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Evolutionary Oseen Model for Generalized Newtonian Fluid with Multivalued Nonmonotone Friction Law

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Abstract. The paper deals with the non-stationary Oseen system of equations for the generalized Newtonian incompressible fluid with multivalued and nonmonotone frictional slip boundary conditions. First, we provide a result on existence of a unique solution to an abstract evolutionary inclusion involving the Clarke subdifferential term for a nonconvex function. We employ a method based on a surjectivity theorem for multivalued L-pseudomonotone operators. Then, we exploit the abstract result to prove the weak unique solvability of the Oseen system.

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1. Introduction

In this paper we investigate the non-stationary Oseen system of equations which describes the flow of a viscous incompressible generalized Newtonian fluid and is governed by nonlinear multivalued and nonmonotone boundary conditions of frictional type. This type of problem occurs when in the non-stationary generalized Navier–Stokes equation the nonlinearity in the convective term is linearized by replacing the first argument by an already computed approximation. Such an approximation is used in implicit time discretization method applied together with a fixed point strategy, see e.g. [5,12] and the references therein.

We consider a nonlinear slip boundary condition which is described by the subdifferential of a nonconvex potential function. In order to deal with the nonconvex potential we exploit the notion of the generalized gradient of Clarke, see [2]. For this reason the weak formulation of the problem takes the form of a parabolic hemivariational inequality. If the potential generating the slip condition is a convex function, then the variational formulation of the problem is a variational inequality, see e.g. [5,9,10,14]. The stationary and non-stationary Oseen equations with homogeneous Dirichlet boundary condition were studied by the Galerkin method in [8] while stationary flow of non-Newtonian fluid with frictional boundary conditions have been recently treated in [23].

The mathematical theory of hemivariational inequalities has started with a pioneering work of Panagiotopoulos [25] and has been extensively developed in the last 30 years mainly because of various applications. We refer to monographs [21,24,26,27] to the wealth of problems which solutions have been possible using the theory of hemivariational inequalities. The hemivariational inequalities which appear in problems of solid mechanics can be found in [11,13,15,18,29] and in problems of fluid mechanics

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in [6,7,19,20]. The basic reference for mathematical analysis of fluid flows governed by Navier–Stokes equations are books by Temam [30] and [31].

In the first part of this paper, we study a class of abstract first order evolutionary subdifferential inclusions in the framework of evolution triple of spaces. For such class we deliver an existence and uniqueness result which is a generalization of earlier contributions in [13,18,22]. The proof of this result is based on a surjectivity theorem for L-pseudomonotone operotors. In the second part, we apply our abstract result to obtain the weak unique solvability of the non-stationary Oseen model.

The paper is organized as follows. In Sect. 2 we briefly recall preliminary material. Section 3 is devoted to the statement and the proof of an existence and uniqueness theorem for an abstract subdifferential inclusion. The physical setting and classical formulation of the Oseen model is given in Sect. 4 and its weak formulation in the form of a hemivariational inequality is provided in Sect. 5. Finally, in Sect. 6 we demonstrate a result on existence of a unique solution to the hemivariational inequality modeling the flow problem, and give examples of the constitutive function and convex and nonconvex potentials.

2. Preliminaries

In this section we shortly recall basic definitions on single-valued and multivalued operators in Banach spaces and on the Clarke subdifferential which are used in the sequel. More details on these topics can be found in monographs [2–4,32].

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space with its topological dual denoted by X^*. The notation $\langle\cdot,\cdot\rangle_{X^*\times X}$ stands for the duality pairing of X^* and X, and the space X endowed with the weak topology is denoted by w-X. Given a set $D \subset X$, we define $\|D\|_X = \sup\{\|d\|_X \mid d \in D\}$.

Consider a multivalued operator $A\colon X \to 2^{X^*}$. It is called bounded if it maps bounded sets into bounded ones. It is called coercive if either its domain $D(A) = \{u \in X \mid Au \neq \emptyset\}$ is bounded or $D(A)$ is unbounded and

$$\lim_{\|u\|_X\to\infty,\ u\in D(A)} \frac{\inf\{\,\langle u^*, u\rangle_{X^*\times X} \mid u^* \in Au\,\}}{\|u\|_X} = +\infty.$$

We recall the notion of pseudomonotonicity of a multivalued operator.

Definition 1. Let $A\colon X \to 2^{X^*}$ be a multivalued operator and $L\colon D(L) \subset X \to X^*$ be a linear and maximal monotone operator. The operator A is called pseudomonotone with respect to L (or L-pseudomonotone) if the following conditions hold

(a) for all $u \in X$ the set Au is a nonempty, bounded, closed, and convex subset of X^*.
(b) A is upper semicontinuous (u.s.c.) from each finite dimensional subspace of X to X^* endowed with the weak topology.
(c) if $\{u_n\} \subset D(L)$, $u_n \to u$ weakly in X, $Lu_n \to Lu$ weakly in X^*, $u_n^* \in Au_n$ is such that $u_n^* \to u^*$ weakly in X^* and $\limsup \langle u_n^*, u_n - u\rangle_{X^*\times X} \le 0$, then $u^* \in Au$ and $\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}$.

We recall the following surjectivity result. It is stated in [4, Theorem 1.3.73] under the hypothesis that X is strictly convex. However, by passing to an equivalent norm on X, we may always assume that X is a strictly convex Banach space.

Theorem 2. *Let X be a reflexive Banach space, let $L\colon D(L) \subset X \to X^*$ be a linear and maximal monotone operator. If $A\colon X \to 2^{X^*}$ is bounded, coercive, and L-pseudomonotone, then $L + A$ is surjective.*

Consider a single-valued operator $A\colon X \to X^*$. The operator A is said to be demicontinuous if for all $w \in X$, the functional $u \mapsto \langle Au, w\rangle_{X^*\times X}$ is continuous, i.e., A is continuous as a mapping from X to w^*-X^*. It is monotone, if for all u, $v \in X$, we have $\langle Au - Av, u - v\rangle_{X^*\times X} \ge 0$. The operator $A\colon X \to X^*$ is said to be bounded if it maps bounded subsets of X into bounded subsets of X^*.

Now, we recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $h\colon E \to \mathbb{R}$ defined on a Banach space E. The generalized directional derivative of h at $x \in E$ in the direction $v \in E$, denoted by $h^0(x; v)$, is defined by

$$h^0(x;v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{h(y+\lambda v)-h(y)}{\lambda}.$$

The generalized gradient of h at $x \in E$, denoted by $\partial h(x)$, is a subset in the dual space E^* given by

$$\partial h(x) = \{\zeta \in E^* \mid h^0(x;v) \geq \langle \zeta, v\rangle_{E^*\times E} \text{ for all } v \in E\}.$$

The basic properties of the generalized directional derivative and the generalized gradient as well as the relations between the generalized directional derivative and classical notions of differentiability can be found in [2,3,21,24].

3. Subdifferential Inclusion of First Order

In this section we study the first order evolutionary inclusion which contains the Clarke subdifferential operator. Our aim is to prove an existence and uniqueness result.

We study the inclusion within the framework of an evolution (Gelfand) triple of spaces $V \subset H \subset V^*$, where V is a reflexive and separable Banach space, H is a separable Hilbert space, the embedding $V \subset H$ is continuous, and V is dense in H. Given $0 < T < \infty$, $2 \leq p < \infty$ and $1/p + 1/q = 1$, we introduce the spaces $\mathcal{V} = L^p(0,T;V)$ and $\mathcal{W} = \{w \in \mathcal{V} \mid w' \in \mathcal{V}^*\}$, where the time derivative $w' = \partial w/\partial t$ is understood in the sense of vector-valued distributions. It follows from reflexivity of V that both $\mathcal{V}$ and its dual space $\mathcal{V}^* = L^q(0,T;V^*)$ are reflexive Banach spaces. It is known that the space $\mathcal{W}$ endowed with the graph norm $\|w\|_{\mathcal{W}} = \|w\|_{\mathcal{V}} + \|w'\|_{\mathcal{V}^*}$ is a separable and reflexive Banach space. Let $\mathcal{H} = L^2(0,T;H)$. Identifying $\mathcal{H}$ with its dual, we obtain the following continuous embeddings $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. It is known that the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous, where $C(0,T;H)$ is the Banach space of all continuous functions on $[0,T]$ with values in H. An important corollary of this embedding is that the values of any function in $\mathcal{W}$ are well defined in H for all $t \in [0,T]$. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is denoted by

$$\langle w, v\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle w(t), v(t)\rangle_{V^*\times V}\, dt \ \text{ for } w \in \mathcal{V}^*,\ v \in \mathcal{V},$$

where $\langle\cdot,\cdot\rangle_{V^*\times V}$ stands for the duality brackets of the pair (V^*, V).

Let $A\colon (0,T)\times V \to V^*$ and $J\colon (0,T)\times V \to \mathbb{R}$. We assume that J is locally Lipschitz in its second argument and we denote by ∂J the Clarke generalized gradient of J with respect to its second argument. Given $f\colon (0,T) \to V^*$ and $w_0 \in V$, we consider the following evolutionary inclusion.

Problem 3. Find $w \in \mathcal{W}$ such that

$$\begin{cases} w'(t) + A(t,w(t)) + \partial J(t,w(t)) \ni f(t) \ \text{ a.e. } t \in (0,T),\\ w(0) = w_0. \end{cases}$$

In the study of Problem 3 we introduce the following definition.

Definition 4. By a solution of Problem 3 we mean a function $w \in \mathcal{W}$ for which there exists $w^* \in \mathcal{V}^*$ such that $w^*(t) \in \partial J(t,w(t))$ for a.e. $t \in (0,T)$, $w'(t) + A(t,w(t)) + w^*(t) = f(t)$ for a.e. $t \in (0,T)$ and $w(0) = w_0$.

We need the following hypotheses on the data.

$\underline{H(A)}$: $A\colon (0,T)\times V \to V^*$ is such that

(1) $A(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$.
(2) $A(t,\cdot)$ is demicontinuous on V for a.e. $t \in (0,T)$.
(3) $\|A(t,v)\|_{V^*} \leq a_0(t) + a_1\|v\|_V^{p-1}$ for all $v \in V$, a.e. $t \in (0,T)$ with $a_0 \in L^q(0,T)$, $a_0 \geq 0$ and $a_1 \geq 0$.
(4) $A(t,\cdot)$ is strongly monotone for a.e. $t \in (0,T)$, i.e., for a constant $m_A > 0$,

$$\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_{V^*\times V} \geq m_A\, \|v_1 - v_2\|_V^p$$

for all $v_1,\ v_2 \in V$, a.e. $t \in (0,T)$.

$\underline{H(J)}$: $J\colon (0,T)\times V\to\mathbb{R}$ is such that

(1) $J(\cdot,v)$ is measurable on $(0,T)$ for all $v\in V$.
(2) $J(t,\cdot)$ is locally Lipschitz on V for a.e. $t\in(0,T)$.
(3) $\|\partial J(t,v)\|_{V^*}\le c_0(t)+c_1\|v\|_V^{p-1}$ for all $v\in V$, a.e. $t\in(0,T)$ with $c_0\in L^q(0,T)$, $c_0\ge 0$, $c_1\ge 0$.
(4) $\partial J(t,\cdot)$ is relaxed monotone for a.e. $t\in(0,T)$, i.e., for a constant $m_J\ge 0$,

$$\langle \partial J(t,v_1)-\partial J(t,v_2),v_1-v_2\rangle_{V^*\times V}\ge -m_J\,\|v_1-v_2\|_V^p$$

for all v_1, $v_2\in V$, a.e. $t\in(0,T)$.

$\underline{(H_1)}$: $f\in\mathcal{V}^*$, $w_0\in V$.

$\underline{(H_2)}$: $m_A>2^{2p-3}\max\{2m_J,c_1\}$.

We have the following existence and uniqueness result.

Theorem 5. *Assume hypotheses $H(A)$, $H(J)$, (H_1) and (H_2). Then Problem 3 has a unique solution.*

Proof. It consists of two parts, in Step 1 we prove the existence, and in Step 2 the uniqueness.

Step 1. Let $\mathcal{A}\colon\mathcal{V}\to\mathcal{V}^*$ and $\mathcal{B}\colon\mathcal{V}\to 2^{\mathcal{V}^*}$ be the Nemitsky operators corresponding to the translations of $A(t,\cdot)$ and $\partial J(t,\cdot)$ by the element w_0,

$$(\mathcal{A}v)(t)=A(t,v(t)+w_0),$$
$$(\mathcal{B}v)(t)=\{\,v^*\in\mathcal{V}^*\mid v^*(t)\in\partial J(t,v(t)+w_0)\ \text{ for a.e. } t\in(0,T)\,\}$$

for $v\in\mathcal{V}$ and a.e. $t\in(0,T)$. Next, we introduce an operator $\mathcal{F}\colon\mathcal{V}\to 2^{\mathcal{V}^*}$ defined by

$$\mathcal{F}v=\mathcal{A}v+\mathcal{B}v\quad\text{for } v\in\mathcal{V}.$$

Define an operator $L\colon D(L)\subset\mathcal{V}\to\mathcal{V}^*$ by

$$Lv=v'\ \text{ for } v\in D(L)$$

with its domain $D(L)=\{w\in\mathcal{W}\mid w(0)=0\}$. The operator L is linear and maximal monotone (see [32, Proposition 32.10]). With these operators, we consider the following inclusion

$$\begin{cases} Lw+\mathcal{F}w\ni f\ \text{ in }\ \mathcal{V}^*,\\ w(0)=0.\end{cases}\tag{1}$$

Then, $w\in\mathcal{W}$ is a solution of Problem 3 if and only if $w-w_0\in\mathcal{W}$ satisfies (1).

In what follows we are going to apply Theorem 2 to prove that problem (1) has a solution. For this goal, we will show that $\mathcal{F}$ has the properties required in Theorem 2.

Claim 1. $\mathcal{F}$ is a bounded operator. Let $v\in\mathcal{V}$ and $v^*\in\mathcal{F}v$. Then $v^*=\mathcal{A}v+w^*$ with $w^*\in\mathcal{B}v$. Using hypotheses $H(A)(3)$ and $H(J)(3)$, we have

$$\|\mathcal{A}v\|_{\mathcal{V}^*}\le\|a_0\|_{L^q(0,T)}+a_1\,\|v+w_0\|_{\mathcal{V}}^{p-1},\tag{2}$$
$$\|w^*\|_{\mathcal{V}^*}\le\|c_0\|_{L^q(0,T)}+c_1\,\|v+w_0\|_{\mathcal{V}}^{p-1}.\tag{3}$$

Combining these inequalities, we immediately deduce that $\mathcal{F}$ is a bounded operator, being the sum of two bounded operators.

Claim 2. $\mathcal{F}$ is a coercive operator. First, by $H(A)(3)$ and (4), we have the following coercivity condition for $A(t,\cdot)$

$$\begin{aligned}\langle A(t,v),v\rangle_{V^*\times V}&=\langle A(t,v)-A(t,0),v\rangle_{V^*\times V}+\langle A(t,0),v\rangle_{V^*\times V}\\ &\ge m_A\,\|v\|_V^p-a_0(t)\|v\|_V\end{aligned}\tag{4}$$

for all $v \in V$, a.e. $t \in (0,T)$. Next, let $v \in \mathcal{V}$ and $v^* \in \mathcal{F}v$. Then $v^* = \mathcal{A}v + w^*$ where $w^* \in \mathcal{B}v$. Using $H(A)(3)$, the inequality (4), the Hölder inequality, and the inequality $|a+b|^p \geq 2^{1-p}|a|^p - |b|^p$ for a, $b \in \mathbb{R}$ and $1 < p < \infty$, we obtain

$$\begin{aligned}\langle \mathcal{A}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \left(\langle A(t,v(t)+w_0), v(t)+w_0\rangle_{V^*\times V} - \langle A(t,v(t)+w_0), w_0\rangle_{V^*\times V}\right) dt \\ &\geq \int_0^T \left(m_A\|v(t)+w_0\|_V^p - a_0(t)\|v(t)+w_0\|_V - \left(a_0(t) + a_1\|v(t)+w_0\|_V^{p-1}\right)\|w_0\|_V\right) dt \\ &\geq 2^{1-p}m_A\|v\|_{\mathcal{V}}^p - a_1 2^{p-2}\|w_0\|_{\mathcal{V}}\|v\|_{\mathcal{V}}^{p-1} - \|a_0\|_{L^q(0,T)}\|v\|_{\mathcal{V}} \\ &\quad - \left(m_A + a_1 2^{p-2}\right)\|w_0\|_{\mathcal{V}}^p - 2\|w_0\|_{\mathcal{V}}\|a_0\|_{L^q(0,T)}.\end{aligned}$$

Since $w^* \in \mathcal{B}v$, we have $w^*(t) \in \partial J(t, v(t)+w_0)$ for a.e. $t \in (0,T)$. Using $H(J)(3)$ and (4), we obtain

$$\begin{aligned}\langle w^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &\geq \int_0^T \left(-m_J\|v(t)+w_0\|_V^p - c_0(t)\|v(t)+w_0\|_V\right. \\ &\quad \left. - (c_0(t) + c_1\|v(t)+w_0\|_V^{p-1})\|w_0\|_V\right) dt \\ &\geq -m_J\, 2^{p-1}\|v\|_{\mathcal{V}}^p - c_1 2^{p-2}\|w_0\|_{\mathcal{V}}\|v\|_{\mathcal{V}}^{p-1} - \|c_0\|_{L^q(0,T)}\|v\|_{\mathcal{V}} \\ &\quad - 2^{p-2}(2m_J + c_1)\|w_0\|_{\mathcal{V}}^p - 2\|c_0\|_{L^q(0,T)}\|w_0\|_{\mathcal{V}}.\end{aligned}$$

Hence

$$\begin{aligned}\langle v^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle \mathcal{A}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle w^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq \left(2^{1-p}m_A - m_J 2^{p-1}\right)\|v\|_{\mathcal{V}}^p \\ &\quad - 2^{p-2}\left(a_1 + c_1\right)\|w_0\|_{\mathcal{V}}\|v\|_{\mathcal{V}}^{p-1} - \left(\|a_0\|_{L^q(0,T)} + \|c_0\|_{L^q(0,T)}\right)\|v\|_{\mathcal{V}} - c_4\end{aligned}$$

with $c_4 = (m_A + a_1 2^{p-2} + m_J 2^{p-1} + c_1 2^{p-2})\|w_0\|_{\mathcal{V}}^p + 2\left(\|c_0\|_{L^q(0,T)} + \|a_0\|_{L^q(0,T)}\right)\|w_0\|_{\mathcal{V}}$. On the other hand, from $H(J)(3)$, by the Hölder inequality, we deduce

$$\begin{aligned}\langle w^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &\leq \int_0^T (c_0(t) + c_1\|v(t)+w_0\|_V^{p-1})\|v(t)\|_V\, dt \\ &\leq c_1 2^{p-2}\|v\|_{\mathcal{V}}^p + c_1 2^{p-2}\|w_0\|_{\mathcal{V}}^{p-1}\|v\|_{\mathcal{V}} + \|c_0\|_{L^q(0,T)}\|v\|_{\mathcal{V}}.\end{aligned}$$

In a consequence, we have

$$\begin{aligned}\langle v^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle \mathcal{A}v, v\rangle_{\mathcal{V}^*\times\mathcal{V}} + \langle w^*, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq \left(2^{1-p}m_A - c_1 2^{p-2}\right)\|v\|_{\mathcal{V}}^p \\ &\quad - a_1 2^{p-2}\|w_0\|_{\mathcal{V}}\|v\|_{\mathcal{V}}^{p-1} - \left(\|a_0\|_{L^q(0,T)} + c_1 2^{p-2}\|w_0\|_{\mathcal{V}}^{p-1} + \|c_0\|_{L^q(0,T)}\right)\|v\|_{\mathcal{V}} - c_5\end{aligned}$$

with $c_5 = (m_A + a_1 2^{p-2})\|w_0\|_{\mathcal{V}}^p + 2\|w_0\|_{\mathcal{V}}\|a_0\|_{L^q(0,T)}$. Thus, it is clear from (H_2) that $\mathcal{F}$ is a coercive operator.

Claim 3. $\mathcal{F}$ is a L-pseudomonotone operator. First, we show the following properties of the operator $\mathcal{A}$.

$$\mathcal{A}\colon \mathcal{V} \to \mathcal{V}^* \text{ is demicontinuous,} \tag{5}$$

$$\langle \mathcal{A}v_1 - \mathcal{A}v_2, v_1 - v_2\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq m_A\|v_1 - v_2\|_{\mathcal{V}}^p \text{ for all } v_1, v_2 \in \mathcal{V}. \tag{6}$$

For a proof of (5), let $v_n \to v$ in $\mathcal{V}$. By passing to a subsequence if necessary, by [21, Theorem 2.39], we have $v_n(t) \to v(t)$ in V for a.e. $t \in (0,T)$ and $\|v_n(t)\|_V \leq h(t)$ for a.e. $t \in (0,T)$ with $h \in L^p(0,T)$. Exploiting $H(A)(2)$, we deduce

$$A(t, v_n(t)) \to A(t, v(t)) \text{ weakly in } V^*, \text{ a.e. } t \in (0,T).$$

Hence, we have $\langle A(t,v_n(t)), \varphi(t)\rangle_{V^*\times V} \to \langle A(t,v(t)), \varphi(t)\rangle_{V^*\times V}$ for all $\varphi \in \mathcal{V}$, a.e. $t \in (0,T)$. We use $H(A)(3)$ and apply the Lebesgue dominated convergence theorem, see e.g. [21, Theorem 2.38], to obtain

$$\lim \int_0^T \langle A(t,v_n(t)), \varphi(t)\rangle_{V^*\times V}\, dt = \int_0^T \langle A(t,v(t)), \varphi(t)\rangle_{V^*\times V}\, dt.$$

Thus $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$. The standard argument shows that the entire sequence $\{\mathcal{A}v_n\}$ converges weakly in $\mathcal{V}^*$ to $\mathcal{A}v$. This concludes the proof of property (5).+

The strong monotonicity property for $\mathcal{A}$ in (6) follows directly from hypothesis $H(A)(4)$.

We now prove that the operator $\mathcal{F}$ is pseudomonotone with respect to L, that is, it satisfies conditions (a)–(c) of Definition 1.

(a) For every $v \in \mathcal{V}$, the set $\mathcal{F}v$ is a nonempty, bounded, closed, and convex in $\mathcal{V}^*$. The fact that values of the operator $\mathcal{F}$ are nonempty and convex follows from the well known property (see [2, Proposition 2.1.2]) that values of $\partial J(t,\cdot)$ are nonempty and convex subsets of V^* for a.e. $t \in (0,T)$. From (2) and (3), it follows that $\|v^*\|_{\mathcal{V}^*} \le \overline{d}_0 + \overline{d}_1\|v + w_0\|_{\mathcal{V}}^{p-1}$ for all $v^* \in \mathcal{F}v = \mathcal{A}v + \mathcal{B}v$, $v \in \mathcal{V}$ with $\overline{d}_0$, $\overline{d}_1 \ge 0$. Hence, the set $\mathcal{F}v$ is bounded in $\mathcal{V}^*$ for all $v \in \mathcal{V}$. The set $\mathcal{B}v$ is also closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$. Indeed, let $v \in \mathcal{V}$, $v_n^* \in \mathcal{V}^*$, $v_n^* \in \mathcal{B}v$, $v_n^* \to v^*$ in $\mathcal{V}^*$. Passing to a subsequence if necessary, we may suppose that $v_n^*(t) \to v^*(t)$ in V^* for a.e. $t \in (0,T)$. We have

$$v_n^*(t) \in \partial J(t, v(t) + w_0) \ \text{ a.e. } t \in (0,T)$$

and the latter is a closed subset of V^*. Thus $v^*(t) \in \partial J(t, v(t) + w_0)$ for a.e. $t \in (0,T)$, i.e., $v^* \in \mathcal{B}v$, which proves the closedness of the set $\mathcal{B}v$. Hence, the set $\mathcal{F}v$ is closed in $\mathcal{V}^*$ for all $v \in \mathcal{V}$, which concludes the proof of (a).

(b) The operator $\mathcal{F}$ is u.s.c. from $\mathcal{V}$ into $2^{\mathcal{V}^*}$, where $\mathcal{V}^*$ is endowed with the weak topology. In order to show this property, we apply [3, Proposition 4.1.4]. To this end, we prove that the weak inverse image $\mathcal{F}^-(D) = \{v \in \mathcal{V} \mid \mathcal{F}v \cap D \ne \emptyset\}$ is a closed subset of $\mathcal{V}$, for every weakly closed set $D \subset \mathcal{V}^*$. Let $\{v_n\} \subset \mathcal{F}^-(D)$ be such that $v_n \to v$ in $\mathcal{V}$. We may assume, passing to a subsequence if necessary, that

$$v_n(t) \to v(t) \ \text{ in } V, \text{ a.e. } t \in (0,T). \tag{7}$$

Therefore, there exists $v_n^* \in \mathcal{F}v_n \cap D$ for $n \in \mathbb{N}$, that is,

$$v_n^* = \mathcal{A}v_n + w_n^* \tag{8}$$

with $w_n^* \in \mathcal{B}v_n$ and $v_n^* \in D$. Since $\{v_n\}$ is bounded in $\mathcal{V}$ and the operators $\mathcal{A}$ and $\mathcal{B}$ are bounded (cf. Claim 1), we know that $\{v_n^*\}$ and $\{w_n^*\}$ are both bounded in $\mathcal{V}^*$. Thus, at least for subsequences, we may suppose that

$$v_n^* \to v^*, \qquad w_n^* \to w^* \ \text{ weakly in } \mathcal{V}^*$$

with v^*, $w^* \in \mathcal{V}^*$. Since D is weakly closed in $\mathcal{V}^*$, we have $v^* \in D$. By the definition of the operator $\mathcal{B}$, we have

$$w_n^*(t) \in \partial J(t, v_n(t) + w_0) \ \text{ a.e. } t \in (0,T). \tag{9}$$

Taking into account the convergences (7) and $w_n^* \to w^*$ weakly in $\mathcal{V}^*$, and the fact that $\partial J(t,\cdot)$ is u.s.c. with closed and convex values, we can apply a convergence theorem found in [1, p. 60] to the inclusion (9) and deduce $w^*(t) \in \partial J(t, v(t) + w_0)$ a.e. $t \in (0,T)$. Hence, $w^* \in \mathcal{B}v$.

By the demicontinuity of the operator $\mathcal{A}$ [cf. (5)], we have $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$. Passing to the limit in (8), we obtain $v^* = \mathcal{A}v + w^*$, where $w^* \in \mathcal{B}v$ and $v^* \in D$. Therefore, $v^* \in \mathcal{F}v \cap D$, which implies $v^* \in \mathcal{F}^-(D)$. This proves that $\mathcal{F}^-(D)$ is closed in $\mathcal{V}$ and concludes the proof of condition (b).

(c) The condition (c) of Definition 1 holds. Let $\{v_n\} \subset D(L)$, $v_n \to v$ weakly in $\mathcal{W}$, $v_n^* \in \mathcal{F}v_n$, $v_n^* \to v^*$ weakly in $\mathcal{V}^*$ and

$$\limsup \langle v_n^*, v_n - v\rangle_{\mathcal{V}^* \times \mathcal{V}} \le 0. \tag{10}$$

We prove that $v^* \in \mathcal{F}v$ and

$$\langle v_n^*, v_n\rangle_{\mathcal{V}^* \times \mathcal{V}} \to \langle v^*, v\rangle_{\mathcal{V}^* \times \mathcal{V}}. \tag{11}$$

First, we observe that the operator $\mathcal{F}\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ is strongly monotone. Indeed, by $H(J)(4)$, we have

$$\begin{aligned}\langle w_1^* - w_2^*, v_1 - v_2\rangle_{\mathcal{V}^* \times \mathcal{V}} &= \int_0^T \langle w_1^*(t) - w_2^*(t), v_1(t) - v_2(t)\rangle_{V^* \times V}\, dt \\ &\ge -m_J \int_0^T \|v_1(t) - v_2(t)\|_V^p\, dt = -m_J \|v_1 - v_2\|_{\mathcal{V}}^p\end{aligned}$$

for all $w_i^* \in \mathcal{B}v_i$, $v_i \in \mathcal{V}$, $i = 1, 2$. This inequality together with (6) imply

$$\langle v_1^* - v_2^*, v_1 - v_2 \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq (m_A - m_J) \|v_1 - v_2\|_{\mathcal{V}}^p$$

for all $v_i^* \in \mathcal{F}v_i$, $v_i \in \mathcal{V}$, $i = 1, 2$. From (H_2), it follows that the operator $\mathcal{F}$ is strongly monotone.

Next, we prove that $v_n \to v$ in $\mathcal{V}$. From the strong monotonicity of $\mathcal{F}$, we have

$$(m_A - m_J) \|v_n - v\|_{\mathcal{V}}^p \leq \langle v_n^* - \eta, v_n - v \rangle_{\mathcal{V}^* \times \mathcal{V}}$$

for all $v_n^* \in \mathcal{F}v_n$, $\eta \in \mathcal{F}v$. Taking lim sup in the last inequality and using (10), we obtain

$$\begin{aligned} 0 &\leq (m_A - m_J) \liminf \|v_n - v\|_{\mathcal{V}}^p \leq (m_A - m_J) \limsup \|v_n - v\|_{\mathcal{V}}^p \\ &\leq \limsup \langle v_n^*, v_n - v \rangle_{\mathcal{V}^* \times \mathcal{V}} - \lim \langle \eta, v_n - v \rangle_{\mathcal{V}^* \times \mathcal{V}} \leq 0, \end{aligned}$$

implying $v_n \to v$ in $\mathcal{V}$.

Using the strong convergence of v_n to v in $\mathcal{V}$, and passing to a subsequence if necessary, we may assume

$$v_n(t) \to v(t) \ \text{ in } V, \ \text{a.e. } t \in (0, T). \tag{12}$$

Consequently, from $v_n^* \in \mathcal{F}v_n$, we have

$$v_n^* = \mathcal{A}v_n + w_n^* \tag{13}$$

with $w_n^* \in \mathcal{B}v_n$, and thus

$$w_n^*(t) \in \partial J(t, v_n(t) + w_0) \ \text{ a.e. } t \in (0, T).$$

By the boundedness of the operator $\mathcal{B}$ (cf. Claim 1), we can assume that $w_n^* \to w^*$ weakly in $\mathcal{V}^*$ with $w^* \in \mathcal{V}^*$. Similarly as in the proof of condition (b), we use the convergences (12) and $w_n^* \to w^*$ weakly in $\mathcal{V}^*$, and apply the convergence theorem of [1, p. 60] to obtain

$$w^*(t) \in \partial J(t, v(t) + w_0) \ \text{ a.e. } t \in (0, T).$$

Hence, $w^* \in \mathcal{B}v$. By the demicontinuity of the operator $\mathcal{A}$ [cf. (5)], we obtain $\mathcal{A}v_n \to \mathcal{A}v$ weakly in $\mathcal{V}^*$. Passing to the limit in (13), we get $v^* = \mathcal{A}v + w^*$. Since $w^* \in \mathcal{B}v$, we have $v^* \in \mathcal{F}v$. From $v_n^* \to v^*$ weakly in $\mathcal{V}^*$ and $v_n \to v$ in $\mathcal{V}$, we deduce (11), which concludes the proof of condition (c).

Having established Claims 1–3 and noting that the operator L is linear and maximal monotone, we are in a position to apply Theorem 2. We deduce that the problem (1) has at least one solution $w \in D(L)$. Then, $w + w_0 \in \mathcal{W}$ is a solution of Problem 3. This concludes the proof of the existence part of the theorem.

Step 2. We prove the uniqueness part of a solution to Problem 3. Assume $w_1, w_2 \in \mathcal{W}$ are two solutions. Then, there are $w_i^* \in \mathcal{V}^*$ such that

$$\begin{cases} w_i'(s) + A(t, w_i(s)) + w_i^*(s) = f(s) \ \text{ a.e. } s \in (0, T), \\ w_i^*(s) \in \partial J(s, w_i(s)) \ \text{ a.e. } s \in (0, T), \\ w_i(0) = w_0 \end{cases}$$

for $i = 1, 2$. We subtract the two equations for w_1 and w_2, take the result in duality with $w_1(t) - w_2(t)$, integrate from 0 to t, and note that $w_1(0) - w_2(0) = 0$ to obtain

$$\begin{aligned} \frac{1}{2} \|w_1(t) - w_2(t)\|_H^2 &+ \int_0^t \langle A(s, w_1(s)) - A(s, w_2(s)), w_1(s) - w_2(s) \rangle_{V^* \times V} \, ds \\ &+ \int_0^t \langle w_1^*(s) - w_2^*(s), w_1(s) - w_2(s) \rangle_{V^* \times V} \, ds = 0 \end{aligned}$$

for all $t \in [0, T]$. From hypotheses $H(A)(4)$ and $H(J)(4)$, we obtain

$$\frac{1}{2} \|w_1(t) - w_2(t)\|_H^2 + (m_A - m_J) \int_0^t \|w_1(s) - w_2(s)\|_V^p \, ds \leq 0$$

for all $t \in [0, T]$. Hence, by the smallness condition in (H_2), it follows that $w_1 = w_2$ on $[0, T]$, i.e., a solution to Problem 3 is unique. □

Remark 6. It can be shown that the hypothesis $H(J)(4)$ is equivalent to the following condition

$$J^0(t, v_1; v_2 - v_1) + J^0(t, v_2; v_1 - v_2) \leq m_J \|v_1 - v_2\|_V^p \tag{14}$$

for all $v_1,\ v_2 \in V$, a.e. $t \in (0, T)$. The latter has been recently used with $p = 2$ in the literature to prove the uniqueness of the solution to the variational–hemivariational inequality. We refer to [21,29] for examples of nonconvex functions which satisfy the condition (14). Furthermore, we note that when $J(t, \cdot)$ is convex, then (14) holds with $m_J = 0$, i.e., the condition (14) simplifies to the monotonicity of the (convex) subdifferential.

4. Physical Setting and Classical Formulation

In this section we introduce the physical setting of the fluid flow problem and provide the classical description of the Oseen model.

The general physical setting is as follows. A viscous incompressible generalized Newtonian fluid occupies an open, bounded and connected set Ω in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma = \partial\Omega$ supposed to be Lipschitz continuous. We denote by $\boldsymbol{\nu} = (\nu_i)$ the unit outward normal vector on Γ, by $\boldsymbol{x} = (x_i) \in \overline{\Omega}$ the position vector, and by $t \in (0, T)$ the time, where $0 < T < \infty$. We also assume that the boundary Γ is composed of two sets $\overline{\Gamma}_D$ and $\overline{\Gamma}_C$, with disjoint relatively open sets Γ_D and Γ_C such that $|\Gamma_D| > 0$.

We deal with the following non-stationary Oseen model which is used for the flow of incompressible fluid. The non-stationary flow of an incompressible generalized Newtonian fluid may be described by the following conservation laws (cf. e.g. [17] for further details)

$$\begin{aligned} \boldsymbol{u}' - \operatorname{Div} \boldsymbol{S} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad &\text{in } \ \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{u} = 0 \quad &\text{in } \ \Omega \times (0, T). \end{aligned} \tag{15}$$

Here $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and $\pi = \pi(\boldsymbol{x}, t)$ denote the velocity field and the pressure, respectively, and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ is the external (gravity) force field. The expression $(\boldsymbol{b} \cdot \nabla)\boldsymbol{u} = \left(\sum_{j=1}^d b_j \frac{\partial u_i}{\partial x_j}\right)_{i=1,d}$ denotes the convective term, and the solenoidal (divergence free) condition $\operatorname{div} \boldsymbol{u} = \nabla \cdot \boldsymbol{u} = 0$ in Ω states that the motion of the fluid is incompressible. Here, $\boldsymbol{b}$ is a given convection field which has to be divergence-free. The symbols Div and div denote the divergence operators for tensor and vector valued functions $\boldsymbol{S} \colon \Omega \to \mathbb{S}^d$ and $\boldsymbol{u} \colon \Omega \times (0, T) \to \mathbb{R}^d$ defined by

$$\operatorname{Div} \boldsymbol{S} = (S_{ij,j}) \ \text{ and } \ \operatorname{div} \boldsymbol{u} = (u_{i,i}),$$

and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\boldsymbol{x}$. From time to time, we suppress the explicit dependence of the quantities on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$.

The total stress tensor in the fluid is given by $\boldsymbol{\sigma} = -\pi\,\boldsymbol{I} + \boldsymbol{S}$ in Ω, where $\boldsymbol{I}$ denotes the identity matrix and $\boldsymbol{S} \colon \Omega \to \mathbb{S}^d$ is the extra (viscous) part of the stress tensor. The symmetric part of the velocity gradient $\boldsymbol{D} \colon \Omega \to \mathbb{S}^d$ is given by $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top)$. We assume that the extra stress tensor $\boldsymbol{S}$ is related with the symmetric part of the velocity gradient $\boldsymbol{D}$ by means of a constitutive law $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}))$ in Ω. Also, we mention that for $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})) = \boldsymbol{D}(\boldsymbol{u})$ the Eq. (15) reduces to the linear Oseen system.

We complement the above system with boundary conditions. Our main interest lies in the contact and slip frictional boundary conditions on the surface Γ_C. On the part Γ_D of the boundary, the fluid adheres to the wall, and therefore, we consider, for simplicity, the homogeneous Dirichlet condition

$$\boldsymbol{u} = \boldsymbol{0} \ \text{ on } \ \Gamma_D \times (0, T).$$

On the part Γ_C, we decompose the velocity vector into the normal and tangential parts. We denote by u_ν and $\boldsymbol{u}_\tau$ the normal and the tangential components of $\boldsymbol{u}$ on the boundary Γ_C, i.e., $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$. Similarly, for an extra stress tensor field $\boldsymbol{S}$, we define its normal and tangential components by $S_\nu = (\boldsymbol{S}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{S}_\tau = \boldsymbol{S}\boldsymbol{\nu} - S_\nu \boldsymbol{\nu}$, respectively. We assume that there is no flux condition through Γ_C, so that the normal component of the velocity on this part of the boundary satisfies

$$u_\nu = 0 \quad \text{on} \quad \Gamma_C \times (0,T).$$

The tangential components of the stress tensor and the velocity are assumed to satisfy the following multivalued friction law

$$-\boldsymbol{S}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_C \times (0,T),$$

where $j\colon \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ is the prescibed function. Finally, the problem is suplemented by the initial condition

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega.$$

Under these notation, the classical formulation of the Oseen model for flow of incompressible fluid reads as follows.

Problem P. Find a velocity field $\boldsymbol{u}\colon \Omega \times (0,T) \to \mathbb{R}^d$, an extra stress tensor $\boldsymbol{S}\colon \Omega \to \mathbb{S}^d$, and a pressure $\pi\colon \Omega \times (0,T) \to \mathbb{R}$ such that

$$\boldsymbol{u}' - \operatorname{Div} \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u})) + (\boldsymbol{b} \cdot \nabla)\,\boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{in} \quad \Omega \times (0,T), \tag{16}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{17}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D \times (0,T), \tag{18}$$

$$u_\nu = 0 \quad \text{on} \quad \Gamma_C \times (0,T), \tag{19}$$

$$-\boldsymbol{S}_\tau \in \partial j(\boldsymbol{u}_\tau) \quad \text{on} \quad \Gamma_C \times (0,T), \tag{20}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \tag{21}$$

5. Weak Formulation

In this section we present the variational formulation of Problem P. To this end, we introduce some additional notation and state the hypotheses on the data. We will treat the problem in the case $d = 2$ and $d = 3$.

We use the symbol $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \text{for all } \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively.

Let $2 \le p < \infty$ and consider the following spaces

$$\widetilde{V} = \{\, \boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^d) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} = 0 \text{ on } \Gamma_D,\ v_\nu = 0 \text{ on } \Gamma_C \},$$
$$V = \text{ closure of } \widetilde{V} \text{ in } W^{1,p}(\Omega;\mathbb{R}^d)$$

and

$$\widetilde{H} = \{\, \boldsymbol{v} \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^d) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ v_\nu = 0 \text{ on } \Gamma_C \},$$
$$H = \text{ closure of } \widetilde{V} \text{ in } L^2(\Omega;\mathbb{R}^d).$$

The space V is equipped with the norm $\|\boldsymbol{v}\| = \|\boldsymbol{v}\|_{W^{1,p}(\Omega;\mathbb{R}^d)}$ for $\boldsymbol{v} \in V$. On V we introduce also the norm given by $\|\boldsymbol{v}\|_V = \|\boldsymbol{D}(\boldsymbol{v})\|_{L^p(\Omega;\mathbb{S}^d)}$ for $\boldsymbol{v} \in V$. From the Korn inequality $c_K\|\boldsymbol{v}\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \le \|\boldsymbol{D}(\boldsymbol{v})\|_{L^p(\Omega;\mathbb{S}^d)}$ for $\boldsymbol{v} \in V$ with $c_K > 0$ (cf. [7, Theorem 4]), it follows that $\|\cdot\|_{W^{1,p}(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_V$ are the equivalent norms on V. Moreover, V is a reflexive separable Banach space, H is a separable Hilbert space, the embedding $V \subset H$ is continuous and V is dense in H. This means that (V, H, V^*) is an evolution triple of spaces. Recall that in this setting, the space H is identified with its dual and we have $V \subset H \subset V^*$ with dense and continuous embeddings.

Next, analogously as in Sect. 3, we define the spaces $\mathcal{V} = L^p(0,T;V)$, $\mathcal{V}^* = L^q(0,T,V^*)$ and $\mathcal{W} = \{w \in \mathcal{V} \mid w' \in \mathcal{V}^*\}$. We also introduce the space $Y = L^p(\Gamma_C;\mathbb{R}^d)$ and the trace operator $\gamma\colon V \to Y$. Its norm is denoted by $\|\gamma\| = \|\gamma\|_{\mathcal{L}(V;Y)}$. For $\boldsymbol{v} \in V$, we use the same symbol $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary. Note that $v_\nu = 0$ and $\boldsymbol{v}_\tau = \gamma \boldsymbol{v}$ on Γ_C for all $\boldsymbol{v} \in V$.

Furthermore, we also recall that the following Green formulas hold (cf. [21, Theorems 2.24 and 2.25])

$$\int_\Omega \boldsymbol{S} : \boldsymbol{D}(\boldsymbol{v})\,dx + \int_\Omega \mathrm{Div}\,\boldsymbol{S}\cdot\boldsymbol{v}\,dx = \int_{\partial\Omega} \boldsymbol{S}\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma \tag{22}$$

for smooth tensor $\boldsymbol{S}\colon \Omega \to \mathbb{S}^d$ and field $\boldsymbol{v}\colon \Omega \to \mathbb{R}^d$, and

$$\int_\Omega \boldsymbol{w}\cdot\nabla\psi\,dx + \int_\Omega \mathrm{div}\;\boldsymbol{w}\,\psi\,dx = \int_{\partial\Omega} w_\nu\,\psi\,d\Gamma \tag{23}$$

for smooth vector fields $\boldsymbol{w}\colon \Omega \to \mathbb{R}^d$ and $\psi\colon \Omega \to \mathbb{R}$.

In the study of Problem P, we will assume that the constitutive function $\boldsymbol{S}$ and the nonconvex potential j satisfy the following hypotheses.

$\underline{H(S)}:$ $\quad \boldsymbol{S}\colon \Omega\times\mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) $\boldsymbol{S}(\cdot,\boldsymbol{D})$ is measurable on Ω for all $\boldsymbol{D}\in\mathbb{S}^d$ and $\boldsymbol{S}(\boldsymbol{x},\cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $\boldsymbol{x}\in\Omega$.
(ii) $\|\boldsymbol{S}(\boldsymbol{x},\boldsymbol{D})\|_{\mathbb{S}^d} \le a_S(1+\|\boldsymbol{D}\|_{\mathbb{S}^d}^{p-1})$ for all $\boldsymbol{D}\in\mathbb{S}^d$, a.e. $\boldsymbol{x}\in\Omega$ with $a_S>0$.
(iii) $\boldsymbol{S}(\boldsymbol{x},\cdot)$ is strongly monotone for a.e. $\boldsymbol{x} \in \Omega$, i.e., there exists $m_S > 0$ such that $(\boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}_1)-\boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}_2)):(\boldsymbol{D}_1-\boldsymbol{D}_2) \ge m_S\,\|\boldsymbol{D}_1-\boldsymbol{D}_2\|_{\mathbb{S}^d}^p$ for all $\boldsymbol{D}_1$, $\boldsymbol{D}_2\in\mathbb{S}^d$, a.e. $\boldsymbol{x}\in\Omega$.

$\underline{H(j)}:$ $\quad j\colon \Gamma_C\times\mathbb{R}^d\to\mathbb{R}$ is such that

(i) $j(\cdot,\boldsymbol{\xi})$ is measurable on Γ_C for all $\boldsymbol{\xi}\in\mathbb{R}^d$.
(ii) $j(\boldsymbol{x},\cdot)$ is locally Lipschitz on $\mathbb{R}^d$ for a.e. $\boldsymbol{x}\in\Gamma_C$.
(iii) $\|\partial j(\boldsymbol{x},\boldsymbol{\xi})\|_{\mathbb{R}^d} \le c_{j0}+c_{j1}\|\boldsymbol{\xi}\|_{\mathbb{R}^d}^{p-1}$ for all $\boldsymbol{\xi}\in\mathbb{R}^d$, a.e. $\boldsymbol{x}\in\Gamma_C$ with c_{j0}, $c_{j1}\ge 0$.
(iv) $\partial j(\boldsymbol{x},\cdot)$ is relaxed monotone for a.e. $\boldsymbol{x}\in\Gamma_C$, i.e., there exist $m_j\ge 0$ such that $(\partial j(\boldsymbol{x},\boldsymbol{\xi}_1)-\partial j(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\ge -m_j\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathbb{R}^d}^p$ for all $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2\in\mathbb{R}^d$, a.e. $\boldsymbol{x}\in\Gamma_C$.

The exponent p and the external body force $\boldsymbol{f}$ satisfy the following assumptions.

$\underline{H(p)}:$ $\quad 2\le p<\infty$.

$\underline{H(f)}:$

$$\text{if } d=2, \text{ then } \begin{cases} \boldsymbol{f}\in L^r(\Omega;\mathbb{R}^2) \text{ for some } r\in(1,\infty), \text{ if } p=2. \\ \boldsymbol{f}\in L^1(\Omega;\mathbb{R}^2), \text{ if } p>2. \end{cases}$$

$$\text{if } d=3, \text{ then } \begin{cases} \boldsymbol{f}\in L^{\frac{3p}{4p-3}}(\Omega;\mathbb{R}^3), \text{ if } p\in[2,3). \\ \boldsymbol{f}\in L^r(\Omega;\mathbb{R}^3) \text{ for some } r\in(1,\infty), \text{ if } p=3. \\ \boldsymbol{f}\in L^1(\Omega;\mathbb{R}^3), \text{ if } p>3. \end{cases}$$

Finally, the convection field and the initial condition satisfy the following hypothesis.

$\underline{(H_0)}:$ $\quad \boldsymbol{b}$, $\boldsymbol{u}_0\in V$.

We now turn to the variational formulation of Problem P. In what follows, we assume that $\boldsymbol{u}$, $\boldsymbol{S}$ and π are sufficiently smooth functions which solve (16)–(21). Let $\boldsymbol{v}\in V$ and $t\in(0,T)$. We multiply the Eq. (16) by $\boldsymbol{v}$, integrate over Ω and use the Green formula (22) to find that

$$\begin{aligned}\int_\Omega \boldsymbol{u}'(t)\cdot\boldsymbol{v}\,dx &+ \int_\Omega \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}(t))):\boldsymbol{D}(\boldsymbol{v})\,dx - \int_{\partial\Omega}\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}(t)))\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma \\ &+ \int_\Omega((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}(t))\cdot\boldsymbol{v}\,dx + \int_\Omega \nabla\pi(t)\cdot\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,dx.\end{aligned} \tag{24}$$

Note that hypotheses $H(\boldsymbol{f})$ and $H(p)$ guarantee that the integrals $\int_\Omega ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}(t))\cdot\boldsymbol{v}\,dx$ and $\int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,dx$ are well defined. Exploiting the Green formula (23) and conditions $\operatorname{div}\boldsymbol{v}=0$ in Ω, $\boldsymbol{v}=0$ on Γ_D, and $v_\nu=0$ on Γ_C, we obtain

$$\int_\Omega \nabla\pi(t)\cdot\boldsymbol{v}\,dx = -\int_\Omega \pi(t)\operatorname{div}\boldsymbol{v}\,dx + \int_{\Gamma_D}\pi(t)\,v_\nu\,d\Gamma + \int_{\Gamma_C}\pi(t)\,v_\nu\,d\Gamma = 0.$$

Next, from conditions $\boldsymbol{v}=0$ on Γ_D and $v_\nu=0$ on Γ_C, it follows

$$\int_{\partial\Omega}\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}(t)))\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma = \int_{\Gamma_D}\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}(t)))\boldsymbol{\nu}\cdot\boldsymbol{v}\,d\Gamma + \int_{\Gamma_C}(S_\nu v_\nu + \boldsymbol{S}_\tau\cdot\boldsymbol{v}_\tau)\,d\Gamma$$
$$= \int_{\Gamma_C}\boldsymbol{S}_\tau\cdot\boldsymbol{v}_\tau\,d\Gamma.$$

Hence and from (24), we deduce

$$\int_\Omega \boldsymbol{u}'(t)\cdot\boldsymbol{v}\,dx + \int_\Omega \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}(t))):\boldsymbol{D}(\boldsymbol{v})\,dx + \int_\Omega((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}(t))\cdot\boldsymbol{v}\,dx$$
$$-\int_{\Gamma_C}\boldsymbol{S}_\tau\cdot\boldsymbol{v}_\tau\,d\Gamma = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,dx.$$

Using (20) and (21) we obtain the following variational formulation of Problem P.

Problem P_V. *Find a velocity field $\boldsymbol{u}\in\mathcal{W}$ such that*

$$\int_\Omega \boldsymbol{u}'(t)\cdot\boldsymbol{v}\,dx + \int_\Omega \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u}(t))):\boldsymbol{D}(\boldsymbol{v})\,dx + \int_\Omega((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}(t))\cdot\boldsymbol{v}\,dx$$
$$+\int_{\Gamma_C} j^0(\boldsymbol{x},\boldsymbol{u}_\tau(t);\boldsymbol{v}_\tau)\,d\Gamma \geq \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,dx \quad \text{for all} \quad \boldsymbol{v}\in V, \ \text{a.e. } t\in(0,T),$$
$$\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}) \quad \text{a.e.} \quad \boldsymbol{x}\in\Omega.$$

We have the following existence and uniqueness result whose proof will be provided in the next section.

Theorem 7. *Assume $H(S)$, $H(j)$, $H(p)$, $H(f)$ and (H_0), and the following smallness condition*

$$m_S > 2^{2p-3}\frac{\|\gamma\|^p}{c_K^p}\max\{c_{1j}, 2m_j\} \tag{25}$$

holds. Then Problem P_V has a unique solution.

6. Proof of Theorem 7

We will apply an abstract result of Theorem 5. We introduce operators B, $C\colon V\to V^*$ as follows

$$\langle B\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} = \int_\Omega \boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u})):\boldsymbol{D}(\boldsymbol{v})\,dx, \tag{26}$$

$$\langle C\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} = \int_\Omega((\boldsymbol{b}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}\,dx \tag{27}$$

for all $\boldsymbol{u}$, $\boldsymbol{v}\in V$. We will prove that the operator $A=B+C\colon V\to V^*$ satisfies hypothesis $H(A)$. Note that A is independent of $t\in(0,T)$.

First, we establish some properties of the operator B. From $H(S)$(ii) and the Hölder inequality, we have

$$\langle B\boldsymbol{u},\boldsymbol{v}\rangle_{V^*\times V} \leq a_S\int_\Omega\|\boldsymbol{D}(\boldsymbol{v})\|_{\mathbb{S}^d}\,dx + a_S\int_\Omega\|\boldsymbol{D}(\boldsymbol{u})\|_{\mathbb{S}^d}^{p-1}\|\boldsymbol{D}(\boldsymbol{v})\|_{\mathbb{S}^d}\,dx$$
$$\leq \left(a_S|\Omega|^{\frac{1}{q}} + a_S\|\boldsymbol{D}(\boldsymbol{u})\|_{L^p(\Omega;\mathbb{S}^d)}^{p-1}\right)\|\boldsymbol{D}(\boldsymbol{v})\|_{L^p(\Omega;\mathbb{S}^d)}$$
$$= \left(a_S|\Omega|^{\frac{1}{q}} + a_S\|\boldsymbol{u}\|_V^{p-1}\right)\|\boldsymbol{v}\|_V$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$. This implies that the operator $B \colon V \to V^*$ is well defined and $\|B\boldsymbol{u}\|_{V^*} \le a_S\big(|\Omega|^{\frac{1}{q}} + \|\boldsymbol{u}\|_V^{p-1}\big)$ for all $\boldsymbol{u} \in V$, which implies the boundedness of B. Furthermore, condition $H(S)$(iv) implies

$$\begin{aligned}
&\langle B\boldsymbol{u}_1 - B\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{V^*\times V}\\
&\quad = \int_\Omega (\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_1)) - \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_2))) \colon (\boldsymbol{D}(\boldsymbol{u}_1) - \boldsymbol{D}(\boldsymbol{u}_2))\, dx\\
&\quad \ge m_S\, c_K^p\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^p
\end{aligned} \tag{28}$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$, which means that the operator B is strongly monotone.

Next, we show that the operator $B \colon V \to V^*$ is continuous. To this end, let $\boldsymbol{u}_n$, $\boldsymbol{u} \in V$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ in V, as $n \to \infty$. Hence $\boldsymbol{D}(\boldsymbol{u}_n) \to \boldsymbol{D}(\boldsymbol{u})$ in $L^p(\Omega; \mathbb{S}^d)$. From [3, Proposition 2.2.41], by passing to a subsequence if necessary, we have $\boldsymbol{D}(\boldsymbol{u}_n)(\boldsymbol{x}) \to \boldsymbol{D}(\boldsymbol{u})(\boldsymbol{x})$ in $\mathbb{S}^d$ for a.e. $\boldsymbol{x} \in \Omega$, as $n \to \infty$, and $\|\boldsymbol{D}(\boldsymbol{u}_n)(\boldsymbol{x})\|_{\mathbb{S}^d} \le \eta(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$ with $\eta \in L^p(\Omega)$. By assumption $H(S)$(i), we obtain $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)(\boldsymbol{x})) \to \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})(\boldsymbol{x}))$ in $\mathbb{S}^d$ for a.e. $\boldsymbol{x} \in \Omega$. Hypothesis $H(S)$(ii) and the elementary inequality $|x + y|^r \le 2^{r-1}(|x|^r + |y|^r)$ for x, $y \in \mathbb{R}$ and $1 \le r < \infty$ imply

$$\begin{aligned}
&\|\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)(\boldsymbol{x})) - \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})(\boldsymbol{x}))\|_{\mathbb{S}^d}^q\\
&\quad \le 2^{q-1} a_S^q \left((1 + \|\boldsymbol{D}(\boldsymbol{u}_n)(\boldsymbol{x})\|_{\mathbb{S}^d}^{p-1})^q + (1 + \|\boldsymbol{D}(\boldsymbol{u})(\boldsymbol{x})\|_{\mathbb{S}^d}^{p-1})^q\right)\\
&\quad \le c\left(2 + \eta^p(x) + \|\boldsymbol{D}(\boldsymbol{u})(\boldsymbol{x})\|_{\mathbb{S}^d}^p\right) \ \text{ with } \ c > 0
\end{aligned}$$

for a.e. $\boldsymbol{x} \in \Omega$. Hence, by the Lebesgue dominated convergence theorem, we infer

$$\|\boldsymbol{S}(\cdot, \boldsymbol{D}(\boldsymbol{u}_n)(\cdot)) - \boldsymbol{S}(\cdot, \boldsymbol{D}(\boldsymbol{u})(\cdot))\|_{L^q(\Omega;\mathbb{S}^d)}^q \to 0, \ \text{ as } \ n \to +\infty.$$

By the Hölder inequality, we have

$$\begin{aligned}
\langle B\boldsymbol{u}_n - B\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} &= \int_\Omega \big(\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)(\boldsymbol{x})) - \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})(\boldsymbol{x}))\big) : \boldsymbol{D}(\boldsymbol{v})(\boldsymbol{x})\, dx\\
&\le \|\boldsymbol{S}(\cdot, \boldsymbol{D}(\boldsymbol{u}_n)(\cdot)) - \boldsymbol{S}(\cdot, \boldsymbol{D}(\boldsymbol{u})(\cdot))\|_{L^q(\Omega;\mathbb{S}^d)} \|\boldsymbol{D}(\boldsymbol{v})\|_{L^p(\Omega;\mathbb{S}^d)}\\
&= \|\boldsymbol{S}(\cdot, \boldsymbol{D}(\boldsymbol{u}_n)(\cdot)) - \boldsymbol{S}(\cdot, \boldsymbol{D}(\boldsymbol{u})(\cdot))\|_{L^q(\Omega;\mathbb{S}^d)} \|\boldsymbol{v}\|_V
\end{aligned}$$

for all $\boldsymbol{v} \in V$. Hence, it follows that $B\boldsymbol{u}_n$ converges to $B\boldsymbol{u}$ in V^*, as $n \to \infty$. This proves that the operator B is continuous.

Summing up, the operator $B \colon V \to V^*$ is well defined, bounded, strongly monotone, and continuous.

Now, we establish some properties of the linear operator $C \colon V \to V^*$. First, we observe that from the following continuous embeddings

for $p \ge d$, we have $V \subset L^{\overline{r}}(\Omega; \mathbb{R}^d)$ for any $\overline{r} \in (1, \infty)$,
for $p \in (\frac{3d}{d+2}, d)$, we have $V \subset L^{\frac{dp}{d-p}}(\Omega; \mathbb{R}^d) \subset L^{\frac{2p}{p-1}}(\Omega; \mathbb{R}^d)$,

we get

$$\begin{aligned}
\langle C\boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} &\le \|\boldsymbol{b}\|_{L^{\frac{2p}{p-1}}(\Omega;\mathbb{R}^d)} \|\nabla \boldsymbol{u}\|_{L^p(\Omega;\mathbb{R}^{d\cdot d})} \|\boldsymbol{v}\|_{L^{\frac{2p}{p-1}}(\Omega;\mathbb{R}^d)}\\
&\le K\|\boldsymbol{b}\|_V \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V \quad \text{for all } \ \boldsymbol{u}, \boldsymbol{v} \in V \ \text{ with } K > 0.
\end{aligned}$$

This implies that operator $C \colon V \to V^*$ is well defined and continuous, so it is bounded. Moreover, we note that

$$\begin{aligned}
\langle C\boldsymbol{u}, \boldsymbol{u}\rangle_{V^*\times V} &= \int_\Omega ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{u}\, dx = \int_\Omega \sum_{i,j=1}^d b_i \frac{\partial u_j}{\partial x_i} u_j\, dx = \int_\Omega \sum_{i,j=1}^d b_i \frac{\partial}{\partial x_i}\frac{u_j^2}{2}\, dx\\
&= -\frac{1}{2}\int_\Omega \operatorname{div} \boldsymbol{b} \sum_{j=1}^d u_j^2\, dx + \frac{1}{2}\int_{\Gamma_C} b_\nu \sum_{j=1}^d u_j^2\, dx = 0
\end{aligned} \tag{29}$$

for all $\boldsymbol{u} \in \widetilde{V}$. Then, we use density of $\widetilde{V}$ in V, and we get (29) for all $\boldsymbol{u} \in V$. As C is linear, it follows that

$$\langle C\boldsymbol{u}_1 - C\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{V^*\times V} = \langle C(\boldsymbol{u}_1 - \boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{V^*\times V} = 0 \tag{30}$$

for $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$. We deduce that operator $C\colon V \to V^*$ is bounded, monotone and continuous.

From properties established for operators B and C, we deduce that operator A is demicontinuous, strongly monotone with $m_A = m_S c_K^p$ and

$$\|A\boldsymbol{u}\|_{V^*} \le c\,(1 + \|\boldsymbol{u}\|_V^{p-1}) + K\,\|\boldsymbol{b}\|_V\|\boldsymbol{u}\|_V \quad \text{for all} \quad \boldsymbol{u} \in V.$$

Hence A safisfies conditions $H(A)$.

Next, we introduce the functional $J\colon V \to \mathbb{R}$ by

$$J(\boldsymbol{v}) = \int_{\Gamma_C} j(\boldsymbol{x}, \boldsymbol{v}_\tau(\boldsymbol{x}))\,dx \quad \text{for all} \quad \boldsymbol{v} \in V. \tag{31}$$

We will verify that J satisfies hypothesis $H(J)$. Note that J is independent of t. By hypotheses $H(j)$(i) and (ii), it is clear that for the functional J defined by (31), conditions $H(J)(1)$ and (2) hold. From [21, Theorem 3.47] and the relation $\partial J(\boldsymbol{v}) \subset \int_{\Gamma_C} \partial j(\boldsymbol{v}_\tau)\,dx$ for all $\boldsymbol{v} \in V$, we deduce that the hypothesis $H(J)(3)$ is satisfied with $c_0(t) = c_{j0}|\Gamma_C|^{\frac{1}{q}}$ and $c_1 = c_{j1}\|\gamma\|^p$. Next, from $H(j)$(iv), [21, Theorem 3.47], and the relation $|\xi_\tau| \le \|\boldsymbol{\xi}\|$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, we have

$$\begin{aligned}\langle \partial J(\boldsymbol{v}_1) - \partial J(\boldsymbol{v}_2), \boldsymbol{v}_1 - \boldsymbol{v}_2\rangle_{V^*\times V} &= \int_{\Gamma_C} (\partial j(\boldsymbol{x}, \boldsymbol{v}_{1\tau}) - \partial j(\boldsymbol{x}, \boldsymbol{v}_{2\tau}))\cdot(\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau})\,dx\\ &\ge -m_j \int_{\Gamma_C} \|\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}\|_{\mathbb{R}^d}^p\,dx = -m_j\|\gamma\|^p\,\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V^p\end{aligned}$$

for all $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in V$ with $m_J = m_j\|\gamma\|^p$. Hence, the subdifferential of $\partial J(t,\cdot)$ is relaxed monotone, which implies condition $H(J)(4)$. Conditions (H_1) and (H_2) are consequences of hypotheses $H(f)$ and (H_0), and (25), respectively.

Applying Theorem 5, we deduce that the following problem has the unique solution: find $\boldsymbol{u} \in \mathcal{W}$ such that

$$\begin{cases} \boldsymbol{u}'(t) + A(\boldsymbol{u}(t)) + \partial J(\boldsymbol{u}(t)) \ni \boldsymbol{f}(t) \quad \text{a.e. } t \in (0,T),\\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{32}$$

By definition of the Clarke subdifferential, the problem (32) is equivalent to: find $\boldsymbol{u} \in \mathcal{W}$ such that

$$\begin{cases} \langle \boldsymbol{u}'(t) + A(\boldsymbol{u}(t)), \boldsymbol{v}\rangle_{V^*\times V} + J^0(\boldsymbol{u}(t);\boldsymbol{v}) \ge \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V}\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T),\\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases} \tag{33}$$

From (31) and [21, Theorem 3.47(iv)], we have

$$J^0(\boldsymbol{v};\boldsymbol{z}) \le \int_{\Gamma_C} j^0(\boldsymbol{x}, \boldsymbol{v}_\tau(\boldsymbol{x}); \boldsymbol{z}_\tau(\boldsymbol{x}))\,d\Gamma \quad \text{for all} \quad \boldsymbol{v}, \boldsymbol{z} \in V.$$

Exploiting this relation in (33), we deduce that $\boldsymbol{u} \in \mathcal{W}$ is a solution to Problem P_V.

Finally, we show that a solution to Problem P_V is unique. Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in \mathcal{W}$ be solutions to Problem P_V, that is

$$\langle \boldsymbol{u}_i', \boldsymbol{v} - \boldsymbol{u}_i\rangle_{V^*\times V} + \langle A\boldsymbol{u}_i, \boldsymbol{v} - \boldsymbol{u}_i\rangle_{V^*\times V} + \int_{\Gamma_C} j^0(\boldsymbol{u}_{i\tau}; \boldsymbol{v}_\tau - \boldsymbol{u}_{i\tau})\,d\Gamma \ge \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_i\rangle_{V^*\times V}$$

for all $\boldsymbol{v} \in V$ and $i = 1, 2$ where $A = B + C : V \to V^*$ is given by (26) and (27). Taking $\boldsymbol{v} = \boldsymbol{u}_2$ in the inequality for $i = 1$, and $\boldsymbol{v} = \boldsymbol{u}_1$ in the inequality for $i = 2$, then adding them, we get

$$\begin{aligned}&\langle \boldsymbol{u}_1' - \boldsymbol{u}_2', \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{V^*\times V} + \langle A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle_{V^*\times V}\\ &\quad\le \int_{\Gamma_C} (j^0(\boldsymbol{u}_{1\tau}; \boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau}) + j^0(\boldsymbol{u}_{2\tau}; \boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}))\, d\Gamma.\end{aligned}$$

From hypothesis $H(j)$(iv), we deduce

$$\begin{aligned}&j^0(\boldsymbol{u}_{1\tau}; \boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau}) + j^0(\boldsymbol{u}_{2\tau}; \boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau})\\ &\quad= \sup_{\boldsymbol{\xi}\in\partial j(\boldsymbol{u}_{1\tau})} \boldsymbol{\xi}\cdot(\boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau}) + \sup_{\boldsymbol{\zeta}\in\partial j(\boldsymbol{u}_{2\tau})} \boldsymbol{\zeta}\cdot(\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau})\\ &\quad\le -\inf_{\boldsymbol{\xi}\in\partial j(\boldsymbol{u}_{1\tau})} \boldsymbol{\xi}\cdot(\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}) - \inf_{\boldsymbol{\zeta}\in\partial j(\boldsymbol{u}_{2\tau})} (-\boldsymbol{\zeta})\cdot(\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau})\\ &\quad= -\inf_{\boldsymbol{\xi}\in\partial j(\boldsymbol{u}_{1\tau}),\,\boldsymbol{\zeta}\in\partial j(\boldsymbol{u}_{2\tau})} (\boldsymbol{\xi} - \boldsymbol{\zeta})\cdot(\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}) \le m_j\, \|\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}\|^p_{\mathbb{R}^d}.\end{aligned}$$

Now, by the boundedness of the trace operator γ, we have

$$\begin{aligned}&\int_{\Gamma_C} j^0(\boldsymbol{u}_{1\tau}; \boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau}) + j^0(\boldsymbol{u}_{2\tau}; \boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau})\, d\Gamma\\ &\quad\le m_j \int_{\Gamma_C} \|\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}\|^p_{\mathbb{R}^d}\, d\Gamma \le m_j \|\gamma\|^p \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^p_V.\end{aligned} \tag{34}$$

From this inequality, (28) and (30), we obtain

$$m_S\, c_K^p \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^p_V \le m_j \|\gamma\|^p \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^p_V.$$

Finally, using the smallness condition (25) in the last inequality, it follows that $\boldsymbol{u}_1 = \boldsymbol{u}_2$. This completes the proof of the theorem. □

We conclude this section with examples of the constitutive function $\boldsymbol{S}$ and the potential j.

Example 8. The following examples of the constitutive function $\boldsymbol{S}$ can be found in [16,17]:

$$\begin{aligned}\boldsymbol{S}^{(1)}(\boldsymbol{x}, \boldsymbol{D}) &= \kappa_1\, \|\boldsymbol{D}\|^{p-2}_{\mathbb{S}^d} \boldsymbol{D},\\ \boldsymbol{S}^{(2)}(\boldsymbol{x}, \boldsymbol{D}) &= \kappa_1\, (1 + \|\boldsymbol{D}\|_{\mathbb{S}^d})^{p-2} \boldsymbol{D},\\ \boldsymbol{S}^{(3)}(\boldsymbol{x}, \boldsymbol{D}) &= \kappa_1\, (1 + \|\boldsymbol{D}\|^2_{\mathbb{S}^d})^{\frac{p-2}{2}} \boldsymbol{D},\\ \boldsymbol{S}^{(3+i)}(\boldsymbol{x}, \boldsymbol{D}) &= \kappa_0\, \boldsymbol{D} + \boldsymbol{S}^{(i)}(\boldsymbol{x}, \boldsymbol{D})\end{aligned}$$

for $\boldsymbol{D} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$, where $i = 1, 2, 3$, κ_0, κ_1 are suitable positive viscosity parameters, and $2 \le p < \infty$. In what follows, we consider a general constitutive function $\boldsymbol{S} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ of the form

$$\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}) = h(\|\boldsymbol{D}\|_{\mathbb{S}^d})\boldsymbol{D} \quad \text{for} \quad \boldsymbol{D} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \tag{35}$$

where $h : [0, \infty) \to \mathbb{R}$. Fluids which are characterized by the constitutive law (35) are called generalized Newtonian fluids (even if they are non-Newtonian ones). We recall that if $h(r) = h_0$ for $r \ge 0$, $h_0 > 0$ is a given viscosity constant of the fluid, then (35) reduces to $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}) = h_0\boldsymbol{D}$ which represents the linear Stokes' law, and (15) turns into the well known Navier–Stokes system. An incompressible fluids described by Stokes' law are called Newtonian fluids. Fluids that can not be characterized by the Stokes' law are usually called non-Newtonian fluids, cf. [16,17,28] and the references therein.

We provide conditions on the function h in (35) under which the constitutive function $\boldsymbol{S}$ satisfies $H(S)$. The following properties can be proved in a similar way as in [6, Lemma 21]. Let $H(p)$ hold, $h : [0, \infty) \to \mathbb{R}$ and $\boldsymbol{S}$ be given by (35).

(1) If h is continuous, then $H(S)(i)$ holds.
(2) If $|h(r)| \le (a + br)^{p-2}$ for $r \ge 0$ with $a \ge 0$ and $b > 0$, then $H(S)$(ii) holds with $a_S = \max\{2^{p-2}b^{-1}a^{p-1}, (1 + 2^{p-2})b^{p-2}\}$.

(3) If $h\colon [0,\infty) \to [M,\infty)$ is nondecreasing, where $M > 0$, then $H(S)$(iii) holds with $m_S = M$.

Next, we provide a concrete example of the multivalued frictional law of the form (20) with a convex potential j.

Example 9. Let $\boldsymbol{u}_0 \in \mathbb{R}^d$ be a given velocity of the moving part of boundary Γ_C and a nonnegative function $g \in L^\infty(\Gamma_C)$ be a friction coefficient. Consider the convex potential $j\colon \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ defined by $j(\boldsymbol{x},\boldsymbol{\xi}) = g(\boldsymbol{x})\|\boldsymbol{\xi} - \boldsymbol{u}_0\|_{\mathbb{R}^d}$ for $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_C$. This choice of j leads to a threshold slip condition considered in [5,9,10]. It is easy to calculate that

$$\partial j(\boldsymbol{x},\boldsymbol{\xi}) = g(\boldsymbol{x}) \times \begin{cases} \overline{B}(\boldsymbol{0},1) & \text{if } \boldsymbol{\xi} = \boldsymbol{u}_0, \\ \dfrac{\boldsymbol{\xi} - \boldsymbol{u}_0}{\|\boldsymbol{\xi} - \boldsymbol{u}_0\|_{\mathbb{R}^d}} & \text{if } \boldsymbol{\xi} \neq \boldsymbol{u}_0 \end{cases}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_C$, where $\overline{B}(\boldsymbol{0},1)$ denotes the closed unit ball in $\mathbb{R}^d$. Note that the function j satisfies hypothesis $H(j)$ with $c_{j0} = \|g\|_{L^\infty(\Gamma_C)}$, $c_{j1} = 0$ and $m_j = 0$. Then the condition (20) has the following form

$$\begin{cases} \boldsymbol{u}_\tau = \boldsymbol{u}_0 \implies \|\boldsymbol{S}_\tau\|_{\mathbb{R}^d} \le g(\boldsymbol{x}), \\ \boldsymbol{u}_\tau \neq \boldsymbol{u}_0 \implies -\boldsymbol{S}_\tau = g(\boldsymbol{x}) \, \dfrac{\boldsymbol{u}_\tau - \boldsymbol{u}_0}{\|\boldsymbol{u}_\tau - \boldsymbol{u}_0\|_{\mathbb{R}^d}}. \end{cases}$$

This latter is the well known Tresca friction law on Γ_C, cf. [21, Sect. 6.3] for a detailed discussion. The interpretation of the above law is the following. In the case where the velocity of the fluid equals the velocity of the moving boundary, the tangential stress is below a certain threshold value. In turn, if the slip between the velocity of the fluid and the velocity of boundary occurs, then the friction force is directed opposite to the slip velocity and its magnitude is determined by the slip rate value according to the function g.

The multivalued condition (20) covers various versions of nonmonotone threshold condition. An example of this law with a nonconvex potential j is provided below.

Example 10. Let $a \in L^\infty(\Gamma_C)$ be a given function such that $0 \le a(\boldsymbol{x}) < 1$ for a.e. $\boldsymbol{x} \in \Gamma_C$. Consider the nonconvex potential $j\colon \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$j(\boldsymbol{x},\boldsymbol{\xi}) = (a(\boldsymbol{x}) - 1)e^{-\|\boldsymbol{\xi}\|_{\mathbb{R}^d}} + a(\boldsymbol{x})\|\boldsymbol{\xi}\|_{\mathbb{R}^d} \quad \text{for } \ \boldsymbol{\xi} \in \mathbb{R}^d, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \tag{36}$$

It is clear that the function j in (36) is nonconvex and its generalized gradient is given by

$$\partial j(\boldsymbol{x},\boldsymbol{\xi}) = \begin{cases} \overline{B}(\boldsymbol{0},1) & \text{if } \ \boldsymbol{\xi} = 0, \\ \Big((1 - a(\boldsymbol{x}))e^{-\|\boldsymbol{\xi}\|_{\mathbb{R}^d}} + a(\boldsymbol{x})\Big) \dfrac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_{\mathbb{R}^d}} & \text{if } \ \boldsymbol{\xi} \neq 0 \end{cases}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Gamma_C$. By a direct calculation, we have

$$\|\partial j(\boldsymbol{x},\boldsymbol{\xi})\|_{\mathbb{R}^d} \le 1 \ \text{ for all } \ \boldsymbol{\xi} \in \mathbb{R}^d, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C,$$

i.e., j satisfies $H(j)(3)$ with $c_{j0} = 1$ and $c_{j1} = 0$. Furthermore, function $j(\boldsymbol{x},\cdot)$ satisfies for a.e. $\boldsymbol{x} \in \Gamma_C$ the relaxed monotonicity condition $H(j)(4)$ with constant $m_j = 1$.

We observe from definition (36) that condition (20) reduces to the threshold slip law of the form

$$\begin{cases} \|\boldsymbol{S}_\tau\|_{\mathbb{R}^d} \le 1 & \text{if } \ \boldsymbol{u}_\tau = 0, \\ -\boldsymbol{S}_\tau = \Big((1 - a(\boldsymbol{x}))e^{-\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d}} + a(\boldsymbol{x})\Big) \dfrac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|_{\mathbb{R}^d}} & \text{if } \ \boldsymbol{u}_\tau \neq 0 \end{cases} \tag{37}$$

on $\Gamma_C \times (0,T)$. We also remark that in the particular case $a \equiv 1$ in (36), the function j reduces to the convex potential and the associated law (37) becomes the condition decribed in Example 9.

We conclude that the multivalued condition (20) incorporates nonmonotone multivalued relations which are useful in applications, cf. [21, Sect. 3.3] and [24, Sect. 1.2]. More examples of multivalued boundary conditions which can be cast in the framework of variational and hemivariational inequalities can be found in [6,21]. For examples of the constitutive function $\boldsymbol{S}$ which satisfy $H(S)$, see [6,16,17].

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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