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Integral Equations and Operator Theory

Regular Functions on the Scaled Hypercomplex Numbers

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Abstract. In this paper, we study the regularity of $\mathbb{R}$-differentiable functions on open connected subsets of the scaled hypercomplex numbers $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ by studying the kernels of suitable differential operators $\{\nabla_t\}_{t\in\mathbb{R}}$, up to scales in the real field $\mathbb{R}$.

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1. Introduction

In this paper, we study differentiation on scaled hypercomplex numbers scaled by an arbitrary quantity $t \in \mathbb{R}$. Roughly speaking, scaled hypercomplex numbers are the ordered pairs of complex numbers under an arbitrary fixed real number. We let $\mathbb{C}^2$ be the usual 2-dimensional Hilbert space over the complex field $\mathbb{C}$, and understand each vector $(a, b) \in \mathbb{C}^2$ as a hypercomplex number $(a, b) \in \mathbb{H}_t$, inducing the algebraic triple,

$$\mathbb{H}_t = \left(\mathbb{C}^2,\ +,\ \cdot_t\right),$$

with the usual vector addition $(+)$ on $\mathbb{C}^2$, and the t-scaled vector multiplication $(\cdot_t)$,

$$(a_1, b_1) \cdot_t (a_2, b_2) = \left(a_1a_2 + tb_1\overline{b_2},\ a_1b_2 + b_1\overline{a_2}\right),$$

for all $(a_l, b_l) \in \mathbb{C}^2$, for $l = 1, 2$, forms a well-defined unital ring with its unity (or, the $(\cdot_t)$-identity) $(1, 0)$, where $\overline{z}$ mean the conjugates of $z \in \mathbb{C}$ (e.g., see [2]).

From the Hilbert-space representation $\left(\mathbb{C}^2, \pi_t\right)$ of the t-scaled hypercomplex ring $\mathbb{H}_t$, introduced in [2], a hypercomplex number $h = (a, b) \in \mathbb{H}_t$ is realized to be a (2×2)-matrix, or a Hilbert-space operator acting on $\mathbb{C}^2$,

$$\pi_t(h) \overset{\text{denote}}{=} [h]_t \overset{\text{def}}{=} \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

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where $M_2(\mathbb{C})$ is the matricial algebra (which is $*$-isomorphic to the operator C^*-algebra $B(\mathbb{C}^2)$ of all bounded linear operators acting on the Hilbert space $\mathbb{C}^2$) over $\mathbb{C}$, for $t \in \mathbb{R}$. The construction of such rings $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ provides the generalized structures of well-known quaternions (e.g., [5–8,12,15,18,21]), and split-quaternions (e.g., [4,9,14]). Indeed, the ring $\mathbb{H}_{-1}$ is nothing but the noncommutative field $\mathbb{H}$ of all quaternions, and the unital ring $\mathbb{H}_1$ is the ring of all split-quaternions (e.g., [1–3]). The algebra, spectral theory, operator theory, and free probability on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ are studied in [1,2], under the above representation $(\mathbb{C}^2, \pi_t)$. Different from the approaches of [1,2], we study those on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ by defining suitable bilinear forms $\{\langle,\rangle_t\}_{t\in\mathbb{R}}$ on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, in [3]. In the approaches of [3], the pairs $\{(\mathbb{H}_t, \langle,\rangle_t)\}_{t<0}$ form (definite) inner product spaces over $\mathbb{R}$, meanwhile, the pairs $\{(\mathbb{H}_t, \langle,\rangle_t)\}_{t\geq 0}$ become indefinite semi-inner product spaces over $\mathbb{R}$, inducing the complete semi-normed spaces $\{(\mathbb{H}_t, \|.\|_t)\}_{t\in\mathbb{R}}$, having their semi-norms,

$$\|h\|_t = \sqrt{|\langle h, h\rangle_t|}, \ \ \forall h \in \mathbb{H}_t, \ \forall t \in \mathbb{R},$$

where $|.|$ is the absolute value on $\mathbb{R}$. (e.g., [3]). Meanwhile, it is considered in [3] that each t-scaled hypercomplex number $h \in \mathbb{H}_t$ is regarded as a multiplication operator M_h acting on $(\mathbb{H}_t, \|.\|_t)$,

$$M_h(h') = h \cdot_t h' \in (\mathbb{H}_t, \|.\|_t), \quad \forall h' \in \mathbb{H}_t,$$

with its adjoint,

$$M^*_{(a,b)} = M_{(\overline{a},-b)}, \ \ \forall (a,b) \in \mathbb{H}_t,$$

where $(\overline{a}, -b) \overset{\text{denote}}{=} (a,b)^\dagger$ is the hypercomplex-conjugate in $\mathbb{H}_t$, i.e., $M_h^* = M_{h^\dagger}$ in the operator space $B_{\mathbb{R}}(\mathbb{H}_t)$ of all bounded linear operators acting on $\mathbb{H}_t$ "over $\mathbb{R}$," for all scales $t \in \mathbb{R}$, which is a Banach space equipped with the operator semi-norm,

$$\|T\| = sup\{\|Th\|_t : \|h\|_t = 1\}, \ \forall T \in B_{\mathbb{R}}(\mathbb{H}_t).$$

Furthermore, the subset,

$$\mathcal{M}_t \overset{\text{def}}{=} \{M_h \in B_{\mathbb{R}}(\mathbb{H}_t) : h \in \mathbb{H}_t\},$$

of $B_{\mathbb{R}}(\mathbb{H}_t)$ forms a complete semi-normed $*$-algebra over $\mathbb{R}$ of the adjointable operators $\{M_h\}_{h\in\mathbb{R}}$ in $B_{\mathbb{R}}(\mathbb{H}_t)$ (e.g., see [3]), for $t \in \mathbb{R}$. i.e., case-by-case, we understand the set $\mathbb{H}_t$ of t-scaled hypercomplex numbers as a unital ring $\mathbb{H}_t$ with its unity $(1,0)$ algebraically; or, as a complete semi-normed $\mathbb{R}$-vector space $(\mathbb{H}_t, \|.\|_t)$, which is either a $\mathbb{R}$-Hilbert space if $t < 0$, or an indefinite semi-inner product $\mathbb{R}$-space if $t \geq 0$, analytically; or as a complete semi-normed $*$-algebra $\mathcal{M}_t$ over $\mathbb{R}$, operator-algebra-theoretically, for all $t \in \mathbb{R}$.

In this paper, we regard $\mathbb{H}_t$ as a complete semi-normed $\mathbb{R}$-vector space, and act a differential operators ∇_t on the $\mathbb{R}$-differentiable functions on $\mathbb{H}_t$, where

$$\nabla_t = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - j_t\frac{sgn(t)\,\partial}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn(t)\,\partial}{\sqrt{|t|}\partial x_4}, \text{ if } t \neq 0,$$

and

$$\nabla_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_0\frac{\partial}{\partial x_3} + k_0\frac{\partial}{\partial x_4}, \text{ if } t = 0, \tag{1.1}$$

for an arbitrarily fixed scale $t \in \mathbb{R}$, where

$$sgn\,(t) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0, \end{cases}$$

for all $t \in \mathbb{R} \setminus \{0\}$, and

$$i^2 = -1, \text{ and } j_t^2 = k_t^2 = t, \tag{1.2}$$

satisfying the commuting diagrams,

$$\begin{tikzcd} & i \arrow[dl, "1"'] & \\ j_t \arrow[rr, "1"'] & & k_t \arrow[ul, "-t"'] \end{tikzcd}, \quad \text{and} \quad \begin{tikzcd} & i \arrow[dr, "-1"] & \\ j_t \arrow[ur, "t"] & & k_t \arrow[ll, "-1"] \end{tikzcd}. \tag{1.3}$$

Here, the first diagram of (1.3) means that

$$ij_t = k_t, \; j_tk_t = -ti, \text{ and } k_ti = j_t,$$

while the second diagram of (1.3) means that

$$ik_t = -j_t, \; k_tj_t = ti, \text{ and } j_ti = -k_t,$$

for $t \in \mathbb{R}$.

We study the left, or right t-regular functions contained in the kernel, $\ker\nabla_t$ of ∇_t, and the t-harmonic functions determined by the t-regular functions, by defining t-scaled Laplacians,

$$\varDelta_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2}, \quad \text{if } t \neq 0,$$

and

$$\varDelta_0 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 0\left(\frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \text{ if } t = 0,$$

for all $t \in \mathbb{R}$.

2. Scaled Hypercomplex Numbers

In this section, we review scaled hypercomplex numbers. For details, e.g., see [1–3].

2.1. Scaled Hypercomplex Rings

Fix an arbitrarily scale $t \in \mathbb{R}$. Define an operation $(\cdot_t)$ on $\mathbb{C}^2$ by

$$(a_1, b_1) \cdot_t (a_2, b_2) \overset{\text{def}}{=} \left(a_1a_2 + tb_1\overline{b_2},\; a_1b_2 + b_1\overline{a_2}\right), \tag{2.1.1}$$

for $(a_l, b_l) \in \mathbb{C}^2$, for all $l = 1, 2$.

Proposition 2.1. *The algebraic triple* $\left(\mathbb{C}^2, +, \cdot_t\right)$ *forms a unital ring with its unity* $(1, 0)$, *where* $(+)$ *is the usual vector addition on* $\mathbb{C}^2$, *and* $(\cdot_t)$ *is the operation* (2.1.1).

Proof. See [1] for details. □

One can understand these unital rings $\left\{\left(\mathbb{C}^2, +, \cdot_t\right)\right\}_{t\in\mathbb{R}}$ as topological rings, since the operations $(+)$ and $\{(\cdot_t)\}_{t\in\mathbb{R}}$ are continuous on $\mathbb{C}^2$.

Definition 2.2. For $t \in \mathbb{R}$, the ring $\mathbb{H}_t \overset{\text{denote}}{=} \left(\mathbb{C}^2, +, \cdot_t\right)$ is called the t-scaled hypercomplex ring.

For $t \in \mathbb{R}$, define an injective map,

$$\pi_t : \mathbb{H}_t \rightarrow M_2\left(\mathbb{C}\right),$$

by

$$\pi_t\left((a, b)\right) = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}, \ \forall\, (a, b) \in \mathbb{H}_t. \tag{2.1.2}$$

Such an injective map π_t of (2.1.2) satisfies that

$$\pi_t\left(h_1 + h_2\right) = \pi_t\left(h_1\right) + \pi_t\left(h_2\right),$$

and

$$\pi_t\left(h_1 \cdot_t h_2\right) = \pi_t\left(h_1\right)\pi_t\left(h_2\right), \tag{2.1.3}$$

in $M_2\left(\mathbb{C}\right)$, for all $h_1, h_2 \in \mathbb{H}_t$, where $\pi_t\left(h_1\right)\pi_t\left(h_2\right)$ is the usual matricial multiplication (e.g., see [1] for details).

Proposition 2.3. *The pair* $\left(\mathbb{C}^2,\ \pi_t\right)$ *forms an injective Hilbert-space representation of our t-scaled hypercomplex ring $\mathbb{H}_t$, where π_t is an action* (2.1.2).

Proof. It is shown by (2.1.3), and by the continuity of π_t (e.g., [1,2]). □

By the injectivity of π_t, one can understand $\mathbb{H}_t$ as its realization $\pi_t\left(\mathbb{H}_t\right)$ as matrices of $M_2\left(\mathbb{C}\right)$.

Definition 2.4. The subset,

$$\pi_t\left(\mathbb{H}_t\right) = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2\left(\mathbb{C}\right) : (a, b) \in \mathbb{H}_t \right\}, \tag{2.1.4}$$

of $M_2\left(\mathbb{C}\right)$, denoted by $\mathcal{H}_2^t$, is called the t-scaled (hypercomplex-)realization of $\mathbb{H}_t$ (in $M_2\left(\mathbb{C}\right)$) for $t \in \mathbb{R}$. For convenience, we denote the realization $\pi_t\left(h\right)$ of $h \in \mathbb{H}_t$ by $\left[h\right]_t$ in $\mathcal{H}_2^t$.

If $\mathbb{H}_t^{\times} \overset{\text{denote}}{=} \mathbb{H}_t \setminus \left\{(0, 0)\right\}$, where $(0, 0)$ is the $(+)$-identity of $\mathbb{H}_t$, then, it is the maximal monoid,

$$\mathbb{H}_t^{\times} \overset{\text{denote}}{=} \left(\mathbb{H}_t^{\times},\ \cdot_t\right),$$

in $\mathbb{H}_t$, with its $(\cdot_t)$-identity $(1, 0)$, the unity of $\mathbb{H}_t$ (e.g., [2]).

Definition 2.5. The monoid $\mathbb{H}_t^{\times} = \left(\mathbb{H}_t^{\times},\ \cdot_t\right)$ of $\mathbb{H}_t$ is called the t-scaled hypercomplex monoid.

2.2. Invertible Hypercomplex Numbers of $\mathbb{H}_t$

For an arbitrarily fixed $t \in \mathbb{R}$, let $\mathbb{H}_t$ be the corresponding t-scaled hypercomplex ring, isomorphic to its t-scaled realization $\mathcal{H}_2^t$ of (2.1.4). Observe that, for any $(a, b) \in \mathbb{H}_t$, one has

$$det\left([(a,b)]_t\right) = det\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 - t\,|b|^2. \tag{2.2.1}$$

where det is the determinant, and $|.|$ is the modulus on $\mathbb{C}$.

Recall that an algebraic triple $(X, +, \cdot)$ is a noncommutative field, if it is a unital ring, containing $(X^{\times}, \cdot)$ as a non-abelian group (e.g., [2,3]).

Lemma 2.6. *If $(a, b) \in \mathbb{H}_t$, then $|a|^2 \neq t\,|b|^2$ in $\mathbb{C}$, if and only if (a, b) is invertible in $\mathbb{H}_t$ with its inverse,*

$$(a,b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\,|b|^2}, \frac{-b}{|a|^2 - t\,|b|^2}\right) \text{ in } \mathbb{H}_t,$$

satisfying

$$\left[(a,b)^{-1}\right]_t = [(a,b)]_t^{-1} \text{ in } \mathcal{H}_2^t. \tag{2.2.2}$$

Moreover, we have the algebraic characterization,

$$t < 0 \text{ in } \mathbb{R} \Longleftrightarrow \mathbb{H}_t \textit{ is a noncommutative field.} \tag{2.2.3}$$

The proof is straightfoward; see [1].

By the proof of the above proposition, $\{\mathbb{H}_s\}_{s<0}$ are noncommutative fields, but, $\{\mathbb{H}_t\}_{t\geq 0}$ cannot be noncommutative fields by (2.2.3). For any scale $t \in \mathbb{R}$, the t-scaled hypercomplex ring $\mathbb{H}_t$ is decomposed by

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}$$

with

$$\mathbb{H}_t^{inv} = \left\{(a,b) : |a|^2 \neq t\,|b|^2\right\}, \tag{2.2.4}$$

and

$$\mathbb{H}_t^{sing} = \left\{(a,b) : |a|^2 = t\,|b|^2\right\},$$

where $\sqcup$ is the disjoint union.

Proposition 2.7. *The subset $\mathbb{H}_t^{inv}$ of (2.2.4) is a non-abelian group in the monoid $\mathbb{H}_t^{\times}$. Meanwhile, the subset $\mathbb{H}_t^{\times sing} \overset{denote}{=} \mathbb{H}_t^{sing} \setminus \{(0,0)\}$ forms a semigroup without identity in $\mathbb{H}_t^{\times}$.*

Proof. Note that, in general, $\det(AB) = \det(A)\det(B)$, for all $A, B \in M_n(\mathbb{C})$, for all $n \in \mathbb{N}$. See [2] for details. $\square$

Definition 2.8. The block $\mathbb{H}_t^{inv}$ of (2.2.4) is called the group-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$), and the other algebraic block $\mathbb{H}_t^{\times sing}$ of the above proposition is called the semigroup-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$).

By (2.2.3), if $t < 0$ in $\mathbb{R}$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is empty in $\mathbb{H}_t^{\times}$, and hence,

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \Longleftrightarrow \mathbb{H}_t = \mathbb{H}_t^{inv} \cup \{(0,0)\},$$

Meanwhile, if $t \geq 0$ in $\mathbb{R}$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is non-empty, and is properly contained in the t-scaled monoir $\mathbb{H}_t^{\times}$, satisfying (2.2.4).

2.3. Scaled-Hypercomplex Conjugation

In this section, we define a suitable $\mathbb{R}$-adjoint on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, as in [3]. For a scale $t \in \mathbb{R}$, define a unary operation $(\dagger)$ on the t-scaled hypercomplex ring $\mathbb{H}_t$ by

$$(a,b)^{\dagger} \overset{\text{def}}{=} (\overline{a},\ -b), \quad \forall\, (a,b) \in \mathbb{H}_t. \tag{2.3.1}$$

This operation $(\dagger)$ of (2.3.1) is indeed a well-defined unary operation on $\mathbb{H}_t$, inducing the equivalent operation, also denoted by $(\dagger)$, on the t-scaled realization $\mathcal{H}_2^t$ of $\mathbb{H}_t$,

$$[(a,b)]_t^{\dagger} \overset{\text{def}}{=} \left[(a,b)^{\dagger}\right]_t = [(\overline{a}, -b)]_t\,, \tag{2.3.2}$$

for all $(a,b) \in \mathbb{H}_t$. Since the action $\pi_t : \mathbb{H}_t \to \mathcal{H}_2^t$ and the operation $(\dagger)$ of (2.3.1) are bijective, the function (2.3.2) is also a well-defined bijection on $\mathcal{H}_2^t$.

Proposition 2.9. *The bijection $(\dagger)$ of* (2.3.1) *is an adjoint on $\mathbb{H}_t$ over $\mathbb{R}$.*

See [3] for details.

Note that the $\mathbb{R}$-adjoint $(\dagger)$ of (2.3.1) (or, of (2.3.2)) is free from the choice of scales $t \in \mathbb{R}$. So, we call $(\dagger)$ the hypercomplex-conjugate.

If $h = (a,b) \in \mathbb{H}_t$, then one obtains that

$$[h]_t^{\dagger}\,[h]_t = \left[\left(|a|^2 - t\,|b|^2,\ 0\right)\right]_t = [h]_t\,[h]_t^{\dagger}, \tag{2.3.3}$$

for all $h = (a,b) \in \mathbb{H}_t$, for "all" $t \in \mathbb{R}$.

2.4. The Normalized Trace τ on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ Over $\mathbb{R}$

Recall that the t-scaled realization $\mathcal{H}_2^t$ of the t-scaled hypercomplex ring $\mathbb{H}_t$ is an embedded ring of $M_2(\mathbb{C})$. So, the normalized trace $\tau = \frac{1}{2}tr$ on $M_2(\mathbb{C})$ is naturally restricted to $\tau\mid_{\mathcal{H}_2^t}$, also denoted by τ, on $\mathcal{H}_2^t$, where tr is the usual trace on $M_2(\mathbb{C})$. Observe that, for any $[(a,b)]_t \in \mathcal{H}_2^t$, one has

$$\tau\left([(a,b)]_t\right) = \frac{1}{2}tr\left(\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}\right) = \frac{1}{2}\left(a + \overline{a}\right),$$

i.e.,

$$\tau\left([(a,b)]_t\right) = \mathrm{Re}\,(a)\,, \quad \forall\, (a,b) \in \mathbb{H}_t. \tag{2.4.1}$$

Remark and Discussion. Since $\mathcal{H}_2^t$ is a sub-structure of $M_2(\mathbb{C})$, the linear functional $\tau = \frac{1}{2}tr$ on $M_2(\mathbb{C})$ is well-restricted to that on $\mathcal{H}_2^t$. However, note that, since $\tau(\mathcal{H}_2^t) \subseteq \mathbb{R}$ by (2.4.1), it becomes a linear functional "over $\mathbb{R}$." So, in the following text, if we mention "τ is a linear functional," then it actually means that "this restriction is linear over $\mathbb{R}$." The construction of such a

$\mathbb{R}$-linear functional on $\mathcal{H}_2^t$ is motivated by free probability (e.g., [17,20]), and such a free-probabilistic model is considered in detail in [1–3].□

By (2.4.1), one can define a ($\mathbb{R}$-)linear functional, also denoted by τ, on $\mathbb{H}_t$, by

$$\tau\left((a,b)\right) \overset{\text{def}}{=} \operatorname{Re}\left(a\right), \quad \forall\left(a,b\right) \in \mathbb{H}_t. \tag{2.4.2}$$

By using this linear functional τ of (2.4.2) on $\mathbb{H}_t$, we define a form,

$$\langle , \rangle_t : \mathbb{H}_t \times \mathbb{H}_t \to \mathbb{R},$$

by

$$\langle h_1, h_2 \rangle_t \overset{\text{def}}{=} \tau\left(h_1 \cdot_t h_2^{\dagger}\right), \quad \forall h_1, h_2 \in \mathbb{H}_t, \tag{2.4.3}$$

where the linear functional τ in (2.4.3) is in the sense of (2.4.2), whose range is contained in $\mathbb{R}$ in $\mathbb{C}$. Since the hypercomplex-conjugation ($\dagger$) is bijective, this form (2.4.3) is a well-defined function.

Lemma 2.10. *The function* $\langle , \rangle_t$ *of* (2.4.3) *is a bilinear form on* $\mathbb{H}_t$ *"over* $\mathbb{R}$*."*

Proof. By the straightforward computations, one has

$$\langle h_1 + h_2, h_3 \rangle_t = \langle h_1, h_2 \rangle_t + \langle h_2, h_3 \rangle_t,$$
$$\langle h_1, h_2 + h_3 \rangle_t = \langle h_1, h_2 \rangle_t + \langle h_1, h_3 \rangle_t,$$

and

$$\langle rh_1, h_2 \rangle_t = r\left\langle h_1, h_2 \right\rangle_t, \ \langle h_1, rh_2 \rangle_t = r\left\langle h_1, h_2 \right\rangle_t,$$

for all $h_1, h_2, h_3 \in \mathbb{H}_t$, and $r \in \mathbb{R}$, where

$$rh = (r, 0) \cdot_t h, \ \ \forall h \in \mathbb{H}_t, \ r \in \mathbb{R}.$$

See [3] for details. □

The above lemma shows that the forms $\{\langle , \rangle_t\}_{t \in \mathbb{R}}$ of (2.4.3) are well-determined bilinear forms on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, induced by the linear functionals τ of (2.4.2) and the hypercomplex-conjugate ($\dagger$) of (2.3.1).

Lemma 2.11. *For all* $h_1, h_2 \in \mathbb{H}_t$*, we have*

$$\langle h_1, h_2 \rangle_t = \langle h_2, h_1 \rangle_t \, in \mathbb{R}. \tag{2.4.4}$$

Proof. Observe that, for any $h_1, h_2 \in \mathbb{H}_t$,

$$\langle h_1, h_2 \rangle_t = \tau\left(h_1 \cdot_t h_2^{\dagger}\right) = \operatorname{Re}\left(h_1 \cdot_t h_2^{\dagger}\right)$$

since τ of (2.4.1) is a trace

$$= \operatorname{Re}\left(h_2 \cdot_t h_1\right) = \tau\left(h_2 \cdot_t h_1^{\dagger}\right) = \langle h_2, h_1 \rangle_t.$$

□

The above lemma shows our bilinear form $\langle , \rangle_t$ of (2.4.3) is symmetric by (2.4.4).

Lemma 2.12. *If $h_1, h_2 \in \mathbb{H}_t$, then*

$$\left|\langle h_1, h_2\rangle_t\right|^2 \leq \left|\langle h_1, h_1\rangle_t\right|^2 \left|\langle h_2, h_2\rangle_t\right|^2, \tag{2.4.5}$$

where $|.|$ is the absolute value on $\mathbb{R}$.

Proof. See [3] for details. □

Observe now that if $h = (a, b) \in \mathbb{H}_t$, then

$$\langle h, h\rangle_t = \tau\left((a, b) \cdot_t (a, b)^\dagger\right) = \mathrm{Re}\left(|a|^2 - t\,|b|^2\right),$$

by (2.4.1) and (2.4.2), implying that

$$\langle h, h\rangle_t = |a|^2 - t\,|b|^2 = det\left([h]_t\right),$$

and hence,

$$\langle h, h\rangle_t = 0 \Longleftrightarrow det\left([h]_t\right) = 0 \text{ in } \mathbb{R},$$

if and only if

$$h = (a, b) \in \mathbb{H}_t, \text{ with } |a|^2 = t\,|b|^2. \tag{2.4.6}$$

Proposition 2.13. *Let $h = (a, b) \in \mathbb{H}_t$. Then $\langle h, h\rangle_t = 0$, if and only if $|a|^2 = t\,|b|^2$, if and only if $det\left([h]_t\right) = 0$, if and only if h is not invertible in $\mathbb{H}_t$, if and only if $h \in \mathbb{H}_t^{sing}$. i.e.,*

$$\langle h, h\rangle_t = 0, \Longleftrightarrow h \in \mathbb{H}_t^{sing}, \text{ in } \mathbb{H}_t. \tag{2.4.7}$$

Proof. The relation (2.4.7) is shown by (2.4.6). □

If either h_1 or h_2 is contained in $\mathbb{H}_t^{sing}$, then $h_1 \cdot_t h_2^\dagger \in \mathbb{H}_t^{sing}$ in $\mathbb{H}_t$, since

$$det\left(\left[h_1 \cdot_t h_2^\dagger\right]_t\right) = det\left([h_1]_t\,[h_2]_t^\dagger\right),$$

implying that

$$det\left(\left[h_1 \cdot_t h_2^\dagger\right]_t\right) = \left(det\left([h_1]_t\right)\right)\left(det\left([h_2]_t^\dagger\right)\right) = 0.$$

Definition 2.14. For a vector space X over $\mathbb{R}$, a bilinear form $\langle,\rangle : X \times X \to \mathbb{R}$ is called a (definite) semi-inner product on X over $\mathbb{R}$, if (i)

$$\langle x_1, x_2\rangle = \langle x_2, x_1\rangle, \ \forall x_1, x_2 \in X,$$

and (ii) $\langle x, x\rangle \geq 0$, for all $x \in X$. In such a case, the pair $(X, \langle,\rangle)$ is called a semi-inner product space over $\mathbb{R}$. If a semi-inner product $\langle,\rangle$ on a $\mathbb{R}$-vector space X satisfies an additional condition (iii) $\langle x, x\rangle = 0$, if and only if $x = 0_X$, the zero vector of X, then it is called an (definite) inner product on X over $\mathbb{R}$, and the pair $(X, \langle,\rangle)$ is said to be an inner-product space over $\mathbb{R}$.

Definition 2.15. For a vector space X over $\mathbb{R}$, a bilinear form $\langle,\rangle : X \times X \to \mathbb{R}$ is called an indefinite semi-inner product on X over $\mathbb{R}$, if (i)

$$\langle x_1, x_2\rangle = \langle x_2, x_1\rangle, \ \ \forall x_1, x_2 \in X,$$

and (ii) $\langle x, x\rangle \in \mathbb{R}$, for all $x \in X$. In such a case, the pair $(X, \langle,\rangle)$ is called an indefinite-semi-inner product space over $\mathbb{R}$.

If an indefinite semi-inner product $\langle , \rangle$ on a $\mathbb{R}$-vector space X satisfies an additional condition; $\langle x, y\rangle = 0$ for "all" $y \in X$, if and only if $x = 0_X$, then it is called an indefinite inner product on X over $\mathbb{R}$, and the pair $(X, \langle , \rangle)$ is said to be an indefinite-inner product space over $\mathbb{R}$. i.e., an indefinite semi-inner product is non-degenerated as above, then it becomes an indefinite inner product.

By the above lemmas and proposition, one obtains the following result.

Theorem 2.16. *If $t < 0$ in $\mathbb{R}$, then the bilinear form $\langle , \rangle_t$ of (2.4.3) is a continuous inner product on $\mathbb{H}_t$ over $\mathbb{R}$, i.e.,*

$$t < 0 \Longrightarrow (\mathbb{H}_t, \langle , \rangle_t) \; is\, a\, \mathbb{R} - inner\, product\, space. \tag{2.4.8}$$

Meanwhile, if $t \geq 0$, then $\langle , \rangle_t$ is a continuous indefinite semi-inner product on $\mathbb{H}_t$ over $\mathbb{R}$, i.e.,

$$t \geq 0 \Longrightarrow (\mathbb{H}_t, \langle , \rangle_t) \; is\, a\, \mathbb{R} - indefinite\, semi - inner\, product\, space. \tag{2.4.9}$$

However, if $t > 0$, then $(\mathbb{H}_t, \langle , \rangle_t)$ is a $\mathbb{R}$-indefinite inner product space, meanwhile, if $t = 0$, then it is a $\mathbb{R}$-indefinite semi-inner product space.

Proof. Assume first that a given scale t is negative in $\mathbb{R}$. Then the bilinear form $\langle , \rangle_t$ forms a semi-inner product by the symmetry (2.4.4), and

$$\langle (a, b) , (a, b)\rangle_t = |a|^2 - t\,|b|^2 \geq 0,$$

since $t < 0$, for all $(a, b) \in \mathbb{H}_t$. Also, recall that, if $t < 0$, then $\mathbb{H}_t = \mathbb{H}_t^{inv} \cup \{(0, 0)\}$, equivalently, $\mathbb{H}_t^{sing} = \{(0, 0)\}$. It implies that

$$\langle h, h\rangle_t = 0 \Longleftrightarrow h \in \mathbb{H}_t^{sing} \Longleftrightarrow h = (0, 0) \in \mathbb{H}_t.$$

i.e., $\langle , \rangle_t$ is an inner product on $\mathbb{H}_t$ over $\mathbb{R}$, whenever $t < 0$ in $\mathbb{R}$. The continuity of $\langle , \rangle_t$ is guaranteed by (2.4.5). So, if $t < 0$, then the pair $(\mathbb{H}_t, \langle , \rangle_t)$ is not only a $\mathbb{R}$-semi inner product space, but also a $\mathbb{R}$-inner product space. Therefore, the relation (2.4.8) holds.

Assume now that a scale t is nonnegative in $\mathbb{R}$. Then the bilinear form $\langle , \rangle_t$ forms an indefinite semi-inner product by the symmetry (2.4.4), and

$$\langle h, h\rangle_t = det\left([h]_t\right) \in \mathbb{R},$$

since $t \geq 0$, for all $h \in \mathbb{H}_t$. Note that if $t > 0$, then the semigroup-part $\mathbb{H}_t^{\times sing}$ is not empty in $\mathbb{H}_t$, satisfying

$$\langle h, h\rangle_t = 0 \Longleftrightarrow h \in \mathbb{H}_t^{sing},$$

by (2.4.7). However, if

$$\langle h, q\rangle_t = 0, \; \forall q \in \mathbb{H}_t,$$

then $h = 0$ in $\mathbb{H}_t$. i.e., the indefinite semi-inner product $\langle , \rangle_t$ is non-degenerated. So, the form $\langle , \rangle_t$ becomes an indefinite inner product on $\mathbb{H}_t$, whenever $t > 0$. Meanwhile, if $t = 0$, then it is not non-degenerated. So, $(\mathbb{H}_0, \langle , \rangle_0)$ is a $\mathbb{R}$-indefinite semi-inner product space. In conclusion, the statement (2.4.9) holds. $\square$

The above theorem shows that $\{\mathbb{H}_t\}_{t<0}$ are $\mathbb{R}$-inner product spaces, meanwhile, $\{\mathbb{H}_t\}_{t\geq 0}$ form $\mathbb{R}$-indefinite semi-inner product space, by (2.4.8) and (2.4.9), respectively.

Definition 2.17. Let X be a vector space over $\mathbb{R}$, and $\|.\|_X : X \to \mathbb{R}$, a function satisfying (i) $\|x\|_X \geq 0$, for all $x \in X$, and (ii) $\|rx\|_X = |r| \, \|x\|_X$, for all $r \in \mathbb{R}$ and $x \in X$, and (iii)

$$\|x_1 + x_2\|_X \leq \|x_1\|_X + \|x_2\|_X \, , \; \forall x_1, x_2 \in X.$$

Then the function $\|.\|_X$ is called a semi-norm on X over $\mathbb{R}$, and the pair $(X, \|.\|_X)$ is said to be a semi-normed space over $\mathbb{R}$. If a semi-norm $\|.\|_X$ satisfies an additional condition (iv) $\|x\|_X = 0$, if and only if $x = 0_X$ in X, then it is called a norm on X over $\mathbb{R}$, and the pair $(X, \|.\|_X)$ is said to be a normed space over $\mathbb{R}$.

Let $(\mathbb{H}_t, \langle , \rangle_t)$ be either a $\mathbb{R}$-inner product space (if $t < 0$), or a $\mathbb{R}$-indefinite semi-inner product space (if $t \geq 0$), for an arbitrary scale $t \in \mathbb{R}$. Define a function,

$$\|.\|_t : \mathbb{H}_t \to \mathbb{R},$$

by

$$\|h\|_t \overset{\text{def}}{=} \sqrt{|\langle h, h \rangle_t|} = \sqrt{|det\,([h]_t)|} = \sqrt{|\tau\,([h \cdot_t h^\dagger])|}, \tag{2.4.10}$$

for all $h \in \mathbb{H}_t$, where $|.|$ is the absolute value on $\mathbb{R}$. Then it is not hard to check that

$$\|h\|_t \geq 0, \text{ since } |\langle h, h \rangle_t| = \left| |a|^2 - t\,|b|^2 \right| \geq 0,$$

and

$$\|rh\|_t = \sqrt{|\langle rh, rh \rangle_t|} = \sqrt{r^2}\sqrt{|\langle h, h \rangle_t|} = |r| \, \|h\|_t \, ,$$

for all $r \in \mathbb{R}$ and $h \in \mathbb{H}_t$, and

$$\|h_1 + h_2\|_t^2 = |\langle h_1 + h_2, h_1 + h_2 \rangle_t| \leq \left(\|h_1\|_t + \|h_2\|_t\right)^2,$$

by (2.4.5) and the bilinearity of $\langle , \rangle_t$, implying that

$$\|h_1 + h_2\|_t \leq \|h_1\|_t + \|h_2\|_t \, , \text{ in } \mathbb{R},$$

for all $h_1, h_2 \in \mathbb{H}_t$ (See [3] for details).

Theorem 2.18. *The pair* $(\mathbb{H}_t, \|.\|_t)$ *of the t-scaled hypercomplex ring* $\mathbb{H}_t$ *and the function* $\|.\|_t$ *of* (2.4.10) *forms a complete* $\mathbb{R}$*-semi-normed space. In particular,*

$$t < 0 \Longrightarrow (\mathbb{H}_t, \|.\|_t) \; is\, a\, complete\, \mathbb{R} - \; normed\, space, \tag{2.4.11}$$

and

$$t \geq 0 \Longrightarrow (\mathbb{H}_t, \|.\|_t) \; is\, a\, complete\, \mathbb{R} - \; semi - normed\, space. \tag{2.4.12}$$

Proof. By the very above paragraph, the map $\|.\|_t$ of (2.4.10) is a semi-norm on $\mathbb{H}_t$ over $\mathbb{R}$. In particular, the $\mathbb{R}$-semi-normed space $(\mathbb{H}_t, \|.\|_t)$ is complete by (2.4.5) and (2.4.10), since $\mathbb{H}_t$ is a subspace of the finite-dimensional $\mathbb{R}$-vector space $\mathbb{R}^4$. More precisely, if $t < 0$ in $\mathbb{R}$, then this semi-norm $\|.\|_t$ satisfies the additional condition,

$$\|h\|_t = 0 \Longleftrightarrow \langle h, h \rangle_t = 0 \Longleftrightarrow h \in \mathbb{H}_t^{sing} \Longleftrightarrow h = (0,0),$$

implying that $\|.\|_t$ forms a norm on $\mathbb{H}_t$ over $\mathbb{R}$. So, the statement (2.4.11) holds true. Meanwhile if $t \geq 0$, then this semi-norm $\|.\|_t$ cannot be a norm, because,

$$\|h\|_t = 0 \Longleftrightarrow h \in \mathbb{H}_t^{sing},$$

by (2.4.7). Therefore, the statement (2.4.12) holds. □

Recall that complete inner product spaces (over $\mathbb{R}$, or over $\mathbb{C}$) are said to be Hilbert spaces (over $\mathbb{R}$, respectively, over $\mathbb{C}$). So, the statement (2.4.11) can be re-stated by that: if $t < 0$, then $(\mathbb{H}_t, \langle , \rangle_t)$ is a Hilbert space over $\mathbb{R}$ (or, a $\mathbb{R}$-Hilbert space).

3. Scaled Hypercomplex $\mathbb{R}$-Spaces $\mathbb{H}_t$

In the rest of this paper, we understand the t-scaled hypercomplex ring $\mathbb{H}_t$ as the $\mathbb{R}$-vector space $\mathbb{H}_t \overset{\text{denote}}{=} (\mathbb{H}_t, \langle , \rangle_t)$, which is either a $\mathbb{R}$-Hilbert space if $t < 0$, or a complete $\mathbb{R}$-indefinite semi-inner product space (where the completeness is up to that of the corresponding complete $\mathbb{R}$-semi-normed space, $(\mathbb{H}_t, \|.\|_t)$ induced by the semi-norm $\|.\|_t$ of (2.4.10)) if $t \geq 0$. Remark that if $t > 0$, then $\mathbb{H}_t$ forms a $\mathbb{R}$-indefinite inner product space, while if $t = 0$, then it is a $\mathbb{R}$-indefinite semi-inner product space. In this section, we consider a suitable setting of $\mathbb{H}_t$ to study the differentiation of the functions acting on it. More precisely, to study how an operator,

$$D_t = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_t\frac{\partial}{\partial x_3} + k_t\frac{\partial}{\partial x_4}$$

under the conditions (1.2) and (1.3), acts on the set of $\mathbb{R}$-differentiable functions, we investigate an equivalent setting for $\mathbb{H}_t$, which is different from that of Sect. 2, notationally.

Let

$$a = x + yi, \text{ and } b = u + vi$$

be complex numbers with $i = \sqrt{-1}$ and $x, y, u, v \in \mathbb{R}$, and let $(a, b) \in \mathbf{X}_t$, whose realization is

$$[(a,b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} x + yi & tu + tvi \\ & \\ u - vi & x - yi \end{pmatrix}, \tag{3.1}$$

in the t-scaled realization $\mathcal{H}_2^t$ of $\mathbb{H}_t$. Then it is identified with

$$x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + u\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix}, \tag{3.2}$$

by (3.1), where

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [(1,0)]_t\,, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = [(i,0)]_t\,,$$

and

$$\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} = [(0,1)]_t\,, \quad \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} = [(0,i)]_t\,,$$

in $\mathcal{H}_2^t$. It means that every realization T of $\mathcal{H}_2^t$ is expressed by

$$T = x\,[(1,0)]_t + y\,[(i,0)]_t + u\,[(0,1)]_t + v\,[(0,i)]_t\,,$$

for some $x, y, u, v \in \mathbb{R}$. i.e., the t-scaled realization $\mathcal{H}_2^t$ is spanned by

$$\{[(1,0)]_t\,,\ [(i,0)]_t\,,\ [(0,1)]\,,\ [(0,i)]_t\}\,,$$

over $\mathbb{R}$.

Lemma 3.1. *For $t \in \mathbb{R}$, the $\mathbb{R}$-vector space $\mathbb{H}_t$ is spanned by the subset,*

$$\mathbf{B}_t = \{(1,0)\,,\ (i,0)\,,\ (0,1)\,,\ (0,i)\}\,, \tag{3.3}$$

over $\mathbb{R}$, in the sense that: for every $h \in \mathbb{H}_t$, there exist $x, y, u, v \in \mathbb{R}$, such that

$$h = x\,(1,0) + y\,(i,0) + u\,(0,1) + v\,(0,i)\,,$$

where $r\,(a,b) = (r,0) \cdot_t (a,b) = (ra, rb)$, for all $r \in \mathbb{R}$ and $(a,b) \in \mathbb{H}_t$. i.e.,

$$\mathbb{H}_t = \mathrm{span}_{\mathbb{R}}\,(\mathbf{B}_t)\,.$$

Proof. By the injective Hilbert-space representation $\left(\mathbb{C}^2, \pi_t\right)$, the t-scaled hypercomplex ring $\mathbb{H}_t$ is isomorphic to the t-scaled realization $\mathcal{H}_2^t$. And since $\mathcal{H}_2^t$ and $\mathbb{H}_t$ are isomorphic as $\mathbb{R}$-vector spaces. Therefore, the subset $\mathbf{B}_t$ of (3.3) spans $\mathbb{H}_t$ over $\mathbb{R}$, by (3.2). □

The $\mathbb{R}$-basis $\mathbf{B}_t$ of (3.3) satisfies the following properties in $\mathbf{X}_t$.

Proposition 3.2. *By $i\,(t)$, $j(t)$ and $k(t)$, we denote the spanning vectors $(i,0)$, $(0,1)$ and $(0,i)$ of $\mathbf{B}_t$, where $\mathbb{H}_t = span_{\mathbb{R}}\mathbf{B}_t$. Then*

$$\left\{ \begin{array}{c} i\,(t)^2 = (-1,0)\,,\ j(t)^2 = (t,0) = k(t)^2, \\ \\ i\,(t)\,j(t) = k(t),\ j(t)k(t) = -ti\,(t)\,,\ \text{and } k(t)i\,(t) = j(t), \\ \\ k(t)j(t) = ti\,(t)\,,\ j(t)i\,(t) = -k(t),\ \text{and } i\,(t)\,k(t) = -j(t). \end{array} \right\} \tag{3.4}$$

Proof. By the above lemma, equivalently, one has that

$$\mathcal{H}_2^t = \mathrm{span}_{\mathbb{R}}\,(\{\mathbf{1},\ \mathbf{i},\ \mathbf{j}_t,\ \mathbf{k}_t\})\,,$$

where

$$\mathbf{1} = [(1,0)]_t\,,\ \ \mathbf{i} = [(i,0)]_t\,,$$

and

$$\mathbf{j}_t = [(0,1)]_t\,,\ \text{and}\ \mathbf{k}_t = [(0,i)]_t\,.$$

So, to prove the formulas of (3.4), it suffices to show that

$$\left\{ \begin{array}{c} \mathbf{i}^2 = \left[(-1,0)\right]_t, \ \mathbf{j}_t^2 = \left[(t,0)\right]_t = \mathbf{k}_t^2, \\ \\ \mathbf{i}\mathbf{j}_t = \mathbf{k}_t, \ \mathbf{j}_t\mathbf{k}_t = -t\mathbf{i}, \text{ and } \mathbf{k}_t\mathbf{i} = \mathbf{j}_t, \\ \\ \mathbf{k}_t\mathbf{j}_t = t\mathbf{i}, \ \mathbf{j}_t\mathbf{i} = -\mathbf{k}_t, \text{ and } \mathbf{i}\mathbf{k}_t = -\mathbf{j}_t, \end{array} \right\}$$

in $\mathcal{H}_2^t$, respectively. And they are proven by the straightforward computations. For instance,

$$\mathbf{k}_t^2 = \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \left[(t,0)\right]_t = t\mathbf{1};$$

$$\mathbf{i}\mathbf{j}_t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} = \left[(0,i)\right]_t = \mathbf{k}_t,$$

$$\mathbf{i}\mathbf{k}_t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ -1 & 0 \end{pmatrix} = -\left[(0,1)\right]_t = -\mathbf{j}_t,$$

$$\mathbf{k}_t\mathbf{j}_t = \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ti & 0 \\ 0 & -ti \end{pmatrix} = t\left[(i,0)\right]_t = t\mathbf{i},$$

and

$$\mathbf{j}_t\mathbf{k}_t = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -ti & 0 \\ 0 & ti \end{pmatrix} = -t\left[(i,0)\right]_t = -t\mathbf{i}.$$

etc.. Therefore, the formulas of (3.4) are shown. $\square$

By the above proposition, one can conclude that

$$i\,(t)^2 = (-1,0)\,, \text{ and } j(t)^2 = (t,0) = k(t)^2, \tag{3.5}$$

and the following commuting diagrams hold;

$$\begin{tikzcd} & i\,(t) \arrow[dl, "1"'] & \\ j(t) \arrow[rr, "1"'] & & k(t) \arrow[ul, "-t"'] \end{tikzcd} ,$$

and

$$\begin{tikzcd} & i\,(t) \arrow[dr, "-1"] & \\ j(t) \arrow[ur, "t"] & & k(t) \arrow[ll, "-1"] \end{tikzcd} , \tag{3.6}$$

where, the first diagram of (3.6) illustrates that

$$i\,(t)\,j(t) = k(t), \ j(t)k(t) = -ti\,(t)\,, \text{ and } k(t)i\,(t) = j(t),$$

meanwhile the second diagram of (3.6) illustrates that

$$i\,(t)\,k(t) = -j(t), \ k(t)j(t) = ti\,(t)\,, \text{ and } j(t)i\,(t) = -k(t),$$

for $t \in \mathbb{R}$.

If we define a vector space $\mathscr{H}_t$ over $\mathbb{R}$, by

$$\mathscr{H}_t = \left\{x + yi + uj_t + vk_t : x, y, u, v \in \mathbb{R}\right\},$$

i.e.,

$$\mathscr{H}_t = \text{span}_{\mathbb{R}} \{1, i, j_t, k_t\},$$

where $i = \sqrt{-1}$, j_t and k_t are certain imaginary numbers (as spanning vectors) satisfying (1.2) and (1.3), then the morphism,

$$x\,(1,0) + y\,(i,0) + uj(t) + vk(t) \in \mathbf{X}_t \mapsto x + yi + uj_t + vk_t \in \mathscr{H}_t,$$

is a well-defined bijection preserving the formulas of (3.4) to the relations of (1.2) and (1.3). Since $\mathbb{H}_t = \text{span}_{\mathbb{R}} \mathbf{B}_t$, and

$$\mathscr{H}_t = \text{span}_{\mathbb{R}} (\{1, i, j_t, k_t\}),$$

this bijection becomes a $\mathbb{R}$-vector-space-isomorphism. More precisely, we have the following structure theorem of the set $\mathscr{H}_t$ of (3.7).

Theorem 3.3. *Let $\mathscr{H}_t$ be the vector space* (3.7) *spanned by* $\{1, i, j_t, k_t\}$*, satisfying the conditions* (1.2) *and* (1.3)*, over* $\mathbb{R}$*. Then*

$$\mathscr{H}_t \, \textit{is a unital ring with its unity} 1 = 1 + 0i + 0j_t + 0k_t, \tag{3.8}$$

algebraically, and

$$\mathscr{H}_t \, \textit{is a complete} \, \mathbb{R} - \textit{semi} - \textit{normed space}, \, \textit{isomorphic to} \, \mathbb{H}_t, \tag{3.9}$$

analytically, especially, it is a $\mathbb{R}$-Hilbert space if $t < 0$, while it is a complete $\mathbb{R}$-indefinite semi-inner product space if $t \geq 0$, and

$$\mathscr{H}_t \, \textit{is a complete semi} - \textit{normed} * - \textit{algebra over} \, \mathbb{R}, \tag{3.10}$$

operator-algebraically, especially, it becomes a C^*-algebra over $\mathbb{R}$ if $t < 0$, meanwhile it forms a complete semi-normed $*$-algebra over $\mathbb{R}$ if $t \geq 0$.

Proof. By the bijection,

$$x\,(1,0) + y\,(i,0) + uj(t) + vk(t) \in \mathbb{H}_t \longmapsto x + yi + uj_t + vk_t \in \mathscr{H}_t,$$

two mathematical structures $\mathbb{H}_t$ and $\mathscr{H}_t$ are isomorphic algebraically, and analytically, because

$$\mathbb{H}_t = \text{span}_{\mathbb{R}} \{(1,0), (i,0), j_t, k_t\}, \text{ and } \mathscr{H}_t = \text{span}_{\mathbb{R}} \{1, i, j_t, k_t\},$$

over $\mathbb{R}$. So, the statements (3.8) and (3.9) hold, since $\mathbb{H}_t$ is a unital ring algebraically, and a complete $\mathbb{R}$-semi-normed space analytically.

In [3], we showed that, on $\mathbb{H}_t$, one can define the operator space $B_{\mathbb{R}}(\mathbb{H}_t)$ of all bounded linear transformation (simply, operators) acting on this $\mathbb{R}$-vector space $\mathbb{H}_t$ "over $\mathbb{R}$," equipped with the operator "semi-norm,"

$$\|T\| \overset{\text{def}}{=} sup \{\|T\,(h)\|_t : \|h\|_t = 1\},$$

for all $T \in B_{\mathbb{R}}(\mathbb{H}_t)$. Then one can canonically define multiplication operators $M_h \in B_{\mathbb{R}}(\mathbb{H}_t)$ with their symbols $h \in \mathbb{H}_t$, by

$$M_h\,(\eta) \overset{\text{def}}{=} h \cdot_t \eta \in \mathbb{H}_t \quad \forall \eta \in \mathbb{H}_t.$$

And the subset,

$$\mathcal{M}_t = \{M_h \in B_{\mathbb{R}}(\mathbb{H}_t) : h \in \mathbb{H}_t\},$$

of $B_{\mathbb{R}}(\mathbb{H}_t)$ forms a well-defined complete semi-normed $*$-algebra over $\mathbb{R}$. In particular, the operator semi-norm $\|.\|$ becomes an operator "norm," if $t < 0$ (because if $t < 0$, then $\|.\|_t$ is a norm induced by the inner product $\langle , \rangle_t$), meanwhile, it is a operator "semi-norm," if $t \geq 0$ (because if $t \geq 0$, then $\|.\|_t$ forms a semi-norm induced by the indefinite semi-inner product $\langle , \rangle_t$). See [3] for details. Since $\mathbb{H}_t$ is isomorphic to $\mathscr{H}_t$, this complete semi-normed $*$-algebra $\mathcal{M}_t$ is isomorphic to $\mathbb{H}_t$, by a $*$-isomorphism,

$$M_h \in \mathcal{M}_t \longmapsto h \in \mathbb{H}_t \overset{\text{iso}}{=} \mathscr{H}_t,$$

operator-algebraically. Therefore, the statement (3.10) holds, too. $\square$

The above theorem characterizes the algebraic, analytic, and operator-algebraic structures of the set $\mathscr{H}_t$ of (3.7), for $t \in \mathbb{R}$. So, one can re-define the set $\mathbb{H}_t$ of t-scaled hypercomplex numbers by the equivalent set $\mathscr{H}_t$, i.e.,

$$\mathbb{H}_t \overset{\text{def}}{=} \left\{ x + yi_t + uj_t + vk_t \;\middle|\; \begin{array}{c} x,\ y,\ u,\ v \in \mathbb{R}, \\ i_t = i = \sqrt{-1} \in \mathbb{C}, \text{ and} \\ i_t^2 = -1,\ j_t^2 = t = k_t^2, \\ ij_t = k_t,\ j_tk_t = -ti,\ k_ti = j_t, \\ j_ti = -k_t,\ ik_t = -j_t,\ k_tj_t = ti \end{array} \right\}.$$

For convenience, we call $\mathbb{H}_t$, the t-scaled hypercomplexes from now on. By the above theorem, the t-scaled hypercomplexes $\mathbb{H}_t$ is a unital ring with its unity,

$$1 = 1 + 0i_t + 0j_t + 0k_t = (1, 0) \in \mathbb{H}_t,$$

algebraically, and a complete $\mathbb{R}$-semi-normed space analytically (in general, for all $t \in \mathbb{R}$), and a complete semi-normed $*$-algebra over $\mathbb{R}$ operator-algebraically (in particular, it forms a C^*-algebra over $\mathbb{R}$ if $t < 0$). It is not difficult to check that each hypercomplex number $h = x + yi_t + uj_t + vk_t$ of $\mathbb{H}_t$ has its hypercomplex conjugate,

$$h^\dagger = x - yi_t - uj_t - vk_t, \text{ in } \mathbb{H}_t,$$

and one can have a well-defined $\mathbb{R}$-linear functional τ on $\mathbb{H}_t$,

$$\tau(h) = x.$$

So, under this new setting, one can define the real part,

$$\mathrm{Re}\,(x + yi + uj_t + vk_t) = x,$$

and the imaginary part,

$$\mathrm{Im}\,(x + yi_t + uj_t + vk_t) = yi_t + uj_t + vk_t,$$

for all $x + yi_t + uj_t + vk_t \in \mathbb{H}_t$, which are understood to be

$$(x + yi) + (u + vi)\, j_t, \text{ since } ij_t = k_t,$$

where $i = \sqrt{-1} = i_t$, for all $t \in \mathbb{R}$. Recall that, in [3], we generalized complex numbers $\mathbb{C}$, and hyperbolic numbers $\mathcal{D}$ in a similar way.

Definition 3.4. Let $t \in \mathbb{R}$, and $\mathbb{H}_t$, the corresponding t-scaled hypercomplex ring. The subring,

$$\mathbb{D}_t = \{(x, u) \in \mathbb{H}_t : x, u \in \mathbb{R}\},$$

is called the t-scaled hyperbolic (sub)ring, and all elements of $\mathbb{D}_t$ are called t-scaled hyperbolic numbers.

Note that, by definition, this t-scaled hyperbolic ring $\mathbb{D}_t$ forms a closed subspace of the $\mathbb{R}$-vector space $(\mathbb{H}_t, \langle , \rangle_t)$, and it becomes a closed $*$-subalgebra of the complete semi-normed $*$-algebra $\mathbb{H}_t$ over $\mathbb{R}$. By our new notational setting, the t-scaled hyperbolic ring $\mathbb{D}_t$ of [3] can be re-defined by

$$\mathbb{D}_t = \{x + 0i_t + uj_t + 0k_t \in \mathbb{H}_t : x, u \in \mathbb{R}\}, \tag{3.11}$$

in the t-scaled hypercomplexes $\mathbb{H}_t$. It is not hard to check that $\mathbb{D}_{-1}$ is isomorphic to the complex field,

$$\mathbb{C} = \left\{x + ui : x, u \in \mathbb{C},\ i^2 = -1\right\},$$

and $\mathbb{D}_1$ is isomorphic to the classical hyperbolic numbers,

$$\mathcal{D} = \left\{x + uj : x, u \in \mathbb{R},\ j^2 = 1\right\},$$

algebraically, and analytically (e.g., see [3] for details).

Theorem 3.5. *Let $h = x + uj_t \in \mathbb{D}_t$. Then there exists $e^{j_t\theta} \in \mathbb{D}_t$, such that*

$$h = \|h\|_t \, e^{j_t\theta}, \quad \text{in} \quad \mathbb{D}_t,$$

where

$$\left\|e^{j_t\theta}\right\|_t = 1, \ \text{ with } \ \theta = Arg\,((x, u)) \in [0, 2\pi], \tag{3.12}$$

where $[0, 2\pi]$ is the closed interval in $\mathbb{R}$, and $Arg\,((x, u))$ is the argument of the vector $(x, u) \in \mathbb{R}^2$, and

$$e^{j_t\theta} = \begin{cases} cos\left(\sqrt{|t|}\theta\right) + j_t\left(\frac{sin\left(\sqrt{|t|}\theta\right)}{\sqrt{|t|}}\right) & \text{if } t < 0 \\ \\ \pm 1 + uj_t, & \text{if } t = 0 \\ \\ cosh\left(\sqrt{t}\theta\right) + j_t\left(\frac{sinh\left(\sqrt{t}\theta\right)}{\sqrt{t}}\right) & \text{if } t > 0. \end{cases}$$

Proof. In [3], we showed that if $(x, u) \in \mathbb{D}_t$ in $\mathbb{H}_t$, then it is expressed by

$$(x, u) = \|(x, u)\|_t \, exp^{j_t\theta}, \quad \text{in} \quad \mathbb{D}_t,$$

where

$$exp^{j_t\theta} = \begin{cases} \left(cos\left(\sqrt{|t|}\theta\right),\ \frac{sin\sqrt{|t|}\theta}{\sqrt{|t|}}\right) & \text{if } t < 0 \\ \\ (\pm 1, u),\ \text{for all } u \in \mathbb{R} & \text{if } t = 0 \\ \\ \left(cosh\left(\sqrt{t}\theta\right),\ \frac{sinh\left(\sqrt{t}\theta\right)}{\sqrt{t}}\right) & \text{if } t \geq 0, \end{cases} \tag{3.13}$$

where $\theta = Arg\left((x,u)\right) \in [0, 2\pi]$ is the argument of the point (x,u) in $\mathbb{R}^2$. So, the t-scaled polar decomposition (3.12) holds by (3.11) and (3.13). $\square$

As in [3], if we construct the unit set $\mathbb{T}_t = \{h \in \mathbb{D}_t : \|h\|_t = 1\}$ of all units in the t-scaled hyperbolic ring $\mathbb{D}_t$, then

$$\mathbb{T}_t = \left\{e^{j_t\theta} : \theta \in \mathbb{R}\right\},$$

where $e^{j_t\theta}$ are in the sense of (3.12) in $\mathbb{D}_t$.

4. Nonzero-Scaled Regular Functions

Let $t \in \mathbb{R}$ be a fixed scale, and

$$\mathbb{H}_t = \mathrm{span}_{\mathbb{R}}\left(\{1, i_t, j_t, k_t\}\right),$$

the t-scaled hypercomplexes, where $i_t = \sqrt{-1} = i$, j_t and k_t satisfy (1.2) and (1.3). We consider functions,

$$f : \mathbb{H}_t \to \mathbb{H}_t,$$

in the t-scaled hypercomplex variable,

$$w = x_1 + x_2 i_t + x_3 j_t + x_4 k_t, \text{ with } x_1, x_2, x_3, x_4 \in \mathbb{R}.$$

In particular, we are interested in the case where such functions f are $\mathbb{R}$-differentiable in an open connected set $\Omega \subset \mathbb{H}_t$. Our study is motivated by the main results of [4], [5], [6], [8], and [9].

4.1. Motivation: Hyperholomorphic Functions on $\mathbb{H}_{-1}$

For more about hyperholomorphic theory on the (-1)-scaled hypercomplexes $\mathbb{H}_{-1}$, see e.g., [5], [6] and [8]. Recall that the (-1)-scaled hypercomplexes $\mathbb{H}_{-1}$ is the noncommutative field of the quaternions (e.g., see Sects. 2 and 3 above, or [1], [2] and [3]). Recall that the quaternions $\mathbb{H}_{-1}$ is noncommutative for the (-1)-scaled multiplication $(\cdot_{-1})$, and hence, the hyperholomorphic property is considered in two ways from the left, and from the right. Also, in this quaternionic case, the differential operator D_{-1} is called the Cauchy-Fueter (differential) operator (e.g., see [5], [6] and [8]).

Define two operators D_{-1} and $D_{-1}^{\dagger}$ by

$$D_{-1} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_{-1}\frac{\partial}{\partial x_3} + k_{-1}\frac{\partial}{\partial x_4},$$

and

$$D_{-1}^{\dagger} = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}i - \frac{\partial}{\partial x_3}j_{-1} - \frac{\partial}{\partial x_4}k_{-1},$$

on the set of all $\mathbb{R}$-differentiable quaternionic functions.

Definition 4.1. A $\mathbb{R}$-differentiable quaternionic function $f : \mathbb{H}_{-1} \to \mathbb{H}_{-1}$ is left-hyperholomorphic on an open subset Ω of $\mathbb{H}_{-1}$, if

$$D_{-1}f \overset{\text{denote}}{=} \frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2} + j_{-1}\frac{\partial f}{\partial x_3} + k_{-1}\frac{\partial f}{\partial x_4} = 0;$$

and f is said to be right-hyperholomorphic on Ω, if

$$fD_{-1} \overset{\text{denote}}{=} \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} i + \frac{\partial f}{\partial x_3} j_{-1} + \frac{\partial f}{\partial x_4} k_{-1} = 0,$$

in a $\mathbb{H}_{-1}$-variable $w = x_1 + x_2 i + x_3 j_{-1} + x_4 k_{-1}$ with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. We simply say that f is hyperholomorphic on Ω, if it is both left and right hyperholomorphic on Ω.

Readers can verify that we later define the (left, and right) scaled-hyperholomorphic property of $\mathbb{R}$-differentiable functions on $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$ similarly as above. In this section, we concentrate on the left-hyperholomorphic functions on the quaternions $\mathbb{H}_{-1}$.

It is well-known that

$$D_{-1}^{\dagger} D_{-1} = \Delta_{-1},$$

with

$$\Delta_{-1} \overset{\text{def}}{=} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}. \tag{4.1.1}$$

Thus, hyperholomorphic functions are harmonic on $\mathbb{H}_{-1}$.

Proposition 4.2. *A left hyperholomorphic function $f : \mathbb{H}_{-1} \to \mathbb{H}_{-1}$ on an open connected subset U of $\mathbb{H}_{-1}$ is harmonic on U, i.e.,*

$$\Delta_{-1} f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} + \frac{\partial^2 f}{\partial x_4^2} = 0, \text{ on } U. \tag{4.1.2}$$

Proof. If f is left-hyperholomorphic on $\mathbb{H}_{-1}$, then

$$\Delta_{-1} f = D_{-1}^{\dagger} D_{-1} f = D_{-1}^{\dagger} \left(D_{-1} f \right) = 0,$$

by (4.1.1). So, the harmonicity (4.1.2) holds for f. □

It is shown that every left-hyperholomorphic function f on an open subset containing 0 is expressed by the form,

$$f(w) = f(0) + \sum_{n=1}^{3} \zeta_{n+1}(w) \mathcal{R}_{n+1} f(w),$$

with

$$\zeta_{n+1}(w) = x_{n+1} - x_1 e_{n+1}, \tag{4.1.3}$$

and

$$\mathcal{R}_{n+1} f(w) = \int_0^1 \frac{\partial f}{\partial x_{n+1}} (tw)\, dt,$$

where

$$e_2 = i, \quad e_3 = j_{-1}, \quad \text{and} \quad e_4 = k_{-1}$$

(e.g., see [5], [6] and [8]). Note that the functions ζ_2, ζ_3, and ζ_4 of (4.1.3) are both left and right entire hyperholomorphic (on $\mathbb{H}_{-1}$), called the Cauchy-Fueter polynomials on the quaternions $\mathbb{H}_{-1}$. Are the above harmonicity (4.1.2) and the expansion (4.1.3) generalized for arbitrary scales $t \in \mathbb{R}$, under a certain "scaled" hyperholomorphic property?

Independently, in [4] and [9]., the hyperholomorphic property (called the left, or right regularity there), and the harmonicity of $\mathbb{R}$-differentiable functions on open connected subsets of the split-quaternions $\mathbb{H}_1$ is studied and characterized, under slightly different differential-operator and Laplacian settings based on [9] and [14].

4.2. Discussion and Assumption

Let $\mathbb{H}_t$ be the t-scaled hypercomplexes, for a scale $t \in \mathbb{R}$.

Definition 4.3. Suppose $\mathcal{T}_t$ is the semi-norm topology on $\mathbb{H}_t = \mathbf{X}_t$, the t-hypercomplex $\mathbb{R}$-space induced by the semi-norm $\|.\|_t$ of (2.4.10) under (3.7). Define a set,

$$\mathcal{F}_{t.U} \overset{\text{def}}{=} \{f : \mathbb{H}_t \to \mathbb{H}_t \,|f \text{ is } \mathbb{R}\text{-differentiable on } U \in \mathcal{T}_t\},$$

and the set $\mathcal{F}_t \overset{\text{def}}{=} \underset{U \in \mathcal{T}_t}{\cup} \mathcal{F}_{t,U}$.

Note that each $\mathbb{R}$-differentiable-function family $\mathcal{F}_{t,U}$ is understood to be

$$\mathcal{F}_{t,U} = \underset{k \in \Lambda}{\sqcup} \mathcal{F}_{t,U_k}, \text{ if } U = \underset{k \in \Lambda}{\sqcup} U_k \in \mathcal{T}_t, \tag{4.2.1}$$

where $\sqcup$ is the disjoint union, and $\{U_k\}_{k\in\Lambda} \subset \mathcal{T}_t$ for an index set Λ, where $\{U_k\}_{k\in\Lambda}$ are connected in $\mathcal{T}_t$. i.e., each $\mathbb{R}$-differentiable-function family $\mathcal{F}_{t,U}$ for $U \in \mathcal{T}_t$ is the disjoint union of the collection $\{\mathcal{F}_{t,U_k}\}_{k\in\Lambda}$ of $\mathbb{R}$-differentiable-function families on open connected subsets $\{U_k\}_{k\in\Lambda}$ of $\mathbb{H}_t$, as in (4.2.1). Of course, if U is both open and connected in $\mathcal{T}_t$, then the index set Λ in (4.2.1) becomes a singleton set, and hence, $\mathcal{F}_{t,U}$ becomes itself in the sense of (4.2.1). **Assumption**. From below, if we take a $\mathbb{R}$-differentiable-function family $\mathcal{F}_{t,U}$, then the open subset $U \in \mathcal{T}_t$ of $\mathbb{H}_t$ is automatically regarded as a connected subset, for convenience. $\square$

It is not difficult to check that the $\mathbb{R}$-differentiable-function family $\mathcal{F}_{t,U}$ of $U \in \mathcal{T}_t$ forms a well-defined $\mathbb{R}$-vector space. Also, the $\mathbb{R}$-differentiable-function family $\mathcal{F}_t$ forms a $\mathbb{R}$-vector space in the sense that: if $r_1, r_2 \in \mathbb{R}$ and $f_1, f_2 \in \mathcal{F}_t$, and hence, if $r_1 f_1 \in \mathcal{F}_{t,U_1}$ and $r_2 f_2 \in \mathcal{F}_{t,U_2}$, for $U_1, U_2 \in \mathcal{T}_t$, then

$$r_1 f_1 + r_2 f_2 \in \mathcal{F}_{t,U_1 \cap U_2}, \text{ in } \mathcal{F}_t.$$

Before proceeding our works, we discuss the difficulties to maintain similar settings of Sect. 4.1 for the cases where we replace the (-1)-scaled hypercomplexes (which is the quaternions) $\mathbb{H}_{-1}$ to a general t-scaled hypercomplexes $\mathbb{H}_t$, for $t \in \mathbb{R} \setminus \{-1\}$. Naturally, we want to extend the results (4.1.2) and (4.1.3) of Sect. 4.1 to those on $\mathbb{H}_t$, for any scales $t \in \mathbb{R}$. However, if we use the operators,

$$D_t = \frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2} + j_t\frac{\partial f}{\partial x_3} + k_t\frac{\partial f}{\partial x_4},$$

and

$$D_t^{\dagger} = \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} i - \frac{\partial f}{\partial x_3} j_t - \frac{\partial f}{\partial x_4} k_t, \tag{4.2.2}$$

for a fixed $t \in \mathbb{R}$, then one "cannot" have

$$D_t^{\dagger} D_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}$$

in general, like in Sect. 4.1, especially, where $t \neq -1$ in $\mathbb{R}$ (e.g., see Sect. 4.3 below). In other words, one cannot have a natural Laplacian-like operator from the operators (4.2.2). So, we need new settings for studying the hyper-holomorphic property, and harmonicity of $\mathcal{F}_{t,U}$ on $\mathbb{H}_t$.

Also, if a given scale t is zero, i.e., $t = 0$ in $\mathbb{R}$, then the analysis of $\mathcal{F}_{0,U}$ on $\mathbb{H}_0$, for $U \in \mathcal{T}_0$, seems totally different from those on $\{\mathbb{H}_t\}_{t \in \mathbb{R} \setminus \{0\}}$. So, in the rest of Sect. 4 below, we first concentrate on the cases where $t \neq 0$ in $\mathbb{R}$. The case where $t = 0$ will be considered in Sect. 5, independently.

Assumption. In the following Sects. 4.3, 4.4 and 4.5, we assume $t \in \mathbb{R} \setminus \{0\}$.$\square$

4.3. The Scaled Regularity

As we have seen in Sect. 4.1, the (-1)-scaled differential operator D_{-1} satisfies that

$$D_{-1}^{\dagger} D_{-1} = \Delta_{-1} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2},$$

by (4.1.1), allowing us to study harmonicity independent from the imaginary $\mathbb{R}$-basis elements $\{i, j_{-1}, k_{-1}\}$ of $\mathbb{H}_{-1}$. In this section, let's construct ideas and approaches to extend the results of Sect. 4.1 to the general cases where $t \in \mathbb{R} \setminus \{0\}$. Throughout this section, we let $t \in \mathbb{R} \setminus \{0\}$ as we assumed in Sect. 4.2, and $\mathbb{H}_t$, the corresponding t-scaled hypercomplexes.

First, consider the case where we have operators D_t and $D_t^{\dagger}$ on $\mathcal{F}_t$ by

$$D_t = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_t\frac{\partial}{\partial x_3} + k_t\frac{\partial}{\partial x_4},$$

and

$$D_t^{\dagger} \overset{\text{def}}{=} \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i - \frac{\partial}{\partial x_3} j_t - \frac{\partial}{\partial x_4} k_t, \tag{4.3.1}$$

as in Sect. 4.1. Observe that

$$\begin{aligned}
D_t^{\dagger} D_t &= \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i - \frac{\partial}{\partial x_3} j_t - \frac{\partial}{\partial x_4} k_t\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_t\frac{\partial}{\partial x_3} + k_t\frac{\partial}{\partial x_4}\right) \\
&= \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} i \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1} j_t \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1} k_t \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_2} i \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i^2 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2} i j_t \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2} i k_t \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_3} j_t \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} j_t i \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} j_t^2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3} j_t k_t \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_4} k_t \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4} k_t i \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4} k_t j_t \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} k_t^2 \frac{\partial}{\partial x_4}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} i^2 - \frac{\partial}{\partial x_2} i j_t \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2} i k_t \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_3} j_t i \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} j_t^2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3} j_t k_t \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_4} k_t i \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4} k_t j_t \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} k_t^2 \frac{\partial}{\partial x_4} \\
&= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_3} (-ti) \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_4} (ti) \frac{\partial}{\partial x_3} - t \frac{\partial^2}{\partial x_3^2} - t \frac{\partial^2}{\partial x_4^2},
\end{aligned} \tag{4.3.2}
$$

by (3.5) and (3.6), i.e.,

$$
D_t^\dagger D_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + t \frac{\partial}{\partial x_3} i \frac{\partial}{\partial x_4} - t \frac{\partial}{\partial x_4} i \frac{\partial}{\partial x_3} - t \frac{\partial^2}{\partial x_3^2} - t \frac{\partial^2}{\partial x_4^2}, \tag{4.3.3}
$$

by (4.3.2).

Proposition 4.4. *If* $D_t = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + j_t \frac{\partial}{\partial x_3} + k_t \frac{\partial}{\partial x_4}$ *on* $\mathcal{F}_{t,U}$*, for* $U \in \mathcal{T}_t$*, then*

$$
D_t^\dagger D_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + t \frac{\partial}{\partial x_3} i \frac{\partial}{\partial x_4} - t \frac{\partial}{\partial x_4} i \frac{\partial}{\partial x_3} - t \frac{\partial^2}{\partial x_3^2} - t \frac{\partial^2}{\partial x_4^2}, \text{ on } U. \tag{4.3.4}
$$

Proof. The proof is done by the straightforward computations (4.3.3). □

Motivated by (4.3.4), we define new differential operators ∇_t and $\nabla_t^\dagger$ by

$$
\nabla_t = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} - j_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} - k_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4},
$$

and

$$
\nabla_t^\dagger = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t, \tag{4.3.5}
$$

on $\mathcal{F}_{t,U}$, for any $U \in \mathcal{T}_t$, where

$$
\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_l} = \left(\frac{sgn\,(t)}{\sqrt{|t|}} \right) \frac{\partial}{\partial x_l}, \quad \text{for} \quad l = 3, 4,
$$

where

$$
sgn\,(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0, \end{cases} \tag{4.3.6}
$$

for all $t \in \mathbb{R} \setminus \{0\}$. i.e., by (4.3.5) and (4.3.6),

$$
\nabla_t \overset{\text{def}}{=} \begin{cases} \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} - j_t \frac{\partial}{\sqrt{t}\partial x_3} - k_t \frac{\partial}{\sqrt{t}\partial x_4} & \text{if } t > 0 \\ \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + j_t \frac{\partial}{\sqrt{|t|}\partial x_3} + k_t \frac{\partial}{\sqrt{|t|}\partial x_4} & \text{if } t < 0. \end{cases}
$$

Lemma 4.5. *Let* ∇_t *and* $\nabla_t^\dagger$ *be the operators* (4.3.6) *on* $\mathcal{F}_{t,U}$. *Then*

$$\nabla_t^\dagger \nabla_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2}, \tag{4.3.7}$$

on $\mathcal{F}_{t,U}$.

Proof. Consider that,

$$\begin{aligned}
\nabla_t^\dagger \nabla_t = &\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t\right) \\
&\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - j_t\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn\,(t)}{\sqrt{|t|}\partial x_4}\right) \\
= &\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} i\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1} j_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} - \frac{\partial}{\partial x_1} k_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} \\
&-\frac{\partial}{\partial x_2} i \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i^2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2} i j_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} + \frac{\partial}{\partial x_2} i k_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} \\
&+\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t \frac{\partial}{\partial x_1} + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t i \frac{\partial}{\partial x_2} - \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t^2 \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} \\
&-\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t k_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} \\
&+\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t \frac{\partial}{\partial x_1} + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t i \frac{\partial}{\partial x_2} - \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t j_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} \\
&-\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t^2 \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} \\
= &\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2} i^2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2} i j_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} + \frac{\partial}{\partial x_2} i k_t \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} \\
&+\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} j_t i \frac{\partial}{\partial x_2} - \frac{1}{|t|}\frac{\partial}{\partial x_3} j_t^2 \frac{\partial}{\partial x_3} - \frac{1}{|t|}\frac{\partial}{\partial x_3} j_t k_t \frac{\partial}{\partial x_4} \\
&+\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4} k_t i \frac{\partial}{\partial x_2} - \frac{1}{|t|}\frac{\partial}{\partial x_4} k_t j_t \frac{\partial}{\partial x_3} - \frac{1}{|t|}\frac{\partial}{\partial x_4} k_t^2 \frac{\partial}{\partial x_4}
\end{aligned}$$

since$(sgn\,(t))\,(sgn\,(t)) = 1 in\,\{\pm 1\}$

$$\begin{aligned}
= &\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2} i^2 \frac{\partial}{\partial x_2} - \frac{1}{|t|}\frac{\partial}{\partial x_3} j_t^2 \frac{\partial}{\partial x_3} - \frac{1}{|t|}\frac{\partial}{\partial x_3} j_t k_t \frac{\partial}{\partial x_4}, \\
&-\frac{1}{|t|}\frac{\partial}{\partial x_4} k_t j_t \frac{\partial}{\partial x_3} - \frac{1}{|t|}\frac{\partial}{\partial x_4} k_t^2 \frac{\partial}{\partial x_4} \\
= &\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{t}{|t|}\frac{\partial^2}{\partial x_3^2} - \frac{1}{|t|}\frac{\partial}{\partial x_3}\,(-ti)\,\frac{\partial}{\partial x_4} - \frac{1}{|t|}\frac{\partial}{\partial x_4}\,(ti)\,\frac{\partial}{\partial x_3} - \frac{t}{|t|}\frac{\partial^2}{\partial x_4^2} \\
= &\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} + sgn\,(t)\,\frac{\partial}{\partial x_3}\,(i)\,\frac{\partial}{\partial x_4} \\
&-sgn\,(t)\,\frac{\partial}{\partial x_4}\,(i)\,\frac{\partial}{\partial x_3} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2}
\end{aligned}$$

since $\frac{t}{|t|} = sgn\,(t)$

$$= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2},$$

by (3.5) and (3.6), implying that

$$\nabla_t^\dagger \nabla_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2}.$$

□

The above lemma shows that if we define a t-scaled Laplacian operator $\varDelta_t$ on $\mathcal{F}_t$ by

$$\varDelta_t \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial}{\partial x_3^2} - \frac{sgn\,(t)\,\partial}{\partial x_4^2},$$

then we have the following factorization of $\varDelta_t$.

Theorem 4.6. *If* $\varDelta_t \stackrel{def}{=} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn(t)\partial^2}{\partial x_3^2} - \frac{sgn(t)\partial^2}{\partial x_4^2}$ *on* $\mathcal{F}_t$*, then*

$$\varDelta_t = \nabla_t^\dagger \nabla_t. \tag{4.3.8}$$

Proof. The formula (4.3.8) is shown by (4.3.7). □

Note that the above factorization (4.3.8) shows that if $t < 0$, then

$$\nabla_t^\dagger \nabla_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2},$$

meanwhile, if $t > 0$, then

$$\nabla_t^\dagger \nabla_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2},$$

for all $t \in \mathbb{R} \setminus \{0\}$. For example, if $t = -1$, then this formula (4.3.8) becomes

$$\varDelta_{-1} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} = \nabla_{-1}^\dagger \nabla_{-1},$$

as in Sect. 4.1. Meanwhile, if $t = 1$, then the formula (4.3.8) goes to

$$\varDelta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} = \nabla_1^\dagger \nabla_1,$$

as in [4]. Thus, the factorization (4.3.8) can be re-stated as

$$\varDelta_t = \begin{cases} \varDelta_{-1} & \text{if } t < 0 \\ \varDelta_1 & \text{if } t > 0. \end{cases}$$

We now consider how the new differential operator ∇_t of (4.3.6) acts on the functions,

$$\zeta_2 = x_2 - x_1 i,\ \zeta_3 = x_3 - x_1 j_t,\ \text{and}\ \zeta_4 = x_4 - x_1 k_t.$$

Observe that

$$\nabla_t \zeta_2 = \zeta_2 \nabla_t = \frac{\partial \zeta_2}{\partial x_1} + i\frac{\partial \zeta_2}{\partial x_2} - j_t \frac{sng\,(t)\,\partial \zeta_2}{\sqrt{|t|}\partial x_3} - k_t \frac{sgn\,(t)\,\partial \zeta_2}{\sqrt{|t|}\partial x_4}$$

$$
\begin{aligned}
&= \frac{\partial(x_2 - x_1 i)}{\partial x_1} + i\frac{\partial\,(x_2 - x_1 i)}{\partial x_2} - j_t\frac{sgn\,(t)\,\partial\,(x_2 - x_1 i)}{\sqrt{|t|}\partial x_3} \\
&\quad - k_t\frac{sgn\,(t)\,\partial\,(x_2 - x_1 i)}{\sqrt{|t|}\partial x_4} \\
&= (-i) + (1)\,i - (0)\,j_t - (0)\,k_t = -i + i = 0; \qquad (4.3.9)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_t\zeta_3 = \zeta_3\nabla_t &= \frac{\partial\,(x_3 - x_1 j_t)}{\partial x_1} + i\frac{\partial\,(x_3 - x_1 j_t)}{\partial x_2} \\
&\quad - j_t\frac{sgn\,(t)\,\partial\,(x_3 - x_1 j_t)}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn\,(t)\,\partial\,(x_3 - x_1 j_t)}{\sqrt{|t|}\partial x_4} \\
&= (-j_t) + (0)\,i - \left(\frac{sgn\,(t)}{\sqrt{|t|}}\right) j_t - (0)\,k_t = \left(-1 - \frac{sgn\,(t)}{\sqrt{|t|}}\right) j_t; \qquad (4.3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_t\zeta_4 = \zeta_4\nabla_t &= \frac{\partial\,(x_4 - x_1 k_t)}{\partial x_1} + i\frac{\partial\,(x_4 - x_1 k_t)}{\partial x_2} \\
&\quad - j_t\frac{sgn\,(t)\,\partial\,(x_4 - x_1 k_t)}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn\,(t)\,\partial\,(x_4 - x_1 k_t)}{\sqrt{|t|}\partial x_4} \\
&= (-k_t) + (0)\,i - (0)\,j_t - \left(\frac{sgn\,(t)}{\sqrt{|t|}}\right) k_t = \left(-1 - \frac{sgn\,(t)}{\sqrt{|t|}}\right) k_t; \qquad (4.3.11)
\end{aligned}
$$

Proposition 4.7. *Let ∇_t be the differential operator* (4.3.6)*, and let $\{\zeta_l\}_{l=2}^4 \subset \mathcal{F}_{t,\mathbb{H}_t}$ be the functions introduced as in the very above paragraph. Then*

$$\nabla_t\zeta_2 = \zeta_2\nabla_t = 0,$$

and

$$\nabla_t\zeta_3 = \zeta_3\nabla_t = \rho j_t, \quad \nabla\zeta_4 = \zeta_4\nabla_t = \rho k_t, \tag{4.3.12}$$

where

$$\rho = -1 - \frac{sgn\,(t)}{\sqrt{|t|}}, \quad \text{in} \quad \mathbb{R}.$$

Proof. The formula (4.3.12) is proven by (4.3.9), (4.3.10) and (4.3.11). □

The formulas in (4.3.12) show how the operator ∇_t acts on the Fueter-like functions $\{\zeta_l\}_{l=2}^4$ in $\mathcal{F}_{t,\mathbb{H}_t}$.

Definition 4.8. Let ∇_t be the operator (4.3.6) on $\mathcal{F}_{t,U}$, for $U \in \mathcal{T}_t$ in $\mathbb{H}_t$, and $f \in \mathcal{F}_{t,U}$. If

$$\nabla_t f = \frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2} - j_t\frac{sgn\,(t)\,\partial f}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn\,(t)\,\partial f}{\sqrt{|t|}\partial x_4} = 0,$$

then f is said to be left t(-scaled)-regular on U. If

$$f\nabla_t = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}i - \frac{sgn\,(t)\,\partial f}{\sqrt{|t|}\partial x_3}j_t - \frac{sgn\,(t)\,\partial f}{\sqrt{|t|}\partial x_4}k_t = 0,$$

then f is said to be right t(-scaled)-regular on U. If $f \in \mathcal{F}_{t,U}$ is both left and right t-regular, then it is said to be t(-scaled)-regular.

A function $f \in \mathcal{F}_{t,U}$ is called a t(-scaled)-harmonic function, if

$$\Delta_t f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2 f}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2 f}{\partial x_4^2} = 0,$$

where

$$\Delta_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2}.$$

is in the sense of (4.3.8).

By (4.3.12), one can realize that $\zeta_2 = x_2 - x_1 i$ is t-regular, but, $\zeta_3 = x_3 - x_1 j_t$ and $\zeta_4 = x_4 - x_1 k_t$ are neither left nor right t-regular, in general, especially, if $t \neq -1$ in $\mathbb{R} \setminus \{0\}$. As in Sect. 4.1, we concentrate on the left t-regular functions. We finish this section with the following theorem.

Theorem 4.9. *Let $f \in \mathcal{F}_{t,U}$, for $U \in \mathcal{T}_t$ in $\mathbb{H}_t$. If*

$$f \text{ is left } t\text{-regular,}$$

then

$$f \text{ is } t\text{-harmonic on } U. \tag{4.3.13}$$

Proof. Suppose f is left t-regular in $\mathcal{F}_{t,U}$, i.e., $\nabla_t f = 0$. Then $\Delta_t f = 0$, because

$$\Delta_t f = \nabla_t^{\dagger} (\nabla_t f) = \nabla_t^{\dagger} (0) = 0,$$

implying that it is t-harmonic on U. □

4.4. Certain t-Regular Functions on $\mathbb{H}_t$

Throughout this section, we automatically assume that a given scale t is non-zero, i.e., $t \in \mathbb{R} \setminus \{0\}$. Let ∇_t and $\nabla_t^{\dagger}$ be the operators (4.3.6),

$$\nabla_t = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - j_t\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4},$$

and

$$\nabla_t^{\dagger} = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}i + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3}j_t + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4}k_t,$$

on $\mathcal{F}_{t,U}$, for any $U \in \mathcal{T}_t$ in the t-scaled hypercomplexes $\mathbb{H}_t$, and let

$$\Delta_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2},$$

on $\mathcal{F}_{t,U}$, satisfying

$$\Delta_t = \nabla_t^{\dagger}\nabla_t, \quad \text{on} \quad \mathcal{F}_{t,U},$$

by (4.3.8), where $i = \sqrt{-1} = i_s \in \mathbb{H}_s$, for all $s \in \mathbb{R}$. Motivated by (4.3.12) and (4.3.13), define entire $\mathbb{R}$-differentiable functions $\{\eta_l\}_{l=2}^4$ on $\mathbb{H}_t$ by

$$\eta_2\,(w) = x_2 - x_1 i,$$

and

$$\eta_3(w) = x_3 + \frac{sgn(t)\, x_1}{\sqrt{|t|}} j_k, \quad \eta_4(w) = x_4 + \frac{sgn(t)\, x_1}{\sqrt{|t|}} k_t, \tag{4.4.1}$$

in a $\mathbb{H}_t$-variable $w = x_1 + x_2 i + x_3 j_t + x_4 k_t$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$.

Observe first that

$$\eta_2 = x_2 - x_1 i \text{ in } \mathcal{F}_{t,\mathbb{H}_t},$$

is t-regular in $\mathcal{F}_{t,\mathbb{H}_t}$, i.e.,

$$\eta_2 \nabla_t = 0 = \nabla_t \eta_2, \quad \text{on} \quad \mathbb{H}_t, \tag{4.4.2}$$

by (4.3.12), since η_2 is identical to the function ζ_2 introduced in Sect. 4.3.

Lemma 4.10. *If $\eta_2 = x_2 - x_1 i \in \mathcal{F}_{1,\mathbb{H}_t}$ is in the sense of* (4.4.1)*, then*

$$\eta_2 \text{ is a } t\text{-harmonic } t\text{-regular function on } \mathbb{H}_t. \tag{4.4.3}$$

Proof. The function η_2 is t-regular on $\mathbb{H}_t$ by (4.4.2). Therefore, by (4.3.13), it is t-harmonic on $\mathbb{H}_t$, too. □

Let's denote the quantity $\sqrt{|t|}$ by ρ and $sgn(t)$ by $s_t \in \{\pm\}$, and let $\eta_3 = x_3 + \frac{s_t x_1}{\rho} j_t \in \mathcal{F}_{t,\mathbb{H}_t}$ be in the sense of (4.4.1). Observe that

$$\begin{aligned}
\nabla_t \eta_3 &= \frac{\partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\partial x_1} + i \frac{\partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\partial x_2} \\
&\quad - j_t \frac{s_t \partial\left(x_3 + \frac{s x_{t1}}{\rho} j_t\right)}{\rho \partial x_3} - k_t \frac{s_t \partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\rho \partial x_4} \\
&= \left(\frac{s_t}{\rho} j_t\right) + i(0) - j_t\left(\frac{s_t}{\rho}\right) - k_t(0) = \frac{s_t}{\rho} j_t - \frac{s_t}{\rho} j_t = 0,
\end{aligned}$$

and

$$\begin{aligned}
\eta_3 \nabla_t &= \frac{\partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\partial x_1} + \frac{\partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\partial x_2} i - \frac{s_t \partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\rho \partial x_3} j_t \\
&\quad - \frac{s_t \partial\left(x_3 + \frac{s_t x_1}{\rho} j_t\right)}{\rho \partial x_4} k_t \\
&= \left(\frac{s_t}{\rho} j_t\right) + (0) i - \left(\frac{s_t}{\rho}\right) j_t - (0) k_t = \frac{s_t}{\rho} j_t - \frac{s_t}{\rho} j_t = 0;
\end{aligned} \tag{4.4.4}$$

also, if $\eta_4 = x_4 + s_t \rho x_1 k_t \in \mathcal{F}_{1,\mathbb{H}_t}$ is in the sense of (4.4.1), then

$$\begin{aligned}
\nabla_t \eta_4 &= \frac{\partial\left(x_4 + \frac{s_t x_1}{\rho} k_t\right)}{\partial x_1} + i \frac{\partial\left(x_4 + \frac{s_t x_1}{\rho} k_t\right)}{\partial x_2} \\
&\quad - j_t \frac{s_t \partial\left(x_4 + \frac{s_t x_1}{\rho} k_t\right)}{\rho \partial x_3} - k_t \frac{s_t \partial\left(x_4 + \frac{s_t x_1}{\rho} k_t\right)}{\rho \partial x_4} \\
&= \left(\frac{s_t}{\rho} k_t\right) + i(0) - j_t(0) - k_t\left(\frac{s_t}{\rho}\right) = \frac{s_t}{\rho} k_t - \frac{s_t}{\rho} k_t = 0,
\end{aligned}$$

and

$$\begin{aligned}\eta_4 \nabla_t &= \frac{\partial\left(x_4 + \frac{s_t x_1}{\rho} k_t\right)}{\partial x_1} + \frac{\partial\left(x_4 + \frac{s_t x_1}{\rho} k_t\right)}{\partial x_2} i \\ &\quad - \frac{s_t \partial\left(x_4 + \frac{x_1}{\rho} k_t\right)}{\rho \partial x_3} j_t - \frac{s_t \partial\left(x_4 + \frac{x_1}{\rho} k_t\right)}{\rho \partial x_4} k_t \\ &= \left(\frac{s_t}{\rho} k_t\right) + (0)\, i - (0)\, j_t - \left(\frac{s_t}{\rho}\right) k_t = \frac{s_t}{\rho} k_t - \frac{s_t}{\rho} k_t = 0. \quad (4.4.5)\end{aligned}$$

Lemma 4.11. *If* $\eta_3 = x_3 + \frac{sng(t)x_1}{\sqrt{|t|}} j_t$*, and* $\eta_4 = x_4 + \frac{sgn(t)x_1}{\sqrt{|t|}} k_t$ *are in the sense of* (4.4.1) *in* $\mathcal{F}_{1,\mathbb{H}_t}$*, then*

$$\eta_3 \text{ and } \eta_4 \text{ are } t\text{-harmonic } t\text{-regular on } \mathbb{H}_t. \quad (4.4.6)$$

Proof. The functions $\eta_3, \eta_4 \in \mathcal{F}_{1,\mathbb{H}_t}$ are both left and right t-regular on $\mathbb{H}_t$, by (4.4.4) and (4.4.5). Therefore, the functions η_3 and η_4 are t-harmonic on $\mathbb{H}_t$, too, by (4.3.13). □

By the above two lemmas, one obtains the following result.

Theorem 4.12. *The functions* $\{\eta_l\}_{l=2}^4$ *of* (4.4.1) *are* t*-harmonic* t*-regular functions on* $\mathbb{H}_t$*.*

Proof. The proof is done by (4.4.3) and (4.4.6). □

The following corollary is an immediate consequence of the above theorem.

Corollary 4.13. *Let* $f = s_1\eta_2 + s_2\eta_3 + s_3\eta_4 \in \mathcal{F}_{t,\mathbb{H}_t}$ *be a* $\mathbb{R}$*-linear combination of the* t*-regular functions* $\{\eta_l\}_{l=2}^4$ *of* (4.4.1)*, where* $s_1, s_2, s_3 \in \mathbb{R}$*. Then* f *is* t*-harmonic* t*-regular on* $\mathbb{H}_t$*.*

Proof. Let $f = \sum\limits_{n=1}^{3} s_n \eta_{n+1} \in \mathcal{F}_{1,\mathbb{H}_t}$ be a $\mathbb{R}$-linear combination of $\{\eta_l\}_{l=2}^4$, for $s_1, s_2, s_3 \in \mathbb{R}$. Then

$$\nabla_t f = \sum_{n=1}^{3} s_n \left(\nabla_t \eta_{n+1}\right) = 0,$$

and

$$f \nabla_t = \sum_{n=1}^{3} s_n \left(\eta_{n+1} \nabla_t\right) = 0,$$

by (4.4.2), (4.4.4) and (4.4.5). Therefore, this function f is t-regular on $\mathbb{H}_t$. So, by (4.3.13), this function f is t-harmonic on $\mathbb{H}_t$, too. □

4.5. Left t-Regular Functions of $\mathcal{F}_t$

In this section, we consider left t-regular functions in $\mathcal{F}_t$ more in detail. As in Sects. 4.3 and 4.4, throughout this section, we assume that $t \in \mathbb{R} \setminus \{0\}$. Recall that the functions,

$$\eta_2(w) = x_2 - x_1 i,$$

and

$$\eta_3(w) = x_3 + \frac{sgn(t)\, x_1}{\sqrt{|t|}} j_t, \quad \eta(w) = x_4 + \frac{sgn(t)\, x_1}{\sqrt{|t|}} k_t,$$

of (4.4.1) in a $\mathbb{H}_t$-variable $w = x_1 + x_2 i + x_3 j_t + x_4 k_t$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$, are t-harmonic t-regular functions on $\mathbb{H}_t$. We first study t-regular functions induced by $\{\eta_l\}_{l=2}^4$. To do that we define a new operation $(\times)$ on the t-scaled hypercomplexes $\mathbb{H}_t$.

Definition 4.14. Let $h_1, ..., h_N \in \mathbb{H}_t$, for $N \in \mathbb{N}$. Then the symmetrized product of $h_1, ..., h_N$ is defined by a new hypercomplex number,

$$\underset{n=1}{\overset{N}{\times}} h_n \overset{\text{denote}}{=} h_1 \times ... \times h_N \overset{\text{def}}{=} \frac{1}{N!} \sum_{\sigma \in S_N} h_{\sigma(1)} \cdot_t h_{\sigma(2)} \cdot_t ... \cdot_t h_{\sigma(N)}, \quad (4.5.1)$$

where S_N is the symmetric (or, permutation) group over $\{1, ..., N\}$, where $(\cdot_t)$ is the t-scaled multiplication (2.1.1) on $\mathbb{H}_t$.

Notation. From below, if there are no confusions, we denote the operation $(\cdot_t)$ simply by $(\cdot)$ for convenience. i.e., the above definition (4.5.1) can be re-written to be

$$\underset{n=1}{\overset{N}{\times}} h_n = \frac{1}{N!} \sum_{\sigma \in S_N} h_{\sigma(1)} h_{\sigma(2)} ... h_{\sigma(N)}, \quad (4.5.2)$$

for $h_1, ..., h_N \in \mathbb{H}_t$, for $t \in \mathbb{R} \setminus \{0\}$.□

Remark. The above symmetrized product $(\times)$ of (4.5.1) is well-defined for "all" $t \in \mathbb{R}$, including the case where $t = 0$. As we assumed, we restrict our interests to the cases where $t \neq 0$ in $\mathbb{R}$ in this section, but we emphasize that the above symmetrized product of (4.5.1) is also well-defined where $t = 0$ (See Sect. 5 below).□

Remark that, actually, the above definition (4.5.1) of the symmetrized product implies the following case. If we consider

$$h^{(n)} \overset{\text{denote}}{=} \underbrace{h \times h \times ... \times h}_{n\text{-times}}, \quad \text{for } n \in \mathbb{N},$$

then, all permutations σ of S_n induce exactly same element, so,

$$h^{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \left(h_{\sigma(1)} ... h_{\sigma(n)} \right), \text{ with } h_{\sigma(j)} = h, \ \forall j,$$

in $\mathbb{H}_t$, implying that

$$h^{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} ... h_{\sigma(n)} = \frac{1}{n!} (n! h^n) = h^n, \quad (4.5.3)$$

where $h^n = \underbrace{hh.......h}_{n\text{-times}} \overset{\text{denote}}{=} \underbrace{h \cdot_t h \cdot_t ... \cdot_t h}_{n\text{-times}}$, for all $n \in \mathbb{N}$, by (4.5.2). So, if we consider "mutually distinct" h_1 and h_2 in $\mathbb{H}_t$, and

$$h_1^{(n)} \times h_2 \quad \text{in} \quad \mathbb{H}_t,$$

then

$$h_1^{(n)} \times h_2 = \underbrace{h_1 \times ... \times h_1}_{n\text{-times}} \times h_2,$$

satisfying

$$h_1^{(n)} \times h_2 = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}} \left((h_1)_{\sigma(1)} ... (h_1)_{\sigma(n)} \right) (h_2)_{\sigma(n+1)},$$

and hence,

$$h_1^{(n)} \times h_2 = \frac{n!}{(n+1)!} \sum_{\sigma \in S_{n+1}} \left(h_1^n \right) (h_2)_{\sigma(n+1)},$$

by (4.5.3). More generally, if $h_1, ..., h_N \in \mathbb{H}_t$ are "mutually distinct," and $n_1, ..., n_N \in \mathbb{N}$, then

$$\underset{j=1}{\overset{N}{\times}} h_j^{(n_j)} = \frac{\prod_{j=1}^{N} (n_j!)}{\left(\sum_{j=1}^{N} n_j \right)!} \left(\sum_{\sigma \in S_{\sum_{j=1}^{N} n_j}} h_{\sigma(1)} ... h_{\sigma\left(\sum_{j=1}^{N} n_j\right)} \right), \tag{4.5.4}$$

by (4.5.3), where $h_{\sigma(j)} \in \{h_1, ..., h_N\}$, for all $j \in \left\{ 1, ..., \sum_{j=1}^{n} n_j \right\}$.

Proposition 4.15. *Let $h_1, ..., h_N$ be mutually distinct elements of the t-scaled hypercomplexes $\mathbb{H}_t$, for $N \in \mathbb{N} \setminus \{1\}$, and let $n_1, ..., n_N \in \mathbb{N}$. If*

$$n \overset{denote}{=} \sum_{j=1}^{N} n_j = n_1 + ... + n_N \in \mathbb{N},$$

then

$$\underset{j=1}{\overset{N}{\times}} h^{(n_j)} = \frac{\prod_{j=1}^{N} (n_j!)}{n!} \left(\sum_{\sigma \in S_n} h_{\sigma(1)} h_{\sigma(2)} ... h_{\sigma(n)} \right),$$

with

$$h_j^{(k)} = \underbrace{h_j \times h_j \times ... \times h_j}_{k\text{-times}} = h_j^k, \ \forall j = 1, ..., N, \tag{4.5.5}$$

for all $k \in \{1, ..., n\}$, where $h_{\sigma(j)} \in \{h_1, ..., h_N\}$, for all $j = 1, ..., n$.

Proof. The refined computation (4.5.5) of the definition (4.5.1) is obtained by (4.5.4). □

Let $f_1, ..., f_N : \mathbb{H}_t \to \mathbb{H}_t$ be functions for $N \in \mathbb{N}$. Then, similar to (4.5.1) (expressed by (4.5.2)), one can define a symmetrized-product function of them by

$$\underset{n=1}{\overset{N}{\times}} f_n = \frac{1}{N!} \sum_{\sigma \in S_N} f_{\sigma(1)} f_{\sigma(2)} ... f_{\sigma(N)},$$

where

$$\left(f_{\sigma(1)} f_{\sigma(2)} ... f_{\sigma(N)}\right)(h) = f_{\sigma(1)}(h) \cdot_t ... \cdot_t f_{\sigma(N)}(h), \tag{4.5.6}$$

simply denoted by

$$\left(f_{\sigma(1)} f_{\sigma(2)} ... f_{\sigma(N)}\right)(h) = f_{\sigma(1)}(h) ... f_{\sigma(N)}(h),$$

as in (4.5.2), for all $\sigma \in S_N$ and $h \in \mathbb{H}_t$. Also, similar to (4.5.3), one write

$$f_j^{(n)} = \underbrace{f_j \times f_j \times ... \times f_j}_{n\text{-times}} = f_j^n, \ \ \forall n \in \mathbb{N},$$

in terms of (4.5.6). Then, for any $h \in \mathbb{H}_t$, the image $\left(\underset{n=1}{\overset{N}{\times}} f_n^{(n_j)}\right)(h)$ is expressed similar to (4.5.5), if we replace $h_{\sigma(j)}$ to $f_{\sigma(j)}(h)$ in (4.5.5). Without loss of generality, let's axiomatize that

$$f^{(0)} = 1, \text{ the constant 1-function on } \mathbb{H}_t,$$

for all functions $f : \mathbb{H}_t \to \mathbb{H}_t$.

Definition 4.16. Let $\mathbf{n} \overset{\text{denote}}{=} (n_1, n_2, n_3) \in \mathbb{N}_0^3$ be a triple of numbers in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\{\eta_l\}_{l=2}^4$ be the t-harmonic t-regular functions (4.4.1). Define a function $\eta^{\mathbf{n}}$ by

$$\eta^{\mathbf{n}} \overset{\text{def}}{=} \frac{1}{\mathbf{n}!} \left(\eta_2^{(n_1)} \times \eta_3^{(n_2)} \times \eta_4^{(n_3)}\right),$$

where

$$\mathbf{n}! = (n_1!)(n_2!)(n_3!) \in \mathbb{N}, \tag{4.5.7}$$

and

$$\eta_{l+1}^{(n_l)} = \underbrace{\eta_{l+1} \times \eta_{l+1} \times ... \times \eta_{l+1}}_{n_l\text{-times}} = \eta_{l+1}^{n_l}, \ \forall l = 1, 2, 3.$$

For an arbitrary $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$, let $\eta^{\mathbf{n}} : \mathbb{H}_t \to \mathbb{H}_t$ be a function (4.5.7). By the very construction, this function $\eta^{\mathbf{n}}$ is contained in $\mathcal{F}_{1,\mathbb{H}_t}$. Observe that, if we let

$$e_2 = -i, \ e_3 = \frac{sgn(t)}{\sqrt{|t|}} j_t, \text{ and } e_4 = \frac{sgn(t)}{\sqrt{|t|}} k_t,$$

making

$$\eta_l = x_l + x_1 e_l, \ \ \forall l = 2, 3, 4,$$

then

$$(\mathbf{n}!) \nabla_t \eta^{\mathbf{n}} = (\mathbf{n}!) \nabla_t \left(\frac{1}{\mathbf{n}!} \left(\eta_2^{\times n_1} \times \eta_3^{\times n_2} \times \eta_4^{\times n_3}\right)\right)$$

$$= \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} e_l \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)}$$

$$- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} e_l \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)}, \quad (4.5.8)$$

where

$$N = n_1 + n_2 + n_3 \text{ in } \mathbb{N}.$$

Now, note that

$$0 = \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} x_l \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)}$$

$$- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} x_l \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)}, \quad (4.5.9)$$

by the formula (7) in the page 99 of [8]. Let's multiply x_1 to the formula (4.5.8). Then

$$x_1 (\mathbf{n}!) \nabla_t \eta^{\mathbf{n}} = x_1 (n!) \nabla_t \eta^{\mathbf{n}} + 0$$

$$= x_1 \left(\sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} e_l \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)} \right.$$

$$\left. - \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} e_l \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)} \right) + 0$$

by (4.5.8)

$$= \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} (x_1 e_l) \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)}$$

$$- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} (x_1 e_l) \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)}$$

$$+ \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} x_l \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)}$$

$$- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} x_l \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)}$$

by (4.5.9)

$$= \left(\sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\ldots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} (x_1 e_l) \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)} \right.$$

$$+ \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\dots,N\}, \sigma(k)=l} x_l \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)} \Bigg)$$

$$+ \Bigg(- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\dots,N\}, \sigma(k)=l} (x_1 e_l) \, \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)}$$

$$- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\dots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \times \eta_{\sigma(k-1)} x_l \eta_{\sigma(k+1)} \cdots \eta_{\sigma(N)} \Bigg)$$

$$= \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\dots,N\}, \sigma(k)=l} \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} \eta_l \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)}$$

$$- \sum_{\sigma \in S_N} \sum_{l=2}^{4} \sum_{k \in \{1,\dots,N\}, \sigma(k)=l} \eta_l \eta_{\sigma(1)} \cdots \eta_{\sigma(k-1)} \eta_{\sigma(k+1)} \cdots \times \eta_{\sigma(N)}$$

$$= 0, \tag{4.5.10}$$

under the sum over S_N.

Lemma 4.17. *For any* $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$, *the function* $\eta^{\mathbf{n}}$ of (4.5.7) is t-regular on $\mathbb{H}_t$, i.e.,

$$\nabla_t \eta^{\mathbf{n}} = 0 = \eta^{\mathbf{n}} \nabla_t. \tag{4.5.11}$$

Proof. The first equality of the formula (4.5.11) is obtained by (4.5.10). The second equality is obtained similarly by (4.5.10), by the definition of the symetrized product (4.5.2) and (4.5.7), with help of the t-regularity (4.4.3) of $\{\eta_l\}_{l=2}^{4}$. □

By the above lemma, we obtain the following result.

Theorem 4.18. *Let* $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$, *and let* $\eta^{\mathbf{n}} \in \mathcal{F}_{t,\mathbb{H}_t}$ *be the function* (4.5.7). *Then it is a t-harmonic t-regular function, i.e.,*

$$\eta^{\mathbf{n}} \nabla_t = \nabla_t \eta^{\mathbf{n}} = 0, \text{ and } \Delta_t \eta^{\mathbf{n}} = 0, \text{ on } \mathbb{H}_t. \tag{4.5.12}$$

Proof. The t-regularity of $\eta^{\mathbf{n}}$ on $\mathbb{H}_t$ is shown by (4.5.11), for all $\mathbf{n} \in \mathbb{N}_0^3$. Thus, by (4.3.13), the functions $\{\eta^{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{N}_0^3}$ are t-harmonic on $\mathbb{H}_t$. Therefore, the relation (4.5.12) holds. □

By the above theorem, all functions $\left\{\eta^{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^3\right\}$ are t-harmonic t-regular functions on $\mathbb{H}_t$.

Now, recall the complete semi-norm $\|.\|_t$ of (2.4.10) on the t-scaled hypercomplexes $\mathbb{H}_t$,

$$\|h\|_t = \sqrt{|\langle h, h \rangle_t|} = \sqrt{|\tau(hh^{\dagger})|} = \sqrt{\left| |a|^2 - t\,|b|^2 \right|},$$

for all $h = a + bj_t \in \mathbb{H}_t$, regarded as $(a, b) \in \mathbb{H}_t$ with $a, b \in \mathbb{C}$ (in the sense of Sect. 2) satisfying (2.4.11) and (2.4.12). i.e., if $t < 0$, then it is a complete norm on $\mathbb{H}_t$, while if $t > 0$, then it is a complete semi-norm on $\mathbb{H}_t$. If we understand,

$$\eta_2 = x_2 - x_1 i,\ \eta_3 = x_3 + \frac{sgn\,(t)\,x_1}{\sqrt{|t|}} j_t,\ \eta_4 = x_4 + \frac{sgn\,(t)\,x_1}{\sqrt{|t|}} k_t,$$

as their images of $\mathbb{H}_t$, then they are regarded as

$$\eta_2 = (x_2 - x_1 i,\ 0)\,,\ \eta_3 = \left(x_3,\ \frac{sgn\,(t)\,x_1}{\sqrt{|t|}} \right),$$

and

$$\eta_4 = \left(x_4,\ \frac{sgn\,(t)\,x_1}{\sqrt{|t|}} i \right),$$

in “the t-scaled hypercomplex ring $\mathbb{H}_t$ in the sense of Sect. 2.” So, one can compute their norms on $\mathbb{H}_t$,

$$\|\eta_2\|_t = \sqrt{\left| |x_2 - x_1 i|^2 - t\,|0|^2 \right|} = \sqrt{\left| x_2^2 + (-x_1)^2 \right|} = \sqrt{x_1^2 + x_2^2},$$

$$\|\eta_3\|_t = \sqrt{\left| |x_3|^2 - t \left| \frac{sgn\,(t)\,x_1}{\sqrt{|t|}} \right|^2 \right|} = \sqrt{|x_3^2 - sgn\,(t)\,x_1^2|},$$

and

$$\|\eta_4\|_t = \sqrt{\left| |x_4|^2 - t \left| \frac{sgn\,(t)\,x_1}{\sqrt{|t|}} i \right|^2 \right|} = \sqrt{|x_4^2 - sgn\,(t)\,x_1^2|}. \quad (4.5.13)$$

Lemma 4.19. *Let* $\eta^{\mathbf{n}} \in \mathcal{F}_{t,\mathbb{H}_t}$ *be a function* (4.5.7) *for* $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$. *Then*

$$\|\eta^{\mathbf{n}}\|_t \le \left(\sqrt{x_2^2 + x_1^2} \right)^{n_1} \left(\sqrt{|x_3^2 - sgn\,(t)\,x_1^2|} \right)^{n_2} \left(\sqrt{|x_4^2 - sgn\,(t)\,x_1^2|} \right)^{n_3}. \quad (4.5.14)$$

Proof. By (4.5.7), for any $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$,

$$\|\eta^{\mathbf{n}}\|_t = \left\| \frac{1}{\mathbf{n}!} \left(\eta_2^{(n_1)} \times \eta_3^{(n_2)} \times \eta_4^{(n_3)} \right) \right\|_t$$

$$= \left\| \frac{1}{(n_1!)\,(n_2!)\,(n_3!)} \left(\frac{(n_1!)\,(n_2!)\,(n_3!)}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} h_{\sigma(2)} \ldots h_{\sigma(n)} \right) \right\|_t$$

where $h_{\sigma(l)} \in \{\eta_2, \eta_3, \eta_4\}$, for all $\sigma \in S_n$, with $n = n_1 + n_2 + n_3 \in \mathbb{N}$

$$= \|\eta_2^{n_1} \eta_3^{n_2} \eta_4^{n_3}\|_t \le \|\eta_2\|_t^{n_1} \, \|\eta_3\|_t^{n_2} \, \|\eta_4\|_t^{n_3}$$

by (4.5.5)

$$= \left(\sqrt{x_2^2 + x_1^2} \right)^{n_1} \left(\sqrt{|x_3^2 - sgn\,(t)\,x_1^2|} \right)^{n_2} \left(\sqrt{|x_4^2 - sgn\,(t)\,x_1^2|} \right)^{n_3},$$

by (4.5.13). Therefore, the inequality (4.5.14) holds. □

The above lemma shows that, for any arbitrarily fixed $\mathbf{n} \in \mathbb{N}_0^3$, the corresponding function $\eta^{\mathbf{n}}$ is bounded by (4.5.14).

Theorem 4.20. *Let $t \neq 0$ in $\mathbb{R}$, and $f \in \mathcal{F}_{t,U}$, a $\mathbb{R}$-differentiable function, where $U \in \mathcal{T}_t$ containing $0 = 0 + 0i + 0j_t + 0k_t \in \mathbb{H}_t$. If f is $\mathbb{R}$-analytic on U, then*

$$f \text{ is left } t\text{-regular on } U,$$

if and only if

$$f = f(0) + \sum_{\mathbf{n} \in \mathbb{N}^3} \eta^{\mathbf{n}} f_{\mathbf{n}}, \tag{4.5.15}$$

with

$$f_{\mathbf{n}} = \frac{1}{\mathbf{n}!} \frac{\partial^{n_1+n_2+n_3} f}{\partial x_2^{n_1} \partial x_3^{n_2} \partial x_4^{n_3}}(0), \ \forall \mathbf{n} \in \mathbb{N}^3.$$

Proof. Clearly, if $f = f(0) + \sum_{\mathbf{n} \in \mathbb{N}^3} \eta^{\mathbf{n}} f_{\mathbf{n}}$, then it is left t-regular, by the t-regularity (4.5.12) of $\left\{\eta^{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^3\right\}$.

Suppose f is left t-regular in $\mathcal{F}_{t,U}$, satisfying $\nabla_t f = 0$. Then

$$f(w) - f(0) = \sum_{n=2}^{4} (\eta_n(w))\left((R_n f)(w)\right),$$

where

$$(R_n f)(w) = \int_0^1 \frac{\partial f(tw)}{\partial x_n} dt, \ \forall n = 2, 3, 4, \tag{4.5.16}$$

in a $\mathbb{H}_t$-variable $w = x_1 + x_2 i + x_3 j_t + x_4 k_t$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Indeed,

$$\frac{df(tw)}{dt} = x_1 \frac{\partial f(tw)}{\partial x_1} + \sum_{l=2}^{4} x_l \frac{\partial f(tw)}{\partial x_l},$$

identical to

$$\frac{df(tw)}{dt} = x_1 \left(-i \frac{\partial}{\partial x_2} + j_t \frac{sgn(t)\,\partial}{\sqrt{|t|}\partial x_3} + k_t \frac{sgn(t)\,\partial}{\sqrt{|t|}\partial x_4}\right) f(tw) + \sum_{l=2}^{4} x_l \frac{\partial f(tw)}{\partial x_l},$$

implying that

$$\frac{df(tw)}{dt} = \sum_{l=2}^{4} \eta_l(w) \frac{\partial f(tw)}{\partial x_l}.$$

Thus, by iterating (4.5.16), one can get that

$$f = f(0) + \sum_{\mathbf{n} \in \mathbb{N}^3} \eta^{\mathbf{n}} f_{\mathbf{n}}$$

where

$$f_{\mathbf{n}} = \frac{1}{\mathbf{n}!} \frac{\partial^{n_1+n_2+n_3} f}{\partial x_2^{n_1} \partial x_3^{n_2} \partial x_4^{n_3}}(0), \ \forall \mathbf{n} \in \mathbb{N}^3.$$

Note that, such an iteration can be done by the assumption that f is $\mathbb{R}$-analytic on U, by the boundedness condition (4.5.14). Therefore, the characterization (4.5.15) holds true. □

Remark that, if a given scale t is negative, i.e., $t < 0$ in $\mathbb{R}$, then the $\mathbb{R}$-regularity automatically implies the $\mathbb{R}$-analyticity by (4.3.5), (4.3.6) and (4.5.14). So, if $t < 0$, then the above theorem holds without the $\mathbb{R}$-analyticity assumption for $f \in \mathcal{F}_{t,U}$.

5. 0-Regular Functions on $\mathbb{H}_0$

In Sect. 4, we studied the t-regularity and the t-hamonicity of $\mathbb{R}$-differentiable functions of $\mathcal{F}_{t,U}$, for $U \in \mathcal{T}_t$ in the t-scaled hypercomplexes $\mathbb{H}_t$, where a given scale is nonzero, i.e., $t \in \mathbb{R} \setminus \{0\}$. To do that, we defined the operators ∇_t and $\nabla_t^{\dagger}$ on $\mathcal{F}_t$ by

$$\nabla_t = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - j_t\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3} - k_t\frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4},$$

and

$$\nabla_t^{\dagger} = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}i + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_3}j_t + \frac{sgn\,(t)\,\partial}{\sqrt{|t|}\partial x_4}k_t, \tag{5.1}$$

satisfying

$$\Delta_t = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_3^2} - \frac{sgn\,(t)\,\partial^2}{\partial x_4^2} = \nabla_t^{\dagger}\nabla_t, \tag{5.2}$$

in a $\mathbb{H}_t$-variable $w = x_1 + x_2 i + x_3 j_t + x_4 k_t$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$, where $i = \sqrt{-1} = i_0$ in $\mathbb{H}_0$. We showed in Sect. 4 that the functions,

$$\eta_2\,(w) = x_2 - x_1 i,\ \eta_3\,(w) = x_3 + \frac{sgn\,(t)\,x_1}{\sqrt{|t|}}j_t,\ \eta_4 + \frac{sgn\,(t)\,x_1}{\sqrt{|t|}}, \tag{5.3}$$

are t-harmonic t-regular in $\mathcal{F}_{t,\mathbb{H}_t}$, furthermore, all functions of $\mathcal{F}_{t,\mathbb{H}_t}$ formed by

$$\eta^{\mathbf{n}} = \frac{1}{\mathbf{n}!}\left(\eta_2^{(n_1)} \times \eta_3^{(n_2)} \times \eta_4^{(n_3)}\right),\ \text{with } \mathbf{n}! = \prod_{l=1}^{3}(n_l!) \tag{5.4}$$

are t-harmonic t-regular in $\mathcal{F}_{t,\mathbb{H}_t}$, for all $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$, characterizing the left t-regularity (4.5.13) on $\mathcal{F}_t$, whenever $t \neq 0$ in $\mathbb{R}$.

In this section, we consider the case where the given scale t is zero in $\mathbb{R}$, i.e., we study 0-regularity on $\mathcal{F}_0$ acting on the 0-scaled hypercomplexes $\mathbb{H}_0$. Recall that if

$$h = a + bi + cj_0 + dk_0 \in \mathbb{H}_0,\ \text{with } a, b, c, d \in \mathbb{R},$$

then

$$h = (a + bi) + (c + di)\,j_0 = z_1 + z_2 j_0,$$

with

$$z_1 = a + bi,\ \text{and } z_2 = c + di,\ \text{in } \mathbb{C},$$

realized to be

$$[(z_1, z_2)]_0 = \begin{pmatrix} z_1 & 0 \cdot z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}, \text{ in } \mathcal{H}_2^0,$$

where

$$\mathbb{H}_0 = \operatorname{span}_{\mathbb{R}} \{1, i, j_0, k_0\}.$$

So, the $\mathbb{R}$-basis elements $\{i, j_0, k_0\}$ of $\mathbb{H}_0$ satisfy

$$i^2 = -1, \;\; j_0^2 = 0 = k_0^2, \tag{5.5}$$

and the commuting diagrams,

$$\begin{tikzcd} & i \arrow[dl, "1"'] & \\ j_0 \arrow[rr, "1"'] & & k_0 \arrow[ul, "-0=0"'] \end{tikzcd}, \quad \text{and} \quad \begin{tikzcd} & i \arrow[dr, "-1"] & \\ j_0 \arrow[ur, "0"] & & k_0 \arrow[ll, "-1"] \end{tikzcd},$$

saying that

$$ij_0 = k_0, \; j_0k_0 = -0i = 0, \; k_0i = j_0,$$

and

$$j_0i = -k_0, \; ik_0 = -j_0, \; k_0j_0 = 0i = 0, \tag{5.6}$$

by (3.4), (3.5) and (3.6).

Define first the operators ∇_0 and $\nabla_0^\dagger$ by

$$\nabla_0 = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_0\frac{\partial}{\partial x_3} + k_0\frac{\partial}{\partial x_4},$$

and

$$\nabla_0^\dagger = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}i - \frac{\partial}{\partial x_3}j_0 - \frac{\partial}{\partial x_4}k_0, \tag{5.7}$$

on $\mathcal{F}_{0,U}$, for any $U \in \mathcal{T}_t$ in $\mathbb{H}_0$, similar to the differential operators of (5.1), in a $\mathbb{H}_0$-variable $x_1 + x_2i + x_3j_0 + x_4k_0$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Observe that

$$\begin{aligned}
\nabla_0^\dagger\nabla_0 &= \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}i - \frac{\partial}{\partial x_3}j_0 - \frac{\partial}{\partial x_4}k_0\right)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j_0\frac{\partial}{\partial x_3} + k_0\frac{\partial}{\partial x_4}\right) \\
&= \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_1}i\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}j_0\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1}k_0\frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_2}i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}i^2\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_2}ij_0\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}ik_0\frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_3}j_0\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}j_0i\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}j_0^2\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3}j_0k_0\frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_4}k_0\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4}k_0i\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}k_0j_0\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}k_0^2\frac{\partial}{\partial x_4} \\
&= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_2}ij_0\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}ik_0\frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_3}j_0i\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}j_0^2\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3}j_0k_0\frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_4}k_0i\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}k_0j_0\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}k_0^2\frac{\partial}{\partial x_4}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_3} j_0^2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3} j_0 k_0 \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_4} k_0 j_0 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} k_0^2 \frac{\partial}{\partial x_4} \\
&= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_3} (0) \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_3} (-0i) \frac{\partial}{\partial x_4} \\
&\quad - \frac{\partial}{\partial x_4} (0i) \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} (0) \frac{\partial}{\partial x_4} \\
&= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},
\end{aligned}
\tag{5.8}
$$

by (5.5) and (5.6).

Theorem 5.1. *Let* ∇_0 *be the operator* (5.7) *on* $\mathcal{F}_0$. *Then*

$$
\nabla_0^\dagger \nabla_0 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. \tag{5.9}
$$

Proof. The formula (5.9) is obtained by the computation (5.8). □

One can recognize that the formula (5.9) is similar to the formula (5.2). However, we do not have x_l-depending double-partial-derivatives, for $l = 3, 4$, in (5.9). It shows that, in the 0-scaled case, the Laplacian on $\mathcal{F}_{0,U}$ for an open connected subset $U \subseteq \mathbb{H}_0$ is depending only on the first, and the second $\mathbb{R}$-variables x_1 and x_2. More generally, for any $u_3, u_4 \in \mathbb{R} \setminus \{0\}$, if we define operators D_{u_3,u_4} and $D_{u_3,u_4}^\dagger$ by

$$
D_{u_3,u_4} = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + j_0 \frac{u_3 \partial}{\partial x_3} + k_0 \frac{u_4 \partial}{\partial x_4},
$$

and

$$
D_{u_3,u_4}^\dagger = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} i - \frac{u_3 \partial}{\partial x_3} j_0 - \frac{u_4 \partial}{\partial x_4} k_0, \tag{5.10}
$$

as in (5.1), then, for any dilations (5.10) of ∇_0 and $\nabla_0^\dagger$, one has

$$
D_{u_3,u_4}^\dagger D_{u_3,u_4} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{5.11}
$$

similar to (5.8), by (5.5) and (5.6). It shows that, indeed, the Laplacian Δ_0 on $\mathcal{F}_0$ can be well-defined to be

$$
\Delta_0 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},
$$

by (5.11).

Definition 5.2. Define an operator Δ_0 on $\mathcal{F}_{0,U}$, for any open connected subsets U of the 0-scaled hypercomplexes $\mathbb{H}_0$, by

$$
\Delta_0 \overset{\text{def}}{=} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + 0 \cdot \left(\frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} \right). \tag{5.12}
$$

A function $f \in \mathcal{F}_{0,U}$ is said to be left 0(-scaled)-regular on U, if

$$\nabla_0 f = \frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2} + j_0\frac{\partial f}{\partial x_3} + k_0\frac{\partial f}{\partial x_4} = 0,$$

and it is said to be right 0(-scaled)-regular on U, if

$$f\nabla_0 = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}i + \frac{\partial f}{\partial x_3}j_0 + \frac{\partial f}{\partial x_4}k_0 = 0.$$

If $f \in \mathcal{F}_{0,U}$ is both left and right 0-regular, then f is called a 0-regular function on U. Also, a function $f \in \mathcal{F}_{0,U}$ is said to be 0(-scaled)-harmonic on U, if

$$\Delta_0 f = 0,$$

where Δ_0 is in the sense of (5.12).

By definition, we have the following result.

Theorem 5.3. *Let $f \in \mathcal{F}_{0,U}$, for an open connected subset U of $\mathbb{H}_0$. Then*

$$f\, is\, left\, 0-regular \Longrightarrow f\, is\, 0-harmonic\, on\, U. \tag{5.13}$$

Proof. By (5.9), one has $\Delta_0 = \nabla_0^{\dagger}\nabla_0$, on $\mathcal{F}_{0,U}$. So, if f is left 0-regular on U, then

$$\Delta_0 f = \nabla_0^{\dagger}\left(\nabla_0 f\right) = \nabla_0^{\dagger}\left(0\right) = 0,$$

and hence, it is 0-harmonic on U. □

Similar to (5.3), we let

$$\eta_2\left(w\right) = x_2 - x_1 i,$$

and

$$\eta_3\left(w\right) = x_3 - x_1 j_0,\ \text{and}\ \eta_4\left(w\right) = x_4 - x_1 k_0, \tag{5.14}$$

in a $\mathbb{H}_0$-variable $w = x_1 + x_2 i + x_3 j_0 + x_4 k_0$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Then these functions $\{\eta_l\}_{l=2}^{4}$ are contained in $\mathcal{F}_{0,\mathbb{H}_t}$.

Lemma 5.4. *Let $\{\eta_l\}_{l=2}^{4}$ be in the sense of (5.14). Then they are not only 0-regular, but also, 0-harmonic on the 0-scaled hypercomplexes $\mathbb{H}_0$, i.e.,*

$$\{\eta_l\}_{l=2}^{4}\ are\, 0-harmonic\, 0-regular\, functions\, on\, \mathbb{H}_0. \tag{5.15}$$

Proof. If $\eta_2 = x_2 - x_1 i \in \mathcal{F}_{0,\mathbb{H}_0}$, then

$$\begin{aligned}\nabla_0\eta_2 = \eta_2\nabla_0 &= \frac{\partial\left(x_2 - x_1 i\right)}{\partial x_1} + i\frac{\partial\left(x_2 - x_1 i\right)}{\partial x_2} + j_0\frac{\partial\left(x_2 - x_1 i\right)}{\partial x_3} + k_0\frac{\partial\left(x_2 - x_1 i\right)}{\partial x_4}\\ &= (-i) + i\left(1\right) + j_0\left(0\right) + k_0\left(0\right) = -i + i = 0;\end{aligned}$$

and if $\eta_3 = x_3 - x_1 j_0 \in \mathcal{F}_{0,\mathbb{H}_0}$, then

$$\begin{aligned}\nabla_0\eta_3 = \eta_3\nabla_0 = &\frac{\partial\left(x_3 - x_1 j_0\right)}{\partial x_1} + i\frac{\partial\left(x_3 - x_1 j_0\right)}{\partial x_2}\\ &+ j_0\frac{\partial\left(x_3 - x_1 j_0\right)}{\partial x_3} + k_0\frac{\partial\left(x_3 - x_1 j_0\right)}{\partial x_4}\\ &= (-j_0) + i\left(0\right) + j_0\left(1\right) + k_0\left(0\right) = -j_0 + j_0 = 0;\end{aligned}$$

and if $\eta_4 = x_4 - x_1 k_0 \in \mathcal{F}_{0,\mathbb{H}_0}$, then

$$\nabla_0 \eta_3 = \eta_3 \nabla_0 = \frac{\partial (x_4 - x_1 k_0)}{\partial x_1} + i \frac{\partial (x_4 - x_1 k_0)}{\partial x_2}$$
$$+ j_0 \frac{\partial (x_4 - x_1 k_0)}{\partial x_3} + k_0 \frac{\partial (x_4 - x_1 k_0)}{\partial x_4}$$
$$= (-k_0) + i (0) + j_0 (0) + k_0 (1) = -k_0 + k_0 = 0.$$

Therefore, the functions $\{\eta_l\}_{l=2}^{4}$ of (5.14) are 0-regular on $\mathbb{H}_0$. Therefore, these 0-regular functions $\{\eta_l\}_{l=2}^{4}$ are 0-harmonic by (5.13). $\square$

By (5.15), we consider the symmetrized product $\eta^{\mathbf{n}}$, for $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$, defined by

$$\eta^{\mathbf{n}} = \frac{1}{\mathbf{n}!} \left(\eta_2^{(n_1)} \times \eta_3^{(n_2)} \times \eta_4^{(n_3)} \right) \in \mathcal{F}_{0,\mathbb{H}_0}, \tag{5.16}$$

like (5.4), as in Sect. 4.5, where $(\times)$ is the symmetrized product (4.5.1), expressed to be (4.5.2).

Theorem 5.5. *The symmetrized products $\eta^{\mathbf{n}} \in \mathcal{F}_{0,\mathbb{H}_0}$ of* (5.16) *are* 0*-harmonic* 0*-regular functions on $\mathbb{H}_0$, for all $\mathbf{n} \in \mathbb{N}_0^3$, i.e.,*

$$\eta^{\mathbf{n}} \, are \, 0 - harmonic \, 0 - regular \, functions \, on \, \mathbb{H}_0, \forall \mathbf{n} \in \mathbb{N}_0^3. \tag{5.17}$$

Proof. The proof of the 0-regularity of $\eta^{\mathbf{n}}$ are similar to that of (4.5.11). In particular, if we let

$$e_1 = -i, \; e_2 = -j_0, \; \text{and} \, e_3 = -k_0,$$

and if we replace ∇_t to ∇_0, then the formula (4.5.8) is obtained similarly, and hence, the computation (4.5.10) holds by (4.5.9). i.e.,

$$x_1 \mathbf{n}! \nabla_0 \eta^{\mathbf{n}} = 0, \Longrightarrow \nabla_0 \eta^{\mathbf{n}} = 0.$$

So, the symmetrized products $\eta^{\mathbf{n}}$ are 0-regular, for $\mathbf{n} \in \mathbb{N}_0^3$. By the 0-regularity, $\eta^{\mathbf{n}}$ are 0-harmonic on $\mathbb{H}_0$, too, by (5.13). Therefore, the relation (5.17) holds. $\square$

The above theorem shows that all symmetrized products $\eta^{\mathbf{n}}$ of (5.16) are 0-harmonic 0-regular functions on $\mathbb{H}_0$ by (5.17).

Now, consider that if $\{\eta_2, \eta_3, \eta_4\} \subset \mathcal{F}_{0,\mathbb{H}_0}$ are the 0-harmonic 0-regular functions (5.14), then they are understood to be their images,

$$\eta_2 = (x_2 - x_1 i, 0), \; \eta_3 = (x_3, -x_1), \; \eta_4 = (x_4, -x_1 i),$$

as elements of the 0-scaled hypercomplex ring $\mathbb{H}_0$, having their norms,

$$\|\eta_2\|_0 = \sqrt{\left| |x_2 - x_1 i|^2 - 0 \, |0|^2 \right|} = \sqrt{x_2^2 + (-x_1)^2} = \sqrt{x_2^2 + x_1^2},$$

$$\|\eta_3\|_0 = \sqrt{\left| |x_3|^2 - 0 \, |-x_1|^2 \right|} = \sqrt{|x_3^2|} = |x_3|,$$

and

$$\|\eta_4\|_0 = \sqrt{\left| |x_4|^2 - 0 \, |-x_1 i|^2 \right|} = \sqrt{|x_4^2|} = |x_4|. \tag{5.18}$$

Lemma 5.6. *Let* $\eta^{\mathbf{n}} \in \mathcal{F}_{0,\mathbb{H}_t}$ *be a function* (5.16) *for* $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$. *Then*

$$\|\eta^{\mathbf{n}}\|_0 \leq \left(\sqrt{x_1^2 + x_2^2}^{\,n_1}\right)\left(|x_3|^{n_2}\right)\left(|x_4|^{n_3}\right). \tag{5.19}$$

Proof. By (5.16), for any $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$,

$$\|\eta^{\mathbf{n}}\|_0 = \left\| \frac{1}{\mathbf{n}!}\left(\eta_2^{(n_1)} \times \eta_3^{(n_2)} \times \eta_4^{(n_3)}\right) \right\|_0$$

$$= \left\| \frac{1}{(n_1!)(n_2!)(n_3!)} \left(\frac{(n_1!)(n_2!)(n_3!)}{n!} \sum_{\sigma \in S_n} h_{\sigma(1)} h_{\sigma(2)} \ldots h_{\sigma(n)} \right) \right\|_0$$

where $h_{\sigma(l)} \in \{\eta_2, \eta_3, \eta_4\}$, for all $\sigma \in S_n$, with $n = n_1 + n_2 + n_3 \in \mathbb{N}$

$$= \|\eta_2^{n_1} \eta_3^{n_2} \eta_4^{n_3}\|_0 \leq \|\eta_2\|_0^{n_1} \, \|\eta_3\|_0^{n_2} \, \|\eta_4\|_0^{n_3}$$

by (4.5.5)

$$= \sqrt{x_1^2 + x_2^2}^{\,n_1} \, |x_3|^{n_2} \, |x_4|^{n_3},$$

by (5.18). Therefore, the boundedness condition (5.19) holds. □

The above lemma shows that, for any arbitrarily fixed $\mathbf{n} \in \mathbb{N}_0^3$, the corresponding function $\eta^{\mathbf{n}}$ is bounded by (5.19).

Theorem 5.7. *Let* $f \in \mathcal{F}_{0,U}$ *be a* $\mathbb{R}$*-differentiable function, where* U *is an open connected subset of* $\mathbb{H}_0$ *containing* $0 \in \mathbb{H}_0$. *If* f *is* $\mathbb{R}$*-analytic at* $0 \in U$, *then*

$$f \text{ is left } 0\text{-regular on } U,$$

if and only if

$$f = f(0) + \sum_{\mathbf{n} \in \mathbb{N}^3} \eta^{\mathbf{n}} f_{\mathbf{n}}, \tag{5.20}$$

with

$$f_{\mathbf{n}} = \frac{1}{\mathbf{n}!} \frac{\partial^{n_1+n_2+n_3} f}{\partial x_2^{n_1} \partial x_3^{n_2} \partial x_4^{n_3}}(0), \ \forall \mathbf{n} \in \mathbb{N}^3.$$

Proof. If $f = f(0) + \sum_{\mathbf{n} \in \mathbb{N}^3} \eta^{\mathbf{n}} f_{\mathbf{n}}$, then it is left 0-regular, by the 0-harmonic-0-regularity (5.17) of $\left\{\eta^{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^3\right\}$.

Suppose f is left 0-regular in $\mathcal{F}_{0,U}$, satisfying $\nabla_0 f = 0$. Then

$$f(w) - f(0) = \sum_{n=2}^{4} (\eta_n(w))((R_n f)(w)),$$

where

$$(R_n f)(w) = \int_0^1 \frac{\partial f(tw)}{\partial x_n} dt, \ \forall n = 2, 3, 4, \tag{5.21}$$

in the $\mathbb{H}_t$-variable $w = x_1 + x_2 i + x_3 j_t + x_4 k_t$, with $x_1, x_2, x_3, x_4 \in \mathbb{R}$. By iterating (5.21) under the $\mathbb{R}$-analyticity assumption, one can get the function in (5.20), by (5.19). □

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Declarations

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References

[1] Alpay, D., Cho, I.: Operators Induced by Certain Hypercomplex Systems. Opuscula Math. **43**(3), 275–333 (2023)

[2] Alpay, D., Cho, I.: Dynamical Systems of Operators Induced by Scaled Hypercomplex Rings. Adv. Appl. Clifford Alg. **33**, 33 (2023). https://doi.org/10.1007/s00006-023-01272-0

[3] Alpay, D., Cho, I.: On Scaled Hyperbolic Numbers Induced by Scaled Hypercomplex Rings, Pure Appl. Funct. Anal. (2023) To appear

[4] Alpay, D., Luna-Elizarraras, M.E., Shapiro, M., Struppa, D.C.: Gleason's Problem, Rational Functions and Spaces of Left-regular Functions: The Split-Quaternioin Settings. Israel J. Math. **226**, 319–349 (2018)

[5] Alpay, D., Paiva, I.L., Struppa, D.C.: A General Setting for Functions of Fueter Variables: Differentiability, Rational Functions, Fock Module and Related Topics. Israel J. Math. **236**, 207–246 (2020)

[6] Alpay, D., Shapiro, M.: Problème de Gleason et Interpolation pour let Fonctions hyper-analytiques. C. R. Math. Acad. Sci. Paris, **335** (2022)

[7] Curtis, M.L.: Matrix Groups, ISBN: 0-387-90462-X. Published by Springer-Verlag, NY (1979)
[8] Delanghe, R.: On Regular-Analytic Functions with Values in a Clifford Algebra. Math. Ann. **425**, 91–111 (1970)
[9] Frenkel, I., Libine, M.: Split Quaternionic Analysis and Separation of the Series for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C}) / SL(2, \mathbb{R})$. Adv. Math. **228**, 678–763 (2011)
[10] Halmos, P.R.: Book, Linear Algebra Problem., ISBN: 978-0-88385-322-1,: Published by Math. Assoc, Amer (1995)
[11] Halmos, P. R.: Hilbert Space Problem Book, ISBN: 978-038-790685-0, (1982) Published by Springer-Verlag (NY)
[12] Kravchenko, V.: Applied Quaternionic Analysis, ISBN: 3-88538-228-8, (2003) Published by Heldemann Verlag
[13] Kobayashi, M.: Hyperbolic Hopfield Neural Networks with Directional Multistate Activation Function. Neurocomputing **275**, 2217–2226 (2018)
[14] Libine, M.: An Invitation to Split Quaternionic Analysis, in Hypercomplex Analysis and Applications, Trends in Math., Birkhauser / Springer, Basel AG, Basel, 161 - 180 (2011)
[15] Rodman, L.: Topics in Quaternion Linear Algebra, ISBN:978-0-691-16185-3,: Published by Prinston Univ. Press, NJ (2014)
[16] Rozenfeld, B. A.: The History of non-Eucledean Geometry: Evolution of the Concept of a Geometric Spaces, ISBN: 978-038-796458-4, (1988) Published by Springer
[17] R. Speicher, Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory, Amer. Math. Soc., Memoire, ISBN:978-0-8218-0693-7: Published by Amer. Math, Soc (1998)
[18] Sudbery, A., Analysis, Quaternionic: Published by Math. Cambridge Philosop. Soc, Proc (1998). https://doi.org/10.1017/s0305004100053638
[19] Vince, J. A.: Geometric Algebra for Computer Graphics, ISBN: 978-1-84628-996-5, (2008) Published by Springer
[20] Voiculescu, D.V., Dykema, K.J., Nica, A.: Variables, Free Random, ISBN:978-0-8218-1140-5,: Published by Amer. Math, Soc (1992)
[21] Voight, J.: Quaternion Algebra, Available on http://math.dartmouth.edu/~jvoight/quat-book.pdf, (2019) Dept. of Math., Dartmouth Univ

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