can assume that $\varphi(x)$ satisfies (C2E), by replacing $\varphi(x)$ by $\big(\varphi(x) - \varphi(-x)\big)/2$ if necessary. In this case $\nabla \times \varphi(x)$ and $h(x)$ satisfy (C2AE). We thus conclude that $G(x)$ satisfies (C2AE). The modification of I_3 is given in the introduction. $\square$

We can prove Theorem 2.11 in the same way as Theorem 2.5. However, in order to obtain a solution satisfying (C2E), we modify the proof. We replace the space X_j by $\left\{f \in L^4_\sigma(\Omega_j) \,\middle|\, f \text{ satisfies (C2E)}\right\}$ and making use of the fact that since $u(x)$ satisfies (C2E) implies that so does $U[u]$ in Lemma 5.2.

We derive (2.9) from the assumptions of Theorem 2.11. For every integer $j \ge J+1$ we put $a_j = \displaystyle\int_{2^j \le |x| \le 2^{j+1}} |\nabla u(x)|^2\, dx$. Then we have

$$\begin{aligned}\frac{1}{2^{j+1}} &\int_{2^j}^{2^{j+1}} \int_0^{2\pi} \left|\frac{\partial}{\partial\theta} u(r\cos\theta, r\sin\theta)\right|^2 d\theta\, dr \\ &\le 2 \int_{2^j}^{2^{j+1}} \int_0^{2\pi} \left|\frac{\partial}{\partial\theta} u(r\cos\theta, r\sin\theta)\right|^2 \frac{1}{r}\, d\theta\, dr \le 2a_j.\end{aligned}$$

Hence, for every $j \ge J+1$, there exists a number $r_j \in \left[2^j, 2^{j+1}\right]$ such that

$$\int_0^{2\pi} \left|\frac{\partial}{\partial\theta} u(r_j\cos\theta, r_j\sin\theta)\right|^2 d\theta \le 4a_j.$$

In view of the assumption (C2E), we can apply the Sobolev embedding theorem to obtain

$$\sup_{\theta \in [0,2\pi]} |u(r_j\cos\theta, r_j\sin\theta)|^2 \le Ca_j \tag{7.2}$$

with a positive constant C independent of u and j. Next, for every ρ, s such that $2^j \le \rho \le s \le r_j$, we have

$$\begin{aligned}&\left|\int_0^{2\pi} |u(s\cos\theta, s\sin\theta)|^2\, d\theta - \int_0^{2\pi} |u(\rho\cos\theta, \rho\sin\theta)|^2\, d\theta\right| \\ &\quad\le \int_0^{2\pi}\int_\rho^s \left|\frac{\partial}{\partial r}|u(r\cos\theta, r\sin\theta)|^2\right| dr\, d\theta \\ &\quad\le 2\int_0^{2\pi}\int_\rho^s \left|\frac{\partial}{\partial r} u(r\cos\theta, r\sin\theta)\right| |u(r\cos\theta, r\sin\theta)|\, dr\, d\theta\end{aligned}$$

$$\leq 2\left(\int_0^{2\pi}\int_\rho^s \left|\frac{\partial}{\partial r}u(r\cos\theta, r\sin\theta)\right|^2 dr\, d\theta\right)^{1/2}$$
$$\times \left(\int_0^{2\pi}\int_\rho^s |u(r\cos\theta, r\sin\theta)|^2\, dr\, d\theta\right)^{1/2}$$
$$\leq 2^{1-j}\left(\int_{2^j\leq|x|\leq 2^{j+1}} |\nabla u(x)|^2\, dx\right)^{1/2}\left(\int_{2^j\leq|x|\leq 2^{j+1}} |u(x)|^2\, dx\right)^{1/2} \leq C' a_j$$

with a positive constant C' independent of u and j. Hence, for every $\rho \in \left[2^j, 2^{j+1}\right]$, the estimate (7.2) implies

$$\int_0^{2\pi} |u(\rho\cos\theta, \rho\sin\theta)|^2\, dr$$
$$\leq 2\pi C a_j + \left|\int_0^{2\pi} |u(r_j\cos\theta, r_j\sin\theta)|^2\, d\theta - \int_0^{2\pi} |u(\rho\cos\theta, \rho\sin\theta)|^2\, d\theta\right|$$
$$\leq (2\pi C + C')a_j.$$

Since $a_j \to 0$ as $j\to\infty$, we obtain $\int_0^{2\pi} |u(r\cos\theta, r\sin\theta)|^2\, d\theta \to 0$ as $r\to\infty$. The conclusion (2.9) follows from this fact and the fact

$$\int_0^{2\pi} |v(r\cos\theta, r\sin\theta)|^2\, d\theta \to 0 \text{ as } r\to\infty.$$

If the inclusion relation $\operatorname{supp}\nabla\cdot F \subset \{x \mid |x| < 2^K\}$ holds for some $K\in\mathbb{N}$, we can regard $w(x)$ as the solution of the system

$$-\Delta w(x) + \big(w(x)\cdot\nabla\big)w(x) + \nabla\pi(x) = 0 \qquad \text{in } \{x \mid |x| > 2^K\},$$
$$\nabla\cdot w(x) = 0 \qquad \text{in } \{x \mid |x| > 2^K\}.$$

Hence [11, Theorem XII.3.4] and (2.9) imply that $w(x)\to 0$ uniformly as $|x|\to\infty$.

The inequality (2.7) can be proved in the same way as in Theorem 2.5. □

We finally prove Theorem 2.12. In view of Remark 2.13, we may assume $2 < q < \infty$. Then the assumption and Proposition 7.1, together with Lemma 3.1, imply $u(x)\in L^4(\Omega)$, in view of the estimate

$$\left\|u^2\right\|_2 \leq \left\|\frac{u(x)}{|x|^{1-2/q}}\right\|_{2q/(q-2)}\left\||x|^{1-2/q)}(x)\right\|_q \leq C_{\rho,\varepsilon_n}\|\nabla u\|_2\delta_{q,\Omega} < \infty.$$

Next, in the same way as in the proof of Theorem 2.9, we employ (6.14), where $\{\psi_k\}_{k=1}^\infty$ is a sequence of functions in $C_{0,\sigma}^\infty(\Omega)$ such that $\nabla\psi_k \to \nabla u$ in $\left(L^2(\Omega)\right)^4$ and that $|x|^{1-2/q}\psi_k(x) \to |x|^{1-2/q}u(x)$ $*$-weakly in $\left(L^{q,\infty}(\Omega)\right)^2$ given by Proposition 6.1, (iii). The proofs of (6.15), (6.16) and (6.21) hold without change. We can prove (6.17) and (6.18) in the same way, by replacing $\lambda_q(|x|)$ by $|x|^{1-2/q}$ and $L^{q/(q-1)}(\Omega)$ by $L^{q/(q-1),1}(\Omega)$.

For I_5, we have (6.19). From the equality

$$J_1 = \Big(|x|^{1-2/q}\big(u(x)-\psi_k(x)\big), \big(|x|^{2/q-1}u(x)\cdot\nabla\big)u'(x)\Big)$$

and the fact

$$\Big\|\big(|x|^{2/q-1}u(x)\cdot\nabla\big)u'(x)\Big\|_{q/(q-1),1} \leq C\Big\||x|^{2/q-1}u(x)\Big\|_{2q/(q-2),2}\|\nabla u'\|_2$$

together with the weak-$*$ convergence of $|x|^{1-2/q}\psi_k(x)$ in $\big(L^{q,\infty}(\Omega)\big)^2$, we see that $J_1 \to 0$ as $k \to \infty$. In the same way we see that $J_2 \to 0$ as $k \to \infty$. Finally, in view of Proposition 3.5, (i) we can estimate

$$\begin{aligned}|J_3| &\le C\Big\||x|^{1-2/q}\psi_k(x)\Big\|_{q,\infty}\Big\||x|^{2/q-1}\big(u(x)-u'(x)\big)\Big\|_{2q/(q-2),2}\|\nabla(u-\psi_k)\|_2\\ &\le C\Big\||x|^{1-2/q}\psi_k(x)\Big\|_{q,\infty}\big\|\nabla\big(u(x)-u'(x)\big)\big\|_2\|\nabla(u-\psi_k)\|_2.\end{aligned}$$

The weak-$*$ convergence implies the boundedness of $\big\||x|^{1-2/q}\psi_k(x)\big\|_{q,\infty}$. Hence the fact $\nabla\psi_k \to \nabla u$ in $\big(L^2(\Omega)\big)^4$ implies $J_3 \to 0$. From these facts we have (6.20). It follows from Proposition 3.5, (i) that

$$\|\nabla(u-u')\|_2{}^2\left(\frac{3}{4} - C\Big\||x|^{1-2/q}u(x)\Big\|_{q,\infty}\right) \le 0,$$

in the same way as in the proof of Theorem 2.9, where C depends only on q. This estimate implies $u(x) = u'(x)$ if $\big\||x|^{1-2/q}u(x)\big\|_{q,\infty} < C_q = 3/4C$. □

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Compliance with Ethical Standards

Conflict of interest The author declares that he has no conflict of interest.

Appendix A. Proof of Lemma 3.1.

We first choose p_0, p_1, q_0, q_1 so that $p_0 < p < p_1 < \infty$, $q_0 < q < q_1 < \infty$ and $1/p_o + 1/q_0 < 1$, and put $r_{j,k} = p_1 q_k/(p_1 + q_k)$ for $j, k = 0, 1$.

We first consider the case $\beta = \infty$ and $\gamma = \alpha$. It follows from the Hölder inequality $\|fg\|_{r_{j,k}} \le \|f\|_{p_j}\|g\|_{q_k}$ for $j, k = 0, 1$ that the estimate

$$\|fg\|_{p_j q/(p_j+q),\infty} \le C\|f\|_{p_j}\|g\|_{q,\infty}$$

holds for $j = 0, 1$ by real interpolation with respect to q. Applying real interpolation with respect to p, we obtain

$$\|fg\|_{r,\alpha} \le C\|f\|_{p,\alpha}\|g\|_{q,\infty}.$$

The case $\alpha = \infty$ can be proved in the same way. We next consider the case $\gamma = 1$. It suffices to consider the case $\alpha, \beta \in (1, \infty)$. For every $h \in L^{r/(r-1),\infty-}$, we have

$$\|fgh\|_1 \le C\|f\|_{p,\alpha}\|gh\|_{p/(p-1),\alpha/(\alpha-1)}. \tag{A.1}$$

Since

$$\frac{p-1}{p} = 1 - \frac{1}{p} = 1 - \frac{1}{r} + \frac{1}{q} \text{ and } \frac{\alpha-1}{\alpha} = 1 - \frac{1}{\alpha} = \frac{1}{\beta},$$

we have $\|gh\|_{p/(p-1),\alpha/(\alpha-1)} \le C\|g\|_{q,\beta}\|h\|_{r/(r-1),\infty}$. Substituting this estimate into (A.1), we obtain

$$\|fgh\|_1 \le C\|f\|_{p,\alpha}\|g\|_{q,\beta}\|h\|_{r/(r-1),\infty}.$$

Since the norm of $L^{r/(r-1),\infty-}$ is $\|\cdot\|_{r/(r-1),\infty}$, we have $fg \in \left(L^{r/(r-1),\infty-}\right)' = L^{r,1}$ with the estimate

$$\|fg\|_{r,1} \leq C\|f\|_{p,\alpha}\|g\|_{q,\beta}.$$

We finally consider the general case. It suffices to consider the case $\beta < \infty$. Put $\theta = \beta/\alpha(1-\beta)$. Since $\beta/(\beta-1) \leq \alpha \leq \infty$, we have $0 \leq \theta \leq 1$. Then we can write

$$\frac{1}{\alpha} = \theta\left(1 - \frac{1}{\beta}\right) \text{ and } \frac{\theta}{\gamma} = \frac{1}{\beta} + \theta\left(1 - \frac{1}{\beta}\right).$$

Then the estimate

$$\|fg\|_{r,\beta/\{1+\theta(\beta-1)\}} \leq C\|f\|_{p,\beta/\theta(\beta-1)}\|g\|_{q,\beta} \tag{A.2}$$

holds for the case $\theta = 0$ and $\theta = 1$. By complex interpolation we see that (A.2) holds for every $\theta \in [0,1]$. This completes the proof. □

Appendix B. Proof of Proposition 3.5.

Since $C_0^\infty(\Omega)$ is dense in $\dot{H}_0^1(\Omega)$, we may assume that $u(x) \in C_0^\infty(\Omega)$. Put

$$v(x) = \frac{1}{2\pi}\int_0^{2\pi} u(|x|\cos\theta, |x|\sin\theta)\, d\theta$$

and $w(x) = u(x) - v(x)$. Then we have

$$\begin{aligned}\left|\frac{\partial v}{\partial x_j}(r\cos\theta, r\sin\theta)\right| &= \left|\frac{\partial r}{\partial x_j}(r\cos\theta, r\sin\theta)\right| \cdot \left|\frac{dv(r\cos\theta, r\sin\theta)}{dr}\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial u}{\partial r}(r\cos\theta, r\sin\theta)\right| d\theta.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^{2\pi}\left|\frac{\partial v}{\partial x_j}(r\cos\theta, r\sin\theta)\right|^2 d\theta &\leq \frac{1}{2\pi}\left(\int_0^{2\pi}\left|\frac{\partial u}{\partial r}(r\cos\theta, r\sin\theta)\right| d\theta\right)^2 \\ &\leq \int_0^{2\pi}\left|\frac{\partial u}{\partial r}(r\cos\theta, r\sin\theta)\right|^2 d\theta \leq \int_0^{2\pi}|\nabla u(r\cos\theta, r\sin\theta)|^2\, d\theta.\end{aligned}$$

Integrating with respect to r, we see that $\|\nabla v\|_2 \leq \|\nabla u\|_2$. Since $w(x) = u(x) - v(x)$, It follows that $\|\nabla w\|_2 \leq 2\|\nabla w\|_2$. Since $w(x)$ satisfies the condition (7.1) and since $\lambda_{2\rho/(\rho-2)}(t) = t^{2/\rho}\big(\log(\varepsilon_0 + e)\big)^{1/2+1/\rho} \geq t^{2/\rho}$, it follows that

$$\left\|w(x)/\lambda_{2\rho/(\rho-2)}(|x|)\right\|_\rho \leq \left\||x|^{-2/\rho}w(x)\right\|_{\rho,2} \leq 2C_\rho\|\nabla u\|_2. \tag{B.1}$$

We now consider $v(x)$. By the assumption we see that $v(x)$ is radially symmetric, $v(x) = 0$ if $|x| \leq \varepsilon_0$. Hence we can define $f(t)$ defined on $\mathbb{R}$ such that $v(x) = f(\log|x|)$. Then we have $f(t) \equiv 0$ on $(-\infty, \log\varepsilon_0]$. We also have

$$\begin{aligned}\int_{\mathbb{R}^2}\big(\nabla v(x)\big)^2\, dx &\geq 2\pi\int_{\varepsilon_0}^\infty r\left|\frac{\partial}{\partial r}v(r\omega)\right| dr \\ &= 2\pi\int_{\varepsilon_0}^\infty r\left|\frac{\partial f}{\partial t}(\log r)\right|^2\left(\frac{\partial}{\partial r}(\log r)\right)^2 dr = 2\pi\int_{\log\varepsilon_0}^\infty |f'(t)|^2\, dt.\end{aligned}$$

It follows that $f(t) \in \dot{H}_0^1(\mathbb{R}) = \dot{B}_{2,2}^1(\mathbb{R})$ with the norm estimate $\|\nabla f\|_2 \leq \|\nabla v\|_2 \leq \|\nabla u\|_2$. Hence the Sobolev embedding theorem implies that $f \in \dot{B}_{\infty,2}^{1/2}(\mathbb{R}) \subset \dot{B}_{\infty,\rho}^{1/2}(\mathbb{R})$ and

$$\|f\|_{\dot{B}_{\infty,\rho}^{1/2}} \leq C_\rho\|\nabla u\|_2. \tag{B.2}$$

On the other hand, we have

$$
\begin{aligned}
\|f\|_{\dot B^{1/2}_{\infty,\rho}}{}^{\rho} &= \int_0^\infty \left(\frac{\|f(\cdot+t)-f(\cdot)\|_\infty}{t^{1/2}}\right)^\rho \frac{dt}{t} \\
&\geq \int_0^\infty \left(\frac{|f(t+\log\varepsilon_0)|}{t^{1/2}}\right)^\rho \frac{dt}{t} = \int_0^\infty \left(\frac{|v(r\varepsilon_0\omega)|}{(\log|x|)^{1/\rho+1/2}}\right)^\rho \frac{d(\log r)}{dr}\,dr \\
&= \int_0^\infty \left(\frac{|v(r\varepsilon_0\omega)|}{(\log|x|)^{1/\rho+1/2}}\right)^\rho \frac{1}{r}\,dr = C_{\varepsilon_0}\int_{\mathbb{R}^2}\left(\frac{|v(x)|}{|x|^{2/\rho}(\log|x|)^{1/\rho+1/2}}\right)^\rho dx \\
&\geq C_{\varepsilon_0}\int_{\mathbb{R}^2}\left(\frac{|v(x)|}{\lambda_{2\rho/(\rho-2)}(|x|)}\right)^\rho dx.
\end{aligned}
$$

Substituting this estimate into (B.2) we obtain

$$
\left\| v(x)/\lambda_{2\rho(\rho-2)}(|x|)\right\|_\rho \leq C'_{\rho,\varepsilon_0}\|\nabla u\|_2.
$$

The conclusion follows from this estimate and (B.1). □

Appendix C. Proof of Proposition 7.1.

Since $C_0^\infty(\Omega)$ is dense in $\dot H_0^1(\Omega)$, we may assume that $u(x)\in C_0^\infty(\Omega)$. Put $D_k=\{x\in\mathbb{R}^2 \mid 2^{k-1}\leq|x|\leq 2^k\}$ for every $k\in\mathbb{Z}$. Then the assumption implies that $\int_{D_k} u(x)\,dx=0$. Hence, for every $\rho\in[2,\infty)$ and every $p\in\big(2\rho/(2+\rho),\rho\big]$, the Poincaré inequality and the Sobolev embedding theorem imply

$$
2^{k(2/p-2/\rho-1)}\left(\int_{D_k}|u(x)|^\rho\,dx\right)^{1/\rho}\leq C\left(\int_{D_k}|\nabla u(x)|^p\,dx\right)^{1/p}
$$

with a positive constant C depending only on p and ρ. It follows that

$$
\left(\int_{D_k}\big(|x|^{2/p-2/\rho-1}|u(x)|\big)^\rho\,dx\right)^{1/\rho}\leq C\left(\int_{D_k}|\nabla u(x)|^p\,dx\right)^{1/p},
$$

once again the positive constant C depends only on p and ρ. Then we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^2}\big(|x|^{2/p-2/\rho-1}|u(x)|\big)^\rho\,dx\right)^{1/\rho} \\
&\quad= \left(\sum_{k=-\infty}^{\infty}\int_{D_k}\big(|x|^{2/p-2/\rho-1}|u(x)|\big)^\rho\,dx\right)^{1/\rho} \\
&\quad\leq \left\{\sum_{k=-\infty}^{\infty}\left(\int_{D_k}\big(|x|^{2/p-2/\rho-1}|u(x)|\big)^\rho\,dx\right)^{p/\rho}\right\}^{1/p} \\
&\quad\leq C\left\{\sum_{k=-\infty}^{\infty}\int_{D_k}|\nabla u(x)|^p\,dx\right\}^{1/p} = C\left(\int_{\mathbb{R}^2}|\nabla u(x)|^p\,dx\right)^{1/p}. \qquad \text{(C.1)}
\end{aligned}
$$

If $\rho=2$, we put $p=2$. Then (C.1) implies $\big\||x|^{-1}u(x)\big\|_2\leq C\|\nabla u(x)\|_2$.

If $\rho>2$, we put

$$
p_0=\frac{\rho}{\rho-1},\ \rho_0=2,\ p_1=\frac{2\rho}{\rho-1}\ \text{and}\ \rho_1=2\rho.
$$

Then we have $p_0<2<p_1$ and

$$
\frac{2}{p_0}-\frac{2}{\rho_0}=2-\frac{2}{\rho}-1=1-\frac{2}{\rho}=1-\frac{1}{\rho}-\frac{1}{\rho}=\frac{2}{p_1}-\frac{2}{\rho_1}.
$$

Hence (C.1) implies

$$\left\| |x|^{2/p-2/\rho-1} |u(x)| \right\|_{\rho_j} \leq C \|\nabla u(x)\|_{p_j} \text{ for } j = 0, 1. \tag{C.2}$$

Moreover, if we choose $\theta \in (0,1)$ such that $\theta(1/\rho_0 - 1/\rho_1) = 1/\rho_0 - 1/\rho$, then we have

$$\theta\left(\frac{1}{p_0} - \frac{1}{p_1}\right) = \theta\left(1 - \frac{1}{\rho} - \frac{1}{2} + \frac{1}{2\rho}\right) = \theta\left(\frac{1}{2} - \frac{1}{2\rho}\right) = \frac{1}{2} - \frac{1}{\rho} = \frac{1}{p_0} - \frac{1}{2}.$$

It follows that the real interpolation spaces $\left(L^{p_0}(\mathbb{R}^2), L^{p_1}(\mathbb{R}^2)\right)_{\theta,2}$ and $\left(L^{\rho_0}(\mathbb{R}^2), L^{\rho_1}(\mathbb{R}^2)\right)_{\theta,2}$ coincide with $L^{2,2}(\mathbb{R}^2) = L^2(\mathbb{R}^2)$ and $L^{\rho,2}(\mathbb{R}^2)$ respectively. Hence, applying real interpolation to (C.2), we obtain

$$\left\| |x|^{-2/\rho} u(x) \right\|_{\rho,2} \leq C_\rho \|\nabla u(x)\|_2$$

with a positive constant C_ρ. This completes the proof. □

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