Regular Ideals, Ideal Intersections, and Quotients

Jonathan H. Brown, Adam H. Fuller, David R. Pitts
and Sarah A. Reznikoff

Abstract. Let $B \subseteq A$ be an inclusion of C^*-algebras. We study the relationship between the regular ideals of B and regular ideals of A. We show that if $B \subseteq A$ is a regular C^*-inclusion and there is a faithful invariant conditional expectation from A onto B, then there is an isomorphism between the lattice of regular ideals of A and invariant regular ideals of B. We study properties of inclusions preserved under quotients by regular ideals. This includes showing that if $D \subseteq A$ is a Cartan inclusion and J is a regular ideal in A, then $D/(J \cap D)$ is a Cartan subalgebra of A/J. We provide a description of regular ideals in the reduced crossed product of a C*-algebra by a discrete group.

1. Introduction

An inclusion $B \subseteq A$ of C^*-algebras has the *ideal intersection property* if every non-trivial ideal of A has non-trivial intersection with B. The ideal intersection property is useful for obtaining structural results for the inclusion. Here are a few of the many examples.

(i) When B is a maximal abelian subalgebra of A, the ideal intersection property is a key ingredient for establishing the Cuntz–Krieger Uniqueness theorems for graph C^*-algebras (see for example [35, Theorem 3.13]) and for groupoids [10, Theorem 3.1]. Further work on the ideal intersection property in groupoid C*-algebras has been carried out in [6] and [26].

(ii) When there is an action of $\mathbb{T}$ on A, and B is the fixed point algebra, the ideal intersection property of A is used to establish the gauge-invariant uniqueness theorems for graph algebras [22, Theorem 2.3] and Cuntz-Pimsner algebras [23, Theorem 6.2] among others.

(iii) If G is a discrete group acting topologically freely on a locally compact Hausdorff space X, the ideal intersection property for $C(X) \subseteq C(X) \rtimes_r G$ is used to obtain the simplicity result of [2, Theorem 2].

(iv) The ideal intersection property has been used to characterize inclusions for which every pseudo-expectation is faithful [40, Theorem 3.5].

(v) More recently, Pitts [38, Theorem 5.2] used the ideal intersection property to characterize the existence of Cartan envelopes.

The ideal intersection property holds for certain crossed products. Necessary and sufficient conditions for $C(X) \subseteq C(X) \rtimes_r G$ to have the ideal intersection property are given in [24]. Sierakowski [46] shows that $A \subseteq A \rtimes_r G$ has the ideal intersection property for an exact action of discrete group G on C^*-algebra A whose induced action on $\mathrm{Prim}(A)$ is essentially free. Kennedy and Schafhauser [27] further characterize when $A \subseteq A \rtimes_r G$ has the ideal intersection property in terms of outerness and a cohomological obstruction.

Unfortunately, the ideal intersection property does not pass, in general, to quotients (see [41, Example 4.8] or Example 4.6 below). In the setting of amenable groupoid C^*-algebras, that is the inclusion $C_0(G^{(0)}) \subseteq C^*(G)$, the ideal intersection property passes to quotients if and only if there is a correspondence between the lattice of ideals of $C_0(G^{(0)})$ and $C^*(G)$ [7, Corollary 5.9] (for one direction see [43, Corollary 4.9]). Restricting attention to a class of inclusions where the ideal intersection property passes to quotients has been quite productive in the study of ideals of these algebras. For example, Kumjian et al. [29] introduce condition (K) so they could use [43, Corollary 4.9] to study the ideals for this restricted class of graph C^*-algebras. Indeed, for algebras of row-finite directed graphs $C^*(E)$ whose underlying graph E satisfies condition (K), the inclusion $D \subseteq C^*(E)$, where D is the (abelian) algebra generated by the range projections of partial isometries corresponding to finite paths in E, has the property that the ideal intersection property passes to quotients.

Rather than restricting the types of inclusions considered, the paper [9] takes a different tack, instead limiting the types of ideals to *regular ideals*: a main result of [9] shows that the ideal intersection property passes to quotients of graph algebras by regular ideals.

For a commutative C^*-algebra B, regular ideals in B correspond exactly to regular open sets in $\widehat{B}$. More generally, for an arbitrary C^*-algebra B, let $\overline{B}$ be the monotone completion of B and let $Z(\overline{B})$ be the center of $\overline{B}$. Hamana shows that the lattice of regular open sets in $\mathrm{Prim}(B)$ is isomorphic to the projection lattice of $Z(\overline{B})$, which in turn is isomorphic to the lattice of all regular ideals of B, see [20, Lemma 1.4 and Theorem 1.5]. Hamana concludes in [20, p. 526, Remark (b)] that if $I(B)$ is the injective envelope of B, and A is a C^*-algebra with $B \subseteq A \subseteq I(B)$, then the map $J \mapsto J \cap B$ gives an isomorphism of the Boolean algebra of regular ideals of A onto the Boolean algebra of regular ideals of B. This conclusion is similar to our Theorem 3.24 below, but we consider inclusions of the form $B \subseteq A$ having certain regularity properties and a faithful invariant conditional expectation of A onto B instead of assuming that A lies between B and its injective envelope.

One of the main results of the present paper is Theorem 4.2, which shows that if $B \subseteq A$ is an inclusion having a faithful invariant conditional expectation of A onto B, then the ideal intersection property is preserved under quotients by regular ideals. This is a far reaching generalization of the main result in [9], as many inclusions of interest, such as reduced crossed products by discrete groups, come equipped with a faithful invariant conditional expectation. Along the way we give a detailed analysis of regular ideals, culminating in Theorem 3.24, which shows that if $B \subseteq A$ is a regular inclusion with the ideal intersection property and $E : A \to B$ is a faithful invariant conditional expectation, then $J \to J \cap B$ is a one-to-one map from the regular ideals of A to the regular ideals of B.

Establishing that an inclusion is a Cartan inclusion has become important from the lens of classification. Indeed, Barlak and Li [4, Theorem 1.1] show that when A is a separable and nuclear C^*-algebra containing the Cartan MASA B, then A satisfies the UCT. We show in Theorem 4.8 that quotients of Cartan embeddings by regular ideals remain Cartan.

One of the interesting aspects of our work is that the key to many of our results is the *regular ideal intersection property*: $J \cap B = \{0\} \implies J = \{0\}$ for all *regular* ideals J in A. In Sect. 7, we show that for a very large class of examples, the regular ideal intersection property and the ideal intersection property are equivalent.

After we circulated an earlier version of this paper, Exel [13] proved a generalization of Theorem 3.24 using weaker hypotheses than are assumed here. In particular, Exel does not assume the existence of an invariant faithful conditional expectation. Instead Exel assumes that $B \subseteq A$ satisfies an invariance axiom; see [13] for full details.

This paper is organized as follows. We begin with a short section of preliminaries where we introduce regular ideals. In Sect. 3 we analyze the relationship between regular ideals of B and A where $B \subseteq A$ is a regular inclusion. This culminates in Theorem 3.24; this result gives settings in which the Boolean algebras of regular ideals in A are isomorphic to the Boolean algebra of regular and invariant ideals in B. In Sect. 4, we prove our main theorem, Theorem 4.2, which shows that in the presence of a faithful, invariant conditional expectation, the ideal intersection property is inherited by quotients of regular ideals. We use this result to show that the Cartan property passes to quotients by regular ideals. We then specialize to C^*-algebras of exact groupoids in Sect. 5. In particular, Theorem 5.6 gives an explicit description of the regular ideals of the (reduced) C^*-algebra of a twisted exact groupoid G when $C_0(G^{(0)})$ has the ideal intersection property. Section 6 explores some applications of our work, including applications to graph C^*-algebras and reduced crossed products by discrete groups. In Sect. 7, we give classes of regular inclusions for which the regular ideal intersection property implies the ideal intersection property.

2. Preliminaries on Regular Ideals

The focus of this paper is the family of regular ideals in a C^*-algebra. We will recall the definition from [20] after we introduce a piece of notation. Let A be a C*-algebra and let $X \subseteq A$. Denote by $X^\perp$ the set

$$X^\perp = \{a \in A \colon ax = xa = 0 \text{ for all } x \in X\}.$$

If B is a C*-algebra such that $X \subseteq B \subseteq A$ we will write $X^{\perp_B}$ to denote the set determined by the operation restricted to B:

$$X^{\perp_B} = X^\perp \cap B = \{b \in B \colon bx = xb = 0 \text{ for all } x \in X\}.$$

We can now define what it means for an ideal to be *regular*.

Definition 2.1. Let A be a C*-algebra. If $J \subseteq A$ is a subalgebra satisfying $aJ \cup Ja \subseteq A$ for all $a \in A$, we will call J an *algebraic ideal* of A. If $J \subseteq A$ is a closed algebraic ideal we will call J an *ideal* of A.

We use the notation $J \trianglelefteq A$ to say that $J \subseteq A$ is an ideal of A.

Definition 2.2. An ideal $J \trianglelefteq A$ is a *regular ideal* in A if $J = J^{\perp\perp}$.

Hamana [20] gives two characterizations of the regular ideals in a C*-algebra A: one in terms of the topology on the primitive ideal space $\mathrm{Prim}(A)$, and one in terms the monotone closure $\overline{A}$ of A. We end this section with his characterization involving the primitive ideal space, as we require it in the sequel.

Let X be a topological space. For an open set $U \subseteq X$, define

$$U^\perp = (X \backslash U)^\circ.$$

That is, $U^\perp$ is the interior of the complement of U. Recall that an open set $U \subseteq X$ is *regular* if $U = (\overline{U})^\circ$, i.e. U is the interior of its closure. Equivalently, an open set U is regular if $U = U^{\perp\perp}$. The regular open sets of X form a complete Boolean algebra with operations:

(i) $U \wedge V = U \cap V$;
(ii) $U \vee V = (U \cup V)^{\perp\perp}$; and
(iii) $\neg U = U^\perp$;

see, e.g. [19] or [16].

Remark 2.3. The ideal $C_0(U) \subseteq C_0(X)$ is regular if and only if U is a regular open set in X.

Let A be a C^*-algebra and let $\mathrm{Prim}(A)$ be the primitive ideal space. For $R \subseteq A$ define

$$\mathrm{hull}(R) = \{P \in \mathrm{Prim}(A) \colon R \subseteq P\},$$

and for $S \subseteq \mathrm{Prim}(A)$ define

$$\ker(S) = \bigcap_{P \in S} P.$$

Endow $\mathrm{Prim}(A)$ with the usual hull-kernel topology. That is, if $S \subseteq \mathrm{Prim}(A)$ then the closure $\overline{S}$ is given by $\overline{S} = \mathrm{hull\,ker}(S)$. The map $I \mapsto \mathrm{Prim}(A) \backslash \mathrm{hull}(I)$ gives a one-to-one correspondence between ideals $I \trianglelefteq A$ and open subsets of

$\text{Prim}(A)$ [34, Theorem 5.2.7]. When this map is restricted to the regular ideals of A, Hamana establishes the following result.

Proposition 2.4. [c.f. [20, Lemma 1.4]] *Let A be a C^*-algebra. Then the collection of all regular ideals in A forms a complete Boolean algebra with meet, join and negation given by*

  (i) $J \wedge K := J \cap K$;
  (ii) $J \vee K := \langle J \cup K \rangle^{\perp\perp}$;
 (iii) $\neg J := J^{\perp}$.

Furthermore, the map $I \mapsto \text{Prim}(A) \backslash \text{hull}(I)$ on regular ideals of a C^*-algebra A gives a complete Boolean algebra isomorphism from the regular ideals of A to the regular open sets of $\text{Prim}(A)$.

3. Regular Ideals in Regular C^*-Inclusions

Definition 3.1. Let A and B be C*-algebras such that $B \subseteq A$. If B contains an approximate unit for A, we say that $B \subseteq A$ is an *inclusion of* C*-*algebras*. We will sometimes use the notation (A, B) for an inclusion of C^*-algebras (we write the larger algebra first).

Remark 3.2. The condition that B contain an approximate unit for A is sometimes automatic. For example, when B is maximal abelian and $\text{span}\{n \in A : nBn^* \cup n^*Bn \subseteq B\}$ is dense in A, [39, Theorem 2.6] shows that every approximate unit for B is an approximate unit for A.

If $B \subseteq A$ is an inclusion of C*-algebras, we want to know to what extent the regular ideals of A determine the regular ideals of B, and vice versa. In general, one should not expect any strong relationship. However, under some additional hypotheses, we show in Theorem 3.24 that there is an injective map from the regular ideals of A to the invariant regular ideals of B. We will use this map in our main results on quotients by regular ideals, Theorem 4.2 and Theorem 4.8.

We use the following notation extensively. For $I \subseteq A$ we denote the ideal generated by I in A by $\langle I \rangle_A$. If there is no chance of confusion as to where the ideal should lie we will simply write $\langle I \rangle$ in place of $\langle I \rangle_A$.

The following elementary facts will be useful.

Lemma 3.3. *If $J \subseteq A$ is an algebraic ideal, then $J^{\perp} \trianglelefteq A$ is a regular ideal.*

Lemma 3.4. *Let A be a C^*-algebra. Suppose I, $J \subseteq A$ are algebraic ideals satisfying $J^{\perp} \subseteq I$. Then $J^{\perp} = I$ or $I \cap J \neq \{0\}$.*

Proof. Suppose $J^{\perp} \neq I$. Take $x \in I \backslash J^{\perp}$. Since $x \notin J^{\perp}$, we have $xJ \neq \{0\}$ or $Jx \neq \{0\}$. Hence $I \cap J \neq \{0\}$. $\qquad\square$

We now impose some natural extra hypotheses on the inclusions we study. We first show that if there is a faithful conditional expectation from A onto B then $J \cap B$ is a regular ideal of B whenever J is a regular ideal of A.

Lemma 3.5. *Let $B \subseteq A$ be an inclusion of C^*-algebras. Suppose $E\colon A \to B$ is a faithful conditional expectation. Then for any $J \trianglelefteq A$ we have*

$$J^\perp \cap B = E(J)^{\perp_B}.$$

Proof. Take $d \in E(J)^{\perp_B}$ and $a \in J$. Then

$$E(d^*a^*ad) = d^*E(a^*a)d = 0.$$

Since E is faithful, it follows that $ad = 0$. Similarly $da = 0$. Hence $d \in J^\perp$.

Suppose now that $d \in J^\perp \cap B$ and $a \in J$. Then $dE(a) = E(da) = 0$, and $E(a)d = E(ad) = 0$. Hence $d \in E(J)^{\perp_B}$. $\square$

Remark 3.6. While Lemma 3.5 is stated for an ideal $J \trianglelefteq A$, the conclusion holds for any subset $J \subseteq A$.

Proposition 3.7. *Let $B \subseteq A$ be an inclusion of C^*-algebras. Suppose $E\colon A \to B$ is a faithful conditional expectation. If $J \trianglelefteq A$ is a regular ideal of A, then $J \cap B$ is a regular ideal of B.*

Proof. Let $J \trianglelefteq A$ be a regular ideal. Then, by regularity of J and Lemma 3.5, we get

$$J \cap B = J^{\perp\perp} \cap B = E(J^\perp)^{\perp_B}.$$

It follows now by Lemma 3.3 that $J \cap B$ is a regular ideal in B. $\square$

We recall the following definitions.

Definition 3.8. Let $B \subseteq A$ be an inclusion of C*-algebras. An element $n \in A$ is a *normalizer of B* if $nBn^* \subseteq B$ and $n^*Bn \subseteq B$. We denote the set of all normalizers of B in A by $\mathcal{N}(A, B)$. If

$$A = \overline{\text{span}}\mathcal{N}(A, B)$$

then we say B is *regular in A*, or $B \subseteq A$ is a *regular inclusion of* C*-*algebras*.

Remark 3.9. We are now using the term *regular* in two different senses: one refers to ideals and the other to inclusions. The context will make clear which sense we intend. This unfortunate development is the result of historical naming conventions. In the first sense, *regular ideal*, the term goes back to at least Hamana [20], and seems to be derived from the notion of regular open sets in topology. The second, *regular inclusion*, goes back to at least to both Vershik's and Feldman and Moore's early work on Cartan inclusions of von Neumann algebras, see [51] and [15].

Remark 3.10. Let $B \subseteq A$ be an inclusion of C*-algebras. Note that if $n \in \mathcal{N}(A, B)$, then $n^*n \in B$. This follows from the assumption that there is an approximate unit for B that also an approximate unit for A.

Definition 3.11. Let $B \subseteq A$ be a regular inclusion of C*-algebras. Let $N \subseteq \mathcal{N}(A, B)$ be a *-semigroup with dense span in A. If $K \trianglelefteq B$ we say that K is an *N-invariant ideal* if $nKn^* \subseteq K$ for all $n \in N$. When $N = \mathcal{N}(A, B)$ we will simply say that K is an *invariant ideal*.

Remark 3.12. If $K = I \cap B$ for some ideal I in A then K is invariant: indeed, for a normalizer n, $nIn^* \subseteq I$ since I is an ideal and $nBn^* \subseteq B$ since n is a normalizer.

Definition 3.13. Let $B \subseteq A$ be a regular inclusion of C*-algebras. Let $N \subseteq \mathcal{N}(A, B)$ be a *-semigroup with dense span in A. If $E \colon A \to B$ is a conditional expectation, we say that E is an *N-invariant conditional expectation* if $nE(a)n^* = E(nan^*)$ for all $a \in A$ and $n \in N$. When E is $\mathcal{N}(A, B)$-invariant we will simply say that E is an *invariant conditional expectation*.

We have introduced N-invariance for ideals and conditional expectations because in some settings it is unclear whether a given ideal or conditional expectation is invariant under the entire *-semigroup of normalizers. Nevertheless, in many situations, there is a natural *-semigroup N of normalizers such that span N is dense in A, under which an ideal or conditional expectation is invariant. Corollary 3.20 will show that when there is an N-invariant faithful conditional expectation, the N-invariant ideals will necessarily be invariant. This occurs in the crossed product setting.

Example 3.14. Let Γ be a discrete group acting via α on a C*-algebra B and let $A = B \rtimes_r \Gamma$. Let $N = \{b\delta_s \colon b \in B, \ s \in \Gamma\}$. Then N is a *-semigroup of normalizers with dense span in $B \rtimes_r \Gamma$. An ideal K in B is N-invariant in the sense of Definition 3.11 if and only if $\alpha_s(K) = K$ for all $s \in \Gamma$.

The map $E \colon \sum_{i=1}^k b_{s_i}\delta_{s_i} \mapsto b_e\delta_e$ extends to a faithful conditional expectation from $B \rtimes_r \Gamma$ to B, [11, Proposition 4.1.9]. The conditional expectation E is N-invariant.

Let us examine the notion of faithful, invariant conditional expectations for some more classes of examples.

Examples 3.15.   (i) Suppose A is a unital C*-algebra and $B = \mathbb{C}I$. Then $\mathcal{N}(A, B)$ is the collection of all scalar multiples of unitary operators in A, so (A, B) is a regular inclusion. Here, the collection of all conditional expectations may be identified with the state space of A: for a state τ on A, the map E_τ given by $E_\tau(a) = \tau(a)I$ is a conditional expectation. Note that E_τ is faithful precisely when τ is a faithful; and E_τ is invariant precisely when τ is tracial. Thus the existence of a faithful or invariant expectation is not guaranteed. In this setting, there are many conditional expectations, and there is no connection between faithfulness and invariance.

(ii) Suppose $B \subseteq A$ is a regular inclusion with B maximal abelian in A.
    (a) The existence of a conditional expectation is not automatic [37, Example 1.2].
    (b) When a conditional expectation $E : A \to B$, exists, it is always unique and invariant. If A is unital, uniqueness follows from [37,

Theorem 3.5] and invariance from [37, Proposition 3.14]. When A is not unital, consider the unitizations, $\tilde{B} \subseteq \tilde{A}$. By [39, Corollary 2.4(i)], $\mathcal{N}(A, B) \subseteq \mathcal{N}(\tilde{A}, \tilde{B})$. Thus, $\tilde{B} \subseteq \tilde{A}$ is a regular inclusion, and $\tilde{B}$ is maximal abelian in $\tilde{A}$. The unitization $\tilde{E}$ of E (that is, $A \times \mathbb{C} \ni (a, \lambda) \mapsto (E(a), \lambda) \in \tilde{B}$) is therefore an conditional expectation which is invariant and unique. Since $\mathcal{N}(A, B) \subseteq \mathcal{N}(\tilde{A}, \tilde{B})$, it follows that E is also invariant and unique.

(c) It is possible to construct a discrete dynamical system (Γ, X, α) with Γ acting freely on a compact Hausdorff space X where the full and reduced crossed products of $C(X)$ by Γ differ; in this case, $C(X)$ is maximal abelian in the full crossed product $C(X) \rtimes_f \Gamma$, yet the conditional expectation onto $C(X)$ is not faithful, [14, Theorem 3.1 and Remark 3.2].

Definition 3.16. Let $B \subseteq A$ be an inclusion of C*-algebras. We say that B has the *ideal intersection property in A* if whenever $J \trianglelefteq A$ is a non-zero ideal, $J \cap B \neq \{0\}$. We say that B has the *regular ideal intersection property in A* if whenever $J \trianglelefteq A$ is a non-zero regular ideal, $J \cap B \neq \{0\}$. We will also say that the inclusion (A, B) has the ideal intersection property or the regular ideal intersection property.

Other authors use different terminology for the ideal intersection property. For example, an inclusion with the ideal intersection property is called C^*-*essential* in [40] and is said to *detect ideals* in [32]. To the best of our knowledge, the regular ideal intersection property has not appeared elsewhere.

Remark 3.17. It is clear that if B has the ideal intersection property in A then it has the regular ideal intersection property. We will show in Sect. 7 that for a large class of examples the regular ideal intersection property implies the ideal intersection property. Example 7.1 gives an example of a regular inclusion $B \subseteq A$ with the regular ideal intersection property but not the ideal intersection property.

Notation 3.18. Let $B \subseteq A$ be a regular inclusion of C*-algebras with a faithful conditional expectation $E \colon A \to B$. If $K \trianglelefteq B$ is an ideal, denote by J_K the set

$$J_K = \{a \in A \colon E(a^*a) \in K\}.$$

Our aim in the remainder of this section is to prove Theorem 3.24, which shows that when the regular inclusion (A, B) has the regular ideal intersection property and E is faithful and invariant, then $K \mapsto J_K$ defines a bijection between invariant regular ideals of B and regular ideals of A. The following gives a monomorphism of the invariant regular ideals of B into the regular ideals of A.

Proposition 3.19. *Let $B \subseteq A$ be a regular inclusion of C*-algebras, let N be a $*$-semigroup of normalizers with dense span in A, and let $E \colon A \to B$ be a faithful N-invariant conditional expectation. The following statements hold.*

(i) *If $K \trianglelefteq B$ is an N-invariant ideal, then J_K is an ideal in A such that*

$$J_K \cap B = K \subseteq E(J_K).$$

(ii) *If $K \trianglelefteq B$ is an invariant regular ideal, then J_K is a regular ideal satisfying*

$$J_K \cap B = K = E(J_K) \quad and \quad J_K^\perp = J_{K^{\perp_B}}.$$

Proof. (i) Let $K \trianglelefteq B$ be an N-invariant ideal. Take $a \in J_K$ and $b \in A$. Then $a^*b^*ba \leq \|b\|^2 a^*a$ and hence

$$E((ba)^*ba) = E(a^*b^*ba) \leq \|b\|^2 E(a^*a) \in K$$

since conditional expectations are (completely) positive maps. As ideals in C*-algebras are hereditary it follows that $E((ab)^*(ab)) \in K$ and hence $ab \in J_K$. Thus J_K is a left-ideal of A. Now take any $a \in J_K$ and any $n \in N$. Then $E((an)^*(an)) = E(n^*a^*an) = n^*E(a^*a)n \in K$, as E is an N-invariant conditional expectation and K is an N-invariant ideal. Hence $an \in J_K$. As the span of N is dense in A, it follows that J_K is also a right ideal. That $J_K \cap B = K \subseteq E(J_K)$ follows from the definition, so (i) holds.

(ii) Now suppose that $K \trianglelefteq B$ is an N-invariant regular ideal. Note that $E(J_K)$ is an algebraic ideal in B since J_K is an ideal in A. Let us show $K = E(J_K)$.

If $K \neq E(J_K)$ then, by Lemma 3.4, $E(J_K) \cap K^{\perp_B} \neq \{0\}$. Hence there is an $0 \neq a \in J_K$ such that $E(a) \in K^{\perp_B}$. Thus

$$K \ni E(a^*a) \geq E(a)^*E(a) \in K^{\perp_B}.$$

Since K is hereditary, $E(a)^*E(a) \in K \cap K^{\perp_B}$ and hence $E(a) = 0$, contrary to choice of a. Thus $E(J_K) = K$.

It remains to show that J_K is a regular ideal. Let $J_{K^\perp} := \{a \in A : E(a^*a) \in K^{\perp_B}\}$. It suffices to show that $J_K = J_{K^\perp}^\perp$. Take any $a \in J_K$ and $b \in J_{K^\perp}$. Since J_K is a right ideal and $J_{K^\perp}$ is a left ideal, $E((ab)^*ab) \in K \cap K^{\perp_B} = \{0\}$. Since E is faithful it follows that $ab = 0$. Similarly, $ba = 0$. Hence $J_K \subseteq J_{K^\perp}^\perp$.

Now take any $a \in J_{K^\perp}^\perp$ and $b \in K^{\perp_B} \subseteq J_{K^\perp}$. Then $E(a)b = E(ab) = 0$. Thus $E(J_{K^\perp}^\perp) \subseteq (K^{\perp_B})^{\perp_B} = K$; that is, $J_{K^\perp}^\perp \subseteq J_K$. Combining the two inclusions we get $J_K = J_{K^\perp}^\perp$ and hence J_K is a regular ideal. $\square$

Let $B \subseteq A$ be a regular inclusion of C*-algebras, let N be a *-semigroup of normalizers with dense span in A. The following corollary shows that, in the presence of an N-invariant faithful conditional expectation $E\colon A \to B$, the N-invariant ideals of B and the $\mathcal{N}(A,B)$-invariant ideals of B coincide.

Corollary 3.20. *Let $B \subseteq A$ be a regular inclusion of C*-algebras, let N be a *-semigroup of normalizers with dense span in A, and let $E\colon A \to B$ be a faithful N-invariant conditional expectation. An ideal $K \trianglelefteq B$ is N-invariant if and only if it is invariant.*

Proof. Any invariant ideal of B is N-invariant. Suppose that $K \trianglelefteq B$ is N-invariant. By Proposition 3.19 (i), $K = J_K \cap B$. Hence K is an invariant ideal by Remark 3.12. $\square$

We now return to our study of regular ideals.

Corollary 3.21. *Let $B \subseteq A$ be a regular inclusion of C*-algebras, N be a $*$-semigroup of normalizers with dense span in A, and let $E \colon A \to B$ be a faithful N-invariant conditional expectation. Further suppose that B has the regular ideal intersection property in A. If $K \trianglelefteq B$ is an invariant regular ideal, then $\langle K \rangle_A^{\perp\perp} = J_K$.*

Proof. Since $\langle K \rangle_A^{\perp\perp} \trianglelefteq A$ is the smallest regular ideal containing K, Proposition 3.19(ii) gives $\langle K \rangle_A^{\perp\perp} \subseteq J_K$. Let $L := J_K \cap \langle K \rangle_A^{\perp}$ and let $b \in L \cap B$. By Proposition 3.19, $J_K \cap B = K$, so $b \in K$. But $bK = 0$ because $b \in \langle K \rangle_A^{\perp}$. Thus $b = 0$, so $L \cap B = \{0\}$. By Proposition 2.4, L is a regular ideal in A. Since (A, B) has the regular ideal intersection property, we conclude that $L = \{0\}$, that is, $\langle K \rangle_A^{\perp\perp} = J_K$. □

Proposition 3.22. *Let $B \subseteq A$ be a regular inclusion, where B has the regular ideal intersection property in A. Let $E \colon A \to B$ be a faithful conditional expectation. If $J \trianglelefteq A$ is a regular ideal, then $J \cap B = E(J)$.*

Proof. Let $I_1 = J \cap B$ and $I_2 = E(J)$. Then I_1 is an ideal of B, I_2 is an algebraic ideal of B, and $I_1 \subseteq I_2$. The ideal $I_1 \neq \{0\}$ since B has the regular ideal intersection property in A; the algebraic ideal $I_2 \neq \{0\}$ since E is a faithful conditional expectation. Further, I_1 is an invariant ideal, and it is a regular ideal by Proposition 3.7.

Assume $I_1 \neq I_2$. Let $I_3 = I_2 \cap I_1^{\perp_B}$. Then I_3 is an algebraic ideal. Further, $I_3 \neq \{0\}$ by Lemma 3.4. We show

$$\langle I_3 \rangle_A \subseteq (\langle I_1 \rangle_A)^{\perp}. \tag{3.23}$$

First notice that since B is regular in A, for any algebraic ideal $I \subseteq B$, $\langle I \rangle_A$ is the closed linear span of $\{n_1 b n_2 : b \in I, \ n_i \text{ normalizers}\}$.

So let $b \in I_3$, $c \in I_1$ and $\{n_i\}_{i=1}^4 \subseteq \mathcal{N}(A, B)$. Using the facts that $n_4^* n_4 \in B$, $I_1 \trianglelefteq B$ is invariant and $I_3 \subseteq I_1^{\perp_B}$, we obtain

$$((n_1 b n_2)(n_3 c n_4))((n_1 b n_2)(n_3 c n_4))^* = n_1 b(n_2 n_3 c n_4 n_4^* c^* n_3^* n_2^*) b^* n_1^* = 0,$$

so $(n_1 b n_2)(n_3 c n_4) = 0$. Similarly, $(n_3 c n_4)(n_1 b n_2) = 0$. So

$$n_1 b n_2 \in \overline{\text{span}}\{ncm : n, m \in \mathcal{N}(A, B), c \in I_1\}^{\perp} = (\langle I_1 \rangle_A)^{\perp}.$$

The inclusion (3.23) follows.

Since $I_3 \subseteq I_2 = E(J)$ there exists $a \in J$ such that $0 \neq E(a) \in I_3$. Then $0 \neq E(a)a^* \in \langle I_3 \rangle_A \cap J$, showing $\langle I_3 \rangle_A \cap J \neq \{0\}$. By (3.23), the regular ideal $L := J \cap \langle I_1 \rangle_A^{\perp} \neq \{0\}$.

Note that $\langle I_1 \rangle_A^{\perp} \cap B \subseteq I_1^{\perp_B}$. As $J \cap B = I_1$ it follows that $L \cap B = \{0\}$. This contradicts the regular ideal intersection property. Thus $J \cap B = E(J)$. □

Combining Proposition 3.19 and Proposition 3.22 we get the following theorem.

Theorem 3.24. *Let $B \subseteq A$ be a regular inclusion of C*-algebras satisfying the regular ideal intersection property, and let N be a $*$-semigroup of normalizers with dense span in A. Further assume there is an N-invariant faithful*

conditional expectation $E\colon A \to B$. *The invariant regular ideals of B form a Boolean algebra. The map $J \mapsto J \cap B$, with inverse given by $K \mapsto J_K$, is a Boolean algebra isomorphism between the regular ideals of A and the invariant regular ideals of B.*

Proof. Recall that by Corollary 3.20, an ideal $K \trianglelefteq B$ is invariant if and only if it is N-invariant. Let $K_1, K_2 \trianglelefteq B$ be invariant regular ideals. Note that $K_1 \cap K_2$ is an invariant ideal, so Proposition 2.4 shows $K_1 \wedge K_2 = K_1 \cap K_2$ is an invariant regular ideal. For a regular invariant ideal $K \trianglelefteq B$, Proposition 3.19 shows $J_K \trianglelefteq A$ is a regular ideal and $K^{\perp_B} = J_K^\perp \cap B$. But $J_K^\perp \cap B$ is an invariant ideal of B, whence $\neg K = K^{\perp_B}$ is also a regular invariant ideal in B. As $K_1 \vee K_2 = \neg((\neg K_1) \wedge (\neg K_2))$, we conclude that the invariant regular ideals of B form a Boolean algebra.

The result now follows from Proposition 3.19 and Proposition 3.22. $\square$

Remark 3.25. Combining Proposition 2.4 with Theorem 3.24 shows that in the setting of a regular inclusion with the regular ideal intersection property and a faithful, invariant conditional expectation, the invariant regular ideals in B and the regular open sets in $\mathrm{Prim}(A)$ are isomorphic Boolean algebras.

4. Quotients by Regular Ideals

With the results from Sect. 3, we have all the tools we need to prove our main results about quotienting by regular ideals. We first note that regular inclusions are preserved by quotients.

Remark 4.1. Let (A, B) be a regular inclusion and let $J \trianglelefteq A$ be any ideal of A. Then $(A/J, B/(J \cap B))$ is a regular inclusion. Indeed, if (u_λ) is a net in B which is an approximate unit for A, then $(u_\lambda + J)$ is a net in $B/(B \cap J)$ which is an approximate unit for A/J, so $(A/J, B/(J \cap B))$ is an inclusion. Since $\{n + J : n \in \mathcal{N}(A, B)\}$ has dense span in A/J, the inclusion is regular.

Theorem 4.2. *Suppose (A, B) is a regular inclusion with the ideal intersection property. Let N be a $*$-semigroup of normalizers with dense span in A and suppose there is an N-invariant faithful conditional expectation $E\colon A \to B$. Let $J \trianglelefteq A$ be a regular ideal. Then $B/(J \cap B)$ has the ideal intersection property in A/J.*

Proof. For notational purposes, use q for the quotient mapping of A onto A/J, let $\dot{A} := A/J$, and $\dot{B} := B/(B \cap J)$.

Let $I \trianglelefteq \dot{A}$ satisfy $I \cap \dot{B} = \{0\}$ and put $L := q^{-1}(I) \trianglelefteq A$. Note that L contains J. Our task is to show $J = L$. Arguing by contradiction, suppose $J \neq L$. Since J is a regular ideal, Lemma 3.4 shows $L \cap J^\perp$ is a non-zero ideal of A. Define

$$K := L \cap B$$

and note that $K = J \cap B$. Indeed, $J \cap B \subseteq L \cap B$ because $J \subseteq L$, and the reverse inclusion follows from $I \cap \dot{B} = \{0\}$.

By Proposition 3.7, K is a non-zero regular ideal in B. By Proposition 3.19 and Theorem 3.24, $J^\perp \cap B = K^{\perp_B}$. Hence $(L \cap J^\perp) \cap B = K \cap K^{\perp_B} = \{0\}$. This contradicts the ideal intersection property. Thus $L = J$, and I is the zero ideal in A/J. □

Remark 4.3. In this remark, we discuss how Theorem 4.2 can be used to give an alternate proof of [9, Theorem 3.5].

Let $E = \{E^0, E^1, r, s\}$ be a directed graph. We assume that E is row-finite with no sources, that is $0 < r^{-1}(v) < \infty$ for all $v \in E^0$. A sequence of edges $e_1 e_2 \cdots e_n$ is a path in E, if for all i, $r(e_i) = s(e_{i-1})$; we say it is a return path if $r(e_1) = s(e_n)$. We say E satisfies condition (L) if for every return path $e_1 e_2 \cdots e_n$ there exists an i such that $r^{-1}(r(e_i)) \backslash \{e_i\} \neq \emptyset$.

A set of mutually orthogonal projections $\{P_v\}_{v \in E^0}$ and a set of partial isometries $\{S_e\}_{e \in E}$ is a Cuntz–Krieger E-family if

$$P_{s(e)} = S_e^* S_e = \sum_{r(f) = s(e)} S_f S_f^*.$$

The C^*-algebra of E, $C^*(E)$, is the unique C^*-algebra generated by a universal Cuntz–Krieger E-family, $\{p_v, s_e\}$ (for details see [41]). The C^*-algebra $C^*(E)$ comes equipped with a gauge action. More precisely, for each $t \in \mathbb{T}$ the map

$$\gamma_t(s_e) = t s_e \text{ and } \gamma_t(p_v) = p_v$$

for $e \in E^1$ and $v \in E^0$, uniquely defines an automorphism on $\mathrm{C}^*(E)$ so that the map $t \mapsto \gamma_t$ is strongly continuous.

Let D_E be the C^*-subalgebra of $C^*(E)$ generated by

$$\{(s_{e_1} s_{e_2} \cdots s_{e_n})(s_{e_1} s_{e_2} \cdots s_{e_n})^* : n \in \mathbb{N}, e_i \in E^1\}.$$

A consequence of the Cuntz–Krieger Uniqueness theorem [28, Theorem 3.7] is that $D_E \subseteq C^*(E)$ has the ideal intersection property if and only if E satisfies condition (L).

Now if J is an ideal in $C^*(E)$, then by [5, Theorem 4.1], J is generated by a set of vertex projections $\{p_v : v \in H \subseteq E^0\}$ if and only if J is gauge-invariant and in this case there exists a directed graph E/J such that $C^*(E/J) \cong C^*(E)/J$.

Let E be a a directed graph satisfying condition (L). Suppose J is a gauge-invariant regular ideal. By Theorem 4.2,

$$D_E/(J \cap D_E) \subseteq C^*(E)/J \cong C^*(E/J)$$

has the ideal intersection property and thus E/J satisfies condition (L). Thus [9, Theorem 3.5] follows from Theorem 4.2.

In [9, Proposition 3.7] we go on to show that all regular ideals in $C^*(E)$ are gauge-invariant when the graph E has condition (L). The analogous result for higher-rank graph C^*-algebras is proved in [45, Proposition 6.7]. We generalize these results in Theorem 5.6 below (see Sect. 6.1 for details on the relationship to graph algebras).

Theorem 4.2 has implications for quotients of Cartan inclusions by regular ideals, and to this we now turn our attention. Let us first recall the notion of a Cartan inclusion from [44].

Definition 4.4. Let $D \subseteq A$ be a regular inclusion of C*-algebras. We say that D is a *Cartan subalgebra*, or D *is Cartan in* A if

   (i) there is a faithful conditional expectation $E \colon A \to D$;
   (ii) D is maximal abelian in A.

We will also say that (A, D) is a *Cartan inclusion* when these conditions hold.

Remark 4.5. Recall that if D is a Cartan subalgebra of A, then the inclusion $D \subseteq A$ has the ideal intersection property. This well-known fact can be found in several places, e.g. [36, Corollary 3.2] and [37, Theorem 6.1].

Thus for a Cartan inclusion (A, D), Theorem 4.2 implies that if $J \trianglelefteq A$ is a regular ideal, then $D/(J \cap D) \subseteq A/J$ has the ideal intersection property too. It is therefore natural to ask if this quotient inclusion is also Cartan. We prove this in Theorem 4.8 below. The following example shows that this does not hold for arbitrary ideals. Another example for graph algebras is described in [5, Remark 4.5].

Example 4.6. Suppose $\mathbb{Z}$ acts on the closed disk $\overline{\mathbb{D}}$ by irrational rotation and

$$\mathbb{P} := \overline{\mathbb{D}} \backslash \{(0,0)\}.$$

Then $n \cdot x \neq x$ for all $x \in \mathbb{P}$. Notice that $\mathbb{P}$ is dense in $\overline{\mathbb{D}}$ so $C(\overline{\mathbb{D}})$ is Cartan in $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$ (see [44, Example 6.1]).

On the other hand, $\mathbb{P}$ is an open invariant subset of $\overline{\mathbb{D}}$ and so by [42, Proposition II.4.6] we have that $C_0(\mathbb{P}) \rtimes \mathbb{Z}$ is an ideal in $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$. Moreover the quotient $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}/(C_0(\mathbb{P}) \rtimes \mathbb{Z})$ is isomorphic to $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ and $C(\overline{\mathbb{D}})/C_0(\mathbb{P}) \cong \mathbb{C}$. As $\mathbb{C}$ is not maximal abelian in $C(\mathbb{T})$ we have $C(\overline{\mathbb{D}})/C_0(\mathbb{P})$ is not Cartan in $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}/(C_0(\mathbb{P}) \rtimes \mathbb{Z})$. Thus Cartan inclusions are not necessarily preserved by quotients. Notice that $(\overline{\mathbb{P}})^{\circ} = \overline{\mathbb{D}}$ and so $\mathbb{P}$ is not regular in $\overline{\mathbb{D}}$. Hence, by Proposition 3.7, we have $C_0(\mathbb{P}) \rtimes \mathbb{Z}$ is not a regular ideal in $C(\overline{\mathbb{D}}) \rtimes \mathbb{Z}$.

Before proving Theorem 4.8 which shows the quotient of a Cartan inclusion by a *regular* ideal is again a Cartan inclusion, we show that Theorem 4.2 guarantees the existence of a faithful conditional expectation in the quotient.

Lemma 4.7. *Let* (A, B) *be a regular inclusion with the ideal intersection property, let* N *be a* $*$-*semigroup of normalizers with dense span in* A, *and suppose*

$E : A \rightarrow B$ *is a faithful* N-*invariant conditional expectation. If* $J \trianglelefteq A$ *is a regular ideal, then the map*

$$E_{A/J} \colon A/J \rightarrow B/(B \cap J) \quad \textit{given by}$$
$$a + J \mapsto E(a) + (J \cap B)$$

is a well-defined faithful conditional expectation.

Proof. Let $K := J \cap B$. For $a_1, a_2 \in A$, if $a_1 + J = a_2 + J$, then $a_1 - a_2 \in J$, whence $E(a_1 - a_2) \in E(J)$. By Proposition 3.22, $E(J) = K$. Thus $E(a_1) + K = E(a_2) + K$. Therefore, $E_{A/J}$ is well-defined, and an argument using [50, Theorem 1] (adjoining a unit if necessary) shows it is a conditional expectation.

It remains to show that $E_{A/J}$ is faithful. Let

$$L := \{x \in A/J \colon E_{A/J}(x^*x) = 0\}.$$

We shall show that L is an ideal in A/J. That L is a left ideal follows as in the proof of Proposition 3.19. To show L is a right ideal, we use the N-invariance of E: for $a + J \in L$ and $n \in N$,

$$E_{A/J}([(a + J)(n + J)]^*(a + J)(n + J)) = E(n^*a^*an) + K$$
$$= nE(a^*a)n + K$$
$$= 0.$$

This gives $(a+J)(n+J) \in L$. Therefore for any $y \in \operatorname{span} N$, $(a+J)(y+J) \in L$. Since $\operatorname{span} N$ is dense in A by assumption, L is a right ideal.

By definition, $L \cap (B/K) = \{0\}$, and an application of Theorem 4.2 shows $L = \{0\}$. Thus $E_{A/J}$ is a faithful conditional expectation. $\square$

Here is the promised result concerning quotients of Cartan inclusions by regular ideals.

Theorem 4.8. *If D is a Cartan subalgebra of a C^*-algebra A and $J \trianglelefteq A$ is a regular ideal, then $D/(J \cap D)$ is a Cartan subalgebra of A/J.*

Proof. Let $K = D \cap J$. Remark 4.1 shows $(A/J, D/(D \cap J))$ is a regular inclusion.

Since (A, D) is a Cartan inclusion, it has the ideal intersection property (see Remark 4.5). Further, by Example 3.15(ii)(b) the faithful conditional expectation $E \colon A \rightarrow D$ is invariant. We can thus apply Lemma 4.7 to see that there is a faithful conditional expectation $E_{A/J} \colon A/J \rightarrow D/K$.

It remains to show that D/K is a maximal abelian subalgebra of A/J. To this end, take $a \in A$ such that $ad - da \in J$ for all $d \in D$. To complete the proof, we will show

$$a + J = E(a) + J. \tag{4.9}$$

Note that $J^\perp \cap D = K^{\perp_D}$ by Proposition 3.19(ii). If $e \in J^\perp \cap D$ then $ae - ea \in J^\perp \cap J$. Thus $ae - ea = 0$. This shows that

$$ae = ea \text{ for all } e \in K^{\perp_D}.$$

Next we show ea commutes with D. For $d \in D$ and $e \in K^{\perp_D}$, the facts that $ad - da \in J$, D is abelian, and $ae = ea$ yield,

$$0 = e(ad - da) = (ea)d - d(ea).$$

Hence ea commutes with D.

As D is maximal abelian in A we have $ea = ae \in D$, whence $ae = E(ae) = E(a)e$. These considerations are independent of the choice of $e \in K^{\perp_D}$, so

$$(a - E(a))e = 0 \text{ for all } e \in K^{\perp_D}.$$

Let n_1 and n_2 be normalizers of D and take $e \in K^{\perp_D}$. Then

$$((a - E(a))n_1 e n_2)((a - E(a))n_1 e n_2)^* = (a - E(a))n_1 e n_2 n_2^* e^* n_1^* (a - E(a))^*$$
$$= 0,$$

since $n_1 e n_2 n_2^* e^* n_1^* \in K^{\perp_B}$. Thus $(a - E(a))n_1 e n_2 = 0$. Therefore

$$a - E(a) \in \langle K^{\perp_B} \rangle^{\perp}.$$

However, $\langle K^{\perp_B} \rangle^{\perp} = J$ by Proposition 3.19. Hence $a + J = E(a) + J$, establishing (4.9). □

Remark 4.10. The converse of Theorem 4.8 does not hold: it is possible to find a Cartan inclusion (A, D) and a non-regular ideal $J \trianglelefteq A$ so that $D/(J \cap D)$ is Cartan in A/J, see e.g. [9, Example 3.9].

5. C^*-algebras of Twisted Étale Groupoids

We now specialize our study to the C^*-algebras of twisted, Hausdorff, étale groupoids. As a consequence of exactness we will get an explicit description of the regular ideals of such C^*-algebras.

To begin our discussion, we recall a few facts and some notation concerning C^*-algebras from twists. We refer the reader to [8, Section 2] for details.

For a groupoid G we will use $G^{(0)}$ to denote its unit space. We identify $G^{(0)}$ with the objects in G and use $r, s : G \to G^{(0)}$ to denote the range and source maps.

For each subset $\Delta \subseteq G^{(0)}$ there is a subgroupoid

$$G_\Delta := \{\gamma \in G : r(\gamma), s(\gamma) \in \Delta\}.$$

When $G_\Delta = \{\gamma \in G : s(\gamma) \in \Delta\}$, that is, $s(\gamma) \in \Delta \Rightarrow r(\gamma) \in \Delta$, Δ is said to be an *invariant subset* of $G^{(0)}$; we will also refer to such as set as *invariant*.

We assume groupoids are endowed with a locally compact Hausdorff topology in which composition and inversion are continuous. An open set $B \subseteq G$ is a *bisection* if $r|_B$ and $s|_B$ are homeomorphisms onto their images. We say G is étale if it has a basis consisting of bisections. A *twist* is a central extension of a groupoid G

$$G^{(0)} \times \mathbb{T} \xrightarrow{\iota} \Sigma \xrightarrow{q} G.$$

We usually suppress writing the maps ι and q and simply denote a twist by $\Sigma \to G$ or $(\Sigma; G)$. The map q restricts to a homeomorphism between the unit spaces $\Sigma^{(0)}$ and $G^{(0)}$. We will thus identify $\Sigma^{(0)}$ with $G^{(0)}$ without mention of q. A well-known construction produces the *reduced C^*-algebra, $C_r^*(\Sigma; G)$ of* $(\Sigma; G)$; it is the completion of a convolution algebra using the reduced norm. Details may be found in a number of sources; see for example, [44], [47] or [8]. If $\Sigma = G \times \mathbb{T}$ we drop Σ from the notation and refer to this algebra as $C_r^*(G)$.

If H is an open subgroupoid of G, the twist $\Sigma \to G$ determines a central extension

$$H^{(0)} \times \mathbb{T} \to q^{-1}(H) \twoheadrightarrow H$$

and $C_r^*(q^{-1}(H); H)$ embeds in $C_r^*(\Sigma; G)$ [8, Lemma 2.7]. We will use this embedding in the following instances.

(i) If U is an invariant open subset of $G^{(0)}$, then G_U is open in G and $q^{-1}(G_U) = \Sigma_U$. The C*-algebra $C_r^*(\Sigma_U; G_U)$ embeds in $C_r^*(\Sigma; G)$ as a closed ideal. In fact, $C_r^*(\Sigma_U; G_U) = \langle C_0(U) \rangle_{C_r^*(\Sigma;G)}$ by [8, Lemma 2.7].

(ii) Since G is étale and Hausdorff, the unit space $G^{(0)}$ is clopen in G. Further, $C_r^*(q^{-1}(G^{(0)}); G^{(0)})$ embeds in $C_r^*(\Sigma; G)$. As

$$C_r^*(q^{-1}(G^{(0)}); G^{(0)}) \cong C_0(G^{(0)}),$$

we identify $C_0(G^{(0)})$ with $C_r^*(q^{-1}(G^{(0)}); G^{(0)})$. In this sense, we regard $C_0(G^{(0)})$ as a subalgebra of $C_r^*(\Sigma; G)$; furthermore, $(C_r^*(\Sigma; G), C_0(G^{(0)}))$ is a regular inclusion.

In the last instance, there exists a canonical conditional expectation $E : C_r^*(\Sigma; G) \to C_0(G^{(0)})$. This conditional expectation is invariant for the *-semigroup N_{bi} of normalizers that are supported on bisections. By Corollary 3.20 an ideal $C_0(U)$ in $C_0(G^{(0)})$ is invariant in the sense of Definition 3.11 if and only if it is N_{bi}-invariant. Moreover, $C_0(U)$ is an invariant ideal of $C_0(G^{(0)})$ if and only if U is an invariant open set.

Suppose U is an open invariant set in $G^{(0)}$ and $F = G^{(0)} \setminus U$. Since F is a closed invariant subset, G_F is a groupoid with unit space F. Further, $q^{-1}(G_F) = \Sigma_F$. We thus get a twist,

$$F \times \mathbb{T} \to \Sigma_F \twoheadrightarrow G_F.$$

It is not difficult to show that the restriction map, $C_c(\Sigma; G) \ni g \mapsto g|_F \in C_c(\Sigma_F; G_F)$, extends to a *-homomorphism,

$$\varphi : C_r^*(\Sigma; G) \to C^*(\Sigma_F; G_F). \tag{5.1}$$

Thus, with $\theta : C_0(G^{(0)}) \to C_0(F)$ denoting the quotient map $\theta(f) = f|_F$, we obtain the commutative diagram (where the vertical maps are the conditional expectations described above),

$$
\begin{array}{ccccc}
C_r^*(\Sigma_U; G_U) & \longrightarrow & C_r^*(\Sigma; G) & \overset{\varphi}{\twoheadrightarrow} & C_r^*(\Sigma_F; G_F) \\
\downarrow{\scriptstyle E_U} & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle E_F} \\
C_0(U) & \longrightarrow & C_0(G^{(0)}) & \overset{\theta}{\twoheadrightarrow} & C_0(F).
\end{array}
\tag{5.2}
$$

While the bottom row is exact in the middle, it is possible that the top row is not exact [43, Appendix by Skandalis].

Definition 5.3. If the top row of (5.2) is also exact in the middle, we say that G is *inner exact at* U. When G is inner exact for every open invariant set $U \subseteq G^{(0)}$, we say G is *inner exact*, or more simply, *exact*.

By [33, Theorem 3.5], inner exactness is a property of G and is independent of the twist over G. Our next goal is to obtain a description of certain regular ideals in the reduced C^*-algebra of a twist over an exact groupoid, see Theorem 5.6 below.

Let $U \subseteq G^{(0)}$ be an invariant open set. Set

$$J_U := \{x \in C_r^*(\Sigma; G) \colon E(x^*x) \in C_0(U)\}.$$

(This is simplified notation for the ideal $J_{C_0(U)}$ defined in Notation 3.18.)

Proposition 5.4. *Suppose* $\Sigma \to G$ *is a twist,* $U \subseteq G^{(0)}$ *is an invariant open set, and* φ *is defined as in* (5.1). *Then* $\ker \varphi = J_U$.

Proof. Suppose $x \in \ker \varphi$. Then $\theta(E(x^*x)) = E_F(\varphi(x^*x)) = 0$ so $E(x^*x) \in \ker \theta$. Since the bottom row of (5.2) is exact, $E(x^*x) \in C_0(U)$. Thus, $\ker \varphi \subseteq J_U$.

Now suppose $x \in J_U$. Then $E_F(\varphi(x^*x)) = \theta(E(x^*x)) = 0$, so faithfulness of E_F gives $\varphi(x) = 0$. Hence $x \in \ker \varphi$, and so $\ker \varphi = J_U$ as desired. □

This new description of J_U as the kernel of a quotient map allows us to strengthen Corollary 3.21 for twisted groupoid C*-algebras. In the groupoid setting, Corollary 3.21 says that if $C_0(G^{(0)})$ has the regular ideal intersection property in $C_r^*(\Sigma; G)$, and $U \subseteq G^{(0)}$ is an open regular invariant set, then $C_r^*(\Sigma_U; G_U)^{\perp\perp} = J_U$. We now show that for reduced C*-algebras of twists we can drop the regular ideal intersection property.

Proposition 5.5. *Let* $\Sigma \to G$ *be a twist. Let* $U \subseteq G^{(0)}$ *be a regular invariant open set and* $F = G^{(0)} \setminus U$. *Let* $\varphi : C_r^*(\Sigma; G) \to C_r^*(\Sigma_F; G_F)$ *be the* $*$-*homomorphism that extends restriction. Then* $\ker \varphi$ *is a regular ideal and* $C_r^*(\Sigma_U; G_U)^{\perp\perp} = \ker \varphi$.

Proof. By Proposition 5.4, $\ker \varphi = J_U$. Since U is regular and invariant, Proposition 3.19 shows that $\ker \varphi$ is regular. Since $C_r^*(\Sigma_U; G_U) \subseteq \ker \varphi$, we obtain $C_r^*(\Sigma_U; G_U)^{\perp\perp} \subseteq \ker \varphi$.

Aiming at a contradiction, assume $C_r^*(\Sigma_U; G_U)^{\perp\perp} \neq \ker \varphi$. Then, by Lemma 3.4,

$$\ker \varphi \cap C_r^*(\Sigma_U; G_U)^\perp \neq \{0\}.$$

Fix a non-zero positive element $a \in \ker \varphi \cap C_r^*(\Sigma_U; G_U)^\perp$. For $e \in C_0(U)$, $ea = 0$ because $a \in C_r^*(\Sigma_U; G_U)^\perp$. On the other hand, if $f \in C_0(U^\perp)$, then $fE(a) \in C_0(U^\perp) \subseteq C_0(F)$, so

$$fE(a) = \theta(fE(a)) = \theta(E(fa)) = E_F(\varphi(f)\varphi(a)) = 0.$$

Thus for any $h \in C_0(U \cup U^\perp)$

$$hE(a) = 0.$$

But $U \cup U^\perp$ is dense in $G^{(0)}$, so $E(a) = 0$. Faithfulness of E gives $a = 0$, contrary to hypothesis. Thus $\ker \varphi = C_r^*(\Sigma_U; G_U)^{\perp\perp}$, as desired. $\square$

Let $\Sigma \to G$ be a twist and assume that G is exact. We now show that if $(C_r^*(\Sigma; G), C_0(G^{(0)}))$ has the ideal intersection property then the regular ideals of $C_r^*(\Sigma; G)$ have an explicit description in terms of the dynamics of G.

Theorem 5.6. *Let $\Sigma \to G$ be a twist. Suppose that G is exact and $U \subseteq G^{(0)}$ is a regular invariant open set. Then $C_r^*(\Sigma_U; G_U)$ is a regular ideal in $C_r^*(\Sigma; G)$.*

If, in addition, $C_0(G^{(0)})$ has the regular ideal intersection property in $C_r^*(\Sigma; G)$, every regular ideal of $C_r^*(\Sigma; G)$ is of this form.

Proof. Since G is exact, $\ker \varphi = C_r^*(\Sigma_U; G_U) = J_U$. By Proposition 5.5, $C_r^*(\Sigma_U; G_U)$ is a regular ideal in $C_r^*(\Sigma; G)$.

Theorem 3.24 shows that in the presence of the regular ideal intersection property, all regular ideals are of the form $C_r^*(\Sigma_U; G_U)$. $\square$

The following corollary to (non-twisted) groupoid C*-algebras is a special case of Theorem 5.6. We record it, however, as many of the examples which motivated this study are in this setting.

Corollary 5.7. *Let G be an exact locally compact Hausdorff étale groupoid. If $U \subseteq G^{(0)}$ is a regular open invariant set, then $C_r^*(G_U)$ is a regular ideal in $C_r^*(G)$.*

If $C_0(G^{(0)})$ has the ideal intersection property inside $C_r^*(G)$ then all regular ideals of $C_r^*(G)$ are of this form.

In Sect. 6.1, we discuss how Corollary 5.7 applies to graph C*-algebras. More generally, Proposition 6.3 gives an application of Theorem 5.6 for Cartan inclusions in nuclear C*-algebras. In Theorem 6.5, a result analogous to Theorem 5.6 and Corollary 5.7 for reduced crossed products of discrete group actions on (not necessarily abelian) C*-algebras is given. If a discrete group Γ acts on a locally compact Hausdorff space X, then the reduced crossed product $C_0(X) \rtimes_r \Gamma$ is both an example of a reduced groupoid C*-algebra and a reduced crossed-product C*-algebra. Corollary 6.6 gives an application of both Corollary 5.7 and Theorem 6.5 in this setting.

Remark 5.8. A careful reader will note that we did not need that G was exact for Theorem 5.6 or Corollary 5.7, only that Diagram (5.2) is exact for regular open sets $U \subseteq G^{(0)}$.

Our aim in the remainder of this section is to present Example 5.13. This is an example of a groupoid G that is not inner exact but which is inner exact at all open regular invariant sets $U \subseteq G^{(0)}$.

Definition 5.9. Let Γ be a discrete group. A sequence of subgroups (K_n) is an *approximating sequence* for Γ if

(i) each K_n is a normal, finite index subgroup of Γ;
(ii) $K_n \supseteq K_{n+1}$ for all n; and
(iii) $\bigcap_n K_n = \{e\}$.

Given a discrete group Γ with an approximating sequence (K_n) one can construct what is known as a HLS groupoid. These groupoids were introduced (and named for) Higson, Lafforgue and Skandalis [21]. Higson, Lafforgue and Skandalis use HLS groupoids to give a counter-example to the groupoid version of the Baum-Connes conjecture. In so doing, they give an example of a HLS groupoid which is not exact. We give now the definition of a HLS groupoid, as presented by Willett [52].

Definition 5.10. Let Γ be a discrete group with an approximating sequence (K_n). For each n let $\Gamma_n = \Gamma/K_n$, and let $\pi_n \colon \Gamma \to \Gamma_n$ be the quotient map. Let $\Gamma_\infty = \Gamma$ and π_∞ be the identity map on Γ. Let $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ be the one-point compatification of $\mathbb{N}$ and let G be the disjoint union of $\{n\} \times \Gamma_n$ for all $n \in \mathbb{N}^*$. That is,

$$G = \bigsqcup_{n \in \mathbb{N}^*} \{n\} \times \Gamma_n.$$

Equip G with the topology generated by the sets

(i) $\{(n, g)\}$ for $n \in \mathbb{N}$ and $g \in \Gamma_n$; and
(ii) for each $\gamma \in \Gamma$ and $N \in \mathbb{N}$, $\{(n, \pi_n(\gamma)) \colon n \in \mathbb{N} \cup \{\infty\},\ n \geq N\}$.

With this topology, $\{(n, \pi_n(e)) : n \in \mathbb{N}^*\}$ is a clopen set in G, and with its relative topology, it is homeomorphic to $\mathbb{N}^*$. We identify $\{(n, \pi_n(e)) : n \in \mathbb{N}^*\}$ with $\mathbb{N}^*$.

Endow G with a partial product

$$(n, g_1)(m, g_2) = \begin{cases} (n, g_1 g_2) & \text{if } n = m \\ \text{undefined} & \text{when } n \neq m. \end{cases}$$

The inverse operation is given by $(n, g)^{-1} = (n, g^{-1})$. With these operations, G becomes a Hausdorff étale topological groupoid and is called the *HLS groupoid* associated to Γ and (K_n). The unit space of G is $\mathbb{N}^*$.

We will need the following proposition for Example 5.13.

Proposition 5.11. *Let Γ be a discrete group with an approximating sequence (K_n) and let G be the associated HLS groupoid. Assume that $C_r^*(G) = C^*(G)$. Then*

$$C_r^*(G_U) \to C_r^*(G) \to C_r^*(G_{\mathbb{N}^* \setminus U})$$

is exact for all regular open sets $U \subseteq \mathbb{N}^*$.

Proof. This result hinges on two claims.

Claim 1. If $U \subseteq \mathbb{N}$, then $C_r^*(G_U) = C^*(G_U)$.

To prove this claim note that if U is finite then the groupoid G_U is finite, and the result is clear. If U is infinite, then the result follows from [52, Lemma 2.6]. Thus Claim 1 is proved.

Claim 2. If $U \subseteq \mathbb{N}^*$ with $\infty \in U$ and $U \cap \mathbb{N}$ infinite, then $C_r^*(G_U) = C^*(G_U)$.

To prove this claim, note that G_U is the HLS groupoid associated to Γ and the approximating sequence $(K_n)_{n \in U \cap \mathbb{N}}$. By assumption $C_r^*(G) = C^*(G)$, so it follows from [52, Lemma 2.7] that we also have $C_r^*(G_U) = C^*(G_U)$. Thus the second claim holds.

Now, let $U \subseteq \mathbb{N}^*$ be a regular open set. Then U must take one of the following forms:

 (i) $U \subseteq \mathbb{N}$ and U is finite;
 (ii) $\infty \in U$ and $\mathbb{N}^* \setminus U$ is finite;
(iii) $U \subseteq \mathbb{N}$ and both U and $\mathbb{N} \setminus U$ are infinite.

In all cases, Claim 1 and Claim 2 show that we have the commutative diagram

$$
\begin{array}{ccccc}
C^*(G_U) & \longrightarrow & C^*(G) & \longrightarrow & C^*(G_{\mathbb{N}^* \setminus U}) \\
\| & & \| & & \| \\
C_r^*(G_U) & \longrightarrow & C_r^*(G) & \longrightarrow & C_r^*(G_{\mathbb{N}^* \setminus U}).
\end{array}
\tag{5.12}
$$

As the top line of (5.12) is exact, the bottom line is also exact.   □

Example 5.13. Let $\mathbb{F}_2$ be the free group over two generators. Let (K_n) be the approximating sequence for $\mathbb{F}_2$ given in [52, Lemma 2.8], and let G be the associated groupoid. By [1, Proposition 3.5] G is not exact, since $\mathbb{F}_2$ is not amenable. However, by [52, Lemma 2.7 and Lemma 2.8], $C_r^*(G) = C^*(G)$. Hence, by Proposition 5.11 the failure of exactness for G does not happen for open regular invariant sets $U \subseteq G^{(0)}$.

6. Applications

In this section, we give a number of applications.

6.1. Directed Graphs

Let E be a directed graph. In Remark 4.3, we showed that if J is a gauge-invariant regular ideal in $C^*(E)$, then the graph E/J satisfies condition (L) if E does. In this subsection we use Corollary 5.7 to show that if E satisfies condition (L) then all regular ideals in $C^*(E)$ are gauge-invariant recovering [9, Proposition 3.7].

To see this, we use the groupoid G constructed from E in [29] which, among other things, has $C^*(E) \cong C_r^*(G)$ with the isomorphism sending each vertex projection p_v to a characteristic function on a compact open subset of $G^{(0)}$. Further, the groupoid G is amenable [29, Corollary 5.5], and so G is exact, see e.g. [43, Remark 4.5].

Since the gauge-invariant ideals of $C^*(E)$ are precisely the ideals generated by their vertex projections [5, Theorem 4.1], the gauge-invariant ideals are ideals of the form $C_r^*(G_U)$ for open invariant sets $U \subseteq G^{(0)}$. As noted in Example 4.3, D_E has the intersection property in $C^*(E)$ if and only if E satisfies condition (L). Thus Corollary 5.7 generalizes [9, Proposition 3.7]. Similarly, for higher-rank graph C^*-algebras, Corollary 5.7 recovers [45, Proposition 6.7].

In [17] analogous results to the graph C^*-algebra results of [9] are shown in the purely algebraic setting of Leavitt path algebras. In a Leavitt path algebra the correct analogy of a gauge-invariant ideal is a $\mathbb{Z}$-graded ideal. It is shown in [17, Corollary 3.4] that all regular ideals in a Leavitt path algebra are graded, even when the graph does not satisfy condition (L). For graph C^*-algebras, however, condition (L) is needed. Indeed, if E is the graph of one vertex and one edge, then $C^*(E) = C(\mathbb{T})$. Here any open regular subset of $\mathbb{T}$ gives a regular ideal. None of these, other than $\{0\}$ and $C(\mathbb{T})$, are gauge-invariant.

6.2. Transformation Groups

Let Γ be a discrete group acting by homeomorphisms on a compact Hausdorff space X. The action of Γ on X is *topologically free* if the interior of $X_t := \{x \in X : t \cdot x = x\}$ is empty for all non-identity elements $t \in \Gamma$. The ideal intersection property for $(C(X) \rtimes_r \Gamma, C(X))$ is closely related to topological freeness. In fact, when the action of Γ on X is topologically free, S. Kawamura and J. Tomiyama show in [25, Theorem 4.1] that $(C(X) \rtimes_r \Gamma, C(X))$ has the ideal intersection property, and the converse holds when Γ is amenable. We do not know whether the ideal intersection property for $(C(X) \rtimes_r \Gamma, C(X))$ is equivalent to topological freeness of the action of Γ on X for discrete groups Γ if the hypothesis of amenability on Γ is dropped. However, some relaxation of the amenability hypothesis is possible; see the equivalence of statements (iv) and (v) of [40, Theorem 4.6].

Recall that a closed set $F \subseteq X$ is *regular* if it is the closure of its interior; it is easy to see that a closed set F is regular if and only if $X \setminus F$ is a regular open set. We now observe that the restriction of a topologically free action to an invariant regular closed set is again topologically free.

Proposition 6.1. *Let Γ be a discrete group acting topologically freely on a compact Hausdorff space X. If $Y \subseteq X$ is a regular and closed Γ-invariant set, then the restricted action of Γ to Y is also topologically free.*

Proof. Let Y be an invariant regular closed set. We need to show that the interior of Y_s as a subset of Y, $(Y_s)^{\circ_Y}$, is empty for all $e \neq s \in \Gamma$. Suppose $s \in \Gamma$ and $(Y_s)^{\circ_Y} \neq \emptyset$. This means there exists an open set $U \subseteq X$ such that

$U \cap Y \subseteq Y_s$ with $U \cap Y$ nonempty. Since Y is a regular closed set, $U \cap Y^\circ \neq \emptyset$. Thus,

$$\emptyset \neq U \cap Y^\circ \subseteq U \cap Y \subseteq Y_s \subseteq X_s.$$

This shows the interior of X_s is non-empty. Since Γ acts topologically freely on X, we conclude $s = e$. □

Remark 6.2. Proposition 6.1 can be used to construct an alternate proof of Theorem 4.2 for inclusions $C(X) \subseteq C(X) \rtimes_r \Gamma$.

Compare Proposition 6.1 to Example 4.6. In Example 4.6 $X = \overline{\mathbb{D}}$, and $F := X \backslash \mathbb{P}$ is a single point. The action of $\Gamma = \mathbb{Z}$ on this point is (necessarily) trivial, and thus not topologically free. In this case, however, F is not a regular closed set.

6.3. Cartan Embeddings

We used Theorem 4.2 to show that the quotient of a Cartan inclusion by a regular ideal is again a Cartan inclusion, see Theorem 4.8. We have already discussed in Sect. 6.1 that Theorem 5.6 recovers [9, Proposition 6.7]. Recall that condition (L) in [9, Proposition 6.7] is precisely the condition that the subalgebra D_E is a Cartan subalgebra of the graph algebra $C^*(E)$.

Suppose (A, D) is a Cartan inclusion and let $K \trianglelefteq D$ be an invariant regular ideal. If $L \trianglelefteq A$ satisfies $L \cap B = K = E(L)$, then $\langle K \rangle_A \subseteq L \subseteq J_K$. This leads to the obvious question: does $\langle K \rangle_A = J_K$? Equivalently, is $\langle K \rangle_A$ a regular ideal of A? In the graph algebra setting, [9, Proposition 6.7] precisely says that we always have $\langle K \rangle_A = J_K$ for invariant regular ideals K. We generalize this to more general Cartan inclusions now, using Theorem 5.6.

Proposition 6.3. *Let (A, D) be a Cartan inclusion with A nuclear. The map $J \mapsto J \cap D$ is a bijection between the regular ideals of A and the invariant regular ideals of D. The inverse map is given by $K \mapsto \langle K \rangle_A$.*

Proof. By [44] (see also [30, Corollary 7.6]), there is a twist $\Sigma \to G$, with G Hausdorff, étale, and effective so that A is isomorphic to $C_r^*(\Sigma; G)$ via an isomorphism which carries D to $C_0(G^{(0)})$. By [49, Theorem 5.4] G is amenable if and only if A is nuclear. Further, when G is amenable, G is exact, see, e.g. [43, Remark 4.11].

Recall from Remark 4.5 that (A, D) has the ideal intersection property. Thus by applying Theorem 3.24 and Theorem 5.6, we obtain the result. □

6.4. Crossed Products of C^*-Algebras

So far, the applications in this section have dealt with inclusions $B \subseteq A$ where B is abelian. Here we sketch how our arguments can be applied to reduced crossed products of possibly nonabelian C^*-algebras by discrete exact groups to obtain an analogue of Theorem 5.6.

Let Γ be a discrete group acting by automorphisms on a C^*-algebra A. Let $A \rtimes_r \Gamma$ denote the reduced crossed product and let $E_A \colon A \rtimes_r \Gamma \to A$ be the usual faithful conditional expectation [11, Proposition 4.1.9]. Recall from Example 3.14, that E_A is invariant under $N = \{a\delta_s \colon s \in \Gamma\}$. Note that, an

ideal of A is N-invariant if and only if it is invariant under the action of Γ. By Corollary 3.20 it then follows that an ideal in A is invariant under the action of Γ if and only if it is an invariant ideal in the sense of Definition 3.11.

If $K \trianglelefteq A$ is a Γ-invariant ideal we get the following commutative diagram.

$$
\begin{array}{ccccc}
K \rtimes_r \Gamma & \longrightarrow & A \rtimes_r \Gamma & \overset{\varphi}{\twoheadrightarrow} & A/K \rtimes_r \Gamma \\
\downarrow{\scriptstyle E_K} & & \downarrow{\scriptstyle E_A} & & \downarrow{\scriptstyle E_{A/K}} \\
K & \longrightarrow & A & \underset{\theta}{\twoheadrightarrow} & A/K
\end{array}
\tag{6.4}
$$

The bottom line of (6.4) will always be exact. However, it is possible that the top line may not be exact. As with algebras associated with groupoids (see Definition 5.3), if the top line of (6.4) is exact for all invariant ideals $K \trianglelefteq A$ we say that the action of Γ on A is *exact*. In particular, this will happen if Γ is exact. See [46] for a study of exact actions and the ideal structure of $A \rtimes_r \Gamma$.

Theorem 6.5. *Let Γ be a discrete group that acts by an exact action on a C^*-algebra A. If $K \trianglelefteq A$ is an invariant regular ideal, then $K \rtimes_r \Gamma$ is a regular ideal in $A \rtimes_r \Gamma$.*

If in addition $A \subseteq A \rtimes_r \Gamma$ has the ideal intersection property, every regular ideal in $A \rtimes_r \Gamma$ has this form.

Proof. Let K be an invariant regular ideal in A. Since the action of Γ is exact we get that $\ker \varphi = K \rtimes_r \Gamma$. Arguing as in Proposition 5.5 one shows $K \rtimes_r \Gamma = \ker \varphi$ is a regular ideal, giving the first statement.

Additionally, assume that $A \subseteq A \rtimes_r \Gamma$ has the ideal intersection property, and let J_K be as in Notation 3.18. Arguing as in Proposition 5.4 we obtain $J_K = \ker \varphi$. Now Theorem 3.24 gives that all regular ideals are of this form. □

Let Γ be a discrete group acting by homeomorphisms on a compact space X. The reduced crossed-product $C(X) \rtimes_r \Gamma$ is isomorphic to the reduced groupoid C^*-algebra $C_r^*(\Gamma \times X)$, where $\Gamma \times X$ is the transformation group. Thus, crossed products of abelian C^*-algebras are special cases of groupoid C^*-algebras. We have the following corollary which is a special case of both Theorem 6.5 and Corollary 5.7.

Corollary 6.6. *Let Γ be an exact discrete group acting on a compact Hausdorff space X by homeomorphisms. If $U \subseteq X$ is an open regular Γ-invariant subset, then $C_0(U) \rtimes_r \Gamma$ is a regular ideal in $C_0(X) \rtimes_r \Gamma$.*

If the action of Γ on X is topologically free, then all regular ideals of $C_0(X) \rtimes_r \Gamma$ are of this form.

Proof. The groupoid $\Gamma \times X$ is exact since the group Γ is exact. This follows immediately on comparing Definition 5.3 with Definition 1.5 of [46]. If the action of Γ is topologically free, [25, Theorem 4.1] shows $C(X)$ has the ideal intersection property in $C(X) \rtimes \Gamma$. Thus, the result follows by Corollary 5.7 or Theorem 6.5. □

Remark 6.7. We compare Theorem 6.5 to results of Sierakowski [46]. The inclusion $A \subseteq A \rtimes_r \Gamma$ has the *residual intersection property* if $A/J \subseteq A/J \rtimes_r \Gamma$ has the ideal intersection property for every invariant ideal $J \trianglelefteq A$. In [46, Theorem 1.3] it is shown that all ideals of $A \rtimes_r \Gamma$ are of the form $J \rtimes_r \Gamma$ for an invariant ideal $J \trianglelefteq A$ if and only if the action of Γ is exact on A and $A \subseteq A \rtimes_r \Gamma$ has the residual intersection property. Theorem 6.5 shows that the ideal intersection property suffices for regular ideals; the residual ideal intersection property is only needed for non-regular ideals.

The question of when $A \subseteq A \rtimes_r \Gamma$ has the ideal intersection property has been studied by several authors. The ideal intersection property is determined by the action of Γ on some injective objects. However, instead of topological freeness, one must instead look to see if the action is *properly outer*. See [55, Section 2.3] or [27, Definition 2.8] for definitions of properly outer actions. Let $I(A)$ be the injective envelope of A and let $I_\Gamma(A)$ be the Γ-injective envelope of A. We summarize some of the results of Zarikian [55] and Kennedy and Schafhauser [27].

Theorem 6.8. [cf. [55] and [27]] *Let Γ be a discrete group acting on a C^*-algebra A. Then the following are equivalent:*

(i) *$A \subseteq A \rtimes_r \Gamma$ has the ideal intersection property;*
(ii) *$I(A) \subseteq I(A) \rtimes_r \Gamma$ has the ideal intersection property;*
(iii) *$I_\Gamma(A) \subseteq I_\Gamma(A) \rtimes_r \Gamma$ has the ideal intersection property.*

In particular, this will happen if the action of Γ on $I(A)$ is properly outer.

Proof. The equivalence of (i), (ii) and (iii) is [27, Theorem 5.5]. For the second part see [55, Theorem 6.4] or [27, Theorem 6.4]. □

7. Settings Where the Regular Ideal Intersection Property and the Ideal Intersection Property Coincide

For most of our theorems we require the seemingly weaker regular ideal intersection property instead of ideal intersection property. The main purpose of this section is to describe some classes of inclusions for which the ideal intersection property and regular ideal intersection property coincide. As noted in the introduction, the ideal intersection property has become an important tool for obtaining structural results about inclusions, and it seems to us that the equivalence of the ideal intersection property and the regular ideal intersection property is also an interesting structural result in its own right.

We first give an example of an inclusion with the regular ideal intersection property but without the ideal intersection property.

Example 7.1. Let H be a separable Hilbert space. Let $B \subseteq B(H)$ be any unital, non-zero C^*-algebra having trivial intersection with the compact operators, $K(H)$. The only ideals in $B(H)$ are $\{0\}$, $K(H)$, and $B(H)$. Of these, $\{0\}$ and $B(H)$ are regular ideals in $B(H)$, while $K(H)$ is not a regular ideal. As $B(H) \cap B = B \neq \{0\}$, the inclusion $(B(H), B)$ has the regular ideal intersection property, but it does not have the ideal intersection property because $K(H) \cap B = \{0\}$.

We can choose B above so that B is abelian and the inclusion is regular. For a straightforward example, the inclusion $(B(H), B)$ will be regular when $B = \mathbb{C}I$ since unitaries normalize B and every $T \in B(H)$ is a linear combination of four unitary operators.

We recall that a topological space is called *semiregular* if it has a basis of regular open sets.

Proposition 7.2. *Let $B \subseteq A$ be an inclusion of C^*-algebras. Assume that* $\mathrm{Prim}(A)$ *with the hull kernel topology is semiregular. Then $B \subseteq A$ has the ideal intersection property if and only if it has the regular ideal intersection property.*

Proof. Since the ideal intersection property implies the regular ideal intersection property, it suffices to show that for $B \subseteq A$ the regular ideal intersection property implies the ideal intersection property.

Assume $B \subseteq A$ has the regular ideal intersection property and let $I \trianglelefteq A$ be a non-trivial ideal. Then $\mathrm{Prim}(A) \setminus \mathrm{hull}(I)$ is an open subset of $\mathrm{Prim}(A)$. By the semiregularity of $\mathrm{Prim}(A)$ there is a regular open set $U \subseteq \mathrm{Prim}(A) \setminus \mathrm{hull}(I)$. By Proposition 2.4, the ideal $J = \ker(\mathrm{Prim}(A) \setminus U)$ is a regular ideal A, and by [34, Theorem 5.4.7 (3)] $J \subseteq I$. By the regular ideal intersection property $J \cap B \neq \{0\}$, and hence $I \cap B \neq \{0\}$. Hence $B \subseteq A$ has the ideal intersection property. $\qquad\square$

Corollary 7.3. *Let $B \subseteq A$ be an inclusion of C^*-algebras. Assume that* $\mathrm{Prim}(A)$ *with the hull kernel topology is Hausdorff. Then $B \subseteq A$ has the ideal intersection property if and only if it has the regular ideal intersection property.*

In particular, if A is abelian, then $B \subseteq A$ has the ideal intersection property if and only if it has the regular ideal intersection property.

Proof. The primitive ideal space $\mathrm{Prim}(A)$ is always locally compact. Thus if $\mathrm{Prim}(A)$ is Hausdorff then $\mathrm{Prim}(A)$ is semiregular. This follows since locally compact Hausdorff spaces are regular, and regular spaces are semiregular, see [48, pg. 16]. $\qquad\square$

Remark 7.4. By combining Corollary 7.3 with [40, Corollary 3.21] we obtain several checkable characterizations of unital inclusions $B \subseteq A$ with A abelian having the regular ideal intersection property or equivalently, the ideal intersection property.

Remark 7.5. We can apply Corollary 7.3 to show that regular ideal intersection property and ideal intersection property are equivalent for inclusions in a variety of crossed products that are known to have Hausdorff spectrum. For example, Williams characterizes when the crossed product of a transformation group (Γ, X, α) with Γ second countable and abelian has Hausdorff spectrum [54]. If the stability subgroups of (Γ, X, α) are all subgroups of a fixed abelian group, Williams characterizes when the crossed product is continuous trace (and so by definition has Hausdorff spectrum) [53, Theorem 5.1]. Echterhoff uses Williams' result to give conditions that guarantee crossed products are continuous trace for transformation groups (Γ, X, α) with Γ a

Lie group [12, Corollary 3] or Γ is a discrete group [12, Theorem 3]. Moreover, Archbold and an Huef [3, Theorem 3.9] characterize when the crossed product has continuous trace when the action of Γ on $\mathrm{Prim}(A)$ is free.[1]

We state one application using a theorem of Green [18].

Corollary 7.6. *Let Γ be a locally compact group with with a free, proper action on a locally compact Hausdorff space X. Let $B \subseteq C_0(X) \rtimes \Gamma$ be a C^*-subalgebra. Then $B \subseteq C_0(X) \rtimes \Gamma$ has the ideal intersection property if and only if it has the regular ideal intersection property.*

Proof. Note that the reduced crossed product $C(X) \rtimes_r \Gamma$ and the full crossed product $C(X) \rtimes \Gamma$ coincide since the action of Γ is free and proper. The quotient space X/G is Hausdorff and $\mathrm{Prim}(C(X) \rtimes \Gamma) \cong X/G$ by [18, Theorem 14]. The result thus follows from Corollary 7.3. $\square$

The following application of Corollary 7.3 gives a wide variety of examples where the regular ideal intersection property is equivalent to the ideal intersection property.

Theorem 7.7. *Let (A, B) be a regular inclusion with B abelian and assume that B^c, the relative commutant of B in A, is abelian. If (A, B^c) is a Cartan inclusion, then (A, B) has the ideal intersection property if and only if (A, B) has the regular ideal intersection property.*

Remark 7.8. Suppose (A, B) is a regular inclusion with both B and B^c abelian. We wish to observe that $B^c \subseteq A$ is a regular inclusion of C^*-algebras. An easy argument using the partial automorphism θ_n of [39, Corollary 2.3] shows $\mathcal{N}(A, B) \subseteq \mathcal{N}(A, B^c)$. Therefore, B^c is regular in A. Because B contains an approximate unit for A, so does B^c, whence (A, B^c) is a regular inclusion. Thus, in the setting of Theorem 7.7, (A, B^c) is a Cartan inclusion if and only if there is a faithful conditional expectation of A onto B^c.

Proof of Theorem 7.7. To show that the regular ideal intersection property implies the ideal intersection property, we establish the contrapositive. So assume $B \subseteq A$ does not have the ideal intersection property and let $\{0\} \neq J \trianglelefteq A$ be such that $J \cap B = \{0\}$. By hypothesis, $B^c \subseteq A$ has the ideal intersection property. Thus $J \cap B^c \neq \{0\}$. Thus the inclusion $B \subseteq B^c$ does not have the ideal intersection property. By Corollary 7.3, $B \subseteq B^c$ does not have the regular ideal intersection property.

Therefore, we may find a non-trivial regular ideal $K \trianglelefteq B^c$ such that $K \cap B = \{0\}$. By Theorem 3.24, there is a regular ideal $J_K \trianglelefteq A$ such that $J_K \cap B^c = K$. Hence $J_K \cap B = K \cap B = \{0\}$. That is, $J_K \trianglelefteq A$ is a non-trivial regular ideal with trivial intersection with B. Hence $B \subseteq A$ does not have the regular ideal intersection property. $\square$

We can readily apply Theorem 7.7 to graph C*-algebras.

---

[1]We would like to thank the anonymous referee for pointing out the above applications.

Corollary 7.9. *Let E be a row-finite graph. Let D_E be the abelain subalgebra of the graph C^*-algebra $C^*(E)$ described in Remark 4.3. The following are equivalent:*

(i) *E satisfies condition (L);*
(ii) *D_E has the ideal intersection property in $C^*(E)$; and*
(iii) *D_E has the regular ideal intersection property in $C^*(E)$.*

Proof. The commutant of D_E in $C^*(E)$ is a Cartan subalgebra of $C^*(E)$ by [35, Theorem 3.6 and Theorem 3.7]. Thus the result follows from Theorem 7.7 and the Cuntz–Krieger Uniqueness theorem [28, Theorem 3.7]. $\qquad\square$

Our final corollary states that the regular ideal intersection property and the ideal intersection property coincide for many groupoid C^*-algebras of interest, not just graph C*-algebras.

Corollary 7.10. *Suppose G is an étale groupoid and let G' be the isotropy of G. Assume that G' is abelian and the interior of G' is closed in G. Then $C_0(G^{(0)}) \subseteq C_r^*(G)$ has the ideal intersection property if and only if it has the regular ideal intersection property.*

Proof. Let $(G')^\circ$ be the interior of the isotropy of G. By [10, Corollary 4.5], $C^*((G')^\circ)$ is Cartan in $C_r^*(G)$. Let $C_0(G^{(0)})^c$ be the relative commutant of $C_0(G^{(0)})$ inside $C_r^*(G)$. The result will follow from Theorem 7.7 once we show,

$$C_0(G^{(0)})^c = C_r^*((G')^\circ). \tag{7.11}$$

Now Proposition II.4.7 (i) in [42] shows first that $C_r^*((G')^\circ) \subseteq C_0(G^{(0)})^c$ and second that for any $b \in C_0(G^{(0)})^c$ we must have b supported in G'. But then the open support of b must be contained in $(G')^\circ$ and since $(G')^\circ$ is closed we get $b \in C_r^*((G')^\circ)$, that is $C_0(G^{(0)})^c = C_r^*((G')^\circ)$ as desired. $\qquad\square$

Data availibility Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations
Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

[1] Anantharaman-Delaroche, C.: Some remarks about the weak containment property for groupoids and semigroups. arXiv: 1604.01724 (2021)

[2] Archbold, R.J., Spielberg, J.S.: Topologically free actions and ideals in discrete C^*-dynamical systems. Proc. Edinburgh Math. Soc.(2) **37**(1), 119–124 (1994)

[3] Archbold, R., Huef, A.A.: Strength of convergence and multiplicities in the spectrum of a C^*-dynamical system. Proc. Lond. Math. Soc. (3) **96**(3), 545–581 (2008)

[4] Barlak, S., Li, X.: Cartan subalgebras and the UCT problem. Adv. Math. **316**, 748–769 (2017)

[5] Bates, T., Pask, D., Raeburn, I., Szymański, W.: The C^*-algebras of row-finite graphs. N. Y. J. Math. **6**, 307–324 (2000)

[6] Borys, C.: The Furstenberg boundary of a groupoid. arXiv:1904.10062 (2020)

[7] Brown, J., Clark, L.O., Farthing, C., Sims, A.: Simplicity of algebras associated to étale groupoids. Semigroup Forum **88**(2), 433–452 (2014)

[8] Brown, J.H., Fuller, A.H., Pitts, D.R., Reznikoff, S.A.: Graded C^*-algebras and twisted groupoid C^*-algebras. N. Y. J. Math. **27**, 205–252 (2021)

[9] Brown, J.H., Fuller, A.H., Pitts, D.R., Reznikoff, S.A.: Regular ideals of graph algebras. R. Mt. J. Math. **52**(1), 43–48 (2022)

[10] Brown, J.H., Nagy, G., Reznikoff, S., Sims, A., Williams, D.P.: Cartan subalgebras in C^*-algebras of Hausdorff étale groupoids. Integral Equ. Oper. Theory **85**(1), 109–126 (2016)

[11] Brown, N., Ozawa, N.: C^*-Algebras and Finite-Dimensional Approximations, Graduate Studies in Mathematics, vol. 88. American Mathematical Society, Providence (2008)

[12] Echterhoff, S.: On transformation group C^*-algebras with continuous trace. Trans. Am. Math. Soc. **343**(1), 117–133 (1994)

[13] Exel, R.: Regular ideals under the ideal intersection property. arXiv:2301.10073 (2023)

[14] Exel, R., Pitts, D.R., Zarikian, V.: Exotic ideals in free transformation group C^*-algebras. Bull. Lond. Math. Soc. **55**(3), 1489–1510 (2023)

[15] Feldman, J., Moore, C.C.: Ergodic equivalence relations, cohomology, and von Neumann algebras. II. Trans. Am. Math. Soc. **234**(2), 325–359 (1977)

[16] Halmos, P., Givant, S.: Introduction to Boolean Algebras. Undergraduate Texts in Mathematics, Springer, New York (2009)

[17] Gonçalves, D., Royer, D.: A note on the regular ideals of Leavitt path algebras. J. Algebra Appl. **11**(21), 2250225 (2022)

[18] Green, P.: C^*-algebras of transformation groups with smooth orbit space. Pac. J. Math. **72**(1), 71–97 (1977)

[19] Halmos, P.R.: Lectures on Boolean Algebras, Van Nostrand Mathematical Studies No. 1. D. Van Nostrand Co. Inc., Princeton (1963)

[20] Hamana, M.: The centre of the regular monotone completion of a C^*-algebra. J. Lond. Math. Soc. **26**(3), 522–530 (1982)

[21] Higson, N., Lafforgue, V., Skandalis, G.: Counterexamples to the Baum–Connes conjecture. Geom. Funct. Anal. **12**(2), 330–354 (2002)

[22] Huef, A.A., Raeburn, I.: The ideal structure of Cuntz–Krieger algebras. Ergodic Theory Dyn. Syst. **17**(3), 611–624 (1997)

[23] Katsura, T.: A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. I. Fundamental results. Trans. Am. Math. Soc **356**(11), 4287–4322 (2004)

[24] Kawabe, T.: Uniformly recurrent subgroups and the ideal structure of reduced crossed products. arXiv:1701.03413v1 (2017)

[25] Kawamura, S., Tomiyama, J.: Properties of topological dynamical systems and corresponding C^*-algebras. Tokyo J. Math. **13**(2), 251–257 (1990)

[26] Kennedy, M., Kim, S.J., Li, X., Raum, S., Ursu, D.: The ideal intersection property for essential groupoid C*-algebras. arXiv:2107.03980 (2021)

[27] Kennedy, M., Schafhauser, C..: Noncommutative boundaries and the ideal structure of reduced crossed products. Duke Math. J. **168**(17), 3215–3260 (2019)

[28] Kumjian, A., Pask, D., Raeburn, I.: Cuntz–Krieger algebras of directed graphs. Pac. J. Math. **184**(1), 161–174 (1998)

[29] Kumjian, A., Pask, D., Raeburn, I., Renault, J.: Graphs, groupoids, and Cuntz–Krieger algebras. J. Funct. Anal. **144**(2), 505–541 (1997)

[30] Kwaśniewski, B.K., Meyer, R.: Noncommutative Cartan C*-subalgebras. Trans. Am. Math. Soc. **373**(12), 8697–8724 (2020)

[31] Kwaśniewski, B.K., Meyer, R.: Ideal structure and pure infiniteness of inverse semigroup crossed products. J. Noncommut. Geom. **17**(3), 999–1043 (2023)

[32] Kwaśniewski, B.K., Meyer, R.: Aperiodicity: the almost extension property and uniqueness of pseudo-expectations. Int. Math. Res. Not. IMRN **18**, 14384–14426 (2022)

[33] LaLonde, S.M.: Some consequences of stabilization theorem for Fell bundles over exact groupoids. J. Oper. Theory **81**(2), 335–369 (2019)

[34] Murphy, G.J.: C^*-Algebras and Operator Theory. Academic Press, Boston (1990)

[35] Nagy, G., Reznikoff, S.: Abelian core of graph algebras. J. Lond. Math. Soc. **85**(3), 889–908 (2012)

[36] Nagy, G., Reznikoff, S.: Pseudo-diagonals and uniqueness theorems. Proc. Am. Math. Soc. **142**(1), 263–275 (2014)

[37] Pitts, D.R.: Structure for regular inclusions. I. J. Oper. Theory **78**(2), 357–416 (2017)

[38] Pitts, D.R.: Structure for regular inclusions. II: Cartan envelopes, pseudo-expectations and twists. J. Funct. Anal. **281**(1), 108993 (2021)

[39] Pitts, D.R.: Normalizers and approximate units for inclusions of C^*-algebras. Indiana U. Math. J. **72**(5), 1849–1866 (2023)

[40] Pitts, D.R., Zarikian, V.: Unique pseudo-expectations for C^*-inclusions. Illinois J. Math. **59**(2), 449–483 (2015)

[41] Raeburn, I.: Graph algebras. In: CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (2005)

[42] Renault, J.: A Groupoid Approach to C^*-Algebras. Lecture Notes in Mathematics, vol. 793. Springer, Berlin (1980)

[43] Renault, J.: The ideal structure of groupoid crossed product C^*-algebras. With an appendix by Georges Skandalis. J. Oper. Theory **25**(1), 3–36 (1991)

[44] Renault, J.: Cartan subalgebras in C^*-algebras. Irish Math. Soc. Bull. **61**, 29–63 (2008)

[45] Schenkel, T.: Regular ideals of locally-convex higher-rank graph algebras. N. Y. J. Math. **28**, 1581–1595 (2022). (**MR4521031**)

[46] Sierakowski, A.: The ideal structure of reduced crossed products. Münster J. Math. **3**, 237–261 (2010)

[47] Sims, A.: Hausdorff étale groupoids and their C^*-algebras. In: Sims, A., Szabó, G., Williams, D., Perera, F. (eds.) Operator Algebras and Dynamics: Groupoids, Crossed Products, and Rokhlin Dimension. Advanced Courses in Mathematics, pp. 58–120. CRM Barcelona Birkhäuser/Springer, Cham (2020)

[48] Steen, L.A., Seebach, J.A., Jr.: Counterexamples in Topology. Holt, Rinehart and Winston Inc., New York (1970)

[49] Takeishi, T.: On nuclearity of C^*-algebras of Fell bundles over étale groupoids. Publ. Res. Inst. Math. Sci. **50**(2), 251–268 (2014)

[50] Tomiyama, J.: On the projection of norm one in W^*-algebras. Proc. Jpn. Acad. **33**, 608–612 (1957)

[51] Veršik, A.M.: Nonmeasurable decompositions, orbit theory, algebras of operators. Soviet Math. Dokl. **12**, 1218–1222 (1971)

[52] Willett, R.: A non-amenable groupoid whose maximal and reduced C^*-algebras are the same. Münster J. Math. **8**(1), 241–252 (2015)

[53] Williams, D.P.: Transformation group C^*-algebras with continuous trace. J. Funct. Anal. **41**(1), 40–76 (1981)

[54] Williams, D.P.: Transformation group C^*-algebras with Hausdorff spectrum. Illinois J. Math. **26**(2), 317–321 (1982)

[55] Zarikian, V.: Unique expectations for discrete crossed products. Ann. Funct. Anal. **10**(1), 60–71 (2019)

Jonathan H. Brown
Department of Mathematics
University of Dayton
Dayton OH 45469-2316
USA
e-mail: jonathan.henry.brown@gmail.com

Adam H. Fuller (✉)
Department of Mathematics
Ohio University
Athens OH 45701
USA
e-mail: fullera@ohio.edu

David R. Pitts
Department of Mathematics
University of Nebraska-Lincoln
Lincoln NE 68588
USA
e-mail: dpitts2@unl.edu

Sarah A. Reznikoff
Department of Mathematics
Virginia Tech
Blacksburg VA 24061-1026
USA
e-mail: reznikoff@vt.edu