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Integral Equations and Operator Theory

Sufficient Criteria for Stabilization Properties in Banach Spaces

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Abstract. We study abstract sufficient criteria for cost-uniform open-loop stabilizability of linear control systems in a Banach space with a bounded control operator, which build up and generalize a sufficient condition for null-controllability in Banach spaces given by an uncertainty principle and a dissipation estimate. For stabilizability these estimates are only needed for a single spectral parameter and, in particular, their constants do not depend on the growth rate w.r.t. this parameter. Our result unifies and generalizes earlier results obtained in the context of Hilbert spaces. As an application we consider fractional powers of elliptic differential operators with constant coefficients in $L_p(\mathbb{R}^d)$ for $p \in [1, \infty)$ and thick control sets.

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1. Introduction

Let X, U be Banach spaces, $(S_t)_{t\geq 0}$ a C_0-semigroup on X with generator $-A$, $B \in \mathcal{L}(U, X)$, $x_0 \in X$. We consider the control system

$$\dot{x}(t) = -Ax(t) + Bu(t), \qquad t > 0, \qquad x(0) = x_0 \tag{1.1}$$

with a control function $u \in L_r((0, \infty); U)$ for some $r \in [1, \infty]$. In this paper we focus on the question whether the system (1.1) is open-loop stabilizable; that is, there is a control function $u \in L_r((0, \infty); U)$ such that the corresponding mild solution decays exponentially. We give a sufficient condition for cost-uniform open-loop stabilizability which is based on a well-known strategy to prove null-controllability. The system (1.1) is called null-controllable in time $T > 0$ if there is a control function $u \in L_r((0, T); U)$ such that the corresponding solution of (1.1) satisfies $x(T) = 0$. Clearly, null-controllability

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implies stabilizability. We weaken sufficient conditions for null-controllability to obtain more general criteria for stabilizability.

One possible approach to prove null-controllability is a method known as the Lebeau–Robbiano strategy, originating in the seminal work by Lebeau and Robbiano [16], see also [11,17]. Subsequently, this strategy was generalised in various steps to C_0-semigroups on Hilbert spaces, see, e.g., [3,21–23,27], and more recently to C_0-semigroups on Banach spaces, see [4,7]. The essence of this approach is to show an uncertainty principle and a dissipation estimate for the dual system which are valid for an infinite sequence of so-called spectral parameters, and prove that the growth rate in the uncertainty principle is strictly smaller than the decay rate of the dissipation estimate. In Sect. 3 we show that for proving stabilizability in general Banach spaces one can drop the assumption on the growth and decay rate in the estimates. This was first observed in [10,20] in the context of Hilbert spaces. Similar to what was used in a proof in [20], we show that it is sufficient to prove the uncertainty principle and the dissipation estimate only for one single spectral parameter. This leads to a plain condition for stabilizability in Banach spaces which does not involve assumptions on the constant in the uncertainty principle. In particular, one novel observation is that the uncertainty principle and the dissipation estimate are needed only for a particular fixed operator P (in the notion of Proposition 3.1) instead of a whole family. Let us stress that the latter improvement allows to apply our result to models where an uncertainty principle is available only for some spectral parameters as in [18]. We will pursue this application in a forthcoming paper.

In order to prove the sufficient condition for stabilizability we introduce in Sect. 2 two auxiliary concepts, namely α-controllability and a weak observability inequality. Similar to a result in [24] for Hilbert spaces, we show a duality result for these concepts in general Banach spaces. In order to deal with this more general framework, we directly use a separation theorem instead of a Fenchel–Rockafellar duality argument applied in [24].

Finally, in Sect. 4, we verify the sufficient conditions for fractional powers of elliptic differential operators $-A$ with constant coefficients on $L_p(\mathbb{R}^d)$ for $p \in [1, \infty)$ and where $B = \mathbf{1}_E \colon L_p(E) \to L_p(\mathbb{R}^d)$ is the embedding from a so-called thick set $E \subset \mathbb{R}^d$ to $\mathbb{R}^d$. This complements recent results in the Hilbert space $L_2(\mathbb{R}^d)$ for the fractional heat equation and more general Fourier multipliers, see [1,10,12,19,20].

The paper is a result of two independent reseach questions raised by M.E. and D.G., C.S., M.T., which turned out address the same topic.

2. Stabilizability and Related Concepts

Let X, U be Banach spaces, $(S_t)_{t \geq 0}$ a C_0-semigroup on X with generator $-A$, $B \in \mathcal{L}(U, X)$, and $x_0 \in X$. We consider the control system

$$\dot{x}(t) = -Ax(t) + Bu(t), \qquad t > 0, \qquad x(0) = x_0 \tag{2.1}$$

where $u \in L_r((0,\infty);U)$ with some $r \in [1,\infty]$. The unique mild solution of (2.1) is given by Duhamel's formula

$$x(t) = S_t x_0 + \int_0^t S_{t-\tau} Bu(\tau)\,\mathrm{d}\tau, \qquad t > 0.$$

For $t > 0$ the *controllability map* $L_t \in \mathcal{L}(L_r((0,t);U), X)$ is given by

$$L_t u = \int_0^t S_{t-\tau} Bu(\tau)\,\mathrm{d}\tau. \tag{2.2}$$

Definition 2.1. The system (2.1) is called *cost-uniformly open-loop stabilizable with respect to* $L_r((0,\infty);U)$ if there exist $M \geq 1$, $\omega < 0$, and $K \geq 0$ such that for all $x_0 \in X$ there exists $u \in L_r((0,\infty);U)$ such that

$$\|u\|_{L_r((0,\infty);U)} \leq K\|x_0\|_X \quad \text{and}$$
$$\|x(t)\|_X = \|S_t x_0 + L_t u\|_X \leq M\mathrm{e}^{\omega t}\|x_0\|_X, \quad t \geq 0.$$

Remark 2.2. Recall that one says that the system (2.1) is *closed-loop stabilizable* or *stabilizable by feedback* if there exists $F \in \mathcal{L}(X,U)$ such that $-A + BF$ generates an exponentially stable C_0-semigroup. Then F is called *state feedback operator* and the control u given by $u(t) = Fx(t)$ yields an exponentially stable solution x. In Hilbert spaces the existence of a state feedback operator follows from classical Riccati theory, see e.g. [28, Theorem IV.4.4]. The notion of cost-uniform open-loop stabilizability is clearly weaker than closed-loop stabilizability.

Next we introduce two concepts, namely α-controllability and weak observability inequalities, and discuss their close connection to cost-uniform open-loop stabilizability.

2.1. α-Controllability

In this section we define α-controllability and show that for $\alpha \in [0,1)$ it is equivalent to cost-uniform open-loop stabilizability.

Definition 2.3. Let $\alpha \geq 0$ and $T > 0$. The system (2.1) is called *cost-uniformly* α*-controllable in time* T *with respect to* $L_r((0,T);U)$ if there exists $K \geq 0$ such that for all $x_0 \in X$ there exists $u \in L_r((0,T);U)$ such that

$$\|u\|_{L_r((0,T);U)} \leq K\|x_0\|_X \quad \text{and} \quad \|x(T)\|_X = \|S_T x_0 + L_T u\|_X \leq \alpha\|x_0\|_X.$$

Remark 2.4. If the system (2.1) is cost-uniformly α-controllable for all $\alpha > 0$, it is usually called *cost-uniform approximate null-controllable*. For the control system (2.1), the quantity $\|u\|_{L_r((0,T);U)}$ is called *cost*.

Similarly to [24, Lemma 31] (see also [24, Theorem 26]) we obtain the following relationship between cost-uniform α-controllability and cost-uniform open-loop stabilizability.

Proposition 2.5. *The system* (2.1) *is cost-uniformly open-loop stabilizable if and only if there exist* $\alpha \in [0,1)$ *and* $T > 0$ *such that* (2.1) *is cost-uniformly* α*-controllable in time* T.

Proof. Assume that (2.1) is cost-uniformly open-loop stabilizable. Then for all $\alpha \in (0,1)$ there exists $T > 0$ such that $M\mathrm{e}^{\omega T} \leq \alpha$, with M and ω as in Definition 2.1. Hence the solution of (2.1) satisfies $\|x(T)\|_X \leq \alpha\|x_0\|_X$. Moreover the cost $\|u\|_{L_r((0,\infty);U)}$ can be controlled uniformly w.r.t. the initial value x_0. This proves the claim.

We now show the converse and assume that (2.1) is cost-uniformly α-controllable in time T. For $\alpha = 0$ we have $x(T) = 0$ and therefore $x(t) = 0$ for all $t \geq T$, so the statement is trivial. Thus, let $\alpha \in (0,1)$. Let $x_0 \in X$ and $u_0 \in L_r((0,T);U)$ such that $\|u_0\|_{L_r((0,T);U)} \leq K\|x_0\|_X$ and $\|S_T x_0 + L_T u_0\|_X \leq \alpha\|x_0\|_X$. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we recursively define $x_{k+1} = S_T x_k + L_T u_k$ and choose $u_k \in L_r((0,T);U)$ such that

$$\|u_k\|_{L_r((0,T);U)} \leq K\|x_k\|_X \quad \text{and} \quad \|S_T x_k + L_T u_k\|_X \leq \alpha\|x_k\|_X. \tag{2.3}$$

Define $u\colon [0,\infty) \to U$ as the concatenation

$$u(t) = u_k(t - kT) \quad \text{if } t \in [kT,(k+1)T).$$

For $r \in [1,\infty)$, using (2.3) and the fact $\|x_k\|_X \leq \alpha^k \|x_0\|_X$ for all $k \in \mathbb{N}_0$, we have

$$\|u\|^r_{L_r((0,\infty);U)} \leq \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \|u(\tau)\|_U^r \mathrm{d}\tau \leq K^r \frac{1}{1-\alpha^r} \|x_0\|_X^r,$$

and hence $u \in L_r((0,\infty);U)$. For $r = \infty$, since $\alpha < 1$ we similarly estimate

$$\|u\|_{L_\infty((0,\infty);U)} = \sup_{k\in\mathbb{N}_0} \|u_k\|_{L_\infty((0,T);U)} \leq K\|x_0\|_X\,,$$

and therefore also $u \in L_\infty((0,\infty);U)$.

The control u generates a trajectory

$$x(t) = S_t x_0 + \int_0^t S_{t-\tau} B u(\tau)\mathrm{d}\tau, \quad t > 0$$

satisfying $x(kT) = x_k$ for all $k \in \mathbb{N}_0$. Let $M_S \geq 1$ be such that

$$\sup_{t\in[0,T]} \|S_t\|_{\mathcal{L}(X)} \leq M_S.$$

Then for all $k \in \mathbb{N}_0$ and $t \in [kT,(k+1)T)$, by Hölder's inequality, we have

$$\begin{aligned}
\|x(t)\|_X &= \left\| S_{t-kT} x_k + \int_0^{t-kT} S_{t-kT-\tau} B u_k(\tau - kT)\mathrm{d}\tau \right\|_X \\
&\leq M_S\|x_k\|_X + M_S\|B\|_{\mathcal{L}(U,X)} \int_0^T \|u_k(\tau)\|_U \mathrm{d}\tau \\
&\leq M_S\|x_k\|_X + M_S\|B\|_{\mathcal{L}(U,X)} T^{1/r'} \|u_k\|_{L_r((0,T);U)} \\
&\leq M_S(1 + \|B\|_{\mathcal{L}(U,X)} T^{1/r'} K)\alpha^k\|x_0\|_X,
\end{aligned}$$

where $r' \in [1,\infty]$ such that $1/r + 1/r' = 1$ (and $1/\infty = 0$ as usual). Since $\ln\alpha < 0$ and $\alpha^{k+1} = \mathrm{e}^{(k+1)T\frac{\ln\alpha}{T}} \leq \mathrm{e}^{\frac{\ln\alpha}{T}t}$ for $t \in [kT,(k+1)T)$ we infer that

$$\|x(t)\|_X \leq \frac{M_S}{\alpha}(1 + \|B\|_{\mathcal{L}(U,X)} T^{1/r'} K)\mathrm{e}^{\frac{\ln\alpha}{T}t}\|x_0\|_X.$$

Thus, we obtain the assertion with $M = \frac{M_S}{\alpha}(1 + \|B\|_{\mathcal{L}(U,X)} T^{1/r'} K) \geq 1$ and $\omega = \ln \alpha / T < 0$. ☐

Remark 2.6. Let us observe that in [24] the authors are able to show that cost-uniformly α-controllability is equivalent to closed-loop stabilization, see Remark 2.2 for the terminology, using Riccati theory, which, to the best of our knowledge, is not available in our setting. Indeed, although there are instances of extensions of Riccati theory and the use of Riccati operators for generators of C_0-semigroups on Banach spaces [6,13,25], the operator B is always assumed to map a Hilbert space to a Banach space, which does not reflect our assumptions for the system (2.1). The main difficulty in carrying over the theory to the case of B defined on a Banach space is how to make sense of the quadratic functional leading to the state feedback operator.

2.2. Weak Observability Inequalities

In this section, we prove the duality between cost-uniform α-controllability and a weak observability estimate for the dual system.

Definition 2.7. Let X, Y be Banach spaces, $(S_t)_{t\geq 0}$ a semigroup on X, $C \in \mathcal{L}(X, Y)$, $T > 0$, and assume that $[0, T] \ni t \mapsto \|CS_t x\|_Y$ is measurable for all $x \in X$. Let $r \in [1, \infty]$. Then we say that a *weak observability inequality* is satisfied if there exist $K_{\mathrm{obs}} \geq 0$ and $\alpha \geq 0$ such that for all $x \in X$ we have

$$\|S_T x\|_X \leq \begin{cases} K_{\mathrm{obs}} \left(\int_0^T \|CS_t x\|_Y^r \mathrm{d}t \right)^{1/r} + \alpha \|x\|_X & \text{if } r \in [1, \infty), \\ K_{\mathrm{obs}} \sup_{t \in [0,T]} \|CS_t x\|_Y + \alpha \|x\|_X & \text{if } r = \infty. \end{cases} \tag{2.4}$$

We write X' and U' for the dual spaces of X and U, respectively, and $S_T' \in \mathcal{L}(X')$ and $B' \in \mathcal{L}(X', U')$ for the dual operators of S_T and B, respectively.

Theorem 2.8. *Let X, U be Banach spaces, $(S_t)_{t\geq 0}$ a C_0-semigroup on X, $T > 0$, $r \in [1, \infty]$ and $L_T \in \mathcal{L}(L_r((0, T); U), X)$ the controllability map defined in (2.2). Let further $K \geq 0$ and $\alpha \geq 0$. Then the following statements are equivalent:*

(a) *For every $x \in X$ and $\epsilon > 0$ there exists $u \in L_r((0, T); U)$ with*

$$\|u\|_{L_r((0,T);U)} \leq K \|x\|_X \quad \text{and} \quad \|S_T x + L_T u\|_X < (\alpha + \epsilon) \|x\|_X.$$

(b) *For all $x' \in X'$ we have*

$$\|S_T' x'\|_{X'} \leq \begin{cases} K \left(\int_0^T \|B' S_t' x'\|_{U'}^{r'} \mathrm{d}t \right)^{1/r'} + \alpha \|x'\|_{X'} & \text{if } r' \in [1, \infty), \\ K \sup_{t \in [0,T]} \|B' S_t' x'\|_{U'} + \alpha \|x'\|_{X'} & \text{if } r' = \infty, \end{cases}$$

where $r' \in [1, \infty]$ with $1/r + 1/r' = 1$.

In contrast to [24] we refrain to use the Fenchel–Rockafellar duality theorem to prove Theorem 2.8 and instead employ a shorter argument involving the following well-known separation theorem. Note that by means of the separation theorem, also a version of the Fenchel–Rockafellar duality theorem can be shown. However, we prefer to use it directly to show the duality, without making the detour via Fenchel–Rockafellar theory. We cite here a version from [5, Lemma 1.2], for a proof see [8, Theorem I.5.10, Lemma II.4.1].

Lemma 2.9. *Let A, B be convex sets in a Banach space X. Then $A \subset \overline{B}$ if and only if*

$$\sup_{x\in A}\langle x, x'\rangle_{X,X'} \leq \sup_{x\in B}\langle x, x'\rangle_{X,X'} \qquad \textit{for all } x' \in X'.$$

Proof of Theorem 2.8. We consider the convex sets

$$\begin{aligned} A &= \{S_T x : \|x\|_X \leq 1\} \quad \text{and} \\ B &= \{L_T u + \alpha x : \|u\|_{L_r((0,T);U)} \leq K, \|x\|_X \leq 1\}. \end{aligned}$$

We observe that the following three statements are equivalent:

(a) $A \subset \overline{B}$

(b) for all $\epsilon > 0$ and $x_1 \in X$ with $\|x_1\|_X \leq 1$ there exists $u \in L_r((0,T);U)$ with $\|u\|_{L_r((0,T);U)} \leq K$ and $x_2 \in X$ with $\|x_2\|_X \leq 1$ such that

$$\|S_T x_1 + L_T u + \alpha x_2\|_X < \epsilon.$$

(c) for all $\epsilon > 0$ and $x_1 \in X$ with $\|x_1\|_X \leq 1$ there exists $u \in L_r((0,T);U)$ with $\|u\|_{L_r((0,T);U)} \leq K$ such that

$$\|S_T x_1 + L_T u\|_X < \alpha + \epsilon.$$

While (a) $\Leftrightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious, we note that (b) follows from (c) by choosing $x_2 = -(S_T x_1 + L_T u)/(\alpha + \epsilon)$. Since

$$\big\|S_T x/\|x\|_X + L_T u\big\|_X = \frac{1}{\|x\|_X}$$
$$\big\|S_T x + L_T \|x\|_X u\big\|_X$$

for all $x \in X \setminus \{0\}$, we find that (c) (and thus also (a) and (b)) is equivalent to statement (a) of the theorem. Next, for $x' \in X'$ we compute

$$\sup_{x\in A}\langle x, x'\rangle_{X,X'} = \sup_{\|x\|_X\leq 1}\langle S_T x, x'\rangle_{X,X'} = \|S_T' x'\|_{X'}$$

and

$$\begin{aligned} \sup_{x\in B}\langle x, x'\rangle_{X,X'} &= \sup_{\substack{\|u\|_{L_r((0,T);U)}\leq K,\\ \|x\|_X\leq 1}} \langle L_T u + \alpha x, x'\rangle_{X,X'} \\ &= \sup_{\|u\|_{L_r((0,T);U)}\leq K} \langle L_T u, x'\rangle_{X,X'} + \sup_{\|x\|_X\leq 1} \alpha\langle x, x'\rangle_{X,X'} \\ &= K\|L_T' x'\|_{L_r((0,T);U)'} + \alpha\|x'\|_{X'}. \end{aligned}$$

Finally by [26, Theorem 2.1] we have

$$\|L_T' x'\|_{L_r((0,T);U)'} = \begin{cases} \left(\int_0^T \|B' S_t' x'\|_{U'}^{r'} \mathrm{d}t\right)^{1/r'} & \text{if } r' \in [1,\infty), \\ \sup\limits_{t\in[0,T]} \|B' S_t' x'\|_{U'} & \text{if } r' = \infty, \end{cases}$$

where $r' \in [1, \infty]$ such that $1/r + 1/r' = 1$. Hence, we observe that

$$\sup_{x \in A} \langle x, x' \rangle_{X,X'} \leq \sup_{x \in B} \langle x, x' \rangle_{X,X'}$$

is equivalent to statement (b) of the theorem and the claim follows from Lemma 2.9. □

3. Sufficient Conditions for Stabilizability

In this section we give a sufficient condition for weak observability inequalities in terms of an uncertainty principle and a dissipation estimate, similar to [10, 20]. We emphasize that instead of assuming the uncertainty principle and the dissipation estimate for a family $(P_\lambda)_{\lambda>0}$ with certain dependencies of the constants on the "spectral parameter" λ, we need these assumptions to hold only for one single operator P. We will relate our result to Lemma 2.2 in [10] and Theorem 2.1 in [7]. Using duality we give, similar to [20, Theorem 4.1], a sufficient condition for cost-uniform open-loop stabilizability in Banach spaces without any compatible condition between the uncertainty principle and a dissipation estimate.

Proposition 3.1. *Let X and Y be Banach spaces, $C \in \mathcal{L}(X, Y)$, $P \in \mathcal{L}(X)$, $(S_t)_{t\geq 0}$ a semigroup on X, $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$, and assume that for all $x \in X$ the mapping $t \mapsto \|CS_t x\|_Y$ is measurable. Further, let $r \in [1, \infty]$, $T > 0$ and $K_1, K_2 \colon (0, T] \to [0, \infty)$ continuous functions such that for all $x \in X$ and $t \in (0, T]$ we have*

$$\|PS_t x\|_X \leq K_1(t)\|CPS_t x\|_Y, \tag{3.1}$$

and

$$\|(\mathrm{Id} - P)S_t x\|_X \leq K_2(t)\|x\|_X. \tag{3.2}$$

Then there exist $K_{\mathrm{obs}} \geq 0$ and $\alpha \geq 0$ with

$$\forall x \in X: \quad \|S_T x\|_X \leq \begin{cases} K_{\mathrm{obs}} \left(\int_0^T \|CS_t x\|_Y^r \mathrm{d}t \right)^{1/r} + \alpha\|x\|_X & \text{if } r \in [1, \infty), \\ K_{\mathrm{obs}} \sup\limits_{t\in[0,T]} \|CS_t x\|_Y + \alpha\|x\|_X & \text{if } r = \infty. \end{cases} \tag{3.3}$$

Moreover, for all $\delta \in [0, 1)$ we have

$$K_{\mathrm{obs}} \leq \frac{M\mathrm{e}^{\omega_+ T}}{(1-\delta)T^{1/r}} \max_{t\in[\delta T,T]} K_1(t) \quad \text{and}$$
$$\alpha \leq \frac{M\mathrm{e}^{\omega_+ T}}{(1-\delta)T} \int_{\delta T}^T \left(K_1(t)\|C\|_{\mathcal{L}(X,Y)} + 1\right) K_2(t)\mathrm{d}t,$$

where $\omega_+ = \max\{\omega, 0\}$ and $T^{1/r} = 1$ if $r = \infty$.

Proof. Assume we have shown the statement of the proposition in the case $r = 1$, i.e. for all $x \in X$ we have

$$\|S_T x\|_X \leq K_{\mathrm{obs}} \int_0^T \|CS_t x\|_Y \mathrm{d}t + \alpha\|x\|_X.$$

Then, for all $r \in [1, \infty]$ and all $x \in X$ using Hölder's inequality we obtain

$$\|S_T x\|_X \leq K_{\mathrm{obs}} T^{1/r'} \left(\int_0^T \|C S_t x\|_Y^r \mathrm{d}t \right)^{1/r} + \alpha \|x\|_X,$$

where $r' \in [1, \infty]$ is such that $1/r + 1/r' = 1$. Since $T^{-1}T^{1/r'} = T^{-1/r}$ the statement of the proposition follows. Thus, it is sufficient to prove the case $r = 1$.

Let $t \in (0, T]$ and $x \in X$. Using (3.1) and (3.2) we obtain

$$\begin{aligned} \|S_t x\|_X &\leq \|P S_t x\|_X + \|(\mathrm{Id} - P) S_t x\|_X \leq K_1(t) \|C P S_t x\|_Y + \|(\mathrm{Id} - P) S_t x\|_X \\ &\leq K_1(t) \|C S_t x\|_Y + K_1(t) \|C\|_{\mathcal{L}(X,Y)} \|(\mathrm{Id} - P) S_t x\|_X + \|(\mathrm{Id} - P) S_t x\|_X \\ &\leq K_1(t) \|C S_t x\|_Y + \big(K_1(t) \|C\|_{\mathcal{L}(X,Y)} + 1 \big) K_2(t) \|x\|_X. \end{aligned} \tag{3.4}$$

Since $(S_t)_{t \geq 0}$ is a semigroup we get

$$\|S_T x\|_X = \|S_{T-t} S_t x\|_X \leq M \mathrm{e}^{\omega_+ T} \|S_t x\|_X,$$

where $\omega_+ = \max\{\omega, 0\}$. Since $t \mapsto \|C S_t x\|_Y$ is measurable by assumption, integrating (3.4) with respect to $t \in [\delta T, T]$ we obtain

$$\begin{aligned} &\frac{(1-\delta)T}{M \mathrm{e}^{\omega_+ T}} \|S_T x\|_X \\ &\quad \leq \int_{\delta T}^T K_1(t) \|C S_t x\|_Y \mathrm{d}t + \int_{\delta T}^T \big(K_1(t) \|C\|_{\mathcal{L}(X,Y)} + 1 \big) K_2(t) \mathrm{d}t \, \|x\|_X \\ &\quad \leq \max_{t \in [\delta T, T]} K_1(t) \int_{\delta T}^T \|C S_t x\|_Y \mathrm{d}t + \int_{\delta T}^T \big(K_1(t) \|C\|_{\mathcal{L}(X,Y)} + 1 \big) K_2(t) \mathrm{d}t \, \|x\|_X. \end{aligned}$$

The claim now follows by estimating $\int_{\delta T}^T \|C S_t x\|_Y \mathrm{d}t \leq \int_0^T \|C S_t x\|_Y \mathrm{d}t$ and multiplying both sides by $M \mathrm{e}^{\omega_+ T} / (1-\delta) T$. □

The advantage of Proposition 3.1 is the explicit dependence of K_{obs} and α on the functions K_1, K_2 which allows to give conditions to ensure $\alpha \in [0, 1)$. Thus, in order to prove cost-uniform open-loop stabilizabiliy of a system one can combine Theorem 2.8 and Proposition 2.5 in case $\alpha \in [0, 1)$.

Remark 3.2. In Proposition 3.1 we can replace the uncertainty principle in (3.1) by

$$\forall x \in X : \quad \|P S_{T_0} x\|_X \leq \begin{cases} K_1 \left(\int_0^{T_0} \|C P S_t x\|_Y^r \mathrm{d}t \right)^{1/r} & \text{if } r \in [1, \infty), \\ K_1 \sup_{t \in [0,T]} \|C P S_t x\|_Y & \text{if } r = \infty \end{cases}$$

for some $K_1 > 0$ and $0 < T_0 \leq T$. We then obtain (3.3) with

$$\begin{aligned} &K_{\mathrm{obs}} \leq M \mathrm{e}^{\omega_+ T} 2^{1-1/r} C_1 \quad \text{and} \\ &\qquad \alpha \leq M \mathrm{e}^{\omega_+ T} \Big(2^{1-1/r} K_1 \, \|C\|_{\mathcal{L}(X,Y)} \, \|K_2\|_{L_r(0,T_0)} + K_2(T_0) \Big). \end{aligned}$$

The case $r = \infty$ is similar and the term $2^{1-1/r}$ can be set to 1.

Remark 3.3. Let us relate Proposition 3.1 to the results obtained in [10] and [4,7]. By choosing the functions $K_1, K_2\colon (0,T] \to [0,\infty)$ appropriately we can mimic the assumptions of [10, Lemma 2.2] and [7, Theorem 2.1], respectively. For given $T, \lambda > 0$ suppose we have for all $x \in X$ and $t \in (0,T]$ the inequalities (3.1) and (3.2) with

$$K_1(t) = d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1}} \quad \text{and} \quad K_2(t) = d_2 \mathrm{e}^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}}, \tag{3.5}$$

where $d_0, d_1, d_2, d_3, \gamma_1, \gamma_2, \gamma_3 > 0$. Then Proposition 3.1 implies for all $\delta \in (0,1)$ the weak observability inequality (3.3) with

$$\begin{aligned} K_{\mathrm{obs}} &\leq \frac{M d_0}{\delta T^{1/r}} d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1} + \omega_+ T} \quad \text{and} \\ \alpha &\leq M d_2 \left(d_0 \|C\|_{\mathcal{L}(X,Y)} + 1 \right) \mathrm{e}^{-d_3 \lambda^{\gamma_2} (\delta T)^{\gamma_3} + d_1 \lambda^{\gamma_1} + \omega_+ T}. \end{aligned}$$

Imposing conditions on T and λ we can achieve $\alpha \in [0,1)$. We list here only some interesting cases:

(a) Assume $\gamma_1 > \gamma_2$. Choose $\gamma_3 > 1 - \gamma_2/\gamma_1$, $T > 0$ large enough such that

$$\ln \left(M d_2 (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) \right) < \left(\frac{d_3}{2 d_1} \right)^{\frac{\gamma_2}{\gamma_1 - \gamma_2}} \frac{d_3}{2} (\delta T)^{\frac{\gamma_1 \gamma_3}{\gamma_1 - \gamma_2}} - \omega_+ T,$$

and $\lambda = \left(d_3 (\delta T)^{\gamma_3} / (2 d_1) \right)^{1/(\gamma_1 - \gamma_2)}$. Then $\alpha < 1$.

(b) Assume $\gamma_1 = \gamma_2$. Choose $T > \delta (d_1/d_3)^{1/\gamma_3}$ and

$$\lambda > \left(\frac{\ln \left(M d_2 (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1) \right) + \omega_+ T}{d_3 (\delta T)^{\gamma_3} - d_1} \right)^{\frac{1}{\gamma_1}} > 0.$$

Then again $\alpha \in (0,1)$.

(c) Assume $\gamma_1 < \gamma_2$. For given $T > 0$ choose $\lambda > 0$ large enough such that

$$\ln \left(M d_2 (d_0 \|C\| + 1) \right) + \omega_+ T < d_3 \lambda^{\gamma_2} (\delta T)^{\gamma_3} - d_1 \lambda^{\gamma_1}.$$

Then $\alpha \in (0,1)$.

(d) Assume $\gamma_1 < \gamma_2$. Let $\lambda^* > 0$ and suppose there exists $P \in \mathcal{L}(X)$ such that $P_\lambda = P$ for all $\lambda > \lambda^*$, and such that the inequalities (3.1) and (3.2) hold with K_1, K_2 as in (3.5). Then by [7, Theorem 2.1], $\alpha = 0$.

(e) Assume $\omega_+ = 0$. Then for arbitrary $\lambda, \gamma_1, \gamma_2, \gamma_3 > 0$ we can achieve $\alpha \in (0,1)$ by choosing $T > 0$ large enough.

Note that, in contrast to the cases (a) and (b), in (c) we can ensure $\alpha \in (0,1)$ for every $T > 0$ by choosing $\lambda > 0$ appropriately. The cases (a)-(c) are very similar to what was shown in [10, Lemma 2.2], where the inequalities (3.1) and (3.2) with (3.5) where assumed to hold for all $\lambda > 1$. Note that here the assumptions are only needed for some particular $\lambda > 0$.

By restricting to $\gamma_3 = 1$, Proposition 3.1 and the duality in Theorem 2.8 yield the following plain sufficient condition for cost-uniform open-loop stabilizability similar to the Hilbert space result in [20, Theorem 4.1].

Corollary 3.4. *Let X and U be Banach spaces, $B \in \mathcal{L}(U,X)$ and $P \in \mathcal{L}(X)$ such that*

$$\operatorname{Ran}(P) \subset \operatorname{Ran}(PB). \tag{3.6}$$

Further let $(S_t)_{t\geq 0}$ *a* C_0*-semigroup on* X*, and* $M \geq 1$, $\omega \in \mathbb{R}$ *such that* $\|S_t\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{\omega t}$ *for all* $t \geq 0$*. Assume there exist* $M_P \geq 1$ *and* $\omega_P > \omega_+ := \max\{\omega, 0\}$ *such that*

$$\forall x \in X \ \forall t > 0: \quad \|S_t(\mathrm{Id} - P)x\|_X \leq M_P \mathrm{e}^{-\omega_P t}\|x\|_X. \tag{3.7}$$

Then the system (2.1) *is cost-uniformly open-loop stabilizable.*

Proof. We apply Proposition 3.1 to the dual semigroup $(S_t')_{t\geq 0}$ on X', $Y := U'$, $C := B'$, and P replaced by its dual operator P'. Note that $(S_t')_{t\geq 0}$ is exponentially bounded since $(S_t)_{t\geq 0}$ is exponentially bounded. The measurability of $t \mapsto \|B'S_t'x'\|_{U'}$ for all $x' \in X'$ follows from duality and the description of dual norms via the Hahn–Banach theorem. It is well-known, see [5], that (3.6) implies the existence of $K \geq 0$ such that

$$\forall x' \in X': \quad \|P'x'\|_{X'} \leq K\|B'P'x'\|_{U'}.$$

Further (3.7) implies

$$\forall x' \in X' \ \forall t > 0: \quad \|(\mathrm{Id} - P')S_t'x'\|_{X'} \leq M_P\mathrm{e}^{-\omega_P t}\|x'\|_{X'}.$$

Thus, by Proposition 3.1 with $K_1(t) = K$, $K_2(t) = M_P \mathrm{e}^{-\omega_P t}$ and $\delta = (\omega_P + \omega_+)/2\omega_P$ we obtain the weak observability inequality (3.3) for $(S_t')_{t\geq 0}$ for all $T > 0$ and $r' \in [1, \infty]$ with

$$K_{\mathrm{obs}} \leq \frac{2M\mathrm{e}^{\omega_+ T}}{(1 - \frac{\omega_+}{\omega_P})T^{1/r}}K \quad \text{and} \quad \alpha \leq MM_P\left(K\|B\|_{\mathcal{L}(U,X)} + 1\right)\mathrm{e}^{-\frac{1}{2}(\omega_P - \omega_+)T}.$$

For

$$T > \frac{2\ln\left((MM_P\left(K\|B\|_{\mathcal{L}(U,X)} + 1\right)\right)}{\omega_P - \omega_+}$$

we have $\alpha \in [0, 1)$ and the assertion follows from Theorem 2.8 and Proposition 2.5. □

Remark 3.5. The condition $\mathrm{Ran}(P) \subset \mathrm{Ran}(PB)$ for the control operator B does not require any constants. In applications this means that for the corresponding uncertainty principle for the dual system we do not need any assumption on the growth order of the constants in terms of the spectral parameter. An instance of this is when one considers the system (2.1) with H being the harmonic oscillator in $L_2(\mathbb{R}^d)$, i.e. $H = -\Delta + |x|^2$, and B the characteristic function of a measurable subset of $\mathbb{R}^d$ with positive measure. Indeed, it was shown in [2, Theorem 2.1] and in [10, Lemma 3.2] that a spectral inequality with P being any element of the spectral family associated to H is valid under different geometric assumptions on the measurable subset with different growth orders of the constant with respect to the spectral parameter, while the dissipation estimate satisfies an estimate like the one in the corollary above (see, e.g., [10, Eq. (4.17)]).

Remark 3.6. The system (2.1) is called *complete (or rapidly) cost-uniform open-loop stabilizable* if for all $\nu > 0$ the system

$$\dot{x}(t) = -(A + \nu)x(t) + Bu(t), \qquad t > 0, \qquad x(0) = x_0 \tag{3.8}$$

is cost-uniform open-loop stabilizable. Analogously to [20, Theorem 4.1], by Corollary 3.4 we obtain the following sufficient conditions for complete cost-uniform open-loop stabilizability: Let $(P_k)_{k\in\mathbb{N}}$ in $\mathcal{L}(X)$ satisfying (3.6) for all $k\in\mathbb{N}$ and $(M_k)_{k\in\mathbb{N}}$ in $[1,\infty)$, $(\omega_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ with $\omega_k\to\infty$ as $k\to\infty$ such that

$$\forall x\in X\ \forall t>0: \quad \|S_t(\mathrm{Id}-P_k)x\|_X \leq M_k \mathrm{e}^{-\omega_k t}\|x\|_X.$$

Then (2.1) is complete cost-uniform open-loop stabilizable. Indeed, for all $\nu>0$ there exists $k\in\mathbb{N}$ such that $\omega_k>\omega_+ +\nu$ and by Corollary 3.4 the system (3.8) is cost-uniform open-loop stabilizable.

4. Application: Fourier Multipliers and Fractional Powers

We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, which is dense in $L_p(\mathbb{R}^d)$ for all $p\in[1,\infty)$. The space of tempered distributions, i.e. the topological dual space of $\mathcal{S}(\mathbb{R}^d)$, is denoted by $\mathcal{S}'(\mathbb{R}^d)$. We define the Fourier transformation $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}\mathrm{d}x \quad (\xi\in\mathbb{R}^d).$$

By duality, we can extend the Fourier transformation as a bijection on $\mathcal{S}'(\mathbb{R}^d)$ as well.

Let $m\in\mathbb{N}$ and $a\colon\mathbb{R}^d\to\mathbb{C}$,

$$a(\xi) := \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha \quad (\xi\in\mathbb{R}^d),$$

be a polynomial of degree m with coefficients $a_\alpha\in\mathbb{C}$ and assume that a is strongly elliptic, i.e. there exists $c>0$ and $\omega\in\mathbb{R}$ such that

$$\Re a(\xi) \geq c\,|\xi|^m - \omega \quad (\xi\in\mathbb{R}^d).$$

Let $s\in(0,1]$. Then

$$\Re((a(\xi)+\omega)^s) \geq (\Re a(\xi)+\omega)^s \geq c^s\,|\xi|^{sm} \quad (\xi\in\mathbb{R}^d).$$

Let $\tilde{m}\in\mathbb{N}_0$ be the largest integer less than sm, and $b\colon\mathbb{R}^d\to\mathbb{C}$,

$$b(\xi) := \sum_{|\alpha|\leq\tilde{m}} b_\alpha\xi^\alpha \quad (\xi\in\mathbb{R}^d).$$

We consider $a_{s,b} := (a+\omega)^s + b$. Then there exists $\nu\in\mathbb{R}$ such that

$$\Re a_{s,b}(\xi) = \Re(a(\xi)+\omega)^s + \Re b(\xi) \geq c^s\,|\xi|^{sm} - \nu \quad (\xi\in\mathbb{R}^d). \tag{4.1}$$

Note that $a_{s,b}$ may not be differentiable at 0. However, it can be shown that for $t>0$ we have $e^{-ta_{s,b}}\in L_1(\mathbb{R}^d)$ and $\mathcal{F}^{-1}e^{-ta_{s,b}}\in L_1(\mathbb{R}^d)$. Indeed, $e^{-ta_{s,b}}$ decays faster than any polynomial. Thus, $e^{-ta_{s,b}}\in L_1(\mathbb{R}^d)$ and $\mathcal{F}^{-1}e^{-ta_{s,b}}\in C^\infty(\mathbb{R}^d)$. Moreover, the Riemann–Lebesgue lemma yields $\mathcal{F}^{-1}e^{-ta_{s,b}}\in C_0(\mathbb{R}^d)$. Then by subordination techniques (see e.g. [15]), one can show that $f\mapsto\mathcal{F}^{-1}e^{-ta_{s,0}}*f$ yields a bounded operator on $L_1(\mathbb{R}^d)$. By a perturbation argument, also $f\mapsto\mathcal{F}^{-1}e^{-ta_{s,b}}*f$ is bounded on $L_1(\mathbb{R}^d)$. Since this operator is also translation invariant, $\mathcal{F}^{-1}e^{-ta_{s,b}}$ is given by a finite Borel measure (cf. [9, Theorem 2.58]) and therefore $\mathcal{F}^{-1}e^{-ta_{s,b}} \in L_1(\mathbb{R}^d)$.

Taking into account Young's inequality, for $p \in [1,\infty]$ and $t \geq 0$ we define the operator $S_t^{(s),p} \colon L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$ by

$$S_0^{(s),p} f := f, \quad S_t^{(s),p} f := \mathcal{F}^{-1}\mathrm{e}^{-ta_{s,b}} * f \quad (t > 0).$$

It is easy to see that $S^{(s),p}$ is a C_0-semigroup for $p \in [1,\infty)$ and $S^{(s),\infty}$ is a weak* continuous exponentially bounded semigroup.

Definition 4.1. A set $E \subset \mathbb{R}^d$ is called *thick* if E is measurable and there exist $\rho \in (0,1]$ and $L \in (0,\infty)^d$ such that

$$\left| E \cap \Big(\bigtimes_{i=1}^d (0, L_i) + x \Big) \right| \geq \rho \prod_{i=1}^d L_i \quad (x \in \mathbb{R}^d).$$

Let $\eta \in C_{\mathrm{c}}^\infty([0,\infty))$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ for $r \in [0,1/2]$ and $\eta(r) = 0$ for $r \geq 1$. For $\lambda > 0$ we define $\chi_\lambda \colon \mathbb{R}^d \to \mathbb{R}$ by $\chi_\lambda(\xi) = \eta(|\xi|/\lambda)$. Since $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{F}^{-1}\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$ and for all $p \in [1,\infty]$ we define $P_\lambda \colon L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)$ by $P_\lambda f = (\mathcal{F}^{-1}\chi_\lambda) * f$.

Proposition 4.2. *There exists $K \geq 0$ such that for all $s \in (0,1]$, $p \in [1,\infty]$ and all $\lambda > (2^{sm+4}\max\{\nu,0\}/c^s)^{1/(sm)}$, $t \geq 0$ and $f \in L_p(\mathbb{R}^d)$ we have*

$$\|(I - P_\lambda) S_t^{(s),p} f\|_{L_p(\mathbb{R}^d)} \leq K \mathrm{e}^{-2^{-sm-4} c^s t \lambda^{sm}} \|f\|_{L_p(\mathbb{R}^d)}.$$

Proof. (i) We first show the corresponding estimate for $a_{s,b}(\xi) = |\xi|^{sm}$.

The proof is an adaptation of the proof of [4, Proposition 3.2], so we only sketch the details. Let $f \in L_p(\mathbb{R}^d)$. Then

$$(I - P_\lambda) S_t^{(s),p} f = \mathcal{F}^{-1}\big((1-\chi_\lambda)\mathrm{e}^{-t|\cdot|^{sm}}\big) * f.$$

With $\sigma_{t,\lambda} := \big((1 - \chi_{t^{1/sm}\lambda})\mathrm{e}^{-|\cdot|^{sm}}\big)$ we observe

$$\|\mathcal{F}^{-1}\big((1-\chi_\lambda)\mathrm{e}^{-t|\cdot|^{sm}}\big)\|_{L_1(\mathbb{R}^d)} = \left\|\mathcal{F}^{-1}\sigma_{t,\lambda}\right\|_{L_1(\mathbb{R}^d)},$$

so by Young's inequality it suffices to estimate $\left\|\mathcal{F}^{-1}\sigma_{t,\lambda}\right\|_{L_1(\mathbb{R}^d)}$. Using that the inverse Fourier transform maps differentiation to multiplication, for $\alpha \in \mathbb{N}_0^d$ we observe

$$\left|x^\alpha \mathcal{F}^{-1}\sigma_{t,\lambda}(x)\right| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left|\partial_\xi^\alpha \big((1 - \chi_{t^{1/sm}\lambda}(\xi))\mathrm{e}^{-|\xi|^{sm}}\big)\right| \,\mathrm{d}\xi \quad (x \in \mathbb{R}^d).$$

Estimating the derivatives in the integrand for $|\alpha| \leq d+1$, we find $K_1 \geq 0$ such that

$$\left|x^\alpha \mathcal{F}^{-1}\sigma_{t,\lambda}(x)\right| \leq K_1 \mathrm{e}^{-t\lambda^{sm}/2^{sm+2}} \quad (x \in \mathbb{R}^d).$$

Thus, there exists $K \geq 0$ such that

$$\left\|\mathcal{F}^{-1}\sigma_{t,\lambda}\right\|_{L_1(\mathbb{R}^d)} \leq K \mathrm{e}^{-t\lambda^{sm}/2^{sm+2}}$$

and therefore

$$\|(I - P_\lambda) S_t^{(s),p} f\|_{L_p(\mathbb{R}^d)} \leq K \mathrm{e}^{-2^{-sm-2} t \lambda^{sm}} \|f\|_{L_p(\mathbb{R}^d)}.$$

(ii) For the general case, we use a perturbation argument. Let $\tilde{a}(\xi) := c^s |\xi|^{sm}/2$ and denote the corresponding semigroup by $\widetilde{S}$. Then by (i) we have

$$\|(I - P_\lambda)\widetilde{S}_t f\|_{L_p(\mathbb{R}^d)} \leq K\mathrm{e}^{-2^{-sm-3}c^s t\lambda^{sm}} \|f\|_{L_p(\mathbb{R}^d)}.$$

Moreover, $a_{s,b} = (a_{s,b} - \tilde{a}) + \tilde{a}$ and $a_{s,b} - \tilde{a}$ satisfies an estimate similar to (4.1), so the corresponding semigroup $(T_t)_{t\geq 0}$ obeys an exponential bound of the form

$$\|T_t\| \leq M\mathrm{e}^{\nu t} \quad (t \geq 0).$$

Thus, since $S_t^{(s),p} = T_t\widetilde{S}_t$ and Fourier multipliers commute, we arrive at

$$\begin{aligned}\|(I - P_\lambda)S_t^{(s),p} f\|_{L_p(\mathbb{R}^d)} &= \|S_t^{(s),p}(I - P_\lambda)f\|_{L_p(\mathbb{R}^d)} \\ &\leq \|T_t\| \, \|\widetilde{S}_t(I - P_\lambda)f\|_{L_p(\mathbb{R}^d)} \\ &\leq MK\mathrm{e}^{-t(2^{-sm-3}c^s\lambda^{sm}-\nu)} \|f\|_{L_p(\mathbb{R}^d)}.\end{aligned}$$

Now, for $\lambda > (2^{sm+4}\max\{\nu, 0\}/c^s)^{1/(sm)}$ we conclude $2^{-sm-3}c^s\lambda^{sm} - \nu > 2^{-sm-4}c^s\lambda^{sm}$. □

In view of of the Logvinenko–Sereda Theorem, see e.g. [14], and Proposition 4.2, we can apply Proposition 3.1 and obtain various weak observability estimates by the cases in Remark 3.3 with $\gamma_1 = 1$, $\gamma_2 = sm$ and $\gamma_3 = 1$. We state this as a corollary.

Corollary 4.3. *Let $p \in [1, \infty]$, $s \in (0, 1]$.*

(a) *Let $s \leq 1/m$. Then there exists $T > 0$ such that the semigroup $(S_t^{(s),p})_{t\geq 0}$ satisfies a weak observability inequality with some $\alpha \in (0, 1)$.*

(b) *Let $s > 1/m$. Then for all $T > 0$ the semigroup $(S_t^{(s),p})_{t\geq 0}$ satisfies a weak observability inequality with $\alpha = 0$.*

In view of Theorem 2.8, by duality we thus obtain statements on cost-uniform α-controllability and approximate null-controllability, and in view of Proposition 2.5 also for cost-uniform open-loop stabilizability. Note that for the fractional Laplacian $-A = -(-\Delta)^s$ in $L_2(\mathbb{R}^d)$, the system is not null-controllable for $s < 1/2$, cf. [10,12]. For Corollary 4.3(a) even more is true. By invoking that we prove the uncertainty principle and the dissipation estimate for all $\lambda > \lambda_0$ with some $\lambda_0 \geq 0$, we get, by using Remark 3.3(a) for $T > 0$ large enough, that for all $\alpha \in (0, 1)$ there is $T > 0$ such that $(S_t^{(s),p})_{t\geq 0}$ satisfies a weak observability inequality.

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