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Solving Continuous Time Leech Problems for Rational Operator Functions

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Abstract. The main continuous time Leech problems considered in this paper are based on stable rational finite dimensional operator-valued functions G and K. Here stable means that G and K do not have poles in the closed right half plane including infinity, and the Leech problem is to find a stable rational operator solution X such that

$$G(s)X(s) = K(s) \quad (s \in \mathbb{C}_+) \quad \text{and} \quad \sup\{\|X(s)\| : \Re s \geq 0\} < 1.$$

In the paper the solution of the Leech problem is given in the form of a state space realization. In this realization the finite dimensional operators involved are expressed in the operators of state space realizations of the functions G and K. The formulas are inspired by and based on ideas originating from commutant lifting techniques. However, the proof mainly uses the state space representations of the rational finite dimensional operator-valued functions involved. The solutions to the discrete time Leech problem on the unit circle are easier to develop and have been solved earlier; see, for example, Frazho et al. (Indagationes Math 25:250–274 2014).

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1. Introduction

Throughout this paper $\mathcal{U}$, $\mathcal{E}$ and $\mathcal{Y}$ are all finite dimensional Hilbert spaces. Furthermore, $H^\infty(\mathcal{U}, \mathcal{Y})$ is the Hardy space formed by the set of all operator valued functions $F(s)$ mapping $\mathcal{U}$ into $\mathcal{Y}$ that are analytic in the open right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re(s) > 0\}$ and

$$\|F\|_\infty = \sup\{\|F(s)\| : \Re(s) > 0\} < \infty. \tag{1.1}$$

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Moreover, $L^2_+(\mathcal{U})$ is the Hilbert space formed by the set of all Lebesgue measurable functions $g(t)$ with values in $\mathcal{U}$, $t \geq 0$ and such that

$$\|g\|^2_{L^2_+} = \int_0^\infty \|g(t)\|^2_{\mathcal{U}} dt < \infty.$$

Let G in $H^\infty(\mathcal{U}, \mathcal{Y})$ and K in $H^\infty(\mathcal{E}, \mathcal{Y})$ be rational functions. Let T_G and T_K denote the corresponding Wiener–Hopf operators,

$$T_G : L^2_+(\mathcal{U}) \to L^2_+(\mathcal{Y}) \quad \text{and} \quad T_K : L^2_+(\mathcal{E}) \to L^2_+(\mathcal{Y}).$$

To be precise,

$$\begin{aligned}
\big(T_G u\big)(t) &= D_1 u(t) + \int_0^t g_\circ(t-\tau)u(\tau)d\tau \qquad (\text{for } 0 \leq t < \infty)\\
\big(T_K v\big)(t) &= D_2 v(t) + \int_0^t k_\circ(t-\tau)v(\tau)d\tau \qquad (\text{for } 0 \leq t < \infty)\\
G(s) &= D_1 + \big(\mathfrak{L}g_\circ\big)(s) \quad \text{and} \quad K(s) = D_2 + \big(\mathfrak{L}k_\circ\big)(s) \quad (\text{for } s \in \mathbb{C}_+).
\end{aligned} \tag{1.2}$$

Here $\mathfrak{L}g_\circ$ and $\mathfrak{L}k_\circ$ denote the Laplace transform of the functions $g_\circ$ and $k_\circ$, respectively, that is,

$$\begin{aligned}
\big(\mathfrak{L}g_\circ\big)(s) &= \int_0^\infty g_\circ(t)e^{-st}dt \qquad (\text{for } s \in \mathbb{C}_+)\\
\big(\mathfrak{L}k_\circ\big)(s) &= \int_0^\infty k_\circ(t)e^{-st}dt \qquad (\text{for } s \in \mathbb{C}_+).
\end{aligned} \tag{1.3}$$

It is emphasized that in our problems $g_\circ$ and $k_\circ$ are integrable operator valued functions over the interval $[0, \infty)$. Following engineering notation, the time function $g(t)$ is denoted with a lower case, while its Laplace transform $G(s)$ is denoted by a capital.

A function X in $H^\infty(\mathcal{E}, \mathcal{U})$ is called a *solution to the Leech problem associated with data G and K* whenever

$$G(s)X(s) = K(s) \quad (s \in \mathbb{C}_+) \quad \text{and} \quad \|X\|_\infty \leq 1. \tag{1.4}$$

The Leech problem is an example of a metric constrained interpolation problem, the first part of (1.4) is the interpolation condition, and the second part is the metric constraint.

In an unpublished note from the early 1970's (and then eventually published) [20] Leech proved that the problem is solvable if and only if the operator $T_G T_G^* - T_K T_K^*$ is nonnegative. Later the Leech theorem was derived as a corollary of more general results; see, e.g., [21, page 107], [9, Sect. VIII.6], and [1, Sect. 4.7]. But in Leech's work and in the other results just mentioned the problem was solved for H^∞ functions in the open unit disc. In that case, T_G and T_K are Toeplitz operators. Here we are working with H^∞ functions in the open right half plane.

One can use a Cayley transform to obtain a right half plane solution from an open unit disc solution. Moreover, one can directly use the commutant lifting theorem with the unilateral shift on the appropriate H^2 space determined by the multiplication operator $\frac{s-1}{s+1}$, to arrive at state space solutions to the

Leech problem. However, these methods usually lead to cumbersome formulas, that are not natural. Motivated by commutant lifting techniques in the discrete case we will derive a special state space solution for our continuous time Leech problem, which avoids all the complications associated with the Cayley transform.

In what follows it is assumed that G and K are stable rational finite dimensional operator functions. In other words, G and K are rational operator-valued H^∞ functions. In that case, if the Leech problem associated with G and K is solvable, one expects the problem to have a stable rational finite dimensional operator solution as well. However, a priori this is not clear, and the existence of rational solutions in the discrete time case was proved only recently in [25] by reducing the problem to polynomials, in [24] by adapting the lurking isometry method used in [2], and in [15] by using a state space approach.

The special case of Leech's theorem in the discrete time setting, with $\mathcal{E} = \mathcal{Y}$ and K identically equal to the identity operator $I_\mathcal{Y}$ is part of the Toeplitz-corona theorem, which is due to Carlson [7] for $\mathcal{Y} = \mathbb{C}$, and is due to Fuhrmann [16] for an arbitrary finite dimensional $\mathcal{Y}$. The least squares solution of the Toeplitz-corona version of the equation can be found in [13] and a description of all solutions without any norm constraint in [14].

References [22,23] present a nice state space approach based on lossless systems to provide all solutions to a general or two sided continuous time Leech problem. These papers rely mainly on state space and algebraic techniques. Our approach is different, that is, we use operator methods to develop a solution to the Leech problem. Inspired by the central solution for the commutant lifting theorem, we present a solution to the Leech problem by using the corresponding Wiener–Hopf operators; see Theorem 1.1 below. Then we apply classical state space methods to convert this operator formula to a state space solution and prove some stability results. To keep the presentation simple, we only concentrated on one solution.

For an engineering perspective on discrete time Toeplitz-corona and related problems with applications to signal processing we refer to [22,23, 26,27] and the references therein. See [3] for a nice state space presentation of many classical interpolation problems in both the discrete and continuous time. However, [3] does not treat Leech or Bezout type problems.

Now let X in $H^\infty(\mathcal{E},\mathcal{U})$ be a solution to our Leech interpolation problem with data G and K. Since $\|T_X\| = \|X\|_\infty$, we see that T_X is a contraction. Because $GX = K$ is equivalent to $T_G T_X = T_K$, we have

$$T_G T_G^* - T_K T_K^* = T_G\big(I - T_X T_X^*\big)T_G^* \geq 0.$$

In other words, $T_G T_G^* - T_K T_K^*$ is a positive operator.

In this paper we will provide an explicit state space solution when the operator $T_G T_G^* - T_K T_K^*$ is *strictly positive*, that is, $T_G T_G^* - T_K T_K^*$ positive and invertible. In this case $T_G T_G^*$ is strictly positive too.

In general, when M is a positive and invertible operator, then M is called *strictly positive*, and in that case we write $M \gg 0$. Thus $T_G T_G^* - T_K T_K^* \gg 0$ if and only if $T_G T_G^* - T_K T_K^*$ is *strictly positive*.

Assume that the operator T_G is onto, or equivalently, assume that $T_G T_G^*$ is strictly positive. In particular,

$$\Lambda := T_G^* \left(T_G T_G^*\right)^{-1} T_K \tag{1.5}$$

is a well defined operator mapping $L_+^2(\mathcal{E})$ into $\mathfrak{H} = \ker(T_G)^\perp$ and satisfying $T_G \Lambda = T_K$. It is noted that $\|\Lambda\| < 1$, that is, Λ is strictly contractive if and only if $T_G T_G^* - T_K T_K^*$ is strictly positive. Notice that Λ is strictly contractive if and only if

$$\begin{aligned} 1 > r_{spec}\left(\Lambda^*\Lambda\right) &= r_{spec}\left(T_K^*\left(T_G T_G^*\right)^{-1} T_K\right) = r_{spec}\left(\left(T_G T_G^*\right)^{-1} T_K T_K^*\right) \\ &= r_{spec}\left(\left(T_G T_G^*\right)^{-1/2} T_K T_K^* \left(T_G T_G^*\right)^{-1/2}\right). \end{aligned}$$

In other words, Λ is strictly contractive if and only if

$$I - \left(T_G T_G^*\right)^{-1/2} T_K T_K^* \left(T_G T_G^*\right)^{-1/2}$$

is strictly positive. Multiplying both sides by $\left(T_G T_G^*\right)^{1/2}$, we see that Λ is strictly contractive if and only if $T_G T_G^* - T_K T_K^*$ is strictly positive.

Throughout this paper we assume that Λ defined in (1.5) is strictly contractive, or equivalently, $T_G T_G^* - T_K T_K^*$ is assumed to be strictly positive. In fact, we will develop a state space method involving an algebraic Riccati equation to determine when Λ is strictly contractive. Then motivated by the central solution for the commutant lifting theorem, we will compute a solution X for our Leech problem with data G and K. In fact what we shall present is an analog of Theorem IV.4.1 in [10], which uses a central solution based on a discrete time setting. For the non-discrete time setting the null space for the backward shift in Theorem IV.4.1 in [10] is replaced by the Dirac delta function. An explanation of the role of the Dirac delta is given in Sect. 14 "(Appendix 2)". We will directly show that our solution is indeed a solution to our Leech problem. The next theorem is our first main result for the non-discrete case.

Theorem 1.1. *Let G in $H^\infty(\mathcal{U},\mathcal{Y})$ and K in $H^\infty(\mathcal{E},\mathcal{Y})$ be two rational functions. Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, $\Lambda = T_G^*\left(T_G T_G^*\right)^{-1} T_K$ is a strict contraction. Let X be the function defined by*

$$X(s) = U(s)V(s)^{-1} \tag{1.6}$$

where $U(s)$ and $V(s)$ are the functions defined by

$$\begin{aligned} U(s) &= \left(\mathfrak{L}\Lambda\left(I - \Lambda^*\Lambda\right)^{-1}\delta\right)(s) \\ V(s) &= \left(\mathfrak{L}\left(I - \Lambda^*\Lambda\right)^{-1}\delta\right)(s). \end{aligned} \tag{1.7}$$

Here $\delta(t)$ is the Dirac delta function and $\mathfrak{L}$ is the Laplace transform.

(i) *Then X is a solution to our Leech problem. To be precise, X is a function in $H^\infty(\mathcal{E},\mathcal{U})$ satisfying $G(s)X(s) = K(s)$ and $\|X\|_\infty < 1$.*
(ii) *Moreover, $V(s)$ is an invertible outer function, that is, both $V(s)$ and $V(s)^{-1}$ are functions in $H^\infty(\mathcal{E},\mathcal{E})$. In fact, the function*

$$\Theta(s) = V(\infty)^{1/2} V(s)^{-1} \tag{1.8}$$

is the outer spectral factor for the function $I - X^*(-\overline{s})X(s)$*, that is*

$$I - X(-\overline{s})^* X(s) = \Theta(-\overline{s})^* \Theta(s). \tag{1.9}$$

In this paper we shall derive state space formulas for $X(s)$ and $\Theta(s)$. Furthermore, we will directly show that $X(s)$ is indeed a strictly contractive solution to our Leech problem, that is, $G(s)X(s) = K(s)$ and $\|X\|_\infty < 1$ and Θ is the outer spectral factor for $I - X(-\overline{s})^* X(s)$.

The formula for $X(s) = U(s)V(s)^{-1}$ in Eq. (1.7) was motivated by [10, Theorem IV.4.1], which is a consequent of the central solution to the Sz.-Nagy–Foias commutant lifting theorem. It is emphasized that the setting in [10, Theorem IV.4.1] was developed for the discrete time case, that is, when the Hardy spaces are in the open unit disc, and the corresponding operators are Toeplitz operators. Similar type formulas for U and V are also presented in the band method for the discrete time Nehari problem (see [18]). To arrive at $X(s) = U(s)V(s)^{-1}$ in (1.7), we replaced the Toeplitz operators by Wiener–Hopf operators and the kernel of the backward shift by the Dirac delta function. This leads to the formula in (1.7) for our solution to the Leech problem. Of course, there are several issues that arise when one makes these adjustments to arrive at $X(s)$ in (1.7). First and foremost is that the Dirac delta function $\delta(t)$ is not a well defined function. Moreover, we did not present any justification on why replacing a Toeplitz operators with Wiener–Hopf operators and using the Laplace transform instead of a Fourier transform should lead to a solution $X(s) = U(s)V(s)^{-1}$ to our Leech problem with data $G(s)$ and $K(s)$. However, we will put all of these issues to rest. First we will use the formula $X(s) = U(s)V(s)^{-1}$ in (1.7) to derive the state space realization for $X(s)$ in Theorem 3.2. Then, once we have these state space formulas, we will directly verify that $X(s) = U(s)V(s)^{-1}$ in Theorem 3.2 is indeed a solution to our Leech problem. Moreover, we will also directly verify that all the other results in Theorem 3.2 hold. This direct verification of the solution should eliminate any doubt one may have concerning our solution $X(s)$ to the continuous time Leech problem. Finally, it is noted that we can adapt Theorem IV.4.1 in [10] with the corresponding unilateral shifts on H^2 of the right half plane to obtain Theorem 1.1. However, we decided to simply give the result and verify directly that it holds on the Leech problem.

The paper consists of 14 sections including the introduction, which contains the first main theorem (Theorem 1.1). Throughout, beginning in Sect. 2, the emphasis will be on Leech problems that are based on finite dimensional state space realizations. In the third section the two main theorems are presented, and the first one is proved in the fourth section. The Proof of Theorem 1.1 and the proof of the second theorem in Sect. 3 are based on Lemmas 5.1, 6.1 and 7.1, using formulas (5.3), (6.2), and (7.4). Furthermore, the solution X is given by (8.2), and then Sects. 10 and 11 prove Theorem 3.2. The Proof of Theorem 1.1 is presented in Sect. 12. The Appendices 1 and 2 in Sects. 13 and 14 respectively, present classical results that are used in the paper. "Appendix 1" treats a Riccati equation. In "Appendix 2" the definition of the Dirac delta function δ for the specific class of operators that we need is developed. In fact, the results in Sect. 14 are a special case of the general theory of Dirac delta functions.

For completeness assume that $T_G T_G^* - T_K T_K^*$ is a positive operator not necessarily strictly positive, or equivalently, $\|T_G^* y\| \geq \|T_K^* y\|$ for all y in $L^2_+(\mathcal{Y})$. Then there exists a contraction Λ mapping $L^2_+(\mathcal{E})$ into $\mathfrak{H} = \ker(T_G)^\perp$ such that $\Lambda^* T_G^* = T_K$, or equivalently, $T_G \Lambda = T_K$. By choosing the appropriate unilateral shifts one can use the Sz.-Nagy–Foias commutant lifting theorem to show that there exists a function X in $H^\infty(\mathcal{E},\mathcal{U})$ such that

$$\Lambda = P_{\mathfrak{H}} T_X \quad \text{and} \quad \|X\|_\infty = \|T_X\| \leq 1. \tag{1.10}$$

In particular, X is a solution to our Leech problem. So there exists a solution to our Leech problem if and only if $T_G T_G^* - T_K T_K^*$ is positive. Finally, we compute a solution to our Leech problem when $T_G T_G^* - T_K T_K^*$ is strictly positive. The strictly positive hypothesis is due to standard numerical issues with solving Riccati equations.

2. The State Space Setup

Throughout this paper, we assume that G and K are stable rational matrix functions. Given this we will establish a state space method involving a special Riccati equation to determine when the operator $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, $\Lambda = T_G^* \left(T_G T_G^*\right)^{-1} T_K$ is strictly contractive.

To develop a solution to our rational Leech problem with data G and K, we use some classical state space realization theory from mathematical systems theory (see, e.g., Chapter 1 of [8] or Chapter 4 in [4]). For our G and K this means that the matrix function $\begin{bmatrix} G & K \end{bmatrix}$ admits a state space representation of the following form:

$$\begin{bmatrix} G(s) & K(s) \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \end{bmatrix} + C(sI - A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}. \tag{2.1}$$

As expected, I denotes the identity operator. Throughout A is a stable operator on a finite dimensional space $\mathcal{X}$. By *stable* we mean that all the eigenvalues for A are in the open left half plane $\{s \in \mathbb{C} : \Re(s) < 0\}$. Moreover, B_1, B_2, C, D_1 and D_2 operate between the appropriate spaces. Since G and K are stable rational operator valued functions, G and K have no poles in the closed right half plane $\{s \in \mathbb{C} : \Re(s) \geq 0\}$. The realization (2.1) is called *minimal* if there exists no realization of $\begin{bmatrix} G & K \end{bmatrix}$ as in (2.1) with the dimension of the state space $\mathcal{X}$ smaller than the one in the given realization. In that case, the dimension of the state space $\mathcal{X}$ of A is called the *McMillan degree* of $\begin{bmatrix} G & K \end{bmatrix}$. If the realization (2.1) is minimal, then the matrix A is automatically stable.

We will use the realization (2.1) of $\begin{bmatrix} G(s) & K(s) \end{bmatrix}$ to obtain alternative formulas for the functions $U(s)$ and $V(s)$ in (1.7). These alternative state space formulas will be given in Eqs. (6.2) and (5.3) below.

The observability operator W_{obs} mapping $\mathcal{X}$ into $L^2_+(\mathcal{Y})$ is defined by

$$\left(W_{obs} x\right)(t) = C e^{At} x \qquad (x \in \mathcal{X}). \tag{2.2}$$

If the realization is minimal, then the pair $\{C, A\}$ is observable, and thus, the operator W_{obs} is one to one. We do not require the realization (2.1) to be minimal. All we need is that A is stable and the pair $\{C, A\}$ is observable, or equivalently, W_{obs} is one to one. However, from a practical perspective, one would almost always work with a minimal realization. This guarantees that A is stable, makes the state space computations more efficient.

As a first step towards our main result we obtain Theorem 3.1 below, which presents a necessary and sufficient condition for $T_G T_G^* - T_K T_K^*$ to be strictly positive in terms of the operators in (2.1) and related matrices. To accomplish this we need the rational matrix function with values in $\mathcal{Y}$ given by

$$R(s) = G(s)\widetilde{G}(s) - K(s)\widetilde{K}(s). \tag{2.3}$$

Here $\widetilde{G}(s) = G(-\overline{s})^*$ and $\widetilde{K}(s) = K(-\overline{s})^*$. Note that R has no pole on the imaginary axis $\{i\omega : -\infty < \omega < \infty\}$.

In Sect. 13 "(Appendix 1)" we will show that R admits the following state space representation:

$$R(s) = C(sI - A)^{-1}\Gamma + R_0 - \Gamma^*(sI + A^*)^{-1}C^*. \tag{2.4}$$

Here R_0 on $\mathcal{Y}$ and Γ mapping $\mathcal{Y}$ into $\mathcal{X}$ are defined by

$$R_0 = D_1 D_1^* - D_2 D_2^*, \tag{2.5}$$

$$\Gamma = B_1 D_1^* - B_2 D_2^* + \Delta C^*, \tag{2.6}$$

where Δ is the unique solution of the Lyapunov equation

$$A\Delta + \Delta A^* + B_1 B_1^* - B_2 B_2^* = 0. \tag{2.7}$$

Since A is stable, this Lyapunov equation is indeed solvable. The solution is unique and given by

$$\Delta = \int_0^\infty e^{At}\left(B_1 B_1^* - B_2 B_2^*\right) e^{A^* t} dt. \tag{2.8}$$

With our realization for $R(s)$ we associate the following algebraic Riccati equation:

$$A^*Q + QA + (C - \Gamma^*Q)^* R_0^{-1}(C - \Gamma^*Q) = 0. \tag{2.9}$$

For the moment, let us assume that R_0 is strictly positive. Then we say that Q is a *stabilizing solution* to the algebraic Riccati (2.9) if Q is a strictly positive operator solving (2.9) and the operator A_0 on $\mathcal{X}$ defined by

$$A_0 = A - \Gamma C_0, \quad \text{and} \quad C_0 = R_0^{-1}(C - \Gamma^*Q) \tag{2.10}$$

is stable. If a stabilizing solution exists, then it is unique. If Q is a stabilizing solution, then W_0 is the observability mapping $\mathcal{X}$ into $L_+^2(\mathcal{Y})$ corresponding to the pair $\{C_0, A_0\}$ defined by

$$\big(W_0 x\big)(t) = C_0 e^{A_0 t} x \qquad (x \in \mathcal{X}). \tag{2.11}$$

Finally, T_R denotes the non-causal Wiener–Hopf operator determined by R, that is,

$$\big(T_R y\big)(t) = R_0 y(t) + \int_0^t Ce^{A(t-\tau)}\Gamma y(\tau)d\tau + \int_t^\infty \Gamma^* e^{A^*(\tau - t)}C^* y(\tau)d\tau \tag{2.12}$$

where $y(t)$ is a vector in $L^2_+(\mathcal{Y})$.

Remark 2.1. Theorem 13.4 in Sect. 13 "(Appendix 1)" shows that T_R is a strictly positive operator on $L^2_+(\mathcal{Y})$ if and only if $R_0 \gg 0$ and there exists a stabilizing solution to the algebraic Riccati Eq. (2.9). In this case, the unique stabilizing solution Q is given by $Q = W_{obs}^* T_R^{-1} W_{obs}$.

3. Main Theorems

The first main theorem presented in this paper is Theorem 1.1 in the Introduction. The two other main theorems are Theorems 3.1 and 3.2 in the present section. The proofs of these three main theorems are given in different sections. The Proof of Theorem 1.1 is given in Sect. 12, the Proof of Theorem 3.1 is the main issue of Sect. 4, and the Proof of Theorem 3.2 is based on the results of Sects. 5–10 and 13 "(Appendix 1)" and is completed in Sect. 11.

Theorem 3.1. *Let G and K be stable rational matrix functions, and assume that $\begin{bmatrix} G & K \end{bmatrix}$ is given by the minimal realization* (2.1)*. Let R be the function defined in* (2.4)*. Then the operator*

$$T_G T_G^* - T_K T_K^*$$

is strictly positive if and only if the following two conditions hold.

(i) *The operator T_R is strictly positive, or equivalently, R_0 given by* (2.5) *is strictly positive and there exists a stabilizing solution Q to the algebraic Riccati equation* (2.9)*, that is, $Q \gg 0$ and the operator A_0 defined by* (2.10)*, i.e.,*

$$A_0 = A - \Gamma R_0^{-1}(C - \Gamma^* Q) \tag{3.1}$$

is stable.

(ii) *The operator $Q^{-1} - \Delta$ is strictly positive.*

In this case, the Wiener–Hopf operator T_R is strictly positive and the unique stabilizing solution to the algebraic Riccati equation is given by

$$Q = W_{obs}^* T_R^{-1} W_{obs}. \tag{3.2}$$

The inverse of the operator $T_G T_G^* - T_K T_K^*$ is determined by

$$\left(T_G T_G^* - T_K T_K^*\right)^{-1} = T_R^{-1} + W_0 \Delta \big(I - Q\Delta\big)^{-1} W_0^*. \tag{3.3}$$

Finally, $R(s) = \widetilde{\Phi}(s)\Phi(s)$ where Φ is the invertible outer function given by

$$\begin{aligned} \Phi(s) &= R_0^{1/2}\left(I + C_0(sI - A)^{-1}\Gamma\right) \\ \Phi(s)^{-1} &= \left(I - C_0(sI - A_0)^{-1}\Gamma\right) R_0^{-1/2} \\ C_0 &= R_0^{-1}\left(C - \Gamma^* Q\right). \end{aligned} \tag{3.4}$$

Equation (3.2) and the outer spectral factorization $R(s) = \widetilde{\Phi}(s)\Phi(s)$ with Φ in (3.4) is a special case of Theorem 13.4 in Sect. 13 "(Appendix 1)". Using Theorem 13.4, we will directly prove the following result.

Theorem 3.2. *Let G and K be stable rational matrix functions, and let $\begin{bmatrix} G & K \end{bmatrix}$ be given by the minimal (stable) realization (2.1). Furthermore, assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, assume that items (i) and (ii) of Theorem 3.1 hold. Then the following holds.*

(i) *A solution X to the Leech problem with data G and K is given by the following stable state space realization:*

$$\begin{aligned} X(s) &= D_1^\dagger D_2 + C_x \big(sI - \mathbf{A}\big)^{-1} \big(B_2 - B_1 D_1^\dagger D_2\big) \\ C_x &= D_1^\dagger C + \big(I - D_1^\dagger D_1\big) B_1^* Q \big(I - \Delta Q\big)^{-1} \\ \mathbf{A} &= A - B_1 C_x \\ D_1^\dagger &= D_1^* (D_1 D_1^*)^{-1}. \end{aligned} \tag{3.5}$$

The operator $\mathbf{A}$ is stable, and $\|X\|_\infty < 1$. Finally, the McMillan degree of X is less than or equal to the McMillan degree of $\begin{bmatrix} G & K \end{bmatrix}$.

(ii) *Let $\Theta(s)$ be the rational function in $H^\infty(\mathcal{E}, \mathcal{E})$ defined by*

$$\begin{aligned} \Theta(s) &= D_v^{-1/2} - D_v^{-1/2} C_\theta \big(sI - \mathbf{A}\big)^{-1} \big(B_2 - B_1 D_1^\dagger D_2\big) \\ C_\theta &= (D_2^* C_0 + B_2^* Q) \big(I - \Delta Q\big)^{-1} \\ D_v &= I + D_2^* R_0^{-1} D_2. \end{aligned} \tag{3.6}$$

Then Θ is an invertible outer function, that is, both $\Theta(s)$ and $\Theta(s)^{-1}$ are functions in $H^\infty(\mathcal{E}, \mathcal{E})$. Furthermore, Θ is the outer spectral factor for $I - \widetilde{X} X$. To be precise,

$$\widetilde{\Theta}(s) \Theta(s) = I - \widetilde{X}(s) X(s). \tag{3.7}$$

Theorem 3.1 will be proved in the next section. The Proof of Theorem 3.2 will be finished at the end of Sect. 11.

4. Proof of Theorem 3.1

Throughout the section G in $H^\infty(\mathcal{U}, \mathcal{Y})$ and K in $H^\infty(\mathcal{E}, \mathcal{Y})$ are rational functions, and we assume that $\begin{bmatrix} G & K \end{bmatrix}$ is given by the minimal stable realization in (2.1). We first prove three lemmas. The first deals with the rational matrix function $R(s)$ defined by (2.3).

Lemma 4.1. *Let R be the $L^\infty(\mathcal{Y}, \mathcal{Y})$ rational matrix function defined by (2.3). If $T_G T_G^* - T_K T_K^*$ is strictly positive, then T_R is strictly positive.*

Proof. Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive. For each $\alpha \in \mathbb{C}_+$ set $\varphi_\alpha(t) = e^{-\alpha t}$. Notice that $G(s) = D_1 + (\mathfrak{L} g_o)(s)$ where $g_o(t) = C e^{At} B_1$. For

y in $\mathcal{Y}$, we have

$$\begin{aligned}(T_G^* y\varphi_\alpha)(t) &= D_1^* y e^{-\alpha t} + \int_t^\infty g_o(\tau - t)^* y e^{-\alpha\tau} d\tau \\ &= D_1^* y e^{-\alpha t} + \int_t^\infty g_o(\tau - t)^* y e^{-\alpha(\tau - t)} e^{-\alpha t} d\tau \\ &= D_1^* y e^{-\alpha t} + e^{-\alpha t}\int_0^\infty g_o(u)^* y e^{-\alpha u} du \\ &= \Big(D_1^* + \big(\mathfrak{L} g_o^*\big)(\alpha)\Big) y\varphi_\alpha(t) = G(\alpha)^* y\varphi_\alpha(t).\end{aligned}$$

In other words, for $\Re(\alpha) > 0$:

$$\big(T_G^* y\varphi_\alpha\big)(t) = \varphi_\alpha(t) G(\alpha)^* y \qquad (\text{when } \varphi_\alpha(t) = e^{-\alpha t} \text{ and } y \in \mathcal{Y}). \tag{4.1}$$

Using this with the corresponding result for K, we have

$$T_G^* y\varphi_\alpha = \varphi_\alpha G(\alpha)^* y \quad \text{and} \quad T_K^* y\varphi_\alpha = \varphi_\alpha K(\alpha)^* y \qquad (\text{when } y \in \mathcal{Y}). \tag{4.2}$$

Notice that

$$\|\varphi_\alpha\|^2 = (\varphi_\alpha, \varphi_\alpha) = \int_0^\infty e^{-\alpha t} e^{-\overline{\alpha} t} dt = \frac{1}{\alpha + \overline{\alpha}}.$$

Since $T_G T_G^* - T_K T_K^*$ is strictly positive (by assumption), there exists $\eta > 0$ such that $T_G T_G^* - T_K T_K^* \geq \eta I$. Hence for y in $\mathcal{Y}$, we have

$$((T_G T_G^* - T_K T_K^*) y\varphi_\alpha, y\varphi_\alpha) \geq \eta (y\varphi_\alpha, y\varphi_\alpha).$$

This with (4.2) readily implies that

$$\begin{aligned}\frac{\eta\|y\|^2}{\alpha + \overline{\alpha}} &= \eta\|y\varphi_\alpha\|^2 \leq \|T_G^* y\varphi_\alpha\|^2 - \|T_K^* y\varphi_\alpha\|^2 \\ &= \|\varphi_\alpha G(\alpha)^* y\|^2 - \|\varphi_\alpha K(\alpha)^* y\|^2 \\ &= \|\varphi_\alpha\|_{L_+^2}^2 \Big(\|G(\alpha)^* y\|_{\mathcal{U}}^2 - \|K(\alpha)^* y\|_{\mathcal{E}}^2\Big) \\ &= \frac{(G(\alpha)G(\alpha)^* y, y) - (K(\alpha)K(\alpha)^* y, y)}{\alpha + \overline{\alpha}}.\end{aligned}$$

In other words, we have

$$\frac{G(\alpha)G(\alpha)^* - K(\alpha)K(\alpha)^*}{\alpha + \overline{\alpha}} \geq \frac{\eta}{\alpha + \overline{\alpha}} I \qquad (\alpha \in \mathbb{C}_+).$$

Multiplying with $\alpha + \overline{\alpha}$ and taking limits $\alpha \to i\omega$ on the imaginary axis, shows

$$\begin{aligned}R(i\omega) &= G(i\omega)\widetilde{G}(i\omega) - K(i\omega)\widetilde{K}(i\omega) \\ &= G(i\omega)G(i\omega)^* - K(i\omega)K(i\omega)^* \geq \eta I, \qquad (\text{for } -\infty < \omega < \infty).\end{aligned}$$

This is equivalent to T_R being strictly positive. See [21, Chapter 3, Examples and Addenda] and [21, Sect. 6.2, Theorem B]. □

Lemma 4.2. *Let W_{obs} be defined by (2.2), and let Δ be the unique solution of the Lyapunov Eq. (2.7). Then*

$$T_G T_G^* - T_K T_K^* = T_R - W_{obs} \Delta W_{obs}^*. \tag{4.3}$$

In particular, the operator $T_G T_G^* - T_K T_K^*$ is strictly positive if and only if the operator $T_R - W_{obs} \Delta W_{obs}^*$ is strictly positive.

Proof. We first recall some elementary facts concerning Hankel operators. To this end, let

$$(H_G w)(t) = \int_0^\infty g_o(t+\tau) w(\tau) d\tau \qquad (t \geq 0)$$

be the Hankel operator determined by $G(s) = D_1 + \mathfrak{L} g_o(t)$. In a similar way, let H_K be the corresponding Hankel operator mapping $L_+^2(\mathcal{E})$ into $L_+^2(\mathcal{Y})$ determined by K. Let $W_{con,1}$ mapping $L_+^2(\mathcal{U})$ into $\mathcal{X}$ and $W_{con,2}$ mapping $L_+^2(\mathcal{E})$ into $\mathcal{X}$ be the controllability operators defined by

$$W_{con,j} w = \int_0^\infty e^{At} B_j w(t) dt \qquad (\text{for } j = 1, 2).$$

Let P_j be the controllability Gramian defined by $P_j = W_{con,j} W_{con,j}^*$ for $j = 1, 2$. Notice that P_j is the unique solution to the Lyapunov equation

$$A P_j + P_j A^* + B_j B_j^* = 0 \qquad (\text{for } j = 1, 2).$$

Hence $\Delta = P_1 - P_2$. Using $G(s) = D_1 + (\mathfrak{L} g_o)(s)$ where $g_o(t) = C e^{At} B_1$ and the corresponding result $K(s) = D_2 + (\mathfrak{L} k_o)(s)$ where $k_o(t) = C e^{At} B_2$, we see that $H_G = W_{obs} W_{con,1}$ and $H_K = W_{obs} W_{con,2}$. Finally,

$$H_G H_G^* = W_{obs} P_1 W_{obs}^* \quad \text{and} \quad H_K H_K^* = W_{obs} P_2 W_{obs}^*. \tag{4.4}$$

The Wiener–Hopf operators T_{GG^*} and T_{KK^*} are given by the following identities:

$$T_{GG^*} = T_G T_G^* + H_G H_G^* \quad \text{and} \quad T_{KK^*} = T_K T_K^* + H_K H_K^*. \tag{4.5}$$

Using $R = G\widetilde{G} - K\widetilde{K}$, we have

$$\begin{aligned} T_R &= T_G T_G^* - T_K T_K^* + H_G H_G^* - H_K H_K^* \\ &= T_G T_G^* - T_K T_K^* + W_{obs}(P_1 - P_2) W_{obs}^*; \end{aligned} \tag{4.6}$$

see (4.4). This with $\Delta = P_1 - P_2$, yields (4.3). □

Lemma 4.3. *Let M and T be self-adjoint operators on a Hilbert space $\mathfrak{H}$, and let T be strictly positive. Assume that $M = T - WNW^*$, where N is a self-adjoint operator on a Hilbert space $\mathfrak{X}$, and W is a one-to-one operator with a closed range from $\mathfrak{X}$ into $\mathfrak{H}$. Put $Q = W^* T^{-1} W$. Then $Q \gg 0$. Furthermore $M \gg 0$ if and only if $Q^{-1} - N \gg 0$. In this case*

$$M^{-1} = T^{-1} + T^{-1} W N \big(I - QN\big)^{-1} W^* T^{-1}. \tag{4.7}$$

Proof. Multiplying $M = T - WNW^*$ on both sides by $T^{-1/2}$ yields

$$T^{-1/2} M T^{-1/2} = I - T^{-1/2} W N W^* T^{-1/2}.$$

Let $\mathbb{P}$ be the orthogonal projection onto the range of $T^{-1/2} W$. Then

$$T^{-1/2} M T^{-1/2} = I - \mathbb{P} + \mathbb{P} - T^{-1/2} W N W^* T^{-1/2}.$$

Notice that

$$\mathbb{P} = T^{-1/2} W \big(W^* T^{-1} W\big)^{-1} W^* T^{-1/2} = T^{-1/2} W Q^{-1} W^* T^{-1/2}.$$

This readily implies that

$$T^{-1/2}MT^{-1/2} = (I - \mathbb{P}) + T^{-1/2}W\big(Q^{-1} - N\big)W^*T^{-1/2}.$$

The previous equation decomposes $T^{-1/2}MT^{-1/2}$ into the orthogonal sum of two self-adjoint operators. Therefore M is strictly positive if and only if $T^{-1/2}MT^{-1/2}$ is strictly positive, or equivalently, $Q^{-1}-N$ is strictly positive.

Assume that M is strictly positive, or equivalently, $Q^{-1} - N$ is strictly positive. Then

$$\begin{aligned}
M^{-1} &= \big(T - WNW^*\big)^{-1} \\
&= T^{-1}\big(I - WNW^*T^{-1}\big)^{-1} \\
&= T^{-1} + T^{-1}\big(I - WNW^*T^{-1}\big)^{-1}WNW^*T^{-1} \\
&= T^{-1} + T^{-1}WN\big(I - W^*T^{-1}WN\big)^{-1}W^*T^{-1} \\
&= T^{-1} + T^{-1}WN\big(I - QN\big)^{-1}W^*T^{-1}.
\end{aligned}$$

In other words, M^{-1} is given by the formula in (4.7). □

Proof of Theorem 3.1. Assume the operator $T_GT_G^* - T_KT_K^*$ is strictly positive. Then Lemma 4.1 tells us that T_R is strictly positive and item (i) in Theorem 3.1 is fulfilled. Furthermore, applying Lemma 4.3 with

$$M = T_GT_G^* - T_KT_K^*, \quad T = T_R, \quad W = W_{obs} \quad \text{and} \quad N = \Delta,$$

we see that the matrix $Q^{-1} - \Delta$ is strictly positive, and hence item (ii) in Theorem 3.1 is fulfilled. Furthermore, in this case the inversion formula (4.7) yields the formula to compute the inverse of $T_GT_G^* - T_KT_K^*$ in (3.3).

Conversely, assume items (i) and (ii) are satisfied. Item (i) gives $T_R \gg 0$. Then again using Lemma 4.3, we see that item (ii) implies that $T_GT_G^*-T_KT_K^*$ is strictly positive. □

The rest of this paper is devoted to establishing the state space formulas (3.5) for our solution X of the Leech problem and proving both Theorems 1.1 and 3.2.

5. A State Space Realization for $V(s)$

Our solution to the Leech problem is motivated by $X(s) = U(s)V(s)^{-1}$ where $U(s)$ and $V(s)$ are defined in (1.7). The operator Λ is defined by

$$\Lambda = T_G^*\big(T_GT_G^*\big)^{-1}T_K \quad \text{and} \quad T_G\Lambda = T_K. \tag{5.1}$$

Recall that Λ is a strict contraction if and only if $T_GT_G^* - T_KT_K^*$ is strictly positive. The following result provides a state space realization for $V(s)$.

Lemma 5.1. *Assume that $T_GT_G^* - T_KT_K^*$ is strictly positive, or equivalently, that items (i) and (ii) of Theorem 3.1 hold. Consider the function $V(s)$ defined by*

$$V(s) = \big(\mathfrak{L}\,(I - \Lambda^*\Lambda)^{-1}\,\delta\big)(s) \tag{5.2}$$

where $\delta(t)$ is the Dirac delta function. Then a state space realization for $V(s)$ is given by

$$V(s) = (I + D_2^* R_0^{-1} D_2) + (D_2^* C_0 + B_2^* Q)\left(sI - A_0\right)^{-1}\mathbb{B},$$
$$\mathbb{B} = \left(I - \Delta Q\right)^{-1}\Big(B_2 - B_1 D_1^\dagger D_2\Big)\Big(I + D_2^* R_0^{-1} D_2\Big), \tag{5.3}$$

where $D_1^\dagger = D_1^*(D_1 D_1^*)^{-1}$ and Δ is the unique solution of the Lyapunov Eq. (2.7). Because A_0 is stable, V is a function in $H^\infty(\mathcal{E}, \mathcal{E})$.

Recall that A_0 and C_0 are given by (2.10) and that Q solves (2.9). Formula (5.3) for $V(s)$ will play a fundamental role in computing our solution $X(s)$ for the Leech problem.

Proof of Lemma 5.1. By assumption, the operator $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, conditions (i) and (ii) of Theorem 3.1 hold. Hence T_R is strictly positive, the unique stabilizing solution to the algebraic Riccati Eq. (2.9) is given by $Q = W_{obs}^* T_R^{-1} W_{obs}$ and (see Remark 2.1) $Q^{-1} - \Delta$ is also strictly positive. Moreover, Theorem 3.1 shows that

$$T_G T_G^* - T_K T_K^* = T_R - W_{obs}\Delta W_{obs}^*.$$

Furthermore, the inverse is given by

$$\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} = T_R^{-1} + T_R^{-1} W_{obs}\Delta\big(I - Q\Delta\big)^{-1} W_{obs}^* T_R^{-1}.$$

Recall that $\Lambda = T_G^*\big(T_G T_G^*\big)^{-1} T_K$ is a strict contraction. Using this we obtain

$$\begin{aligned}\big(I - \Lambda^*\Lambda\big)^{-1} &= \Big(I - T_K^*\big(T_G T_G^*\big)^{-1} T_K\Big)^{-1} \\ &= I + \Big(I - T_K^*\big(T_G T_G^*\big)^{-1} T_K\Big)^{-1} T_K^*\big(T_G T_G^*\big)^{-1} T_K \\ &= I + T_K^*\Big(I - \big(T_G T_G^*\big)^{-1} T_K T_K^*\Big)^{-1}\big(T_G T_G^*\big)^{-1} T_K.\end{aligned}$$

In other words,

$$\big(I - \Lambda^*\Lambda\big)^{-1} = I + T_K^*\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} T_K. \tag{5.4}$$

This readily implies that

$$\big(I - \Lambda^*\Lambda\big)^{-1} = I + T_K^*\Big(T_R^{-1} + T_R^{-1} W_{obs}\Omega W_{obs}^* T_R^{-1}\Big) T_K$$
$$\Omega = \Delta\big(I - Q\Delta\big)^{-1}.$$

Recall that $W_0 = T_R^{-1} W_{obs}$ where $(W_0 x)(t) = C_0 e^{A_0 t} x$ and $x \in \mathcal{X}$; see Part (iii) of Theorem 13.4 in Sect. 13 "(Appendix 1)" for a discussion of W_0 in the general Riccati setting. In particular, $W_{obs}^* W_0 = Q$. Hence

$$\big(I - \Lambda^*\Lambda\big)^{-1} = I + T_K^*\Big(T_R^{-1} + W_0\Omega W_0^*\Big) T_K.$$

Let us compute $\left((I - \Lambda^*\Lambda)^{-1}\delta\right)(t)$. By employing $R(s) = \Phi(-\overline{s})^*\Phi(s)$, with $\Phi(s)^{-1} = \left(I - C_0(sI - A_0)^{-1}\Gamma\right) R_0^{-1/2}$ [see (3.4)], we obtain

$$T_R^{-1}\delta = T_\Phi^{-1} T_\Phi^{-*}\delta = T_\Phi^{-1} R_0^{-1/2}\delta = R_0^{-1}\delta - W_0\Gamma R_0^{-1}.$$

The calculations involving the Dirac delta function $\delta(t)$ are explained in Sect. 14 "(Appendix 2)". Using this we have

$$\begin{aligned}&\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} T_K\delta = \Big(T_R^{-1} + W_0\Omega W_0^*\Big)T_K\delta\\ &= \Big(T_R^{-1} + W_0\Omega W_0^*\Big)\Big(D_2\delta + W_{obs}B_2\Big)\\ &= T_R^{-1}\delta D_2 + T_R^{-1}W_{obs}B_2 + W_0\Omega W_0^*\Big(D_2\delta + W_{obs}B_2\Big)\\ &= \delta R_0^{-1}D_2 - W_0\Gamma R_0^{-1}D_2 + W_0B_2 + W_0\Omega C_0^* D_2 + W_0\Omega Q B_2\\ &= \delta R_0^{-1}D_2 + W_0\Big(B_2 - \Gamma R_0^{-1}D_2 + \Omega C_0^* D_2 + \Omega Q B_2\Big)\\ &= \delta R_0^{-1}D_2 + W_0\Big(\Omega C_0^* D_2 - \Gamma R_0^{-1}D_2 + (I+\Omega Q)B_2\Big)\end{aligned}$$

In other words,

$$\begin{aligned}&\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} T_K\delta(t) = \delta R_0^{-1}D_2 + W_0\mathbb{B}\\ &\mathbb{B} = \Omega C_0^* D_2 - \Gamma R_0^{-1}D_2 + (I+\Omega Q)B_2\\ &\Omega = \Delta\big(I - Q\Delta\big)^{-1}.\end{aligned} \tag{5.5}$$

Let us simplify the expression for $\mathbb{B}$. To this end,

$$I + \Omega Q = I + \Delta\big(I - Q\Delta\big)^{-1}Q = I + \Delta Q\big(I - \Delta Q\big)^{-1} = \big(I - \Delta Q\big)^{-1}.$$

Using this and (5.5) gives

$$\mathbb{B} = \Big(\Omega C_0^* - \Gamma R_0^{-1}\Big)D_2 + \big(I - Q\Delta\big)^{-1}B_2,$$

and

$$\begin{aligned}&\big(\Omega C_0^* - \Gamma R_0^{-1}\big)D_2 = \big(\Delta(I - Q\Delta)^{-1}(C^* - Q\Gamma) - \Gamma\big)\, R_0^{-1}D_2\\ &= \big(\Omega C^* - (I - \Delta Q)^{-1}\Gamma\big)R_0^{-1}D_2\\ &= \big((I - \Delta Q)^{-1}\Delta C^* - (I - \Delta Q)^{-1}\Gamma\big)R_0^{-1}D_2.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{B} &= \big((I - \Delta Q)^{-1}\Delta C^* - (I - \Delta Q)^{-1}\Gamma\big)R_0^{-1}D_2 + \big(I - \Delta Q\big)^{-1}B_2\\ &= (I - \Delta Q)^{-1}\big((\Delta C^* - \Gamma)R_0^{-1}D_2 + B_2\big).\end{aligned}$$

Next

$$\begin{aligned}&(\Delta C^* - \Gamma)R_0^{-1}D_2 + B_2 = \big(\Delta C^* - (B_1D_1^* - B_2D_2^* + \Delta C^*)\big)R_0^{-1}D_2 + B_2\\ &= (B_2D_2^* - B_1D_1^*)R_0^{-1}D_2 + B_2 = B_2\big(I + D_2^*R_0^{-1}D_2\big) - B_1D_1^*R_0^{-1}D_2.\end{aligned}$$

And therefore

$$\mathbb{B} = (I - \Delta Q)^{-1}\Big(B_2\big(I + D_2^*R_0^{-1}D_2\big) - B_1D_1^*R_0^{-1}D_2\Big). \tag{5.6}$$

Now observe that

$$\begin{aligned} D_1^*R_0^{-1}D_2\big(I+D_2^*R_0^{-1}D_2\big)^{-1} &= D_1^*\big(I+R_0^{-1}D_2D_2^*\big)^{-1}R_0^{-1}D_2 \\ &= D_1^*\big(R_0+D_2D_2^*\big)^{-1}D_2 \\ &= D_1^*\big(D_1D_1^*\big)^{-1}D_2 = D_1^\dagger D_2. \end{aligned}$$

This readily implies that

$$D_1^*R_0^{-1}D_2\big(I+D_2^*R_0^{-1}D_2\big)^{-1} = D_1^\dagger D_2 \tag{5.7}$$

where $D_1^\dagger = D_1^*\big(D_1D_1^*\big)^{-1}$ is the Moore-Penrose inverse of D_1. Using this in (5.6), we obtain the formula for $\mathbb{B}$ in Eq. (5.3) of Lemma 5.1, that we have been looking for, that is,

$$\mathbb{B} = \big(I-\Delta Q\big)^{-1}\Big(B_2 - B_1D_1^\dagger D_2\Big)\Big(I+D_2^*R_0^{-1}D_2\Big). \tag{5.8}$$

Now that we have a formula for $\mathbb{B}$, notice that

$$\left(T_K^*W_0\mathbb{B}\right)(t) = D_2^*W_0(t)\mathbb{B} + \int_t^\infty B_2^*e^{A^*(\tau-t)}C^*C_0e^{A_0\tau}\mathbb{B}d\tau.$$

Further notice

$$\begin{aligned} &\int_t^\infty B_2^*e^{A^*(\tau-t)}C^*C_0e^{A_0\tau}\mathbb{B}d\tau = \int_t^\infty B_2^*e^{A^*(\tau-t)}C^*C_0e^{A_0(\tau-t)}e^{A_0t}\mathbb{B}d\tau \\ &= \int_0^\infty B_2^*e^{A^*\xi}C^*C_0e^{A_0\xi}e^{A_0t}\mathbb{B}d\xi = B_2^*W_{obs}^*W_0e^{A_0t}\mathbb{B} = B_2^*Qe^{A_0t}\mathbb{B}. \end{aligned}$$

For the last equality consult the formulas (13.16) and (13.17) in Sect. 13. Thus we have

$$\left(T_K^*W_0\mathbb{B}\right)(t) = D_2^*W_0(t)\mathbb{B} + B_2^*Qe^{A_0t}\mathbb{B} \quad (t\geq 0). \tag{5.9}$$

This with (5.5) and (5.4) yields

$$\begin{aligned} (I-\Lambda^*\Lambda)^{-1}\delta &= I\delta + T_K^*\Big(T_GT_G^* - T_KT_K^*\Big)^{-1}T_K\delta \\ &= I\delta + T_K^*\Big(\delta R_0^{-1}D_2 + W_0\mathbb{B}\Big) \\ &= \big(I+D_2^*R_0^{-1}D_2\big)\delta + \Big(D_2^*C_0 + B_2^*Q\Big)e^{A_0t}\mathbb{B}. \end{aligned}$$

By taking the Laplace transform of the previous result, we arrive at the state space realization for $V(s) = \big(\mathfrak{L}(I-\Lambda^*\Lambda)^{-1}\delta\big)(s)$ in Eq. (5.3) of Lemma 5.1 that we have been looking for, that is,

$$V(s) = (I+D_2^*R_0^{-1}D_2) + (D_2^*C_0 + B_2^*Q)\left(sI-A_0\right)^{-1}\mathbb{B}. \tag{5.10}$$

This completes the Proof of Lemma 5.1. □

6. A State Space Realization for $U(s)$

The following result provides a state space formula for the function $U(s)$.

Lemma 6.1. *Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, that items (i) and (ii) of Theorem 3.1 hold. Consider the function $U(s)$ defined by*

$$U(s) = \left(\mathfrak{L}\Lambda\left(I - \Lambda^*\Lambda\right)^{-1}\delta\right)(s). \tag{6.1}$$

Then a state space realization for $U(s)$ is given by

$$\begin{aligned} U(s) &= D_1^* R_0^{-1} D_2 + (D_1^* C_0 + B_1^* Q)(sI - A_0)^{-1}\mathbb{B} \\ \mathbb{B} &= \left(I - \Delta Q\right)^{-1}\Big(B_2 - B_1 D_1^\dagger D_2\Big)\Big(I + D_2^* R_0^{-1} D_2\Big). \end{aligned} \tag{6.2}$$

Because A_0 is stable, U is a function in $H^\infty(\mathcal{E},\mathcal{U})$.

Once we have established our state space realizations for $U(s)$ and $V(s)$, we will directly verify that $U(s)V(s)^{-1}$ is a stable rational solution to our Leech problem, that is,

$$X(s) = U(s)V(s)^{-1} \quad \text{and} \quad G(s)X(s) = K(s) \quad \text{and} \quad \|X\|_\infty < 1. \tag{6.3}$$

Proof of Lemma 6.1. By assumption $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, conditions (i) and (ii) of Theorem 3.1 hold, or equivalently, Λ is a strict contraction. By employing (5.4), we have

$$\begin{aligned} \Lambda(I - \Lambda^*\Lambda)^{-1} &= \Lambda + \Lambda T_K^*\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} T_K \\ &= T_G^*\big(T_G T_G^*\big)^{-1} T_K + T_G^*\big(T_G T_G^*\big)^{-1} T_K T_K^*\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} T_K \\ &= T_G^*\big(T_G T_G^*\big)^{-1}\Big(I + T_K T_K^*\big(T_G T_G^* - T_K T_K^*\big)^{-1}\Big) T_K \\ &= T_G^*\big(T_G T_G^*\big)^{-1}\Big(T_G T_G^* - T_K T_K^* + T_K T_K^*\Big)\big(T_G T_G^* - T_K T_K^*\big)^{-1} T_K \\ &= T_G^*\big(T_G T_G^* - T_K T_K^*\big)^{-1} T_K. \end{aligned}$$

In other words,

$$\Lambda(I - \Lambda^*\Lambda)^{-1} = T_G^*\big(T_G T_G^* - T_K T_K^*\big)^{-1} T_K. \tag{6.4}$$

By consulting (5.5), we have

$$\begin{aligned} \Lambda(I - \Lambda^*\Lambda)^{-1}\delta &= T_G^*\Big(T_G T_G^* - T_K T_K^*\Big)^{-1} T_K\delta \\ &= T_G^*\Big(\delta R_0^{-1} D_2 + W_0\mathbb{B}\Big) \\ &= D_1^* R_0^{-1} D_2\delta + T_G^* W_0\mathbb{B}. \end{aligned} \tag{6.5}$$

The last equality follows by using the properties of the Dirac delta function in Sect. 14 "(Appendix 2)". Replacing K with G in (5.9), we obtain

$$\left(T_G^* W_0\mathbb{B}\right)(t) = \Big(D_1^* C_0 + B_1^* Q\Big)e^{A_0 t}\mathbb{B}. \tag{6.6}$$

By taking the Laplace transform with (6.1), we have

$$\begin{aligned} U(s) &= D_1^* R_0^{-1} D_2 + C_u \big(sI - A_0\big)^{-1} \mathbb{B} \\ C_u &= D_1^* C_0 + B_1^* Q. \end{aligned} \tag{6.7}$$

Here $U(s) = \big(\mathfrak{L}\Lambda(I - \Lambda^*\Lambda)^{-1}\delta\big)(s)$. This yields the state space realization for $U(s)$ in Eq. (6.2) above, and completes the Proof of Lemma 6.1. □

7. A State Space Realization for $V(s)^{-1}$

To compute our solution $X(s) = U(s)V(s)^{-1}$ to the Leech problem, we need to take the inverse of $V(s)$. Recall that

$$\begin{aligned} V(s) &= D_v + C_v(sI - A_0)^{-1}\mathbb{B} \\ D_v &= (I + D_2^* R_0^{-1} D_2) \\ C_v &= D_2^* C_0 + B_2^* Q \\ \mathbb{B} &= \big(I - \Delta Q\big)^{-1}\Big(B_2 - B_1 D_1^\dagger D_2\Big) D_v \\ \Xi &= \big(I - \Delta Q\big)^{-1}. \end{aligned} \tag{7.1}$$

Using standard state space techniques,

$$V(s)^{-1} = D_v^{-1} - D_v^{-1} C_v (sI - \mathbb{A})^{-1} \mathbb{B} D_v^{-1}. \tag{7.2}$$

The "feedback operator" $\mathbb{A}$ is defined by

$$\mathbb{A} = A_0 - \mathbb{B} D_v^{-1} C_v = A_0 - \Xi\Big(B_2 - B_1 D_1^\dagger D_2\Big) C_v. \tag{7.3}$$

The following result expresses $V(s)^{-1}$ is a slightly different form.

Lemma 7.1. *Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, that items (i) and (ii) of Theorem 3.1 hold. Consider the function $V(s)$ in (7.1), or equivalently, $V(s) = \big(\mathfrak{L}\,(I - \Lambda^*\Lambda)^{-1}\,\delta\big)(s)$. Then a state space realization for $V(s)^{-1}$ is given by*

$$\begin{aligned} V(s)^{-1} &= D_v^{-1} - D_v^{-1} C_v (I - \Delta Q)^{-1} \big(sI - \mathbf{A}\big)^{-1} \big(B_2 - B_1 D_1^\dagger D_2\big), \\ \mathbf{A} &= \Xi^{-1} \mathbb{A} \Xi = A - B_1 D_1^\dagger C - B_1 \big(I - D_1^\dagger D_1\big) B_1^* Q \Xi. \end{aligned} \tag{7.4}$$

Moreover, the operator $\mathbf{A}$ is stable. In particular, $V(s)$ is an invertible outer function, that is, both $V(s)$ and its inverse $V(s)^{-1}$ are functions in $H^\infty(\mathcal{E}, \mathcal{E})$.

Proof of (7.4). Let us derive the formula for $\mathbf{A}$ in (7.4). Later in Sect. 9, we will show that $\mathbf{A}$ is stable. Recall that Δ is the solution to the Lyapunov equation

$$A\Delta + \Delta A^* + B_1 B_1^* - B_2 B_2^* = 0$$

and that Q is the stabilizing solution of the Riccati Eq. (2.9); see also (13.14) in Sect. 13. To simplify (7.4), let us compute

$$\begin{aligned}\Xi^{-1}A - A\Xi^{-1} &= \big(I-\Delta Q\big)A - A\big(I-\Delta Q\big)\\ &= -\Delta QA + A\Delta Q\\ &= \Delta A^*Q + \Delta C_0^*R_0C_0 - \Delta A^*Q - B_1B_1^*Q + B_2B_2^*Q\\ &= \Delta C_0^*R_0C_0 - B_1B_1^*Q + B_2B_2^*Q.\end{aligned}$$

In other words,

$$\Xi^{-1}A - A\Xi^{-1} = \Delta C_0^*R_0C_0 - B_1B_1^*Q + B_2B_2^*Q. \tag{7.5}$$

Using this we have

$$\begin{aligned}\Xi^{-1}A_0 &= \Xi^{-1}A - \Xi^{-1}\Gamma C_0\\ &= A\Xi^{-1} + \Delta C_0^*R_0C_0 - B_1B_1^*Q + B_2B_2^*Q - \Xi^{-1}\Gamma C_0\\ &= A\Xi^{-1} + \Big(\Delta C_0^*R_0 - \Gamma + \Delta Q\Gamma\Big)C_0 - B_1B_1^*Q + B_2B_2^*Q.\end{aligned}$$

Furthermore, using (2.10) (see also (13.9) in Sect. 13 "(Appendix 1)") and (13.4) yields the following

$$\begin{aligned}\Delta C_0^*R_0 - \Gamma + \Delta Q\Gamma = \Delta\big(C^* - Q\Gamma\big) - \Gamma + \Delta Q\Gamma &= \Delta C^* - \Gamma\\ &= -B_1D_1^* + B_2D_2^*.\end{aligned}$$

Thus,

$$\Xi^{-1}A_0 = A\Xi^{-1} - \big(B_1D_1^* - B_2D_2^*\big)C_0 - \big(B_1B_1^* - B_2B_2^*\big)Q. \tag{7.6}$$

Notice that

$$\Xi^{-1}A_0 = A\Xi^{-1} - B_1\big(D_1^*C_0 + B_1^*Q\big) + B_2\big(D_2^*C_0 + B_2^*Q\big)$$

By applying the definitions of C_u and C_v in (6.7) and (7.1), we have

$$\Xi^{-1}A_0 = A\Xi^{-1} - B_1C_u + B_2C_v. \tag{7.7}$$

Using this we obtain from (7.3) that

$$\begin{aligned}\Xi^{-1}\mathbb{A} &= \Xi^{-1}A_0 - \Big(B_2 - B_1D_1^\dagger D_2\Big)C_v\\ &= A\Xi^{-1} - B_1C_u + B_2C_v - \Big(B_2 - B_1D_1^\dagger D_2\Big)C_v\\ &= A\Xi^{-1} - B_1C_u + B_1D_1^\dagger D_2C_v.\end{aligned}$$

In other words,

$$\Xi^{-1}\mathbb{A} = A\Xi^{-1} - B_1\Big(C_u - D_1^\dagger D_2C_v\Big). \tag{7.8}$$

A formula for $C_u - D_1^\dagger D_2C_v$. To simplify the formula involving $\mathbb{A}$ in (7.8), we need to work on $C_u - D_1^\dagger D_2C_v$. To this end, notice that

$$\begin{aligned}C_u - D_1^\dagger D_2C_v &= D_1^*C_0 + B_1^*Q - D_1^\dagger D_2\big(D_2^*C_0 + B_2^*Q\big)\\ &= D_1^*\big(I - (D_1D_1^*)^{-1}D_2D_2^*\big)C_0 + \big(B_1^* - D_1^\dagger D_2B_2^*\big)Q.\end{aligned}$$

To simplify the last expression, observe that

$$
\begin{aligned}
&D_1^*\big(I-(D_1D_1^*)^{-1}D_2D_2^*\big)C_0\\
&\quad= D_1^*\big(I-(D_1D_1^*)^{-1}D_2D_2^*\big)R_0^{-1}\big(C-\Gamma^*Q\big)\\
&\quad= D_1^*(D_1D_1^*)^{-1}\big(D_1D_1^*-D_2D_2^*\big)R_0^{-1}\big(C-\Gamma^*Q\big)\\
&\quad= D_1^\dagger\big(C-\Gamma^*Q\big).
\end{aligned}
$$

Using this in our previous formula, we have

$$
\begin{aligned}
C_u-D_1^\dagger D_2C_v &= D_1^\dagger\big(C-\Gamma^*Q\big)+\big(B_1^*-D_1^\dagger D_2B_2^*\big)Q\\
&= D_1^\dagger C+B_1^*Q-D_1^\dagger\big(\Gamma^*+D_2B_2^*\big)Q\\
&= D_1^\dagger C+B_1^*Q-D_1^\dagger\big(D_1B_1^*+C\Delta\big)Q\\
&= D_1^\dagger C\Big(I-\Delta Q\Big)+\big(I-D_1^\dagger D_1\big)B_1^*Q.
\end{aligned}
$$

In other words,

$$
C_u-D_1^\dagger D_2C_v = D_1^\dagger C\Big(I-\Delta Q\Big)+\big(I-D_1^\dagger D_1\big)B_1^*Q. \tag{7.9}
$$

Using $\Xi=\big(I-\Delta Q\big)^{-1}$ in (7.8), we obtain the result that we have been looking for, that is,

$$
\Xi^{-1}\mathbb{A}=A\Xi^{-1}-B_1D_1^\dagger C\Xi^{-1}-B_1\big(I-D_1^\dagger D_1\big)B_1^*Q. \tag{7.10}
$$

Multiplying by Ξ on the right, yields the following formula for $\mathbf{A}$ in (7.4):

$$
\mathbf{A}=\Xi^{-1}\mathbb{A}\Xi=A-B_1D_1^\dagger C-B_1\big(I-D_1^\dagger D_1\big)B_1^*Q\Xi. \tag{7.11}
$$

To complete the Proof of Lemma 7.1, it remains to show that $\mathbb{A}$ is stable. This will be proved in Sect. 9. □

Because $\mathbf{A}$ is similar to $\mathbb{A}$, it follows that the operator $\mathbf{A}$ is also stable. In particular, $V(s)^{-1}$ is a function in $H^\infty(\mathcal{E},\mathcal{E})$. Using the state space formula for $V(s)^{-1}$ in (7.2), with $\mathbf{A}$ defined in (7.11), we arrive at the state space formula for $V(s)^{-1}$ in (7.4).

8. The State Space Realization for $X(s)=U(s)V(s)^{-1}$

We are now ready to compute $X(s)=U(s)V(s)^{-1}$, which will turn out to be our solution to the Leech problem. For convenience recall that V and U are given by

$$
\begin{aligned}
U(s) &= D_1^*R_0^{-1}D_2+C_u\big(sI-A_0\big)^{-1}\mathbb{B}\\
V(s) &= D_v+C_v(sI-A_0)^{-1}\mathbb{B}\\
C_u &= D_1^*C_0+B_1^*Q\\
C_v &= D_2^*C_0+B_2^*Q\\
D_v &= (I+D_2^*R_0^{-1}D_2)\\
\mathbb{B} &= \big(I-\Delta Q\big)^{-1}\Big(B_2-B_1D_1^\dagger D_2\Big)D_v.
\end{aligned} \tag{8.1}
$$

Proposition 8.1. *Given the formulas for $U(s)$ and $V(s)$ in (8.1), and set $X(s)=U(s)V(s)^{-1}$. Then a state space realization for X is given by*

$$X(s)=D_1^\dagger D_2+C_x\big(sI-\mathbf{A}\big)^{-1}\big(B_2-B_1D_1^\dagger D_2\big)$$
where (8.2)

$$C_x=D_1^\dagger C+\big(I-D_1^\dagger D_1\big)B_1^*Q\Xi=\big(C_u-D_1^\dagger D_2C_v\big)\Xi \tag{8.3}$$

$$\mathbf{A}=A-B_1C_x. \tag{8.4}$$

Finally, the operator $\mathbf{A}$ is stable. In particular, $X(s)$ is a rational function in $H^\infty(\mathcal{E},\mathcal{U})$.

This establishes the formula for $X(s)$ in (3.5) in Theorem 3.2. In Sect. 11 we will show that $\|X\|_\infty<1$.

Derivation $X(s)$ in Proposition 8.1. To compute $X(s)=U(s)V(s)^{-1}$, first notice that

$$\begin{aligned}D_1^*R_0^{-1}D_2D_v^{-1}&=D_1^*R_0^{-1}D_2\big(I+D_2^*R_0^{-1}D_2\big)^{-1}\\&=D_1^*\big(I+R_0^{-1}D_2D_2^*\big)^{-1}R_0^{-1}D_2=D_1^*\big(R_0+D_2D_2^*\big)^{-1}D_2\\&=D_1^*\big(D_1D_1^*\big)^{-1}D_2=D_1^\dagger D_2.\end{aligned}$$

In other words, the constant term X_∞ of $X(s)$ is

$$D_1^*R_0^{-1}D_2D_v^{-1}=D_1^\dagger D_2. \tag{8.5}$$

By employing standard state space techniques, we obtain

$$V(s)^{-1}=D_v^{-1}-D_v^{-1}C_v(sI-\mathbb{A})^{-1}\mathbb{B}D_v^{-1}. \tag{8.6}$$

This with the definition of $\mathbb{A}=A_0-\mathbb{B}D_v^{-1}C_v$ in (7.3) yields

$$\begin{aligned}X(s)&=\Big(D_1^*R_0^{-1}D_2+C_u\big(sI-A_0\big)^{-1}\mathbb{B}\Big)\Big(D_v^{-1}-D_v^{-1}C_v\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\Big)\\&=D_1^\dagger D_2+C_u\big(sI-A_0\big)^{-1}\mathbb{B}D_v^{-1}-D_1^\dagger D_2C_v\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\\&\quad-C_u\big(sI-A_0\big)^{-1}\mathbb{B}D_v^{-1}C_v\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\\&=D_1^\dagger D_2-D_1^\dagger D_2C_v\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\\&\quad+C_u\big(sI-A_0\big)^{-1}\Big((sI-\mathbb{A})-\mathbb{B}D_v^{-1}C_v\Big)\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\\&=D_1^\dagger D_2+\big(C_u-D_1^\dagger D_2C_v\big)\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}.\end{aligned}$$

Recall that $\Xi=(I-\Delta Q)^{-1}$. Using this we have

$$\begin{aligned}X(s)&=D_1^\dagger D_2+\big(C_u-D_1^\dagger D_2C_v\big)\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\\&=D_1^\dagger D_2+\Big(D_1^\dagger C+\big(I-D_1^\dagger D_1\big)B_1^*Q\Xi\Big)\Xi^{-1}\big(sI-\mathbb{A}\big)^{-1}\mathbb{B}D_v^{-1}\\&=D_1^\dagger D_2+\Big(D_1^\dagger C+\big(I-D_1^\dagger D_1\big)B_1^*Q\Xi\Big)\big(sI-\mathbf{A}\big)^{-1}\Xi^{-1}\mathbb{B}D_v^{-1};\end{aligned}$$

see (7.9) and (7.11). This yields the formula for $X(s)$ in (8.2). In Sect. 9 we will prove that $\mathbf{A}$ is stable.

9. Proof of the Stability of $\mathbb{A}$ using a Lyapunov Equation

In this section, we will directly show that V^{-1} is analytic on the closed right half plane. To accomplish this we will prove that $\mathbb{A}$ is stable. Since $\mathbf{A}$ is similar to $\mathbb{A}$, the operator $\mathbf{A}$ is also stable. This guarantees that $V(s)^{-1}$ is a function in $H^\infty(\mathcal{E}, \mathcal{E})$; see (8.6). Since A_0 is stable, $V(s)$ is also a function in $H^\infty(\mathcal{E}, \mathcal{E})$; see (8.1). In particular, $V(s)$ is an invertible outer function. Because $U(s)$ is in $H^\infty(\mathcal{E}, \mathcal{U})$, the function $X(s) = U(s)V(s)^{-1}$ is a function in $H^\infty(\mathcal{E}, \mathcal{U})$. In Sect. 11 we will show that $\|X\|_\infty < 1$.

Recall

$$\begin{aligned} V(s) &= D_v + C_v(sI - A_0)^{-1}\mathbb{B} \\ V(s)^{-1} &= D_v^{-1} - D_v^{-1}C_v(sI - \mathbb{A})^{-1}\mathbb{B}D_v^{-1}, \end{aligned} \tag{9.1}$$

where

$$\begin{aligned} \mathbb{A} &= A_0 - \mathbb{B}D_v^{-1}C_v = A_0 - \Xi\Big(B_2 - B_1 D_1^\dagger D_2\Big)C_v, \\ \mathbb{B} &= \Xi\Big(B_2 - B_1 D_1^\dagger D_2\Big)D_v, \\ \Xi &= \big(I - \Delta Q\big)^{-1}. \end{aligned} \tag{9.2}$$

Throughout this section Q is the stabilizing solution for the algebraic Riccati Eq. (2.9).

In order to show that $\mathbb{A}$ is stable, we will first establish the following lemma.

Lemma 9.1. *Let $\mathbb{P}$ be the strictly positive operator defined by*

$$\mathbb{P} = Q - Q\Delta Q = Q\Xi^{-1}. \tag{9.3}$$

Then $\mathbb{A}$ satisfies the Lyapunov equation

$$\mathbb{A}^*\mathbb{P} + \mathbb{P}\mathbb{A} + F^*F = 0 \tag{9.4}$$

where the operator F, the strict contraction Z and the isometry E are given by

$$F = \begin{bmatrix} (I - Z^*Z)^{1/2}C_v \\ (I - EE^*)^{1/2}C_u \\ ZC_v - E^*C_u \end{bmatrix}, \tag{9.5}$$

$$Z = \big(D_1D_1^*\big)^{-1/2}D_2 \quad \textit{and} \quad E = D_1^*\big(D_1D_1^*\big)^{-1/2}. \tag{9.6}$$

Proof. Because $Q^{-1} - \Delta$ is strictly positive (see item (ii) in Theorem 3.1), the operator

$$\mathbb{P} = Q - Q\Delta Q = Q\left(Q^{-1} - \Delta\right)Q$$

is also strictly positive. The first step is to prove that

$$C_v^*C_v - C_u^*C_u = A_0^*(Q - Q\Delta Q) + (Q - Q\Delta Q)A_0. \tag{9.7}$$

Using the formulas for C_u and C_v in (8.1) and the definition of Γ in (2.6) we have that

$$\begin{aligned}D_1C_u - D_2C_v &= R_0C_0 + D_1B_1^*Q - D_2B_2^*Q\\ &= C - \Gamma^*Q + \Gamma^*Q - C\Delta Q = C(I-\Delta Q) = C\Xi^{-1}.\end{aligned}$$

Equation (7.7) states that

$$\Xi^{-1}A_0 = A\Xi^{-1} - B_1C_u + B_2C_v.$$

Thus

$$\begin{aligned}(Q - Q\Delta Q)A_0 &= Q\Xi^{-1}A_0 = QA\Xi^{-1} - QB_1C_u + QB_2C_v\\ &= QA\Xi^{-1} - C_u^*C_u + C_0^*D_1C_u + C_v^*C_v - C_0^*D_2C_v\\ &= QA\Xi^{-1} + C_v^*C_v - C_u^*C_u + C_0^*\big(D_1C_u - D_2C_v\big)\\ &= QA\Xi^{-1} + C_v^*C_v - C_u^*C_u + C_0^*C\Xi^{-1}.\end{aligned}$$

Recall that $Q = W_{obs}^*W_0$. This identity is a consequence of the algebraic Riccati Eq. (13.14); see the defination of W_0 in (2.11), the Lyapunov Eqs. (13.20) and (13.23) with Theorem 13.4 in Sect. 13 "(Appendix 1)". Because Q is self-adjoint $Q = W_0^*W_{obs}$. This yields the Lyapunov equation $QA + A_0^*Q + C_0^*C = 0$. Using this equation, we obtain

$$\begin{aligned}(Q - Q\Delta Q)A_0 &= QA\Xi^{-1} + C_v^*C_v - C_u^*C_u - \big(QA + A_0^*Q\big)\Xi^{-1}\\ &= C_v^*C_v - C_u^*C_u - A_0^*Q\Xi^{-1}\\ &= C_v^*C_v - C_u^*C_u - A_0^*(Q - Q\Delta Q).\end{aligned}$$

Therefore (9.7) holds.

Recall [see (9.3)] that $\mathbb{P} = Q - Q\Delta Q = Q\Xi^{-1}$, and observe that

$$\begin{aligned}\mathbb{P}\mathbb{A} &= \mathbb{P}(A_0 - \mathbb{B}D_v^{-1}C_v)\\ &= -A_0^*\mathbb{P} + C_v^*C_v - C_u^*C_u - \mathbb{P}\Xi\Big(B_2 - B_1D_1^\dagger D_2\Big)C_v\\ &= -A_0^*\mathbb{P} + C_v^*C_v - C_u^*C_u - \Big(QB_2 - QB_1D_1^\dagger D_2\Big)C_v\\ &= -A_0^*\mathbb{P} + C_v^*C_v - C_u^*C_u - \Big(C_v^* - C_0^*D_2 - QB_1D_1^\dagger D_2\Big)C_v.\end{aligned}$$

For the last equality in this calculation, we used the formula for C_v in (8.1). So we have

$$\mathbb{P}\mathbb{A} = -A_0^*\mathbb{P} - C_u^*C_u + \Big(C_0^*D_2 + QB_1D_1^\dagger D_2\Big)C_v. \tag{9.8}$$

By taking the adjoint we see that

$$\mathbb{A}^*\mathbb{P} = -\mathbb{P}A_0 - C_u^*C_u + C_v^*\Big(C_0^*D_2 + QB_1D_1^\dagger D_2\Big)^*. \tag{9.9}$$

Note that (9.7) gives

$$0 = \mathbb{P}A_0 + A_0^*\mathbb{P} - C_v^*C_v + C_u^*C_u. \tag{9.10}$$

Adding the equalities (9.8), (9.9) and (9.10) yields

$$\mathbb{A}^*\mathbb{P} + \mathbb{P}\mathbb{A} = -C_v^* C_v - C_u^* C_u + \left(C_0^* D_2 + QB_1 D_1^\dagger D_2\right) C_v + \\ + C_v^* \left(C_0^* D_2 + QB_1 D_1^\dagger D_2\right)^*.$$

Now observe that, using the formula for C_u in (8.1) with $D_1 D_1^\dagger = I$,

$$C_0^* D_2 + QB_1 D_1^\dagger D_2 = C_0^* D_2 + (C_u^* - C_0^* D_1) D_1^\dagger D_2 = C_u^* D_1^\dagger D_2.$$

This yields the Lyapunov equation that we have been looking for, that is,

$$\mathbb{A}^*\mathbb{P} + \mathbb{P}\mathbb{A} + C_v^* C_v + C_u^* C_u = C_u^* D_1^\dagger D_2 C_v + C_v^* \left(D_1^\dagger D_2\right)^* C_u. \quad (9.11)$$

Next we will transform (9.11) into (9.4). Recall that $D_1 D_1^* - D_2 D_2^* = R_0 \gg 0$. Thus

$$I - \left(D_1 D_1^*\right)^{-1/2} D_2 D_2^* \left(D_1 D_1^*\right)^{-1/2} \gg 0.$$

In other words, $Z = \left(D_1 D_1^*\right)^{-1/2} D_2$ is a strict contraction, that is, $\|Z\| < 1$. The operator E given by

$$E = D_1^* \left(D_1 D_1^*\right)^{-1/2}$$

is an isometry. Using, $D_1^\dagger D_2 = EZ$, we see that $D_1^\dagger D_2$ is a strict contraction.

Notice that

$$\begin{aligned} &C_v^* C_v + C_u^* C_u - C_u^* EZC_v - C_v Z^* E^* C_u \\ &\quad = C_v^*(I - Z^* Z) C_v + C_u^*(I - EE^*) C_u \\ &\qquad + C_v^* Z^* Z C_v + C_u^* EE^* C_u - C_u^* EZC_v - C_v Z^* E^* C_u \\ &\quad = C_v^*(I - Z^* Z) C_v + C_u^*(I - EE^*) C_u \\ &\qquad + \left(ZC_v - E^* C_u\right)^* \left(ZC_v - E^* C_u\right). \end{aligned}$$

So we can rewrite the Lyapunov Eq. (9.11) as

$$\begin{aligned} &\mathbb{A}^*\mathbb{P} + \mathbb{P}\mathbb{A} + C_v^*(I - Z^* Z) C_v + C_u^*(I - EE^*) C_u \\ &\quad + \left(ZC_v - E^* C_u\right)^* \left(ZC_v - E^* C_u\right) = 0, \end{aligned} \quad (9.12)$$

which can be rewritten as (9.4). □

Proposition 9.2. *The operator* $\mathbb{A}$ *in* (9.2) *is stable.*

Proof. Recall that $\mathbb{A}$ satisfies the Lyapunov equation

$$\mathbb{A}^*\mathbb{P} + \mathbb{P}\mathbb{A} + F^* F = 0, \quad (9.13)$$

where $\mathbb{P}$ and F are given by (9.3) and (9.5), respectively. Assume that λ is an eigenvalue for $\mathbb{A}$ with eigenvector x, that is, $\mathbb{A}x = \lambda x$. Using this in the Lyapunov Eq. (9.13), we have

$$0 = (\mathbb{A}^*\mathbb{P}x, x) + (\mathbb{P}\mathbb{A}x, x) + (F^* F x, x) = (\overline{\lambda} + \lambda)(\mathbb{P}x, x) + \|Fx\|^2.$$

Notice that $\mathbb{P} = Q\left(Q^{-1} - \Delta\right)Q$ is strictly positive. Hence $(\mathbb{P}x, x) > 0$ and

$$2\Re(\lambda) = -\frac{\|Fx\|^2}{(\mathbb{P}x, x)}.$$

We will prove that $\Re(\lambda)$ is nonzero. Assume that $\Re(\lambda) = 0$, then we have that $Fx = 0$. So in particular $(I - Z^*Z)^{1/2}C_v x = 0$. Because Z is strictly contractive, $(I - Z^*Z)^{1/2}$ is invertible, and thus $C_v x = 0$. Using this with the definition of $\mathbb{A}$, we see that

$$\lambda x = \mathbb{A}x = \big(A_0 - \mathbb{B}D_v^{-1}C_v\big)x = A_0 x.$$

Thus λ is an eigenvalue for the stable operator A_0. But this means that $\Re(\lambda) < 0$, which contradicts the assumption that $\Re(\lambda) = 0$.

We conclude that $\Re(\lambda) < 0$ and therefore λ is a stable eigenvalue for $\mathbb{A}$. This proves that the operator $\mathbb{A}$ is stable. □

Finally, since $\mathbb{A}$ is similar to $\mathbf{A}$, the operator $\mathbf{A}$ is also stable.

10. A Direct Proof of $G(s)X(s) = K(s)$

So far we have derived a state space formula for $X(s)$; see (3.5) in Theorem 3.2. Because $\mathbb{A}$ is stable and $\mathbb{A}$ is similar to $\mathbf{A}$, the operator $\mathbf{A}$ is stable and $X(s)$ is a rational function in $H^\infty(\mathcal{E}, \mathcal{U})$. In this section, we will directly prove that $G(s)X(s) = K(s)$. For convenience recall that

$$\begin{aligned}
X(s) &= D_1^\dagger D_2 + C_x\big(sI - \mathbf{A}\big)^{-1}\big(B_2 - B_1 D_1^\dagger D_2\big)\\
C_x &= D_1^\dagger C + \big(I - D_1^\dagger D_1\big)B_1^* Q \Xi\\
\mathbf{A} &= A - B_1 C_x\\
\Xi &= \big(I - \Delta Q\big)^{-1} \quad \text{and} \quad D_1^\dagger = D_1^*\big(D_1 D_1^*\big)^{-1}.
\end{aligned} \tag{10.1}$$

Let $\mathfrak{S} = (sI - A)^{-1}$ and $\Upsilon = (sI - \mathbf{A})^{-1}$. Then using $D_1 C_x = C$ with $B_1 C_x = A - \mathbf{A}$, we obtain

$$\begin{aligned}
GX &= \Big(D_1 + C\mathfrak{S}B_1\Big)\Big(D_1^\dagger D_2 + C_x\Upsilon\big(B_2 - B_1 D_1^\dagger D_2\big)\Big)\\
&= D_2 + C\mathfrak{S}B_1 D_1^\dagger D_2 + C\Upsilon\big(B_2 - B_1 D_1^\dagger D_2\big)\\
&\quad + C\mathfrak{S}B_1 C_x \Upsilon\big(B_2 - B_1 D_1^\dagger D_2\big)\\
&= D_2 + C\mathfrak{S}B_1 D_1^\dagger D_2 + C\Upsilon\big(B_2 - B_1 D_1^\dagger D_2\big)\\
&\quad + C\mathfrak{S}\Big(A - \mathbf{A}\Big)\Upsilon\Big(B_2 - B_1 D_1^\dagger D_2\Big).
\end{aligned}$$

In other words,

$$\begin{aligned}
GX = D_2 + C\mathfrak{S}B_2 - C\mathfrak{S}\big(B_2 - B_1 D_1^\dagger D_2\big) + C\Upsilon\Big(B_2 - B_1 D_1^\dagger D_2\Big)\\
+ C\mathfrak{S}\Big(A - \mathbf{A}\Big)\Upsilon\Big(B_2 - B_1 D_1^\dagger D_2\Big).
\end{aligned}$$

Using $K = D_2 + C\mathfrak{S}B_2$ with $B_3 = B_2 - B_1 D_1^\dagger D_2$, we have

$$\begin{aligned}
GX &= K - C\mathfrak{S}B_3 + C\Upsilon B_3 + C\mathfrak{S}\big(A - \mathbf{A}\big)\Upsilon B_3\\
&= K - C\mathfrak{S}B_3 + C\Upsilon B_3 + C\mathfrak{S}\big(\Upsilon^{-1} - \mathfrak{S}^{-1}\big)\Upsilon B_3\\
&= K - C\mathfrak{S}B_3 + C\Upsilon B_3 + C\mathfrak{S}B_3 - C\Upsilon B_3 = K.
\end{aligned}$$

Therefore we have

$$G(s)X(s) = K(s).$$

So to show that $X(s)$ is indeed a solution to our Leech problem, it remains to show that $\|X\|_\infty \le 1$. In fact, we will show in the next section that $\|X\|_\infty < 1$.

11. The Outer Spectral Factor for $I - \widetilde{X}(s)X(s)$

In this section, we will show that $\Theta(s) = D_v^{1/2}V(s)^{-1}$ is the invertible outer spectral factor for $I - \widetilde{X}(s)X(s)$, that is, $\widetilde{\Theta}\Theta = I - \widetilde{X}X$. (Recall that for an operator valued H^∞ function we defined $\widetilde{F}(s) = F(-\overline{s})^*$.) The state space representations of U, V, and X are given by

$$\begin{aligned} U(s) &= D_1^* R_0^{-1} D_2 + C_u\big(sI - A_0\big)^{-1}\mathbb{B} \\ V(s) &= D_v + C_v(sI - A_0)^{-1}\mathbb{B} \\ X(s) &= D_1^\dagger D_2 + C_x\big(sI - \mathbf{A}\big)^{-1}\big(B_2 - B_1 D_1^\dagger D_2\big) \end{aligned} \tag{11.1}$$

where

$$\begin{aligned} \mathbf{A} &= A - B_1 C_x, \\ C_u &= D_1^* C_0 + B_1^* Q, \qquad C_v = D_2^* C_0 + B_2^* Q, \\ C_x &= \big(C_u - D_1^\dagger D_2 C_v\big)\Xi, \quad \text{where} \quad \Xi = \big(I - \Delta Q\big)^{-1}, \\ D_v &= (I + D_2^* R_0^{-1} D_2), \\ \mathbb{B} &= \Xi\Big(B_2 - B_1 D_1^\dagger D_2\Big) D_v. \end{aligned} \tag{11.2}$$

Because $V(s)$ is an invertible outer function, $\Theta(s)$ is also an invertible outer function, that is, both $\Theta(s)$ and $\Theta(s)^{-1}$ are functions in $H^\infty(\mathcal{E}, \mathcal{E})$. This sets the stage for the following result.

Proposition 11.1. *Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, that items (i) and (ii) of Theorem 3.1 hold. Consider the function $V(s)$ in (11.1) and set $\Theta(s) = D_v^{1/2}V(s)^{-1}$. Then Θ admits a state space realization of the form*

$$\begin{aligned} \Theta(s) &= D_v^{-1/2} - D_v^{-1/2} C_v \Xi\big(sI - \mathbf{A}\big)^{-1}\big(B_2 - B_1 D_1^\dagger D_2\big) \\ C_v\Xi &= (D_2^* C_0 + B_2^* Q)\Xi. \end{aligned} \tag{11.3}$$

Furthermore, $\Theta(s)$ is the invertible outer spectral factor satisfying

$$\widetilde{\Theta}(s)\Theta(s) = I - \widetilde{X}(s)X(s). \tag{11.4}$$

In particular, because $\Theta(s)$ is an invertible outer function,

$$\|X\|_\infty < 1. \tag{11.5}$$

Proof. The state space realization for $\Theta(s) = D_v^{1/2}V(s)^{-1}$ follows from the state space realization for $V(s)^{-1}$ in (7.4).

To prove that (11.4) holds, set

$$\begin{aligned} \Psi(s) &= (sI - A_0)^{-1} \quad \text{and} \quad \widetilde{\Psi}(s) = -(sI + A_0^*)^{-1} = \Psi(-\overline{s})^* \\ \mathfrak{B} &= \mathbb{B} D_v^{-1} = \Xi\big(B_2 - B_1 D_1^\dagger D_2\big). \end{aligned}$$

Using this notation, the state space formula for $U(s)$ and $V(s)$ in (11.1) and $D_u D_v^{-1} = D_x = D_1^\dagger D_2$, we see that

$$V D_v^{-1} = I + C_v \Psi \mathfrak{B}, \quad \text{and} \quad U D_v^{-1} = D_1^\dagger D_2 + C_u \Psi \mathfrak{B}. \tag{11.6}$$

We claim that

$$D_v^{-1} = D_v^{-1} V(-\overline{s})^* V(s) D_v^{-1} - D_v^{-1} U(-\overline{s})^* U(s) D_v^{-1}. \tag{11.7}$$

To verify this, notice that the formulas for $V(s)$ and $U(s)$ in (11.6) yield

$$\begin{aligned} & D_v^{-1} \tilde{V} V D_v^{-1} - D_v^{-1} \tilde{U} U D_v^{-1} \\ &= (I + \mathfrak{B}^* \widetilde{\Psi} C_v^*)(I + C_v \Psi \mathfrak{B}) - (D_2^* D_1^{\dagger *} + \mathfrak{B}^* \widetilde{\Psi} C_u^*)(D_1^\dagger D_2 + C_u \Psi \mathfrak{B}) \\ &= I - D_2^* (D_1 D_1^*)^{-1} D_2 + \mathfrak{B}^* \widetilde{\Psi}(C_v^* - C_u^* D_1^\dagger D_2) + (C_v - D_2^* D_1^{\dagger *} C_u) \Psi \mathfrak{B} + \mathfrak{B}^* \widetilde{\Psi}\Big(C_v^* C_v - C_u^* C_u\Big) \Psi \mathfrak{B} \\ &= D_v^{-1} + \mathfrak{B}^* \widetilde{\Psi}(C_v^* - C_u^* D_1^\dagger D_2) + (C_v - D_2^* D_1^{\dagger *} C_u) \Psi \mathfrak{B} + \mathfrak{B}^* \widetilde{\Psi}\Big(C_v^* C_v - C_u^* C_u\Big) \Psi \mathfrak{B}. \end{aligned} \tag{11.8}$$

The last equality follows from

$$D_v^{-1} = I - D_2^* (D_1 D_1^*)^{-1} D_2. \tag{11.9}$$

To prove this equality, observe that

$$\begin{aligned} D_v^{-1} &= (I + D_2^* R_0^{-1} D_2)^{-1} = I - D_2^* R_0^{-1} D_2 (I + D_2^* R_0^{-1} D_2)^{-1} \\ &= I - D_2^* (I + R_0^{-1} D_2 D_2^*)^{-1} R_0^{-1} D_2 \\ &= I - D_2^* (R_0 + D_2 D_2^*)^{-1} D_2 = I - D_2^* (D_1 D_1^*)^{-1} D_2. \end{aligned}$$

This yields (11.9).

To complete the proof, it remains to show that the right hand side of (11.8) is equal to D_v^{-1}, or equivalently,

$$0 = \mathfrak{B}^* \widetilde{\Psi}\big(C_v^* - C_u^* D_1^\dagger D_2\big) + \big(C_v - D_2^* D_1^{\dagger *} C_u\big) \Psi \mathfrak{B} + \mathfrak{B}^* \widetilde{\Psi}\Big(C_v^* C_v - C_u^* C_u\Big) \Psi \mathfrak{B}.$$

By consulting the Lyapunov Eq. in (9.7), we obtain

$$\begin{aligned} & \mathfrak{B}^* \widetilde{\Psi}\Big(C_v^* C_v - C_u^* C_u\Big) \Psi \mathfrak{B} = \mathfrak{B}^* \widetilde{\Psi}\Big(A_0^*(Q - Q\Delta Q) + (Q - Q\Delta Q) A_0\Big) \Psi \mathfrak{B} \\ &\quad = \mathfrak{B}^* \widetilde{\Psi}\Big(- \widetilde{\Psi}^{-1}(Q - Q\Delta Q) - (Q - Q\Delta Q)\Psi^{-1}\Big) \Psi \mathfrak{B} \\ &\quad = -\mathfrak{B}^* (Q - Q\Delta Q) \Psi \mathfrak{B} - \mathfrak{B}^* \widetilde{\Psi}(Q - Q\Delta Q) \mathfrak{B}. \end{aligned}$$

Since $Q - Q\Delta Q = Q\Xi^{-1}$, we have

$$\mathfrak{B}^* \widetilde{\Psi}\Big(C_v^* C_v - C_u^* C_u\Big) \Psi \mathfrak{B} = -\mathfrak{B}^* \Xi^{-*} Q \Psi \mathfrak{B} - \mathfrak{B}^* \widetilde{\Psi} Q \Xi^{-1} \mathfrak{B}.$$

Moreover, we also have

$$\mathfrak{B}^* \widetilde{\Psi}\big(C_v^* - C_u^* D_1^\dagger D_2\big) - \mathfrak{B}^* \widetilde{\Psi} Q \Xi^{-1} \mathfrak{B} = \mathfrak{B}^* \widetilde{\Psi}\Big(C_v^* - C_u^* D_1^\dagger D_2 - Q \Xi^{-1} \mathfrak{B}\Big).$$

Now observe that

$$\begin{aligned}
&C_v^* - C_u^* D_1^\dagger D_2 - Q\Xi^{-1}\mathfrak{B} \\
&\quad = C_0^* D_2 + QB_2 - C_0^* D_1 D_1^\dagger D_2 - QB_1 D_1^\dagger D_2 - Q\Xi^{-1}\Xi\big(B_2 - B_1 D_1^\dagger D_2\big) \\
&\quad = C_0^* D_2 + QB_2 - C_0^* D_2 - QB_2 = 0.
\end{aligned}$$

Therefore

$$\mathfrak{B}^*\widetilde{\Psi}\big(C_v^* - C_u^* D_1^\dagger D_2\big) - \mathfrak{B}^*\widetilde{\Psi}Q\Xi^{-1}\mathfrak{B} = 0.$$

Likewise its adjoint

$$\big(C_v - D_2^* D_1^{\dagger *} C_u\big)\Psi\mathfrak{B} - \mathfrak{B}^*\Xi^{-*}Q\Psi\mathfrak{B} = 0.$$

This with (11.8) shows that

$$D_v^{-1} = D_v^{-1}V(-\overline{s})^*V(s)D_v^{-1} - D_v^{-1}U(-\overline{s})^*U(s)D_v^{-1}.$$

Recall that $X(s) = U(s)V(s)^{-1}$. By applying D_v to both sides, we see that

$$D_v = V(-\overline{s})^*V(s) - V(-\overline{s})^*X(-\overline{s})^*X(s)V(s).$$

By taking the appropriate inverse, we arrive at

$$I - X(-\overline{s})^*X(s) = \Theta(-\overline{s})^*\Theta(s) \quad (\text{where } \Theta(s) = D_v^{1/2}V(s)^{-1}).$$

Therefore Θ is the outer spectral factor for $I - X(-\overline{s})^*X(s)$. Finally, because Θ is an invertible outer function, we also have $\|X\|_\infty < 1$. This completes the Proof of Proposition 11.1. □

Since Proposition 11.1 is proved, we have also finished the Proof of Theorem 3.2.

12. Proof of Theorem 1.1

The Proof of Theorem 1.1 concerns rational functions G in $H^\infty(\mathcal{U},\mathcal{Y})$ and K in $H^\infty(\mathcal{E},\mathcal{Y})$, which have the additional property that $T_G T_G^* - T_K T_K^*$ is strictly positive. Furthermore, X is a function in $H^\infty(\mathcal{E},\mathcal{U})$ defined by (1.6) where $U(s)$ and $V(s)$ are the functions defined by (1.7). Moreover, the rational functions G and K admit a minimal stable realization (2.1). Given these data we have to prove items (i) and (ii).

According to Lemma 6.1 the function $U(s)$ has a realization (6.2) and according to Lemma 5.1 the function $V(s)$ has a realization (5.3). Thus it follows from Proposition 8.1 that $X(s) = U(s)V(s)^{-1}$ has a realization (8.2) and $X(s)$ is analytic on $\mathbb{C}_+$. Moreover, by Sect. 10, we have $GX = K$. Proposition 11.1 gives that $\|X\|_\infty < 1$ and completes the proof that X is indeed a solution of the Leech problem for G and K. Thus item (i) is proved. Finally, Proposition 11.1 also shows us that formula (1.9) holds true with Θ

given by (1.8). This tells us that item (ii) of the theorem is proved. Thus items (i) and (ii) are proved, and hence Theorem 1.1 is proved. □

Declarations

Competing interest The authors have no relevant financial or non-financial interests to disclose. The authors have no competing interests to declare that are relevant to the content of this article.

13. Appendix 1: A Riccati Equation

In this section we bring together results from the existing literature on the algebraic Riccati equation and factorizations and extend these results for our use. The literature on algebraic Riccati equations is well established, and can be found in many places. We will quote results from [5, 17, 19]. The literature on stochastic realization theory is also quite useful; see for instance [6, 11, 12]. Recall that, see (2.3),

$$R(s) = G(s)\widetilde{G}(s) - K(s)\widetilde{K}(s), \tag{13.1}$$

where $G(s)$ and $K(s)$ are given by (2.1), that is

$$G(s) = D_1 + C(sI - A)^{-1}B_1 \quad \text{and} \quad K(s) = D_2 + C(sI - A)^{-1}B_2. \tag{13.2}$$

Moreover, A is stable and $\{C, A\}$ is observable.

Lemma 13.1. *Assume that G and K are given by* (13.2) *and R is determined by* (13.1)*. Then a state space formula for $R(s)$ is given by*

$$R(s) = C(sI - A)^{-1}\Gamma + R_0 - \Gamma^*(sI + A^*)^{-1}C^*. \tag{13.3}$$

Here Γ mapping $\mathcal{Y}$ into $\mathcal{X}$ is defined by

$$\Gamma = B_1D_1^* - B_2D_2^* + \Delta C^* \quad \textit{and} \quad R_0 = D_1D_1^* - D_2D_2^*, \tag{13.4}$$

and Δ is the unique solution of the Lyapunov equation

$$A\Delta + \Delta A^* + B_1B_1^* - B_2B_2^* = 0. \tag{13.5}$$

Proof. It is noted that $\Delta = P_1 - P_2$ where P_1 and P_2 are the unique solutions of the Lyapunov equations

$$AP_1 + P_1A^* + B_1B_1^* = 0 \quad \text{and} \quad AP_2 + P_2A^* + B_2B_2^* = 0. \tag{13.6}$$

Since A is stable, the Eq. (13.6) are solvable and the solutions are unique. To see where the state space formula of R comes from notice that (13.6) gives

$$B_1B_1^* = (sI - A)P_1 - P_1(sI + A^*).$$

Thus

$$(sI - A)^{-1}B_1B_1^*(sI + A^*)^{-1} = P_1(sI + A^*)^{-1} - (sI - A)^{-1}P_1.$$

This gives that

$$\begin{aligned} G(s)\widetilde{G}(s) &= \Big(D_1 + C(sI - A)^{-1}B_1\Big)\Big(D_1^* - B_1^*(sI - A^*)^{-1}C^*\Big) \\ &= D_1D_1^* + C(sI - A)^{-1}B_1D_1^* - D_1B_1^*(sI + A^*)^{-1}C^* \\ &\quad - C(sI - A)^{-1}B_1B_1^*(sI + A^*)^{-1}C^* \\ &= D_1D_1^* + C(sI - A)^{-1}B_1D_1^* - D_1B_1^*(sI + A^*)^{-1}C^* \\ &\quad + C(sI - A)^{-1}P_1C^* - CP_1(sI + A^*)^{-1}C^*, \end{aligned}$$

and therefore

$$\begin{aligned} G(s)\widetilde{G}(s) = D_1D_1^* &+ C(sI - A)^{-1}\Big(B_1D_1^* + P_1C^*\Big) \\ &- \Big(CP_1 + D_1B_1^*\Big)(sI + A^*)^{-1}C^*. \end{aligned}$$

A similar calculation shows that

$$\begin{aligned} K(s)\widetilde{K}(s) = D_2D_2^* &+ C(sI - A)^{-1}\Big(B_2D_2^* + P_2C^*\Big) \\ &- \Big(CP_2 + D_2B_2^*\Big)(sI + A^*)^{-1}C^*. \end{aligned}$$

Combining this with the definition of R in (13.1) and the definitions (13.4) of Γ and R_0, we arrive at (13.3). □

The first result that we quote is [5, Theorem 14.8]. We present this theorem with the notation in [5, Theorem 14.8] replaced by the corresponding notation of the current paper. To be precise: the replacing is, with in each case the identifier in [5, Theorem 14.8] mentioned first

$$\begin{aligned} &J \mapsto I, \quad C \mapsto R_0^{-1/2}C, \quad B \mapsto \Gamma R_0^{-1/2} \\ &X \mapsto Q, \quad W(s) \mapsto R_0^{-1/2}R(s)R_0^{-1/2}, \quad L(s) \mapsto R_0^{-1/2}\Phi(s). \end{aligned}$$

Finally, recall that an operator T on $\mathcal{X}$ is *strictly positive*, denoted by $T \gg 0$, if T is positive and its inverse exists and is also a positive operator.

Theorem 13.2. *Let the rational function R be given by* (13.3) *and assume that $R_0 \gg 0$. Then $R_0^{-1/2}R(s)R_0^{-1/2}$ admits a left spectral factorization with respect to the imaginary axis,*

$$R_0^{-1/2}R(s)R_0^{-1/2} = R_0^{-1/2}\Phi\left(-\overline{s}\,\right)^*\Phi(s)R_0^{-1/2}, \quad s \in \mathbb{C}, \tag{13.7}$$

if and only if the algebraic Riccati equation

$$A^*Q + QA - Q\Gamma R_0^{-1}C - C^*R_0^{-1}\Gamma^*Q + Q\Gamma R_0^{-1}\Gamma^*Q + C^*R_0^{-1}C = 0 \quad (13.8)$$

has a stabilizing solution. In that case the unique spectral factor $\Phi(s)$ *is given by*

$$\Phi(s) = R_0^{1/2}\left(I + R_0^{-1}(C - \Gamma^*Q)(sI - A)^{-1}\Gamma\right).$$

Finally, both Φ *and* Φ^{-1} *are rational functions in* $H^\infty(\mathcal{Y}, \mathcal{Y})$*.*

Here *stabilizing solution* means that $Q \gg 0$ and A_0 is stable, where

$$A_0 = A - \Gamma C_0 \quad \text{and} \quad C_0 = R_0^{-1}(C - \Gamma^*Q). \quad (13.9)$$

Clearly, $R_0^{-1/2}$ on both sides of (13.7) can be eliminated. We inserted $R_0^{-1/2}$ in (13.7) to match the hypothesis in [5, Theorem 14.8]. Notice that the Riccati Eq. (13.8) can be rewritten as

$$A^*Q + QA + (C - \Gamma^*Q)^*R_0^{-1}(C - \Gamma^*Q) = 0. \quad (13.10)$$

or, using (13.9), as

$$A^*Q + QA + C_0^*R_0C_0 = 0. \quad (13.11)$$

We do not prove the above theorem, but we will verify that $\widetilde{\Phi}(s)\Phi(s) = R(s)$. To this end first notice that (13.11) is equivalent to

$$-C_0^*R_0C_0 = (sI + A^*)Q - Q(sI - A). \quad (13.12)$$

Next observe that

$$\begin{aligned}\widetilde{\Phi}(s)\Phi(s) &= \left(I - \Gamma^*(sI + A^*)^{-1}C_0^*\right)R_0\left(I + C_0(sI - A)^{-1}\Gamma\right)\\ &= R_0 - \Gamma^*(sI + A^*)^{-1}C_0^*R_0 + R_0C_0(sI - A)^{-1}\Gamma\\ &\quad - \Gamma^*(sI + A^*)^{-1}C_0^*R_0C_0(sI - A)^{-1}\Gamma\end{aligned}$$

Using (13.12) we get

$$\begin{aligned}\widetilde{\Phi}(s)\Phi(s) &= R_0 - \Gamma^*(sI + A^*)^{-1}C_0^*R_0 + R_0C_0(sI - A)^{-1}\Gamma\\ &\quad + \Gamma^*(sI + A^*)^{-1}\Big((sI + A^*)Q - Q(sI - A)\Big)(sI - A)^{-1}\Gamma\\ &= R_0 - \Gamma^*(sI + A^*)^{-1}\Big(C_0^*R_0 + Q\Gamma\Big) + \Big(\Gamma^*Q + R_0C_0\Big)(sI - A)^{-1}\Gamma\\ &= R_0 - \Gamma^*(sI + A^*)^{-1}C^* + C(sI - A)^{-1}\Gamma = R(s).\end{aligned}$$

In other words, $R(s) = \widetilde{\Phi}(s)\Phi(s)$.
Notice the similarity of this computation with the Proof of Lemma 13.1. We mention that Theorem 13.2 also is a special case of [19, Theorem 19.3.1].
Next we quote [17, Theorem XIII.3.1], which is on canonical Wiener–Hopf factorizations. A rational matrix function R by definition has a canonical (left) Wiener–Hopf factorization with respect to the imaginary axis in the complex plane if

$$R(s) = R_-(s)R_+(s), \quad (s \in \mathbb{C})$$

where R_+ (R_-) has all its poles and zeros on the open left (right) half plane and both are invertible at ∞. Notice that a (left) spectral factorization is a special case of a canonical (left) Wiener–Hopf factorization.
We rephrase [17, Theorem XIII.3.1] in terms of the current paper as follows.

Theorem 13.3. *Let T_R be a Wiener–Hopf operator with rational symbol R. Then T_R is invertible if and only if*

(i) $\det R(s) \neq 0$ *for all* $s \in i\mathbb{R}$ *and*
(ii) $R(s)$ *admits a canonical (left) Wiener–Hopf factorization relative to the imaginary axis.*

The following theorem provides the main result of this section and is used on several places in this paper. It shows when T_R is strictly positive for any rational function $R(s)$ given by a state space realization (13.3).

Theorem 13.4. *Let $R(s)$ be any rational function given by the state space realization*

$$R(s) = C(sI - A)^{-1}\Gamma + R_0 - \Gamma^*(sI + A^*)^{-1}C^* \tag{13.13}$$

where $\{C, A\}$ is observable, A is stable and Γ is an operator from $\mathcal{Y}$ into $\mathcal{X}$. Then the following statements are equivalent

(a) T_R *is strictly positive operator on* $L^2_+(\mathcal{Y})$.
(b) $R_0 \gg 0$ *and there exists a stabilizing solution to the algebraic Riccati equation*

$$A^*Q + QA + (C - \Gamma^*Q)^* R_0^{-1}(C - \Gamma^*Q) = 0. \tag{13.14}$$

Here stabilizing solution means that $Q \gg 0$ and A_0, given by (13.9), is stable.

(c) *The function $R(s)$ admits an invertible outer spectral factorization of the form $R(s) = \widetilde{\Phi}(s)\Phi(s)$, where Φ is the invertible outer function. By invertible outer we mean that both $\Phi(s)$ and $\Phi(s)^{-1}$ are functions in $H^\infty(\mathcal{Y}, \mathcal{Y})$.*

Given the above, the following statements hold.

(i) *The spectral factor Φ and its inverse are given by*

$$\begin{aligned}\Phi(s) &= R_0^{1/2}\left(I + C_0(sI - A)^{-1}\Gamma\right)\\ \Phi(s)^{-1} &= \left(I - C_0(sI - A_0)^{-1}\Gamma\right) R_0^{-1/2}.\end{aligned} \tag{13.15}$$

(ii) *The stabilizing solution Q to the algebraic Riccati Eq. (13.14) is unique, and given by*

$$Q = W_{obs}^* T_R^{-1} W_{obs}. \tag{13.16}$$

(iii) *The operator $W_0 = T_R^{-1}W_{obs}$ mapping $\mathcal{X}$ into $L^2_+(\mathcal{Y})$ is determined by*

$$T_R^{-1}W_{obs}x = W_0x = C_0e^{A_0t}x \qquad (x \in \mathcal{X}). \tag{13.17}$$

In particular, the pair $\{C_0, A_0\}$ is observable.

(iv) *The following holds*

$$R(s)C_0(sI - A_0)^{-1} = C(sI - A)^{-1} - \Gamma^*(sI + A^*)^{-1}Q. \tag{13.18}$$

Proof. The proof of this theorem is split into 6 parts.

Part 1. We prove that statement (c) implies statement (a). We have $R = \widetilde{\Phi}\Phi$ and thus $T_R = T_{\widetilde{\Phi}} T_\Phi$, with T_Φ invertible. Since $T_{\widetilde{\Phi}} = T_\Phi^*$, we have that $T_R = T_\Phi^* T_\Phi$ and $T_R^{-1} = T_\Phi^{-1}\left(T_\Phi^{-1}\right)^*$. We conclude that $T_R \gg 0$.

Part 2. Let us show that statement (a) implies statement (c). From Theorem 13.3 we conclude that since T_R is invertible, the function R has a canonical Wiener–Hopf factorization, $R = R_- R_+$. Without loss of generality we may assume that $R_\pm(\infty) = R_0^{1/2}$. Then R_+ and R_- are uniquely determined. Since T_R is strictly positive, we have that $T_R = T_R^* = T_{R_+}^* T_{R_-}^*$ and thus $R(s) = \widetilde{R}_+(s)\widetilde{R}_-(s)$. The uniqueness of the factorization yields that $R_- = \widetilde{R}_+$. Put $\Phi = R_+$ and we conclude that the factorization

$$R(s) = \widetilde{\Phi}(s)\Phi(s) = \Phi(-\bar{s})^*\Phi(s)$$

is a spectral factorization of R.

Part 3. In this part we show that the statements (b) and (c) are equivalent. We apply Theorem 13.2. Note that if we have the spectral factorization $R = \widetilde{\Phi}\Phi$, then $R_0 \gg 0$. But then we also have (13.7) and conclude that (13.8) has a stabilizing solution. Therefore (13.14) has a stabilizing solution.

Conversely, according to Theorem 13.2, statement (b) implies (c).

Part 4. We have established the equivalence of the statements (a), (b) and (c) and note that statement (i) immediately follows from Theorem 13.2.

Part 5. Let us establish the identity in (13.18) in statement (iv). To accomplish this we employ the realization for $R(s)$ in (13.3). Using $\Gamma C_0 = A - A_0$, a standard calculation shows that

$$\begin{aligned} C(sI-A)^{-1}\Gamma C_0(sI-A_0)^{-1} &= C(sI-A)^{-1}(A-A_0)(sI-A_0)^{-1} \\ &= C(sI-A)^{-1}(A-sI+sI-A_0)(sI-A_0)^{-1} \\ &= C(sI-A)^{-1} - C(sI-A_0)^{-1}. \end{aligned} \tag{13.19}$$

As before, let Q be the stabilizing solution of the algebraic Riccati Eq. (13.14). Recall that the operator A_0 is stable and T_R is strictly positive. Using C_0 and A_0 in (13.9), the Riccati Eq. (13.14) can be rewritten as

$$A^*Q + QA_0 + C^*C_0 = 0. \tag{13.20}$$

Using the Lyapunov Eq. (13.20), we have

$$\begin{aligned} &-\Gamma^*(sI+A^*)^{-1}C^*C_0(sI-A_0)^{-1} \\ &\quad= \Gamma^*(sI+A^*)^{-1}\big(A^*Q+QA_0\big)(sI-A_0)^{-1} \\ &\quad= \Gamma^*(sI+A^*)^{-1}\big((sI+A^*)Q - Q(sI-A_0)\big)(sI-A_0)^{-1} \\ &\quad= \Gamma^*Q(sI-A_0)^{-1} - \Gamma^*(sI+A^*)^{-1}Q \\ &\quad= C(sI-A_0)^{-1} - R_0C_0(sI-A_0)^{-1} - \Gamma^*(sI+A^*)^{-1}Q. \end{aligned}$$

The last equality follows from $\Gamma^*Q = C - R_0C_0$; see (13.9). In other words, $\big(R_0 - \Gamma^*(sI+A^*)^{-1}C^*\big)C_0(sI-A_0)^{-1} = C(sI-A_0)^{-1} - \Gamma^*(sI+A^*)^{-1}Q.$ Recall (13.3) that

$$R(s) = C(sI-A)^{-1}\Gamma + R_0 - \Gamma^*(sI+A^*)^{-1}C^*.$$

This with the preceding identity and (13.19) yields the result that we have been looking for, that is,

$$R(s)C_0(sI - A_0)^{-1} = C(sI - A)^{-1} - \Gamma^*(sI + A^*)^{-1}Q.$$

This proves (13.18) and statement (iv).

Part 6. The observability operator for the pair $\{C_0, A_0\}$ is the operator W_0 mapping $\mathcal{X}$ into $L^2_+(\mathcal{Y})$ defined by

$$W_0 x = C_0 e^{A_0 t} x \qquad (\text{for } x \in \mathcal{X}); \tag{13.21}$$

(see also the identity (3.19) in [13]).

Recall that (13.18) has been proved. Note that $C(sI-A)^{-1}$ is a stable rational operator valued function, while $\Gamma^*(sI+A^*)^{-1}Q$ is a rational function, which is analytic on the open left half plane $\{s \in C : \Re(s) < 0\}$. and has the value zero at infinity. Moreover,

$$\big(\mathfrak{L}W_{obs}\big)(s) = C(sI - A)^{-1} \quad \text{and} \quad \big(\mathfrak{L}W_0\big)(s) = C_0(sI - A_0)^{-1}. \tag{13.22}$$

This with (13.18) implies that $T_R W_0 x$ is equal to $W_{obs}x$ for each $x \in \mathcal{X}$. Since T_R is invertible, we also have $W_0 = T_R^{-1} W_{obs}$. Throughout we assumed that $\{C, A\}$ is observable, or equivalently, W_{obs} is one to one. Therefore $W_0 = T_R^{-1} W_{obs}$ is also one to one. The Lyapunov equation in (13.20) implies that

$$Q = \int_0^\infty e^{A^* t} C C_0 e^{A_0 t} dt = W_{obs}^* W_0.$$

In other words,

$$Q = W_{obs}^* W_0. \tag{13.23}$$

By employing $W_0 = T_R^{-1} W_{obs}$, we see that $Q = W_{obs}^* T_R^{-1} W_{obs}$. This proves statements (ii) and (iii). □

14. Appendix 2: The Dirac Delta Function δ

In this section we introduce a simple form of the Dirac delta for a specific class of operators. The continuity properties of these operators allow for an elementary definition of the Dirac delta.

Let $\mathfrak{F}(\mathcal{U}, \mathcal{Y})$ be the Hilbert space formed by the set of all linear operators mapping $\mathcal{U}$ into $\mathcal{Y}$ under the Frobenius or trace norm, that is, if M is in $\mathfrak{F}(\mathcal{U}, \mathcal{Y})$, then $\|M\|^2_{\mathfrak{F}} = \text{trace}(M^*M)$. Moreover, $L^2_+(\mathfrak{F}(\mathcal{U}, \mathcal{Y}))$ is the Hilbert space formed by the set of all square integrable Lebesgue measurable functions over the interval $[0, \infty)$ with values in $\mathfrak{F}(\mathcal{U}, \mathcal{Y})$.

Consider the state space systems $\{A_1, B_1, C_1, D\}$ and $\{A_2, B_2, C_2, D\}$, where for $j = 1, 2$ the operator A_j is a stable operator on $\mathcal{X}_j$, and B_j maps $\mathcal{U}$ into $\mathcal{X}_j$, while C_j maps $\mathcal{X}_j$ into $\mathcal{Y}$ and D maps $\mathcal{U}$ into $\mathcal{Y}$. The state spaces $\mathcal{X}_j$, input space $\mathcal{U}$ and output space $\mathcal{Y}$ are all finite dimensional complex vector spaces of possibly different dimension of the form $\mathbb{C}^\ell$. Let

$$F(s) = D + C_1(sI - A_1)^{-1}B_1 + C_2(sI + A_2)^{-1}B_2. \tag{14.1}$$

We define the corresponding kernel function $k(t)$ by

$$k(t) = \begin{cases} C_1 e^{A_1 t} B_1, & t \geq 0, \\ C_2 e^{-A_2 t} B_2, & t < 0. \end{cases}$$

The corresponding Wiener–Hopf operator T_F mapping $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{U}))$ into the space $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y}))$ is given by

$$(T_F\, h)\,(t) = Dh(t) + \int_0^\infty k(t-\tau)h(\tau)d\tau \qquad (\text{for } 0 \leq t < \infty), \quad (14.2)$$

where $h \in L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{U}))$. Let $\mathfrak{M}_{\mathcal{U},\mathcal{Y}}$ be the linear space consisting of all Wiener–Hopf operators of the form (14.2). Now fix an operator valued function h in the space $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{U}))$. Then h defines a linear transformation $\mathbb{L}_h$ from $\mathfrak{M}_{\mathcal{U},\mathcal{Y}}$ into the space $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y}))$ by $\mathbb{L}_h(T_F) = T_F h$.
If in (14.1) we have $D = 0$, then the Dirac delta function $\mathbb{L}_\delta$ is formally defined by

$$\mathbb{L}_\delta(T_F)(t) = k(t) = C_1 e^{A_1 t} B_1 \qquad (\text{for } 0 \leq t < \infty).$$

Thus we defined the transformation $\mathbb{L}_\delta$ mapping $\mathfrak{M}_{\mathcal{U},\mathcal{Y}}$ into $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y}))$.
The next step is to extend the definition of δ to the case when D is not equal to 0. We extend $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y}))$ to a direct sum

$$\big(\mathfrak{F}(\mathcal{U},\mathcal{Y})\big) \bigoplus L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y})). \quad (14.3)$$

If h is in $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{U}))$, then $T_F h$ is in the second component $L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y}))$ of the previous space. For $F(s)$ given by (14.1), we define the linear map $\mathbb{L}_\delta T_F$ by

$$\big(\mathbb{L}_\delta T_F\big)(t) = D \bigoplus C_1 e^{A_1 t} B_1 \in \big(\mathfrak{F}(\mathcal{U},\mathcal{Y})\big) \bigoplus L^2_+(\mathfrak{F}(\mathcal{U},\mathcal{Y})).$$

We denote this as

$$\big(T_F \delta\big)(t) = D\delta(t) + C_1 e^{A_1 t} B_1 \qquad (t \geq 0). \quad (14.4)$$

Note that the sum is formal here and that the $\delta(t)$ in $D\delta(t)$ emphasizes this. Finally, it is noted that formally, the Laplace transform of $\delta(t)$ equals one, that is, $\big(\mathfrak{L}\delta\big)(s) = 1$.
Let $F(s) = D + C_1(sI - A_1)^{-1}B_1$, with the operator A_1 a stable operator on $\mathcal{X}_1$, and B_1 maps $\mathcal{U}$ into $\mathcal{X}_1$, while C_1 maps $\mathcal{X}_1$ into $\mathcal{Y}$ and D maps $\mathcal{U}$ into $\mathcal{Y}$. Then

$$(T_F^* h)(t) = D^* h(t) + \int_t^\infty B_1^* e^{-A_1^*(t-\tau)} C_1^* h(\tau) d\tau, \quad t \geq 0,$$

and the kernel function k^* for this operator T_F^* therefore is given by

$$k^*(t) = \begin{cases} 0, & t \geq 0 \\ B_1^* e^{-A_1^* t} C_1^*, & t < 0 \end{cases}.$$

We conclude from that $(T_F^* \delta)(t) = D^* \delta(t)$ because $k^*(t) = 0$ for $t \geq 0$.
The Dirac delta function viewed as a limit For the classical approach to the Dirac delta function, for any $a > 0$, let ξ_a be the function defined by

$\xi_a(t) = ae^{-at}$ for $t \geq 0$ and zero otherwise. Another way to define the Dirac delta function is by

$$\delta(t) = \lim_{a\to\infty} \xi_a(t) \qquad (\text{for } a > 0).$$

Notice that $\delta(t) = 0$ for all $t > 0$ and is undefined at $t = 0$. As before, let $F(s) = D + C(sI - A)^{-1}B$ and let us formally define

$$\big(T_F\delta\big)(t) = \lim_{a\to\infty}\left[D\xi_a(t) + \int_0^t Ce^{A(t-\tau)}B\xi_a(\tau)d\tau\right]. \tag{14.5}$$

Then, using this we have

$$\begin{aligned}
\big(T_F\delta\big)(t) &= D\lim_{a\to\infty}\xi_a(t) + \lim_{a\to\infty}\int_0^t Ce^{A(t-\tau)}Bae^{-a\tau}d\tau \\
&= D\delta(t) + Ce^{At}\lim_{a\to\infty}\int_0^t ae^{-(aI+A)\tau}d\tau B \\
&= D\delta(t) + Ce^{At}\lim_{a\to\infty}\Big[-a(aI+A)^{-1}e^{-(aI+A)\tau}\Big]_0^t B \\
&= D\delta(t) + Ce^{At}\lim_{a\to\infty} a(aI+A)^{-1}\Big(I - e^{-at}e^{-At}\Big)B \\
&= D\delta(t) + Ce^{At}B.
\end{aligned}$$

So formally we have

$$\big(T_F\delta\big)(t) = D\delta(t) + Ce^{At}B \quad (\text{when } F(s) = D + C(sI - A)^{-1}B). \tag{14.6}$$

Finally, it is well know and easy to establish that the Laplace transform of $\delta(t)$ equals one, that is, $\big(\mathcal{L}\delta\big)(s) = 1$. (This follows from the fact that $\big(\mathcal{L}\xi_a\big)(s)$ converges to 1 for fixed s.)
We will also check the expression for $\big(T_F^*\delta\big)(t)$. In fact, we already defined that

$$\big(T_F^*\delta\big)(t) = D^*\delta(t) \qquad (\text{when } F(s) = D + C(sI - A)^{-1}B). \tag{14.7}$$

To formally verify this, observe that

$$\begin{aligned}
\big(T_F^*\delta\big)(t) &= \lim_{a\to\infty} D^*\xi_a(t) + \lim_{a\to\infty}\int_t^\infty B^*e^{A(\tau-t)}C^*\xi_a(\tau)d\tau \\
&= D^*\delta(t) + B^*\lim_{a\to\infty}\int_t^\infty e^{A^*(\tau-t)}ae^{-a(\tau-t)}d\tau e^{-at}C^* \\
&= D^*\delta(t) + B^*\lim_{a\to\infty}\int_0^\infty e^{A^*v}ae^{-av}dve^{-at}C^* \\
&= D^*\delta(t) + B^*\Big[\lim_{a\to\infty} a(aI - A^*)^{-1}e^{-at}\Big]C^* \\
&= D^*\delta(t).
\end{aligned}$$

This yields $\big(T_F^*\delta\big)(t) = D^*\delta(t)$ in (14.7).
Finally let $F_j(s) = D_j + C(sI - A_j)^{-1}B_j$ for $j = 1, 2$. Then it is easy to verify that the following holds assuming all the spaces are compatible:

- $T_{F_2}T_{F_1}h = T_{F_2}(T_{F_1}h)$ when h is in $L^2_+(\mathfrak{F}(\mathcal{U}_1, \mathcal{U}_1))$.
- $(T_{F_2}T_{F_1})(\delta) = T_{F_2}(T_{F_1}\delta)$.
- If $F_1(s) = D_1$, then $T_{F_2}T_{F_1}\delta(t) = D_2D_1\delta(t) + D_2C_2e^{A_2t}B_2D_1$.

- For convenience we will denote $D_1\delta$ also as δD_1. So $D_2D_1\delta$, $D_2\delta D_1$ and δD_2D_1 all are notations for the same element in the first component of a direct sum as (14.3).

References

[1] Bakony, M., Woerdeman, H.J.: Matrix completions, moments, and sums of Hermitian squares. Princeton Univ. Press, Princeton (2011)

[2] Ball, J.A., Trent, T.T.: Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna–Pick interpolation in several variables. J. Funct. Anal. **157**, 1–61 (1998)

[3] Ball, J.A., Gohberg, I., Rodman, L.: Interpolation of Rational Matrix Functions, OT 45. Birkhäuser Verlag, Basel (1990)

[4] Bart, H., Gohberg, I., Kaashoek, M.A., Ran, A.C.M.: Factorization of matrix and operator functions: the state space method, OT 178. Birkhäuser Verlag, Basel (2008)

[5] Bart, H., Gohberg, I., Kaashoek, M.A., Ran, A.C.M.: A state space approach to canonical factorization with applications, OT 200. Birkhäuser Verlag, Basel (2010)

[6] Caines, P.E.: Linear Stochastic Systems. Wiley, New York (1988)

[7] Carlson, L.: Interpolation by bounded analytic functions and the corona problem. Ann. Math. **76**, 547–559 (1962)

[8] Corless, M.J., Frazho, A.E.: Linear sytems and control. Marcel Dekker Inc, New York (2003)

[9] Foias, C., Frazho, A.E.: The Commutant Lifting Approach to Interpolation Problems, OT, vol. 44. Birkhäuser Verlag, Basel (1990)

[10] Foias, C., Frazho, A.E., Gohberg, I., Kaashoek, M.A.: Metric constrained interpolation, commutant lifting and systems. Birkhäuser Verlag, Basel (1998)

[11] Frazho, A.E., Bosri, W.: An operator perspective on signals and systems, OT, vol. 204. Birkhäuser Verlag, Basel (2010)

[12] Frazho, A.E., Kaashoek, M.A., Ran, A.C.M.: The non-symmetric discrete algebraic Riccati equation and canonical factorization of rational matrix functions on the unit circle. Integral Equ. Oper. Theory **66**, 215–229 (2010)

[13] Frazho, A.E., Kaashoek, M.A., Ran, A.C.M.: Right invertible multiplication operators and stable rational matrix solutions to an associate Bezout equation, I: the least squares solution. Integral Equ. Oper. Theory **70**, 395–418 (2011)

[14] Frazho, A.E., Kaashoek, M.A., Ran, A.C.M.: Right invertible multiplication operators and stable rational matrix solutions to an associate Bezout equation, II: Description of all solutions. Oper. Matrices **6**, 833–857 (2012)

[15] Frazho, A.E., ter Horst, S., Kaashoek, M.A.: State space formulas for stable rational matrix solutions of a Leech problem. Indagationes Math. **25**, 250–274 (2014)

[16] Fuhrmann, P.: On the corona theorem and its applications to spectral problems in Hilbert space. Trans. Amer. Math. Soc. **132**, 55–66 (1968)

[17] Gohberg, I., Goldberg, S., Kaashoek, M.A.: Classes of Linear Operators, Volume I, OT, vol. 49. Birkhäuser Verlag, Basel (1990)

[18] Gohberg, I., Goldberg, S., Kaashoek, M.A.: Classes of Linear Operators, Volume II, OT, vol. 63. Birkhäuser Verlag, Basel (1993)

[19] Lancaster, P., Rodman, L.: Algebraic Riccati equations. Clarendon Press, Oxford (1995)

[20] Leech, R.B.: Factorization of analytic functions and operator inequalities, (orginally unpublished in the early 1970's). Integral Equ. Oper. Theory **78**(1), 71–73 (2014)

[21] Rosenblum, M., Rovnyak, J.: Hardy classes and operator theory. Oxford Mathematical Monographs, The Clarendon Press, New York (1985)

[22] Stefanovski, J.: H_∞ problem with nonstrict inequality and all solutions: interpolation approach. SIAM J. Control Optim. **53**(4), 1734–1767 (2015)

[23] Stefanovski, J.: Strongly (J, J')-lossless rational matrices and H_∞ problem. Internat. J. Robust Nonlinear Control **28**(15), 4261–4286 (2018)

[24] Ter Horst, S.: Rational matrix solutions to the Leech equation: The Ball-Trent approach revisited. J. Math. Anal. Appl. **408**, 335–344 (2013)

[25] Trent, T.T.: A constructive proof of the Leech theorem for rational matrix functions. Integral Equ. Oper. Theory **75**, 39–48 (2013)

[26] Wahls, S., Boche, H.: Lower bounds on the infima in some $\mathcal{H}_\infty$ optimization problems. IEEE Trans. Automat. Control **57**, 788–793 (2012)

[27] Wahls, S., Boche, H., Pohl, V.: Zero-forcing precoding for frequency selective MIMO channels with H^∞ criterion and causality constraint. Signal Process. **89**, 1754–1761 (2009)

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