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Essential m-dissipativity for Possibly Degenerate Generators of Infinite-dimensional Diffusion Processes

Benedikt Eisenhuth and Martin Grothaus

Abstract. First essential m-dissipativity of an infinite-dimensional Ornstein-Uhlenbeck operator N, perturbed by the gradient of a potential, on a domain $\mathcal{F}C_b^\infty$ of finitely based, smooth and bounded functions, is shown. Our considerations allow unbounded diffusion operators as coefficients. We derive corresponding second order regularity estimates for solutions f of the Kolmogorov equation $\alpha f - Nf = g$, $\alpha \in (0,\infty)$, generalizing some results of Da Prato and Lunardi. Second, we prove essential m-dissipativity for generators $(L_\Phi, \mathcal{F}C_b^\infty)$ of infinite-dimensional degenerate diffusion processes. We emphasize that the essential m-dissipativity of $(L_\Phi, \mathcal{F}C_b^\infty)$ is useful to apply general resolvent methods developed by Beznea, Boboc and Röckner, in order to construct martingale/weak solutions to infinite-dimensional non-linear degenerate stochastic differential equations. Furthermore, the essential m-dissipativity of $(L_\Phi, \mathcal{F}C_b^\infty)$ and $(N, \mathcal{F}C_b^\infty)$, as well as the regularity estimates are essential to apply the general abstract Hilbert space hypocoercivity method from Dolbeault, Mouhot, Schmeiser and Grothaus, Stilgenbauer, respectively, to the corresponding diffusions.

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1. Introduction

The classical Langevin dynamics (compare [23, Chapter 8.1])

$$\begin{aligned} \mathrm{d}X_t &= Y_t \mathrm{d}t \\ \mathrm{d}Y_t &= (-Y_t - D\Phi(X_t))\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \end{aligned} \tag{1.1}$$

describes the evolution of the positions $X_t = (X_t^{(1)}, ..., X_t^{(n)}) \in (\mathbb{R}^d)^n$ and velocities $Y_t = (Y_t^{(1)}, ..., Y_t^{(n)}) \in (\mathbb{R}^d)^n$ of n particles in dimension d, where

$(W_t)_{t\geq 0}$ is a standard Brownian Motion in $(\mathbb{R}^d)^n$. I.e. the velocity of the particles is subjected to friction and a stochastic perturbation. The n-particle potential $\Phi : (\mathbb{R}^d)^n \to \mathbb{R}$, with gradient $D\Phi$, affects the motion of the particles and can be used to model their interactions.

The equation has been studied under various aspects. In order to show exponential convergence to equilibrium of such type of non-coercive evolution equations, Cedric Villani developed the concepts of hypocoercivity, see [25]. Abstract hypocoercivity concepts for quantitative descriptions of convergence rates are introduced in [13]. These are translated to the corresponding stochastic equations, taking domain issues into account, in [15]. In [16] these concepts have been further generalized to the case where only a weak Poincaré inequality is needed. In this case one obtains (sub-)exponential convergence rates. Ergodicity and rate of convergence to equilibrium of the Langevin dynamics with singular potentials are elaborated e.g. in [3] and [17]. Recently, the dynamics and its hypocoercivity behavior is studied on abstract smooth manifolds, see [20]. The latter articles cited above have in common that they study the associated Kolmogorov backward operator. Applying Itô's formula, the Kolmogorov backward operator associated to (1.1), also called Langevin operator, applied to $f \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ (w.l.o.g $n = 1$) is given by

$$L_\Phi f = \Delta_2 f - \langle x, D_2 f\rangle - \langle D\Phi, D_2 f\rangle + \langle y, D_1 f\rangle.$$

Here, $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ denotes the space of compactly supported smooth (infinitely often differentiable) functions from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$, x and y the projection to the spatial and the velocity component, respectively, $\langle\cdot,\cdot\rangle$ the euclidean inner product, Δ_2, D_2 the Laplacian and the gradient in the velocity component and D_1 the gradient in the spatial component. The key observation is the essential m-dissipativity of $(L_\Phi, C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ defined in $L^2(\mathbb{R}^d \times \mathbb{R}^d, \mu^\Phi, \mathbb{R})$, where

$$\mu^\Phi = (2\pi)^{-\frac{d}{2}} e^{-\Phi(x)-\frac{1}{2}y^2} \lambda^d \otimes \lambda^d.$$

We want to emphasize the degenerate structure of the Langevin equation, i.e. the noise only appears in the velocity component. The degeneracy of the equation corresponds to the fact that the Laplacian in the definition of $(L_\Phi, C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ is degenerate, i.e. only acts in the velocity component. As the antisymmetric part of $(L_\Phi, C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ in $L^2(\mathbb{R}^d \times \mathbb{R}^d, \mu^\Phi, \mathbb{R})$ contains first order differential operators in the spatial component and the symmetric part only differential operators in the velocity component, the operator $(L_\Phi, C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d))$ is non-sectorial.

In this article we address an infinite-dimensional generalization of the Langevin operator above. In order to do that let $(U, (\cdot,\cdot)_U)$ and $(V, (\cdot,\cdot)_V)$ be two real separable Hilbert spaces. Consider the real separable Hilbert space $W = U \times V$ with inner product $(\cdot,\cdot)_W$ defined by

$$((u_1, v_1), (u_2, v_2))_W = (u_1, u_2)_U + (v_1, v_2)_V, \quad (u_1, v_1), (u_2, v_2) \in W.$$

Denote by $\mathcal{B}(U)$ and $\mathcal{B}(V)$ the Borel σ-algebra on U and V, on which we consider centered non-degenerate infinite-dimensional Gaussian measures μ_1 and μ_2, respectively. The measures are uniquely determined by their covariance operators $Q_1 \in \mathcal{L}(U)$ and $Q_2 \in \mathcal{L}(V)$. Furthermore, we consider bounded linear operators $K_{12} \in \mathcal{L}(U;V)$, $K_{21} \in \mathcal{L}(V;U)$ and a symmetric bounded linear operator $K_{22} \in \mathcal{L}(V)$. For a given measurable potential $\Phi : U \to (-\infty,\infty]$, which is bounded from below, we set $\rho_\Phi = \frac{1}{c_\Phi} e^{-\Phi}$, where $c_\Phi = \int_U e^{-\Phi} \mathrm{d}\mu_1$ and consider the measure $\mu_1^\Phi = \rho_\Phi \mu_1$ on $(U, \mathcal{B}(U))$. On $(W, \mathcal{B}(W))$ we introduce the product measure

$$\mu^\Phi = \mu_1^\Phi \otimes \mu_2.$$

The infinite-dimensional Langevin operator $(L_\Phi, \mathcal{F}C_b^\infty)$ is defined by

$$\mathcal{F}C_b^\infty \ni f \mapsto L_\Phi f = S_\Phi f - A_\Phi f \in L^2(\mu^\Phi),$$

where for $f \in \mathcal{F}C_b^\infty$, $S_\Phi f$ and $A_\Phi f$ are given by

$$\begin{aligned} S_\Phi f =& \operatorname{tr}[K_{22} D_2^2 f] - (v, Q_2^{-1} K_{22} D_2 f)_V, \\ A_\Phi f =& (u, Q_1^{-1} K_{21} D_2 f)_U + (D\Phi(u), K_{21} D_2 f)_U - (v, Q_2^{-1} K_{12} D_1 f)_V. \end{aligned}$$

Above, $\mathcal{F}C_b^\infty$ is a space of finitely based, smooth and bounded functions, see Definition 2.3 and 4.1, below. Furthermore, u and v denotes the projections of W to U and V, respectively. One of the major challenges in this article is to show essential m-dissipativity of the infinite-dimensional Langevin operator in $L^2(\mu^\Phi)$.

We also address essential m-dissipativity and regularity estimates in $L^2(\mu_1^\Phi)$ for infinite-dimensional Ornstein-Uhlenbeck operators, perturbed by the gradient of a potential Φ. Indeed we fix a possible unbounded linear operator $(C, D(C))$ in U and introduce the operator $(N, \mathcal{F}C_b^\infty)$ in $L^2(\mu_1^\Phi)$ defined by

$$\mathcal{F}C_b^\infty \ni f \mapsto Nf = \operatorname{tr}[CD^2 f] - (u, Q_1^{-1} CDf)_U - (D\Phi, CDf)_U \in L^2(\mu_1^\Phi).$$

In [7], [2] and [8] similar operators and corresponding regularity estimates are studied, but the results are restricted to bounded diffusion operators $(C, D(C))$ as coefficients.

The essential m-dissipativity of $(N, \mathcal{F}C_b^\infty)$ in $L^2(\mu_1^\Phi)$ is useful in various applications. E.g. to study stochastic quantization problems as in [8, Section 4] and to solve stochastic reaction diffusion equations as in [7, Section 5]. In addition, essential m-dissipativity and related regularity estimates of such operators, will be essential for our planed application of the general abstract hypocoercivity method from [15] to our infinite-dimensional setting. For this application it is needed to allow unbounded diffusion operators $(C, D(C))$ as coefficients in the definition of the perturbed Ornstein-Uhlenbeck operator $(N, \mathcal{F}C_b^\infty)$.

The organization of this article is as follows. First, we fix notions and define several important spaces. Then properties of infinite-dimensional Gaussian measures are elaborated, especially the relation between finite and infinite-dimensional Gaussian measures in Lemma 2.1 and the integration by parts formula from Corollary 2.11 are focused. In Theorem 2.9 we use the integration by parts formula to describe Sobolev spaces w.r.t. infinite-dimensional Gaussian measures.

Sect. 3 introduces necessary conditions on $(C, D(C))$ (compare Hypothesis 3.1) to obtain essential m-dissipativity of an Ornstein-Uhlenbeck operator with diffusion operator $(C, D(C))$ as coefficient. In Theorem 3.6 we perturb this Ornstein-Uhlenbeck operator by the gradient of a potential $\Phi : U \to (-\infty, \infty]$, which is in $W^{1,2}(U, \mu_1, \mathbb{R})$ and bounded from below. If $D\Phi$ is strictly bounded by $\frac{1}{2\sqrt{\lambda_1}}$, where λ_1 is the biggest eigenvalue of Q_1, we obtain essential m-dissipativity of $(N, \mathcal{F}C_b^\infty)$ in $L^2(\mu_1^\Phi)$. Note that the restriction to such potentials is due to the possible unboundedness of $(C, D(C))$. In the second part of this section we imitate the strategy used in [7] to derive an infinite-dimensional second order regularity estimate for $f \in \mathcal{F}C_b^\infty$ in terms of $g = \alpha f - Nf$, $\alpha \in (0, \infty)$.

In Sect. 4, we deal with the essential m-dissipativity of $(L_\Phi, \mathcal{F}C_b^\infty)$ in $L^2(\mu^\Phi)$. First, we consider the case where the potential $\Phi = 0$. We decompose our infinite-dimensional Langevin operator into countable finite-dimensional ones, to use arguments for finite-dimensional Langevin operators as described in [21] and [6]. We derive first order regularity estimates needed to add perturbations in terms of Φ. I.e. we consider potentials Φ as in Theorem 4.11 and use the Neumann-Series theorem to obtain essential m-dissipativity of $(L_\Phi, \mathcal{F}C_b^\infty)$ in $L^2(\mu^\Phi)$. During the whole section we assume Hypothesis 4.3, which is the key to the decomposition described above.

Applications of the results, we derived in Sect. 3 and Sect. 4, are discussed in the last section. We propose an infinite-dimensional non-linear degenerate stochastic differential equation, see (5.1). With the results we achieved in this article and the resolvent methods from [1] we plan to solve it. Moreover, we elaborate, how the essential m-dissipativity of $(N, \mathcal{F}C_b^\infty)$ can be used to show hypocoercivity of the semigroup $(T_t)_{t\geq 0}$ generated by the closure of $(L_\Phi, \mathcal{F}C_b^\infty)$ and how hypocoercivity of $(T_t)_{t\geq 0}$ is related to the long time behavior of the process solving (5.1). The main results obtained in this article are summarized in the following list:

- We prove essential m-dissipativity of perturbed Ornstein-Uhlenbeck operators $(N, \mathcal{F}C_b^\infty)$ in $L^2(\mu_1^\Phi)$. We allow possible unbounded diffusions $(C, D(C))$ as coefficients, see Theorem 3.6. There Hypothesis 3.1 is assumed and perturbations by the gradient of a potential Φ, which is bounded from below and in $W^{1,2}(U, \mu_1, \mathbb{R})$, are considered. In addition, an appropriate bound for the gradient of Φ, i.e. $\|D\Phi\|_{L^\infty(\mu_1)} < \frac{1}{2\sqrt{\lambda_1}}$, where λ_1 is the biggest eigenvalue of Q_1 (see Theorem 3.6), is needed.
- Considering potentials $\Phi \in W^{1,2}(U, \mu_1, \mathbb{R})$, which are convex, bounded from below and lower semicontinuous as in Hypothesis 3.9, we provide second order regularity estimate for $f \in \mathcal{F}C_b^\infty$ in terms of $g = \alpha f - Nf$, $\alpha \in (0, \infty)$. Indeed by Theorem 3.11 it holds

  $$\int_U \mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2 \mathrm{d}\mu_1^\Phi \leq 4 \int_U g^2 \mathrm{d}\mu_1^\Phi,$$

  where $(C, D(C))$ is the possible unbounded diffusion coefficient in the definition of $(N, \mathcal{F}C_b^\infty)$.

- Essential m-dissipativity of the infinite-dimensional Langevin operator $(L_\Phi, \mathcal{F}C_b^\infty)$ in $L^2(\mu^\Phi)$ is shown in Theorem 4.11. We consider potentials Φ in $W^{1,2}(U, \mu_1, \mathbb{R})$, which are bounded from below, with bounded gradient and assume Hypothesis 4.3 and 4.10.

2. Notations and Preliminaries

Let U and V be two real separable Hilbert spaces with inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$, respectively. The induced norms are denoted by $\|\cdot\|_U$ and $\|\cdot\|_V$. The set of all linear bounded operators from U to U and from U to V are denoted by $\mathcal{L}(U)$ and $\mathcal{L}(U;V)$. The adjoint of an operator $J \in \mathcal{L}(U;V)$ is denoted by J^*. By $\mathcal{L}^+(U)$ we shall denote the subset of $\mathcal{L}(U)$ consisting of all nonnegative symmetric operators. The subset of operators in $\mathcal{L}^+(U)$ of trace class is denoted by $\mathcal{L}_1^+(U)$ and the set of Hilbert-Schmidt operators by $\mathcal{L}_2(U)$.

Suppose we have $J \in \mathcal{L}^+(U)$. If J is injective it is reasonable to talk about the inverse of $J : U \to J(U)$, which will be denoted by J^{-1}. Due to [22, Proposition 4.4.8.] there exists a unique operator $J^{\frac{1}{2}} \in \mathcal{L}^+(U)$ such that $(J^{\frac{1}{2}})^2 = J$. If J^{-1} exists, so does $(J^{\frac{1}{2}})^{-1}$, in this case we denote $(J^{\frac{1}{2}})^{-1}$ by $J^{-\frac{1}{2}}$. By $\mathcal{B}(U)$ we denote the Borel σ-algebra, i.e. the σ-algebra generated by the open sets in $(U, (\cdot,\cdot)_U)$. The euclidean inner product and induced norm is denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively.

For a given measure space $(\Omega, \mathcal{A}, m)$ and a Banach space Y we denote by $L^p(\Omega, m, Y)$, $p \in [0,\infty]$ the Hilbert space of equivalence classes of $\mathcal{A}$-$\mathcal{B}(Y)$ measurable and p-integrable functions. The corresponding norm is denoted by $\|\cdot\|_{L^p(\Omega,m,Y)}$. If $p = 2$, the norm is induced by an inner product denoted by $(\cdot,\cdot)_{L^2(\Omega,m,Y)}$. In case $(\Omega, \mathcal{A})$ is clear from the context and $Y = \mathbb{R}^n$ for some $n \in \mathbb{N}$, we also write $L^2(m)$ instead of $L^2(\Omega, m, \mathbb{R}^n)$. By λ^n, $n \in \mathbb{N}$, we denote the Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

On the measurable space $(U, \mathcal{B}(U))$ we consider an infinite-dimensional non-degenerate Gaussian measure μ_1 with covariance operator $Q_1 \in \mathcal{L}_1^+(U)$. Since the measure is non-degenerate the operator Q_1 is injective and therefore positive ($(Q_1 u, u)_U > 0$ for all $u \in U$). For the definition and construction of these measures we refer to the first chapter of [10].

In the next lemma we discuss the important relation between finite and infinite-dimensional Gaussian measures. A proof can be found in [10, Corollary 1.19].

Lemma 2.1. *Given $n \in \mathbb{N}$ and elements $l_1, ..., l_n \in U$. The image measure μ_1^n of μ_1 under the map*

$$U \ni u \mapsto \big((l_1, u)_U, ...(l_n, u)_U\big) \in \mathbb{R}^n$$

is the centered Gaussian measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with covariance matrix $Q_{1,n} = ((Q_1 l_i, l_j)_U)_{ij=1,...,n}$.

If $l_1, ..., l_n$ is an orthonormal system of eigenvectors of Q_1 with corresponding eigenvalues $\lambda_1, ..., \lambda_n$, the covariance matrix $Q_{1,n}$ of μ_1^n is given by the diagonal matrix $\mathrm{diag}(\lambda_1, ..., \lambda_n)$.

During this article we have to perform explicit calculations of integrals with respect to Gaussian measures including monomials of order 2 and 4, therefore the following lemma is useful. To proof it, apply Lemma 2.1 and Isserlis formula from [18].

Lemma 2.2. *For $l_1, l_2, l_3, l_4 \in U$ it holds*

$$\int_U (u, l_1)_U (u, l_2)_U \mathrm{d}\mu_1(u) = (Q_1 l_1, l_2)_U \quad \textit{and}$$

$$\int_U (u, l_1)_U (u, l_2)_U (u, l_3)_U (u, l_4)_U \mathrm{d}\mu_1(u)$$
$$= (Q_1 l_1, l_2)_U (Q_1 l_3, l_4)_U + (Q_1 l_1, l_3)_U (Q_1 l_2, l_4)_U + (Q_1 l_1, l_4)_U (Q_1 l_2, l_3)_U.$$

To cover more general situations we consider a measurable potential $\Phi : U \mapsto (-\infty, \infty]$, which is bounded from below. During the paper we will assume more or less restrictive assumptions on the potential. As in the introduction we set $\rho_\Phi = \frac{1}{c_\Phi} e^{-\Phi}$, where $c_\Phi = \int_U e^{-\Phi} \mathrm{d}\mu_1$. On $(U, \mathcal{B}(U))$ we consider the measure μ_1^Φ defined by

$$\mu_1^\Phi = \rho_\Phi \mu_1.$$

I.e. a measures having a density with respect to the infinite-dimensional Gaussian measure μ_1. We fix an orthonormal basis $B_U = (d_i)_{i \in \mathbb{N}}$ of U.

Definition 2.3. For $n \in \mathbb{N}$, set $B_U^n = \operatorname{span}\{d_1, ..., d_n\}$. The orthogonal projection from U to B_U^n is denoted by $\overline{P}_n$ and the corresponding coordinate map by P_n, i.e. we have for all $u \in U$

$$\overline{P}_n(u) = \sum_{i=1}^n (u, d_i)_U d_i \quad \text{and} \quad P_n(u) = \big((u, d_1)_U, ..., (u, d_n)_U\big).$$

Let $C_b^\infty(\mathbb{R}^n)$ be the space of all bounded smooth (infinitely often differentiable) real-valued functions on $\mathbb{R}^n$. The space of finitely based smooth and bounded functions on U, is defined by

$$\mathcal{F}C_b^\infty(B_U) = \{U \ni u \mapsto \varphi(P_m(u)) \in \mathbb{R} \mid m \in \mathbb{N},\ \varphi \in C_b^\infty(\mathbb{R}^m)\}.$$

The subset of functions only depending on n-directions is defined correspondingly by

$$\mathcal{F}C_b^\infty(B_U, n) = \{U \ni u \mapsto \varphi(P_n(u)) \in \mathbb{R} \mid \varphi \in C_b^\infty(\mathbb{R}^n)\}.$$

For later use we define

$$L^2_{B_{U_n}}(U, \mu_1, \mathbb{R}) = \{U \ni u \mapsto f(P_n(u)) \in \mathbb{R} \mid f \in L^2(\mathbb{R}^n, \mu_1^n, \mathbb{R})\},$$

where μ_1^n is the image measure of μ_1 under P_n. Equipping $L^2_{B_{U_n}}(U, \mu_1, \mathbb{R})$ with the inner product from $L^2(U, \mu_1, \mathbb{R})$ we obtain another Hilbert space.

A very useful density result, proved in [7, Lemma 2.2], is stated in the next lemma.

Lemma 2.4. *The function spaces $\mathcal{F}C_b^\infty(B_U)$ and $\mathcal{F}C_b^\infty(B_U, n)$ are dense in $L^2(U, \mu_1, \mathbb{R})$ and $L^2_{B_{U_n}}(U, \mu_1, \mathbb{R})$, respectively.*

Remark 2.5. Given a Frechét differentiable function $f : U \to \mathbb{R}$. For $u \in U$ we denote by $Df(u) \in U$ the gradient of f in u. Analogously for a two times Frechét differentiable function $f : U \to \mathbb{R}$, we identify $D^2 f(u) \in L(U)$ with the second order Frechét derivative in $u \in U$. For $i, j \in \mathbb{N}$ we denote by $\partial_i f(u) = (Df(u), d_i)_U$ the partial derivative in the direction of d_i and by $\partial_{ij} f(u) = (D^2 f(u) d_i, d_j)_U$ the second order partial derivative in the direction of d_i and d_j.

We continue this section with the important integration by parts formula for infinite-dimensional Gaussian measures, with and without densities. We assume that $B_U = (d_i)_{i\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of Q_1 with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}} \subset (0, \infty)$. W.l.o.g. we assume that $(\lambda_i)_{i\in\mathbb{N}}$ is decreasing to zero.

Since Q_1 is injective, the inverse Q_1^{-1} of $Q_1 : U \to Q_1(U)$ exists. Obviously it holds

$$Q_1^{-1} d_i = \frac{1}{\lambda_i} d_i, \quad i \in \mathbb{N},$$

and therefore it is reasonable to define the operator $Q_1^{-\frac{1}{2}}$ on $\bigcup_{n\in\mathbb{N}} B_U^n$ determined by

$$Q_1^{-\frac{1}{2}} d_i = \frac{1}{\sqrt{\lambda_i}} d_i, \quad i \in \mathbb{N}.$$

Theorem 2.6. *For $f, g \in \mathcal{F}C_b^\infty(B_U)$ and $i \in \mathbb{N}$, it holds the integration by parts formula*

$$\int_U \partial_i f g \,\mathrm{d}\mu_1 = -\int_U f \partial_i g \,\mathrm{d}\mu_1 + \int_U (u, Q^{-1} d_i)_U f g \,\mathrm{d}\mu_1. \tag{2.1}$$

Proof. Apply Lemma 2.1, use the standard integration by parts formula and use Lemma 2.1 again. For a more detailed proof see [10, Lemma 10.1].

Fix a possible unbounded linear operator $(C, D(C))$ on U fulfilling the following hypothesis.

Hypothesis 2.7. *1. $(C, D(C))$ is symmetric.*

2. There is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \le m_k$, it holds

$$B_U^n \subset D(C) \quad C(B_U^n) \subset B_U^{m_k}.$$

Remark 2.8. Note that for given $n \in \mathbb{N}$ and $f = \varphi(P_n(\cdot)) \in \mathcal{F}C_b^\infty(B_U)$ one has $Df = \sum_{i=1}^n \partial_i \varphi(P_n(\cdot)) d_i \in B_U^n$. Hence the hypothesis above ensure that expressions like CDf, $Q_1^{-\frac{1}{2}} CDf$ or $Q_1^{-1} CDf$ are well-defined.

Theorem 2.9. *Assume that Hypothesis 2.7 hold. The operators*

$$D : \mathcal{F}C_b^\infty(B_U) \to L^2(U, \mu_1, U)$$
$$CD : \mathcal{F}C_b^\infty(B_U) \to L^2(U, \mu_1, U)$$
$$Q_1^{-\frac{1}{2}} CD : \mathcal{F}C_b^\infty(B_U) \to L^2(U, \mu_1, U)$$

$$(CD, CD^2) : \mathcal{F}C_b^\infty(B_U) \to L^2(U, \mu_1, U) \times L^2(U, \mu_1, \mathcal{L}_2(U))$$

are closable in $L^2(U, \mu_1, \mathbb{R})$.

Proof. As the proof of closability for the first three operators are essentially the same, we restrict ourself to the third and the last. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^\infty(B_U)$ converge to 0 in $L^2(U, \mu_1, \mathbb{R})$ and be such that $Q_1^{-\frac{1}{2}} CDf_n \to F$ in $L^2(U, \mu_1, U)$ as $n \to \infty$. For $m \in \mathbb{N}$, there is some $m_k \in \mathbb{N}$ such that $B_U^m \subset B_U^{m_k}$. By symmetry and the invariance properties of $(C, D(C))$ and the fact that $(d_i)_{i\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of Q_1 it holds

$$(Q_1^{-\frac{1}{2}} CDf_n, d_m)_U = \sum_{l=1}^{\infty} (Q_1^{-\frac{1}{2}} Cd_l, d_m)_U \partial_l f_n = \sum_{l=1}^{m_k} (Q_1^{-\frac{1}{2}} Cd_l, d_m)_U \partial_l f_n.$$

For an arbitrary $g \in \mathcal{F}C_b^\infty(B_U)$ we obtain by the integration by parts formula

$$\begin{aligned} &\int_U (Q_1^{-\frac{1}{2}} CDf_n, d_m)_U g \mathrm{d}\mu_1 \\ &= -\sum_{l=1}^{m_k} (Q_1^{-\frac{1}{2}} Cd_l, d_m)_U \int_U f_n (\partial_l g - (u, Q_1^{-1} d_l)_U g) \mathrm{d}\mu_1. \end{aligned}$$

Observe that g and $\partial_l g - (u, Q_1^{-1} d_l)_U g$ are in $L^2(U, \mu, \mathbb{R})$ and therefore

$$\int_U (F, d_m)_U g \mathrm{d}\mu_1 = 0.$$

By the density of $\mathcal{F}C_b^\infty(B_U)$ in $L^2(U, \mu_1, \mathbb{R})$ we conclude $(F, d_m)_U = 0$ for all $m \in \mathbb{N}$, hence finally $F = 0$. To show that the fourth operator is closable we proceed similarly. Indeed, let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^\infty(B_U)$ converge to 0 in $L^2(U, \mu_1, \mathbb{R})$ and be such that $CDf_n \to F$ in $L^2(U, \mu_1, U)$ and $CD^2 f_n \to A$ in $L^2(U, \mu_1, \mathcal{L}_2(U))$, as $n \to \infty$. As above $F = 0$. Now for $l, m \in \mathbb{N}$ we find a corresponding m_k with $l, m \le m_k$ such that

$$(CD^2 f_n d_l, d_m)_U = \sum_{i=1}^{m_k} (Cd_l, d_i)_U \partial_{li} f_n.$$

Hence for arbitrary $g \in \mathcal{F}C_b^\infty(B_U)$, we obtain by the integration by parts formula

$$\begin{aligned} \int_U (CD^2 f_n d_l, d_m)_U g \mathrm{d}\mu_1 &= \sum_{i=1}^{m_k} (Cd_l, d_i)_U \int_U \partial_{li} f_n g \mathrm{d}\mu_1 \\ &= -\sum_{i=1}^{m_k} (Cd_l, d_i)_U \int_U \partial_i f_n (\partial_l g - (u, Q_1^{-1} d_l)_U g) \mathrm{d}\mu_1 \\ &= -\int_U (d_l, CDf_n)_U (\partial_l g - (u, Q_1^{-1} d_l)_U g) \mathrm{d}\mu_1. \end{aligned}$$

Arguing as in the first part we observe $(Ad_l, d_m)_U = 0$ in $L^2(U, \mu_1, \mathbb{R})$, implying $A = 0$ in $L^2(U, \mu_1, \mathcal{L}_2(U))$.

By Theorem 2.9 it is reasonable to define $W^{1,2}(U, \mu_1, \mathbb{R})$, $W_C^{1,2}(U, \mu_1, \mathbb{R})$, $W^{1,2}_{Q_1^{-\frac{1}{2}} C}(U, \mu_1, \mathbb{R})$ and $W_C^{2,2}(U, \mu_1, \mathbb{R})$ as the domain of the closures of D,

CD, $Q_1^{-\frac{1}{2}}CD$ and (CD, CD^2) in $L^2(U, \mu_1, \mathbb{R})$, respectively. We still denote the closures of the differential operators from the theorem above by D, CD, $Q_1^{-\frac{1}{2}}CD$ and (CD, CD^2). By [10, Proposition 10.6] every bounded function $f : U \to \mathbb{R}$ with bounded Frechét derivative is in $W^{1,2}(U, \mu_1, \mathbb{R})$ and the classical gradient of f coincides with Df in $L^2(U, \mu_1, \mathbb{R})$.

Remark 2.10. Adapting the proof of [11, Lemma 9.2.5] one can show that the integration by parts formula from Theorem 2.6 also holds in the case that $f, g \in W^{1,2}(U, \mu_1, \mathbb{R})$.

Invoking the remark above the following integration by parts formula for the measure μ_1^Φ is valid.

Corollary 2.11. *Assume the potential $\Phi : U \to (-\infty, \infty]$ is bounded from below and in $W^{1,2}(U, \mu_1, \mathbb{R})$. For $f, g \in \mathcal{F}C_b^\infty(B_U)$ and $i \in \mathbb{N}$, it holds the integration by parts formula*

$$\int_U \partial_i f g \mathrm{d}\mu_1^\Phi = -\int_U f \partial_i g \mathrm{d}\mu_1^\Phi + \int_U (u, Q^{-1} d_i)_U f g \mathrm{d}\mu_1^\Phi + \int_U \partial_i \Phi f g \mathrm{d}\mu_1^\Phi.$$

3. Perturbed Ornstein-Uhlenbeck Operators and Corresponding Regularity Estimates

This section is devoted to an infinite-dimensional Ornstein-Uhlenbeck operator, perturbed by the gradient of the potential Φ. As already mentioned in the introduction, such operators naturally occur during the application of the abstract Hilbert space hypocoercivity method. Also an infinite-dimensional regularity estimate is derived in the second part of this section. The proof of such estimates is motivated by the results from [7], where Giuseppe Da Prato and Alessandra Lunardi investigated Sobolev regularity for a class of second order elliptic partial differential equations in infinite-dimensions. As before μ_1 is a centered non-degenerate Gaussian measure on the real separable Hilbert space U with covariance operator $Q_1 \in \mathcal{L}_1^+(U)$. We fix an orthonormal basis of eigenvectors $B_U = (d_i)_{i\in\mathbb{N}}$ of Q_1. W.l.o.g. the corresponding sequence of eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ decreases to zero. We start with the operator $(N_0, \mathcal{F}C_b^\infty(B_U))$, defined in $L^2(U, \mu_1, \mathbb{R})$ by

$$\mathcal{F}C_b^\infty(B_U) \ni f \mapsto N_0 f = \mathrm{tr}[CD^2 f] - (u, Q^{-1}CDf)_U \in L^2(U, \mu_1, \mathbb{R}),$$

where we assume that $(C, D(C))$ is a possible unbounded linear operator on U. Since we allow such unbounded diffusions as coefficients, we cannot use general results from [7] or [11, Section 10]. Assuming Hypothesis 3.1 below, ensures that the expressions $\mathrm{tr}[CD^2 f]$ and $Q_1^{-1}CDf$ are reasonable.

At this point we have to mention that the operator N_0 is well-defined in the sense that two representatives of the same equivalence class yield the same output. To see this, note that the measure μ_1 has full topological support, i.e. the smallest closed measurable set with full measure is U. The proof of this statement can be found in [24], it relies on the fact that we assumed that the Hilbert space U is separable.

During this section we permanently assume the following hypothesis.

Hypothesis 3.1. *1. $(C, D(C))$ is symmetric and positive.*

2. There is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \leq m_k$, it holds

$$B_U^n \subset D(C) \quad C(B_U^n) \subset B_U^{m_k}.$$

For $n \in \mathbb{N}$, set $n^* = \min_{k\in\mathbb{N}}\{m_k | n \leq m_k\}$.

Note that Hypothesis 3.1 adds positivity of $(C, D(C))$ to Hypothesis 2.7. This additional assumption is important to achieve essential m-dissipativity of $(N_0, \mathcal{F}C_b^\infty(B_U))$, which is proved in the theorem below.

Theorem 3.2. *The operator $(N_0, \mathcal{F}C_b^\infty(B_U))$ is*

1. dissipative in $L^2(U, \mu_1, \mathbb{R})$, with

$$(N_0 f, g)_{L^2(U,\mu_1,\mathbb{R})} = \int_U -(CDf, Dg)_U \mathrm{d}\mu_1,$$

for all $f, g \in \mathcal{F}C_b^\infty(B_U)$ and fulfills

2. the dense range condition $\overline{(Id - N_0)(\mathcal{F}C_b^\infty(B_U))} = L^2(U, \mu_1, \mathbb{R})$,

i.e. is essentially m-dissipative in $L^2(U, \mu_1, \mathbb{R})$. The resolvent in $\alpha \in (0, \infty)$ of the closure $(N_0, D(N_0))$ is denoted by $R(\alpha, N_0)$.

Proof. The first item of the statement follows by the integration by parts formula from Theorem 2.6 together with the invariance properties of $(C, D(C))$. For the second statement we fix $n \in \mathbb{N}$, $f = \varphi(P_n(\cdot)) \in \mathcal{F}C_b^\infty(B_U, n)$ and $n^* \in \mathbb{N}$, according to Hypothesis 3.1. Extending $\varphi \in C_b^\infty(\mathbb{R}^n)$ canonically to a function $\tilde{\varphi} \in C_b^\infty(\mathbb{R}^{n^*})$ we can calculate

$$\begin{aligned} &N_0 f(u) \\ &= \sum_{i,j=1}^{n} \partial_{ij}\varphi(P_n(u))(Cd_i, d_j)_U - \sum_{i=1}^{n}\sum_{j=1}^{n^*}(u, d_j)_U (d_j, Q_1^{-1}Cd_i)_U \partial_i \varphi(P_n(u)) \\ &= \sum_{i,j=1}^{n^*} \partial_{ij}\tilde{\varphi}(P_{n^*}(u))(Cd_i, d_j)_U - \sum_{i,j=1}^{n^*}(u, d_j)_U \frac{1}{\lambda_j}(d_j, Cd_i)_U \partial_i \tilde{\varphi}(P_{n^*}(u)). \end{aligned}$$

Hence if we set

$$C_{n^*} = ((Cd_i, d_j)_U)_{ij=1}^{n^*} \quad \text{and} \quad Q_{1,n^*} = ((Q_1 d_i, d_j)_U)_{ij=1}^{n^*} = \mathrm{diag}(\lambda_1, ..., \lambda_{n^*}),$$

we obtain

$$N_0 f(u) = \mathrm{tr}[C_{n^*} D^2 \tilde{\varphi}(P_{n^*}(u))] - \langle P_{n^*}(u), Q_{1,n^*}^{-1} C_{n^*} D\tilde{\varphi}(P_{n^*}(u))\rangle.$$

It is therefore natural to consider the operator $(N_{0,n^*}, C_b^\infty(\mathbb{R}^{n^*}))$ defined by

$$C_b^\infty(\mathbb{R}^{n^*}) \ni \varphi \mapsto N_{0,n^*}\varphi = \mathrm{tr}[C_{n^*} D^2 \varphi] - \langle \cdot, Q_{1,n^*}^{-1} C_{n^*} D\varphi\rangle \in L^2(\mathbb{R}^{n^*}, \mu_1^{n^*}, \mathbb{R}).$$

As the matrix C_{n^*} is symmetric and positive and $-C_{n^*}Q_{1,n^*}^{-1}$ has only negative eigenvalues we can use the argumentation of [21] (compare also Proposition 4.7 below) to obtain that $(N_{0,n^*}, C_b^\infty(\mathbb{R}^{n^*}))$ is essentially m-dissipative in $L^2(\mathbb{R}^{n^*}, \mu_1^{n^*}, \mathbb{R})$, hence $(Id - N_{0,n^*})(C_b^\infty(\mathbb{R}^{n^*}))$ is dense in $L^2(\mathbb{R}^{n^*}, \mu_1^{n^*}, \mathbb{R})$.

Given $\varepsilon > 0$ and $h = g(P_{n^*}(\cdot)) \in L^2_{B_{U_{n^*}}}(U, \mu_1, \mathbb{R})$. As $(Id - N_{0,n^*})(C_b^\infty(\mathbb{R}^{n^*}))$ is dense in $L^2(\mathbb{R}^{n^*}, \mu_1^{n^*}, \mathbb{R})$ we find $\varphi \in C_b^\infty(\mathbb{R}^{n^*})$ such that

$$\|(Id - N_{0,n^*})\varphi - g\|_{L^2(\mathbb{R}^{n^*}, \mu_1^{n^*}, \mathbb{R})} < \varepsilon.$$

Using Lemma 2.1 we obtain

$$\begin{aligned} &\|(Id - N_0)\varphi(P_{n^*}(\cdot)) - h\|_{L^2_{B_{U_{n^*}}}(U,\mu_1,\mathbb{R})} \\ =&\|(Id - N_{0,n^*})\varphi - g\|_{L^2(\mathbb{R}^n, \mu_1^{n^*}, \mathbb{R})} < \varepsilon. \end{aligned}$$

In other words $(Id - N_0)(\mathcal{F}C_b^\infty(B_U, n^*))$ is dense in $L^2_{B_{U_{n^*}}}(U, \mu_1, \mathbb{R})$. Finally we use the result above to show that $(Id - N_0)(\mathcal{F}C_b^\infty(B_U))$ is dense in $L^2(U, \mu_1, \mathbb{R})$. Indeed let $\varepsilon > 0$ and take an element $h \in L^2(U, \mu_1, \mathbb{R})$. By Lemma 2.4 we find $n \in \mathbb{N}$, w.l.o.g. $n = n^*$, and $g \in \mathcal{F}C_b^\infty(B_U, n^*) \subset L^2_{B_{U_{n^*}}}(U, \mu_1, \mathbb{R})$ such that

$$\|h - g\|_{L^2(U,\mu_1,\mathbb{R})} < \frac{\varepsilon}{2}.$$

As $\overline{(Id - N_0)(\mathcal{F}C_b^\infty(B_U, n^*))} = L^2_{B_{U_{n^*}}}(U, \mu_1, \mathbb{R})$ we find $f \in \mathcal{F}C_b^\infty(B_U, n^*)$ such that

$$\|(Id - N_0)f - g\|_{L^2(U,\mu_1,\mathbb{R})} < \frac{\varepsilon}{2}.$$

Hence the triangle inequality yields

$$\|(Id - N_0)f - h\|_{L^2(U,\mu_1,\mathbb{R})} < \varepsilon.$$

Invoking the famous Lumer-Phillips theorem we obtain that the operator $(N_0, \mathcal{F}C_b^\infty(B_U))$ is essentially m-dissipative in $L^2(U, \mu_1, \mathbb{R})$. □

Before we go ahead and perturb $(N_0, \mathcal{F}C_b^\infty(B_U))$ we need an $L^2(\mu_1)$ regularity estimate for the first and second order derivatives of a function $f \in \mathcal{F}C_b^\infty(B_U)$ in terms of $g \in L^2(U, \mu_1, \mathbb{R})$, where

$$\alpha f - N_0 f = g, \tag{3.1}$$

for a given $\alpha \in (0, \infty)$.

Theorem 3.3. *Suppose we have $f \in \mathcal{F}C_b^\infty(B_U)$ and $g = \alpha f - N_0 f$, $\alpha \in (0, \infty)$, as in Equation (3.1) above. It holds $g \in W^{1,2}(U, \mu_1, \mathbb{R})$ and the identities*

$$\int_U \alpha f^2 + (CDf, Df)_U \mathrm{d}\mu_1 = \int_U gf \mathrm{d}\mu_1$$
$$\int_U \alpha(CDf, Df)_U + \|Q_1^{-\frac{1}{2}} CDf\|_U^2 + \mathrm{tr}[(CD^2 f)^2] \mathrm{d}\mu_1 = \int_U (Dg, CDf)_U \mathrm{d}\mu_1,$$

are valid. In particular it holds

$$\int_U \|Q_1^{-\frac{1}{2}} CDf\|_U^2 + \mathrm{tr}[(CD^2 f)^2] \mathrm{d}\mu_1 = \int_U (N_0 f)^2 \mathrm{d}\mu_1.$$

Proof. Due to the definition of N_0 and the fact that F is in $\mathcal{F}C_b^\infty(B_U)$ it is easy to see that g is infinitely often differentiable. As Dg is in $L^2(U,\mu_1,U)$ and has at most linear growth (compare equation (3.2)), an approximation argument shows that g is in $W^{1,2}(U,\mu_1,\mathbb{R})$.

To show the first equation, we multiply (3.1) with f and integrate over U with respect to μ_1. An application of the integration by parts formula from Theorem 2.6 results in

$$\int_U \alpha f^2 + (CDf, Df)_U \mathrm{d}\mu_1 = \int_U gf \mathrm{d}\mu_1.$$

To show the second equation we differentiate (3.1) with respect to the k-th direction yielding

$$\alpha\partial_k f - N_0\partial_k f + (d_k, Q_1^{-1}CDf)_U = \partial_k g \tag{3.2}$$

Now we multiply the equation above with $\partial_l f(d_k, Cd_l)_U$. In order to structure the arguments we treat the resulting terms separately. If we sum over all indices's a direct calculation shows that the first and third term on the left hand side of the equation above is equal to $\alpha(CDf, Df)_U$ and $(CDf, Q_1^{-1}CDf)_U$, respectively. The right hand side of the equation is then equal to $(Dg, CDf)_U$. We also get

$$\begin{aligned}\sum_{k,l=1}^\infty (d_k, Cd_l)_U \int_U -N_0\partial_k f\partial_l f \mathrm{d}\mu_1 &= \sum_{k,l=1}^\infty (d_k, Cd_l)_U \int_U (CD\partial_k f, D\partial_l f)_U \mathrm{d}\mu_1 \\ &= \int_U \mathrm{tr}[(CD^2 f)^2] \mathrm{d}\mu_1,\end{aligned}$$

and therefore the second equation from the statement is shown. Rearranging the terms of the equation we just derived yields

$$\int_U \|Q_1^{-\frac{1}{2}}CDf\|_U^2 + \mathrm{tr}[(CD^2f)^2]\mathrm{d}\mu_1 = \int_U (D(-N_0 f), CDf)_U \mathrm{d}\mu_1.$$

Now it holds

$$\begin{aligned}\int_U (D(-N_0f), CDf)_U \mathrm{d}\mu_1 &= \sum_{k,l=1}^\infty (d_k, Cd_l)_U \int_U \partial_k(-N_0 f)\partial_l f \mathrm{d}\mu_1 \\ &= \sum_{k,l=1}^\infty (d_k, Cd_l)_U \int_U N_0 f(\partial_{kl} f - (u, Q_1^{-1}d_k)_U \partial_l f)\mathrm{d}\mu_1 \\ &= \int_U (N_0 f)^2 \mathrm{d}\mu_1,\end{aligned}$$

where we used that $N_0 f \in W^{1,2}(U,\mu_1,\mathbb{R})$ and Remark 2.10.

Note that the infinite sums in the calculations above are actually finite ones. $\square$

Remark 3.4. Given an arbitrary $f \in \mathcal{F}C_b^\infty(B_U)$. There is some $n \in \mathbb{N}$, s.t. $Df \in B_U^n$. In particular $CDf \in B_U^n$, by Hypothesis 3.1. Therefore

$$\frac{1}{\lambda_1}\|CDf\|_U^2 \leq \|Q_1^{-\frac{1}{2}}CDf\|_U^2. \tag{3.3}$$

Hence by the last equality in Theorem 3.3 we achieve

$$\begin{aligned}\int_U \frac{1}{\lambda_1}\|CDf\|_U^2 + \mathrm{tr}[(CD^2f)^2]\mathrm{d}\mu_1 &\leq \int_U \|Q_1^{-\frac{1}{2}}CDf\|_U^2 + \mathrm{tr}[(CD^2f)^2]\mathrm{d}\mu_1 \\ &= \int_U (N_0 f)^2 \mathrm{d}\mu_1.\end{aligned} \tag{3.4}$$

Corollary 3.5. *It holds*

$$D(N_0) \subset W^{1,2}_{Q_1^{-\frac{1}{2}}C}(U,\mu_1,\mathbb{R}) \cap W^{2,2}_C(U,\mu_1,\mathbb{R})$$

and for all $f \in D(N_0)$ *we have the inequality*

$$\begin{aligned}\int_U \frac{1}{\lambda_1}\|CDf\|_U^2 + \mathrm{tr}[(CD^2f)^2]\mathrm{d}\mu_1 &\leq \int_U \|Q_1^{-\frac{1}{2}}CDf\|_U^2 + \mathrm{tr}[(CD^2f)^2]\mathrm{d}\mu_1 \\ &= \int_U (N_0 f)^2 \mathrm{d}\mu_1.\end{aligned} \tag{3.5}$$

Proof. Given $f \in D(N_0)$. We find a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^\infty(B_U)$ such that $f_n \to f$ and $N_0 f_n \to N_0 f$ in $L^2(U,\mu_1,\mathbb{R})$ as $n \to \infty$. By (In)equality (3.4) $(f_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $W^{1,2}_{Q_1^{-\frac{1}{2}}C}(U,\mu_1,\mathbb{R}) \cap W^{2,2}_C(U,\mu_1,\mathbb{R})$. Hence $f \in W^{1,2}_{Q_1^{-\frac{1}{2}}C}(U,\mu_1,\mathbb{R}) \cap W^{2,2}_C(U,\mu_1,\mathbb{R})$ and the (in)equality of the statement is shown. □

Using (In)equality (3.5) and Neumann-Series theorem we are able to deal with perturbations of $(N_0, \mathcal{F}C_b^\infty(B_U))$ as described in the following theorem.

Theorem 3.6. *Assume that* Φ *is in* $W^{1,2}(U,\mu_1,\mathbb{R})$*, bounded from below and with* $\|D\Phi\|^2_{L^\infty(\mu_1)} < \frac{1}{4\lambda_1}$*. The operator* $(N, \mathcal{F}C_b^\infty(B_U))$ *defined by*

$$\begin{aligned}\mathcal{F}C_b^\infty(B_U) \ni f \mapsto Nf =& \mathrm{tr}[CD^2f] - (u, Q_1^{-1}CDf)_U \\ &- (D\Phi, CDf)_U \in L^2(U,\mu_1^\Phi,\mathbb{R}),\end{aligned}$$

1. *fulfills*
$$(Nf,g)_{L^2(U,\mu_1^\Phi,\mathbb{R})} = \int_U -(CDf,Dg)_U \mathrm{d}\mu_1^\Phi,$$
for all $f,g \in \mathcal{F}C_b^\infty(B_U)$*, in particular is dissipative in* $L^2(U,\mu_1^\Phi,\mathbb{R})$*. Furthermore we have*
2. *the dense range condition* $\overline{(Id - N)(\mathcal{F}C_b^\infty(B_U))} = L^2(U,\mu_1^\Phi,\mathbb{R})$*.*

In particular $(N, \mathcal{F}C_b^\infty(B_U))$ *is essentially m-dissipative in* $L^2(U,\mu_1^\Phi,\mathbb{R})$*. The resolvent in* $\alpha \in (0,\infty)$ *of the closure* $(N, D(N))$ *is denoted by* $R(\alpha, N)$*.*

Proof. The first item of the statement follows by the integration by parts formula from Corollary 2.11 together with the invariance properties of the involved operators. For $f \in L^2(U,\mu_1,\mathbb{R})$ set

$$Tf = -(D\Phi, CDR(1,N_0)f)_U.$$

Since $D(N_0) \subset W^{1,2}_{Q_1^{-\frac{1}{2}}C}(U,\mu_1,\mathbb{R}) \subset W^{1,2}_C(U,\mu_1,\mathbb{R})$ the definition above is reasonable. Using the Cauchy-Schwarz inequality, Inequality (3.5) and the assumption on Φ we observe

$$\begin{aligned}\|Tf\|^2_{L^2(\mu_1)} &= \int_U (D\Phi, CDR(1,N_0)f)^2_U \mathrm{d}\mu_1 \\ &\le \|D\Phi\|^2_{L^\infty(\mu_1)} \int_U \|CDR(1,N_0)f\|^2_U \mathrm{d}\mu_1 \\ &< \frac{1}{4}\int_U (N_0R(1,N_0)f)^2 \mathrm{d}\mu_1 \\ &= \frac{1}{4}\int_U (f - R(1,N_0)f)^2 \mathrm{d}\mu_1 \le \|f\|^2_{L^2(\mu_1)}.\end{aligned}$$

Therefore the linear operator $T : L^2(U,\mu_1,\mathbb{R}) \to L^2(U,\mu_1,\mathbb{R})$ is well-defined with operator norm less than one. Hence by the Neumann-Series theorem we obtain that $(Id - T)^{-1}$ exists in $\mathcal{L}(L^2(U,\mu_1,\mathbb{R}))$. In particular for a given $g \in L^2(U,\mu_1,\mathbb{R})$ we find $f \in L^2(U,\mu_1,\mathbb{R})$ with $f - Tf = g$ in $L^2(U,\mu_1,\mathbb{R})$. Since $(N_0, D(N_0))$ is m-dissipative, there is $h \in D(N_0)$ with $(Id - N_0)h = f$. This yields

$$(Id - N_0)h + (D\Phi, CDh)_U = f + (D\Phi, CDR(1,N_0)f)_U = f - Tf = g.$$

If we can show that $D(N_0) \subset D(N)$ with $Nf = N_0 f - (D\Phi, CDf)_U$ for all $f \in D(N_0)$ the proof is finish by the Lumer-Philipps theorem. Indeed this implies

$$\mathcal{F}C_b^\infty(B_U) \subset L^2(U,\mu_1,\mathbb{R}) \subset (Id - N)(D(N_0)) \subset (Id - N)(D(N)).$$

i.e. the dense range condition, as $\mathcal{F}C_b^\infty(B_U)$ is dense in $L^2(U,\mu_1^\Phi,\mathbb{R})$. So let $f \in D(N_0)$ be given. There is a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^\infty(B_U)$ s.t. $f_n \to f$ and $N_0 f_n \to N_0 f$ in $L^2(U,\mu_1,\mathbb{R})$. As $\rho_\Phi = \frac{1}{c_\Phi}e^{-\Phi}$ is bounded it is easy to see that $f_n \to f$ in $L^2(U,\mu_1^\Phi,\mathbb{R})$. In view of the assumptions on $D\Phi$ and the Inequality (3.5) we can estimate

$$\begin{aligned}&\|N_0 f - (D\Phi, CDf)_U - Nf_n\|^2_{L^2(\mu_1^\Phi)} \\ \le& 2\|N_0(f - f_n)\|^2_{L^2(\mu_1^\Phi)} + 2\int_U (D\Phi, CD(f - f_n))^2_U \mathrm{d}\mu_1^\Phi \\ \le& 2\|\rho_\Phi\|_{L^\infty(\mu_1)}\big(\|N_0(f - f_n)\|^2_{L^2(\mu_1)} + \|D\Phi\|^2_{L^\infty(\mu_1)}\int_U \|CD(f - f_n)\|^2_U \mathrm{d}\mu_1\big) \\ \le& 2\|\rho_\Phi\|_{L^\infty(\mu_1)}\frac{5}{4}\|N_0(f - f_n)\|^2_{L^2(\mu_1)}.\end{aligned}$$

I.e. $Nf_n \to N_0 f - (D\Phi, CDf)_U$ in $L^2(U,\mu_1^\Phi,\mathbb{R})$. Since $(N, D(N))$ is closed by construction we obtain $D(N_0) \subset D(N)$ with $Nf = N_0 f - (D\Phi, CDf)_U$ for all $f \in D(N_0)$ as desired. □

The following lines are devoted to derive a $L^2(U,\mu_1^\Phi,\mathbb{R})$ regularity estimate for the first and second order derivatives of a function $f \in \mathcal{F}C_b^\infty(B_U)$ in terms of $g \in L^2(U,\mu_1,\mathbb{R})$ related via

$$\alpha f - Nf = g, \tag{3.6}$$

for some given $\alpha \in (0,\infty)$.

Theorem 3.7. *Assume that $\Phi : U \to \mathbb{R}$ is Fréchet differentiable, bounded from below and $D\Phi : U \to U$ is Lipschitz continuous. If $f \in \mathcal{F}C_b^\infty(B_U)$, g and $\alpha \in (0,\infty)$ are as in Equation (3.6) it holds*

$$\int_U \alpha f^2 + (CDf, Df)_U \mathrm{d}\mu_1^\Phi = \int_U gf \mathrm{d}\mu_1^\Phi$$

$$\int_U \alpha(CDf, Df)_U + \|Q_1^{-\frac{1}{2}} CDf\|_U^2 + \mathrm{tr}[(CD^2 f)^2] + (D^2\Phi CDf, CDf)_U \mathrm{d}\mu_1^\Phi$$
$$= \int_U (Dg, CDf)_U \mathrm{d}\mu_1^\Phi.$$

In particular we have

$$\int_U \|Q_1^{-\frac{1}{2}} CDf\|_U^2 + \mathrm{tr}[(CD^2 f)^2] + (D^2\Phi CDf, CDf)_U \mathrm{d}\mu^\Phi = \int_U (Nf)^2 \mathrm{d}\mu_1^\Phi.$$

Proof. The first equation follows by multiplying (3.6) with f, an integration over U with respect to μ_1^Φ and an application of the first item in Theorem 3.6. To show the second equation we differentiate (3.6) with respect to the k-th direction resulting in

$$\alpha \partial_k f - N\partial_k f + (d_k, Q_1^{-1} CDf)_U + \sum_{i=1}^\infty (d_i, CDf)_U \partial_{ki}\Phi = \partial_k g.$$

Note that the infinite sum in the line above is actually a finite one. Moreover $\partial_{ki}\Phi$ exists μ_1-a.e., since the Lipschitz continuous function $\partial_i \Phi : U \to \mathbb{R}$ is Gateaux differentiable μ_1-a.e. by [10, Proposition 10.11]. Now we multiply the equation above with $\partial_l f (d_k, Cd_l)_U$. If we sum over all indices's a direct calculation shows that the first and third term as well as the right hand side is equal to $\alpha(CDf, Df)_U$, $\|Q_1^{-\frac{1}{2}} CDf\|_U^2$ and $(Dg, CDf)_U$, respectively. For the second term we calculate

$$\sum_{k,l=1}^\infty (d_k, Cd_l)_U \int_U -N\partial_k f \partial_l f \mathrm{d}\mu_1^\Phi$$
$$= \sum_{k,l=1}^\infty (d_k, Cd_l)_U \int_U (CD\partial_k f, D\partial_l f)_U \mathrm{d}\mu_1^\Phi = \int_U \mathrm{tr}[(CD^2 f)^2] \mathrm{d}\mu_1^\Phi.$$

Moreover we have

$$\sum_{k,l,i=1}^\infty (d_i, CDf)_U \partial_l f (d_k, Cd_l)_U \partial_{ki}\Phi = \sum_{k,i=1}^\infty (d_i, CDf)_U (d_k, CDf)_U \partial_{ki}\Phi$$
$$= (D^2\Phi CDf, CDf)_U,$$

from which we conclude the second equation. As in Theorem 3.3 we can rearrange the terms of the second equation to get

$$\int_U \|Q_1^{-\frac{1}{2}} CDf\|_U^2 + \mathrm{tr}[(CD^2 f)^2] + (D^2\Phi CDf, CDf)_U \mathrm{d}\mu_1^\Phi$$
$$= \int_U (D(-Nf), CDf)_U \mathrm{d}\mu_1^\Phi.$$

Note that $N_0 f, (D\Phi, CDf)_U \in W^{1,2}(U, \mu_1, \mathbb{R})$, by [10, Proposition 10.11] and [10, Proposition 10.9] as well as $\partial_j f \rho_\Phi \in W^{1,2}(U, \mu_1, \mathbb{R})$, $j \in \mathbb{N}$. Using the integration by parts formula from Remark 2.10 we therefore get

$$
\begin{aligned}
&\int_U (D(-Nf), CDf)_U \mathrm{d}\mu_1^\Phi \\
&= \sum_{i,j=1}^{\infty} (Cd_i, d_j)_U \int_U \big(\partial_i(-N_0 f) + \partial_i (D\Phi, CDf)_U\big)\partial_j f \rho_\Phi \mathrm{d}\mu_1 \\
&= \sum_{i,j=1}^{\infty} (Cd_i, d_j)_U \int_U Nf\big(\partial_i(\partial_j f \rho_\Phi) - (u, Q_1^{-1} d_i)_U \partial_j f \rho_\Phi\big)\mathrm{d}\mu_1 \\
&= \int_U (Nf)^2 \mathrm{d}\mu_1^\Phi.
\end{aligned}
$$

Note that the infinite sums in the calculations above are actually finite ones. □

Remark 3.8. Suppose we are in the situation of Theorem 3.7. Using the Cauchy-Schwarz inequality and the first equation in Theorem 3.7 we obtain

$$
\begin{aligned}
\int_U \alpha f^2 + (CDf, Df)_U \mathrm{d}\mu_1^\Phi &\leq \|g\|_{L^2(\mu_1^\Phi)} \|f\|_{L^2(\mu_1^\Phi)} \\
&= \|g\|_{L^2(\mu_1^\Phi)} \|R(\alpha, N) g\|_{L^2(\mu_1^\Phi)} \\
&\leq \frac{1}{\alpha} \|g\|^2_{L^2(\mu_1^\Phi)}.
\end{aligned}
$$

Now additionally suppose that Φ is a convex function. Hence we can estimate using the third equation in Theorem 3.7 and the convexity of Φ

$$
\begin{aligned}
\textstyle\int_U \operatorname{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2 \mathrm{d}\mu_1^\Phi &\leq \textstyle\int_U (Nf)^2 \mathrm{d}\mu_1^\Phi = \int_U (\alpha f - g)^2 \mathrm{d}\mu_1^\Phi \\
&\leq 2 \textstyle\int_U (\alpha f)^2 + g^2 \mathrm{d}\mu_1^\Phi \leq 4 \int_U g^2 \mathrm{d}\mu_1^\Phi. \qquad (3.7)
\end{aligned}
$$

Hypothesis 3.9. *The potential Φ is in $W^{1,2}(U, \mu_1, \mathbb{R})$, convex, bounded from below and lower semicontinuous.*

Remark 3.10. For a potential Φ fulfilling Hypothesis 3.9 one can introduce the so called Yoshida approximation Φ_t, $t > 0$, defined by

$$
\Phi_t(u) = \inf_{x \in U} \big\{\Phi(x) + \frac{\|u - x\|_U^2}{2t}\big\}.
$$

One can show that for all $t > 0$ is the Yoshida approximation $\Phi_t : U \to (-\infty, \infty]$ is convex and Fréchet differentiable with

1. $-\infty < \inf_{x \in U} \Phi(x) \leq \Phi_t(u) \leq \Phi(u)$ for all $u \in U$,
2. $\lim_{t \to 0} \Phi_t(u) = \Phi(u)$ for all $u \in U$,
3. $\|D\Phi_t(u)\|_U \leq \|D\Phi(u)\|_U$ for μ_1-a.e. $u \in U$ and
4. $\lim_{t \to 0} D\Phi_t(u) = D\Phi(u)$ for μ_1-a.e. $u \in U$.

Furthermore $D\Phi_t$ is Lipschitz continuous for all $t > 0$. A proof of these statements can be found in [12].

Theorem 3.11. *Suppose* Φ *fulfills Hypothesis 3.9. For* $f \in \mathcal{F}C_b^\infty(B_U)$ *and* $g = \alpha f - Nf$, $\alpha \in (0,\infty)$, *as in Equation (3.6) we have*

$$\int_U \alpha f^2 + (CDf, Df)_U \mathrm{d}\mu_1^\Phi \leq \frac{1}{\alpha}\int_U g^2 \mathrm{d}\mu_1^\Phi,$$

$$\int_U \mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2 \mathrm{d}\mu_1^\Phi \leq 4\int_U g^2 \mathrm{d}\mu_1^\Phi.$$

Proof. Let $(\Phi_t)_{t>0}$ be the Yoshida approximation of Φ. For all $t>0$, define g_t by

$$\alpha f - \mathrm{tr}[CD^2 f] + (\cdot, Q_1^{-1} CDf)_U - (D\Phi_t, CDf)_U = g_t.$$

By Remark 3.8 we obtain

$$\int_U (\alpha f^2 + (CDf, Df)_U)\rho_{\Phi_t} \mathrm{d}\mu_1 \leq \frac{1}{\alpha}\int_U g_t^2 \rho_{\Phi_t} \mathrm{d}\mu_1,$$

$$\int_U (\mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2)\rho_{\Phi_t} \mathrm{d}\mu_1 \leq 4\int_U g_t^2 \rho_{\Phi_t} \mathrm{d}\mu_1.$$

By Remark 3.10, $\rho_{\Phi_t} = \frac{1}{c_{\Phi_t}} e^{-\Phi_t}$ is bounded by a constant $\theta \in (0,\infty)$ independent of t. In particular $(\mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2)\rho_{\Phi_t}$ and $(\alpha f^2 + (CDf, Df)_U)\rho_{\Phi_t}$ are bounded by $(\mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2)\theta$ and $(\alpha f^2 + (CDf, Df)_U)\theta$, respectively. Since the expressions converge pointwisely to $(\mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2)\rho_\Phi$ and $(\alpha f^2 + (CDf, Df)_U)\rho_\Phi$ we know that the left hand sides of the inequalities above converge to $\int_U (\mathrm{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2)\mathrm{d}\mu_1^\Phi$ and $\int_U \alpha f^2 + (CDf, Df)_U \mathrm{d}\mu_1^\Phi$, respectively. It also holds

$$\begin{aligned} |\int_U g_t^2 \rho_{\Phi_t} - g^2 \rho_\Phi \mathrm{d}\mu_1| &\leq |\int_U g_t^2(\rho_{\Phi_t} - \rho_\Phi)\mathrm{d}\mu_1| + |\int_U (g_t^2 - g^2)\rho_\Phi \mathrm{d}\mu_1| \\ &= |\int_U g_t^2(\rho_{\Phi_t} - \rho_\Phi)\mathrm{d}\mu_1| + |\|g_t\|_{L^2(\mu_1^\Phi)}^2 - \|g\|_{L^2(\mu_1^\Phi)}^2|. \end{aligned}$$

Note that g_t^2 can be bounded independent of t by an μ_1-integrable function, hence the first term in the above inequality goes to zero as t goes to zero by another application of the dominated convergence theorem. The second term also tends to zero. Indeed the Cauchy-Schwarz inequality and the definitions of g and g_t yields

$$\begin{aligned} \int_U (g - g_t)^2 \mathrm{d}\mu_1^\Phi &= \int_U (D\Phi - D\Phi_t, CDf)_U^2 \mathrm{d}\mu_1^\Phi \\ &\leq \int_U \|D\Phi - D\Phi_t\|_U^2 \|CDf\|_U^2 \mathrm{d}\mu_1^\Phi. \end{aligned}$$

Invoking the third and the fourth item of Remark 3.10 and another application of the dominated convergence theorem yields that g_t converges to g in $L^2(U, \mu_1^\Phi, \mathbb{R})$ as t goes to zero. In particular the corresponding norms in $L^2(U, \mu_1^\Phi, \mathbb{R})$ converge. All together this finishes the proof. $\square$

Note that the regularity estimate we derived in Theorem 3.11 relies on Hypothesis 3.9, i.e. not necessarily demanding the restriction $\|D\Phi\|_{L^\infty(\mu_1)}^2 < \frac{1}{4\lambda_1}$, we needed to show Theorem 3.6.

4. The Infinite-dimensional Langevin Operator

The essential m-dissipativity of finite-dimensional Langevin operators have been extensively studied in [4] and [5] for singular potentials and even in a manifold setting in [20]. In this section we want to extend these result to an infinite-dimensional setting, where as in the above references the non-sectorality of L_Φ causes difficulties.

As described in the introduction, we fix two real separable Hilbert spaces $(U, (\cdot,\cdot)_U)$ and $(V, (\cdot,\cdot)_V)$ and consider the real separable Hilbert space $(W, (\cdot,\cdot)_W)$ defined by $W = U \times V$ and

$$((u_1, v_1), (u_2, v_2))_W = (u_1, u_2)_U + (v_1, v_2)_V, \quad (u_1, v_1), (u_2, v_2) \in W.$$

By μ_1 and μ_2 we denote two centered non-degenerate Gaussian measures on $(U, \mathcal{B}(U))$ and $(V, \mathcal{B}(V))$, respectively. The corresponding covariance operators are denoted by $Q_1 \in \mathcal{L}_1^+(U)$ and $Q_2 \in \mathcal{L}_1^+(V)$. We also fix two orthonormal basis $B_U = (d_i)_{i\in\mathbb{N}}$ and $B_V = (e_i)_{i\in\mathbb{N}}$ of eigenvectors with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ and $(\nu_i)_{i\in\mathbb{N}}$ of Q_1 and Q_2, respectively. W.l.o.g. we assume that $(\lambda_i)_{i\in\mathbb{N}}$ and $(\nu_i)_{i\in\mathbb{N}}$ are decreasing to zero. Furthermore we set $B_W = (B_U, B_V)$.

As in Definition 2.3 one can consider the orthogonal projections to B_U^n and B_V^n, $n \in \mathbb{N}$. To avoid an overload of notation we omit to indicate if we project to B_U^n and B_V^n as it is clear from the context.

On $(W, \mathcal{B}(W))$ we consider the product measure $\mu = \mu_1 \otimes \mu_2$. Using the separability of U and V, [19, Lemma 1.2] it holds $\mathcal{B}(W) = \mathcal{B}(U) \otimes \mathcal{B}(V)$. Applying [10, Theorem 1.12] one can check that μ is a non-degenerate centered Gaussian measure with centered non-degenerate covariance operator $Q \in \mathcal{L}_1^+(W)$ defined by

$$W \ni (u, v) \mapsto Q(u, v) = (Q_1 u, Q_2 v) \in W.$$

Definition 4.1. In $L^2(\mu)$ we denote by $\mathcal{F}C_b^\infty(B_W)$ the space of finitely based smooth and bounded functions on W defined by

$$\mathcal{F}C_b^\infty(B_W) \\ = \{W \ni (u, v) \mapsto \varphi(P_m(u), P_m(v)) \in \mathbb{R} \mid m \in \mathbb{N},\ \varphi \in C_b^\infty(\mathbb{R}^m \times \mathbb{R}^m)\}$$

and correspondingly the space of finitely based smooth and bounded functions on W only dependent on the first n directions by

$$\mathcal{F}C_b^\infty(B_W, n) = \{W \ni (u, v) \mapsto \varphi(P_n(u), P_n(v)) \in \mathbb{R} \mid\ \varphi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)\}.$$

Concerning derivatives of sufficient smooth functions $f : W \to \mathbb{R}$ recall the explanation in Remark 2.5. We set $D_1 f = \sum_{i=1}^\infty (Df, (d_i, 0))_W d_i \in U$ and $D_2 f = \sum_{i=1}^\infty (Df, (0, e_i))_W e_i \in V$ as well as $\partial_{i,1} f = (D_1 f, d_i)_U$ and $\partial_{i,2} f = (D_2 f, e_i)_V$. In particular we have

$$Df = \sum_{i=1}^\infty (Df, (d_i, 0))_W (d_i, 0) + \sum_{i=1}^\infty (Df, (0, e_i))_W (0, e_i) = (D_1 f, D_2 f).$$

Analogously we define $D_1^2 f$, $D_2^2 f$ as well as $\partial_{ij,1} f$ and $\partial_{ij,2} f$.

For given $n \in \mathbb{N}$, recall the image measures μ_1^n and μ_2^n from Lemma 2.1 w.r.t. B_U and B_V, respectively and set $\mu^n = \mu_1^n \otimes \mu_2^n$ on $(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n))$.

Remark 4.2. Arguing as in Lemma 2.4 one can show that $\mathcal{F}C_b^\infty(B_W)$ is dense in $L^2(\mu)$.

Moreover we fix operators $K_{12} \in \mathcal{L}(U;V)$, $K_{21} \in \mathcal{L}(V;U)$ and $K_{22} \in \mathcal{L}^+(U)$. Last but not least we consider a measurable potential $\Phi : U \to (-\infty, \infty]$ which is bounded from below and recall the measures μ_1^Φ and μ^Φ. During the whole section we will assume the following hypothesis.

Hypothesis 4.3. *1. K_{22} is symmetric and positive.*

2. $K_{12}^* = K_{21}$ and K_{21} is injective.

3. There is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \leq m_k$, it holds

$$K_{22}(B_V^n) \subset B_V^{m_k}, \quad K_{12}(B_U^n) \subset B_V^{m_k} \quad \text{and} \quad K_{21}(B_V^n) \subset B_U^{m_k}.$$

4. $\Phi \in W^{1,2}(U, \mu_1, \mathbb{R})$.
For $n \in \mathbb{N}$, set $n^* = \min_{k\in\mathbb{N}}\{m_k | n \leq m_k\}$.

We will realize in Remark 4.6, that the invariance properties of K_{12}, K_{21} and K_{22} included in the hypothesis above, ensures that the infinite dimensional Langevin operator $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$, defined below, has a useful decomposability property.

Definition 4.4. We define $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$ in $L^2(\mu^\Phi)$ by

$$\mathcal{F}C_b^\infty(B_W) \ni f \mapsto L_\Phi f = S_\Phi f - A_\Phi f \in L^2(\mu^\Phi),$$

where for $f \in \mathcal{F}C_b^\infty(B_W)$, $S_\Phi f$ and $A_\Phi f$ are given by

$$\begin{aligned} S_\Phi f =& \mathrm{tr}[K_{22} D_2^2 f] - (v, Q_2^{-1} K_{22} D_2 f)_V, \\ A_\Phi f =& (u, Q_1^{-1} K_{21} D_2 f)_U + (D\Phi(u), K_{21} D_2 f)_U - (v, Q_2^{-1} K_{12} D_1 f)_V. \end{aligned}$$

The designation of S_Φ and A_Φ is not accidental, as we see show in the next lemma, that $(S_\Phi, \mathcal{F}C_b^\infty(B_W))$ is symmetric and $(A_\Phi, \mathcal{F}C_b^\infty(B_W))$ antisymmetric.

Remember that $\mathcal{F}C_b^\infty(B_W)$ is dense in $L^2(\mu^\Phi)$ by Remark 4.2 and the fact that Φ is bounded from below. For $f \in \mathcal{F}C_b^\infty(B_W)$, expressions like $Q_2^{-1} K_{22} D_2 f$, $Q_2^{-1} K_{22} D_2 f$ and $Q_1^{-1} K_{21} D_2 f$ are reasonable due to Hypothesis 4.3, compare also Remark 2.8.

Using the integration by parts formula from Corollary 2.11 together with the invariance properties of K_{22}, K_{21} and K_{12} one can derive the following lemma.

Lemma 4.5. *It holds*

(1) $(S_\Phi, \mathcal{F}C_b^\infty(B_W))$ is symmetric and dissipative in $L^2(\mu^\Phi)$.
(2) $(A_\Phi, \mathcal{F}C_b^\infty(B_W))$ is antisymmetric in $L^2(\mu^\Phi)$.

(3) $1 \in \mathcal{F}C_b^\infty(B_W)$ *with* $L_\Phi 1 = 0$ *and in particular* μ^Φ *is invariant for* $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$ *in the sense that*

$$\int_W L_\Phi f \mathrm{d}\mu^\Phi = 0 \quad \text{for all} \quad f \in \mathcal{F}C_b^\infty(B_W).$$

(4) $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$ *is dissipative in* $L^2(\mu^\Phi)$ *and for all* $f, g \in \mathcal{F}C_b^\infty(B_W)$ *it holds*

$$\begin{aligned}&-\int_W L_\Phi f g \mathrm{d}\mu^\Phi \\ &= \int_W (D_2 f, K_{22} D_2 g)_V - (D_1 f, K_{21} D_2 g)_U + (D_2 f, K_{12} D_1 g)_V \mathrm{d}\mu^\Phi.\end{aligned}$$

By [14, Proposition 3.14] densely defined dissipative operators are closable. Since $(L, \mathcal{F}C_b^\infty(B_W))$, $(S_\Phi, \mathcal{F}C_b^\infty(B_W))$ and $(A, \mathcal{F}C_b^\infty(B_W))$ are densely defined dissipative operators in $L^2(\mu^\Phi)$, it is reasonable to denote their closures by $(L_\Phi, D(L_\Phi))$, $(S_\Phi, D(S_\Phi))$ and $(A_\Phi, D(A_\Phi))$. The overall goal is to show essential m-dissipativity of $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$, i.e. m-dissipativity of $(L_\Phi, D(L_\Phi))$ in $L^2(\mu^\Phi)$.

The idea is to start with $\Phi = 0$, and use a similar argumentation as in the previous section to conclude essential m-dissipativity for Φ as in Theorem 4.11. If $\Phi = 0$, the infinite-dimensional measure μ^Φ reduces to the infinite-dimensional centered non-degenerate Gaussian measure μ with covariance operator Q.

Remark 4.6. Given $n \in \mathbb{N}$ with corresponding n^* provided by Hypothesis 4.3. Set

$$\begin{aligned}&K_{22,n^*} = ((K_{22}e_i, e_j)_V)_{ij=1}^{n^*}, \quad K_{12,n^*} = ((K_{12}d_i, e_j)_V)_{ij=1}^{n^*}, \\ &K_{21,n^*} = K_{12,n^*}^*, \quad Q_{1,n^*} = ((Q_1 d_i, d_j)_U)_{ij=1}^{n^*}, \quad Q_{2,n^*} = ((Q_2 e_i, e_j)_V)_{ij=1}^{n^*}.\end{aligned}$$

and consider the matrices

$$\tilde{K}_{22,n^*} = \begin{pmatrix} 0 & 0 \\ 0 & K_{22,n^*} \end{pmatrix} \quad Q_{n^*} = \begin{pmatrix} Q_{1,n^*} & 0 \\ 0 & Q_{2,n^*} \end{pmatrix}$$
$$K_{n^*} = \begin{pmatrix} 0 & K_{21,n} \\ -K_{12,n^*} & -K_{22,n^*} \end{pmatrix}.$$

We have for $f = \varphi(P_n(\cdot), P_n(\cdot)) \in \mathcal{F}C_b^\infty(B_W)$ and $(u, v) \in W$

$$Lf(u, v) = \mathrm{tr}[\tilde{K}_{22,n^*} D^2 \tilde{\varphi}(z_{n^*})] + \langle K_{n^*} Q_{n^*}^{-1} z_{n^*}, D\tilde{\varphi}(z_{n^*})\rangle,$$

where $z_{n^*} = (P_{n^*}(u), P_{n^*}(v))$ and $\tilde{\varphi} \in C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*})$ is the canonical extension of $\varphi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Hence, it is reasonable to consider the operator $(L_{n^*}, C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*}))$ defined for $\tilde{\varphi} \in C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*})$ and $z \in \mathbb{R}^{n^*} \times \mathbb{R}^{n^*}$ by

$$L_{n^*}\varphi(z) = \mathrm{tr}[\tilde{K}_{22,n^*} D^2 \tilde{\varphi}(z)] + \langle K_{n^*} Q_{n^*}^{-1} z, D\tilde{\varphi}(z)\rangle.$$

Invoking the assumptions from Hypothesis 4.3 the following proposition shows that the operator $(L_{n^*}, C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*}))$ is essentially m-dissipative in $L^2(\mu^{n^*})$. The idea of the proof is inspired by the consideration in [21] and

[6]. By $C_b(\mathbb{R}^{n^*}\times\mathbb{R}^{n^*})$ we denote the space of bounded continuous functions from $\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}$ to $\mathbb{R}$.

Proposition 4.7. *For all $n\in\mathbb{N}$, the operator $(L_{n^*}, C_b^\infty(\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}))$ is essentially m-dissipative in $L^2(\mu^{n^*})$.*

Proof. Note that the Markov semigroup $(S_t)_{t\ge 0}$ associated to L_{n^*} can be represented for all $t\in(0,\infty)$, applied to $\varphi\in C_b(\mathbb{R}^{n^*}\times\mathbb{R}^{n^*})$, evaluated at $z\in\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}$ as

$$S_t\varphi(z)=\frac{1}{c_t}\int_{\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}} e^{-\langle Q(t)^{-1}w,w\rangle/4}\varphi(e^{tK_{n^*}Q_{n^*}^{-1}}z-w)\mathrm{d}(\lambda^{n^*}\otimes\lambda^{n^*})(w),$$

where for $t\in(0,\infty)$

$$c_t=(4\pi)^{n^*}(\det Q(t))^{1/2}\ \text{ and }\ Q(t)=\int_{[0,t)} e^{sK_{n^*}Q_{n^*}^{-1}}\tilde{K}_{22,n^*}e^{sQ_{n^*}^{-1}K_{n^*}^*}\mathrm{d}\lambda(s),$$

is a symmetric and nonnegative matrix. Using the first and the second item of Hypothesis 4.3, in particular the positivity of K_{22} and the injectivity of K_{21} it is easy to see that the matrix

$$\left(\tilde{K}_{22,n^*}^{\frac12}\ \ K_{n^*}Q_{n^*}^{-1}\tilde{K}_{22,n^*}^{\frac12}\right)=\begin{pmatrix}0 & 0 & 0 & K_{21,n^*}Q_{2,n^*}^{-1}K_{22,n^*}^{\frac12}\\ 0 & K_{22,n^*}^{\frac12} & 0 & -K_{22,n^*}Q_{2,n^*}^{-1}K_{22,n^*}^{\frac12}\end{pmatrix}$$

has full rank. Hence the so called Kalman rank condition is satisfied and therefore by [26] $\det Q(t)>0$. In particular the representation of $(S_t)_{t\ge0}$ is reasonable. As $\det Q(t)>0$, we know by the arguments from [9, Section 11.3.1] that the existence of an invariant measure ν for $(S_t)_{t\ge0}$ i.e.

$$\int_{\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}} S_t\varphi\mathrm{d}\nu=\int_{\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}}\varphi\mathrm{d}\nu,$$

for every $t\in[0,\infty)$ and $\varphi\in C_b(\mathbb{R}^{n^*}\times\mathbb{R}^{n^*})$ is equivalent to the existence of a nonnegative symmetric matrix P s.t.

$$\langle PQ_{n^*}^{-1}K_{n^*}^*z,z\rangle+\langle\tilde{K}_{22,n^*}z,z\rangle=0\quad\text{for all } z\in\mathbb{R}^{n^*}\times\mathbb{R}^{n^*}.$$

As the equality above is satisfied for $P=Q_{n^*}$ the existence of the invariant measure is justified. Moreover by [9, Section 11.3.3] the invariant measure is unique and given by μ^{n^*}.

The invariance of the measure μ^{n^*} allows us to extend $(S_t)_{t\ge0}$ to a strongly continuous contraction semigroup in $L^2(\mu^{n^*})$. Using [21, Lemma 2.1], we can show that the generator of the strongly continuous contraction semigroup $(S_t)_{t\ge0}$ is given by $L_{n^*}\varphi$ for all $\varphi\in C_b^\infty(\mathbb{R}^{n^*}\times\mathbb{R}^{n^*})$ and that $C_b^\infty(\mathbb{R}^{n^*}\times\mathbb{R}^{n^*})$ is a core for the generator of $(S_t)_{t\ge0}$ in $L^2(\mu^{n^*})$, i.e. the assertion is shown. □

Theorem 4.8. *The operator $(L,\mathcal{F}C_b^\infty(B_W))$ defined by*

$$\begin{aligned}\mathcal{F}C_b^\infty(B)\ni f\mapsto Lf=&\operatorname{tr}[K_{22}D_2^2f]-(v,Q_2^{-1}K_{22}D_2f)_V\\ &-(u,Q_1^{-1}K_{21}D_2f)_U+(v,Q_2^{-1}K_{12}D_1f)_V\in L^2(\mu),\end{aligned}$$

is essentially m-dissipative in $L^2(\mu)$ with m-dissipative closure $(L,D(L))$. The resolvent in $\alpha\in(0,\infty)$, of $(L,D(L))$ is denoted by $R(\alpha,L)$.

Proof. Since by Lemma 4.5 we already know that $(L, \mathcal{F}C_b^\infty(B_W))$ is dissipative it is left to show that $(Id - L)(\mathcal{F}C_b^\infty(B_W))$ is dense in $L^2(\mu)$. Let $\varepsilon > 0$ and $h \in L^2(\mu)$ be given. Remark 4.2 provides $g = \psi(P_n(\cdot), P_n(\cdot)) \in \mathcal{F}C_b^\infty(B_W, n) \subset L^2(\mu)$ such that

$$\|h - g\|_{L^2(\mu)} < \frac{\varepsilon}{2}.$$

Let $\tilde{\psi} \in C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*})$ be the canonical extension of $\psi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Since $(Id - L_{n^*})(C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*}))$ is dense in $L^2(\mu^{n^*})$, by Proposition 4.7, we find a $\varphi \in C_b^\infty(\mathbb{R}^{n^*} \times \mathbb{R}^{n^*})$ such that

$$\|(Id - L_{n^*})\varphi - \tilde{\psi}\|_{L^2(\mu^{n^*})} < \frac{\varepsilon}{2}.$$

Set $f = \varphi(P_{n^*}(\cdot), P_{n^*}(\cdot))$ and use the triangle inequality together with Lemma 2.1 to observe

$$\begin{aligned} \|(Id - L)f - h\|_{L^2(\mu)} &\le \|(Id - L)f - g\|_{L^2(\mu)} + \|h - g\|_{L^2(\mu)} \\ &= \|(Id - L_{n^*})\varphi - \tilde{\psi}\|_{L^2(\mu^{n^*})} + \|h - g\|_{L^2(\mu)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Before we proof the main result of this section, we discuss regularity estimates similar to the ones from Remark 3.8. In contrast to the estimates in Sect. 3 we don't have to deal with unbounded diffusion operators $(C, D(C))$ as coefficients, but the degenerate structure of $(L, \mathcal{F}C_b^\infty(B_W))$ is more challenging. Indeed, we are only able to derive first order regularity results.

Proposition 4.9. *For $f \in D(L)$ and $\alpha \in (0, \infty)$, set $g = \alpha f - Lf$. Then the following equation hold*

$$\int_W \alpha f^2 + \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \mathrm{d}\mu = \int_W fg \mathrm{d}\mu.$$

In particular

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \mathrm{d}\mu \le \frac{1}{2} \int_W f^2 + (Lf)^2 \mathrm{d}\mu \quad \text{and} \tag{4.1}$$

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \mathrm{d}\mu \le \frac{1}{4\alpha} \int_W g^2 \mathrm{d}\mu. \tag{4.2}$$

Proof. Assume $f \in \mathcal{F}C_b^\infty(B_W)$ and $g = \alpha f - Lf$. Now Multiply $g = \alpha f - Lf$ with f, integrate over W w.r.t. μ and use Lemma 4.5 item (iv) to obtain the first identity. Rearranging the terms we obtain

$$\begin{aligned} \int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \mathrm{d}\mu &= \int_W fg - \alpha f^2 \mathrm{d}\mu = \int_W f(g - \alpha f) \mathrm{d}\mu = -\int_W fLf \mathrm{d}\mu. \\ &\le \frac{1}{2} \int_W f^2 + (Lf)^2 \mathrm{d}\mu. \end{aligned}$$

Moreover by completing the square we have

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \mathrm{d}\mu = -\int_W \alpha f^2 - fg \mathrm{d}\mu \le \frac{1}{4\alpha} \int_W g^2 \mathrm{d}\mu.$$

Since $\mathcal{F}C_b^\infty(B_W)$ is dense in the $(L, D(L))$ graph norm the (in)equalities above are also valid for $f \in D(L)$. Note that for $f \in D(L)$ the expression $K_{22}D_2 f$ is understood in the sense of Theorem 2.9, compare also Corollary 3.5. □

Having the regularity estimates at hand we need one last hypothesis to derive our final essential m-dissipativity result. The hypothesis includes condition on the potential Φ and a coercivity assumption on K_{22} in terms of the operator $K_{12}K_{21}$.

Hypothesis 4.10. *1. There is a constant $c_K \in (0, \infty)$ such that*

$$(K_{12}K_{21}v, v)_V \leq c_K (K_{22}v, v)_V \quad \textit{for all } v \in V.$$

2. The potential Φ is bounded from below and $D\Phi$ is bounded.
Note that the second item implies that $\rho_\Phi = \frac{1}{c_\Phi} e^{-\Phi}$ is bounded.

Theorem 4.11. *Suppose Hypothesis 4.10 is valid, then $D(L) \subset D(L_\Phi)$ with*

$$L_\Phi f = Lf - (D\Phi, K_{21}D_2 f)_U, \quad f \in D(L),$$

and the infinite-dimensional Langevin operator $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative in $L^2(\mu^\Phi)$.

Proof. Using Proposition 4.9 and the first item from Hypothesis 4.10 it holds for all $f \in \mathcal{F}C_b^\infty(B_W)$

$$\begin{aligned}\int_W (K_{21}D_2 f, K_{21}D_2 f)_U \mathrm{d}\mu &\leq \int_W c_K (K_{22}D_2 f, D_2 f)_V \mathrm{d}\mu \\ &\leq \frac{c_K}{2} \int_W f^2 + (Lf)^2 \mathrm{d}\mu.\end{aligned}$$

Since $\mathcal{F}C_b^\infty(B_W)$ is dense $D(L)$ w.r.t. the $(L, D(L))$ graph norm the estimate above also holds for $f \in D(L)$. Again, note that for $f \in D(L)$ the expressions $K_{22}D_2 f$ and $K_{21}D_2 f$ are understood in the sense of Theorem 2.9, compare also Corollary 3.5.

Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^\infty(B_W)$ be a sequence converging to $f \in D(L)$ wit respect to the $(L, D(L))$ graph norm. Since Φ is bounded from below it is easy to check that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^2(\mu^\Phi)$. Moreover we can estimate

$$\begin{aligned}&\int_W (L_\Phi f_n - Lf + (D\Phi, K_{21}D_2 f)_U)^2 \mathrm{d}\mu^\Phi \\ &\leq 2\int_W (Lf_n - Lf)^2 \mathrm{d}\mu^\Phi + 2\int_W (D\Phi, K_{21}D_2(f_n - f))_U^2 \mathrm{d}\mu^\Phi \\ &\leq 2\|\rho_\Phi\|_{L^\infty(\mu_1)} \Big(\int_W (Lf_n - Lf)^2 \mathrm{d}\mu \\ &\quad + \frac{c_K}{2}\|D\Phi\|_{L^\infty(\mu_1)}^2 \int_W (f_n - f)^2 + (Lf_n - Lf)^2 \mathrm{d}\mu \Big).\end{aligned}$$

Hence the sequence $(Lf_n)_{n\in\mathbb{N}}$ converges to $Lf - (D\Phi, K_{21}D_2 f)_U$ in $L^2(\mu^\Phi)$. As $(L_\Phi, D(L_\Phi))$ is closed we get $D(L) \subset D(L_\Phi)$ and for all $f \in D(L)$

$$L_\Phi f = Lf - (D\Phi, K_{21}D_2 f)_U.$$

By Lemma 4.5 item (iv) we already know that $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$ is dissipative. In view of the Lumer-Phillips theorem we are left to show the dense range condition. For $f \in L^2(\mu)$ and $\alpha > 0$ set

$$T_\alpha f = -(D\Phi, K_{21} D_2 R(\alpha, L) f)_U.$$

We calculate using the Cauchy-Schwarz inequality, the assumption on Φ, Hypothesis 4.10 and Inequality (4.2)

$$\begin{aligned}\int_W (T_\alpha f)^2 \mathrm{d}\mu &= \int_W (D\Phi, K_{21} D_2 R(\alpha, L) f)_U^2 \mathrm{d}\mu \\ &\le \|D\Phi\|^2_{L^\infty(\mu_1)} \int_W (K_{21} D_2 R(\alpha, L) f, K_{21} D_2 R(\alpha, L) f)_U \mathrm{d}\mu \\ &\le \|D\Phi\|^2_{L^\infty(\mu_1)} \int_W c_K (K_{22} D_2 R(\alpha, L) f, D_2 R(\alpha, L) f)_U \mathrm{d}\mu \\ &\le \|D\Phi\|^2_{L^\infty(\mu_1)} \frac{c_K}{4\alpha} \int_W f^2 \mathrm{d}\mu.\end{aligned}$$

Hence the operator $T_\alpha : L^2(\mu) \to L^2(\mu)$ is well-defined. Moreover, if

$$\|D\Phi\|^2_{L^\infty(\mu_1)} \frac{c_K}{4\alpha} < 1,$$

we can apply Neumann-Series theorem to get $(Id - T_\alpha)^{-1} \in \mathcal{L}(L^2(\mu))$. In particular, for such α, for all $g \in L^2(\mu)$ we find $f \in L^2(\mu)$ with $f - T_\alpha f = g$ in $L^2(\mu)$. Furthermore there is $h \in D(L)$ with $(\alpha - L)h = f$. Therefore,

$$\begin{aligned}(\alpha - L_\Phi)h &= (\alpha - L)h + (D\Phi, K_{21} D_2 h)_U = f + (D\Phi, K_{21} D_2 R(\alpha, L) f)_U \\ &= f - T_\alpha f = g.\end{aligned}$$

This yields $L^2(\mu) \subset (\alpha - L_\Phi)(D(L))$. Since $L^2(\mu)$ is dense in $L^2(\mu^\Phi)$ and $D(L) \subset D(L_\Phi)$ the dense range condition is shown and the proof is finished. $\square$

5. Examples and Outlook

In this section we have a look at certain examples, where the results we derived above are applicable. We consider the following situation, which is inspired by the one in [7, Section 5].

Let $U = V = L^2((0,1), \lambda, \mathbb{R})$, $W = U \times V$ and K_{12}, K_{21}, K_{22} such that they fulfill the properties in Hypothesis 4.3 and 4.10(e.g. $K_{12} = K_{21} = K_{22} = Id$). Moreover, let $(-\Delta, D(\Delta))$ be the negative Dirichlet Laplacian, i.e.

$$\begin{aligned}&D(\Delta) = W_0^{1,2}((0,1), \lambda, \mathbb{R}) \cap W^{2,2}((0,1), \lambda, \mathbb{R}) \subset L^2((0,1), \lambda, \mathbb{R}), \\ &-\Delta x = -x''.\end{aligned}$$

On $(U, \mathcal{B}(U))$ and $(V, \mathcal{B}(V))$ we consider two centered non-degenerate infinite-dimensional Gaussian measures μ_1 and μ_2 with covariance operators

$$Q_1 = Q_2 = -\Delta^{-1} : L^2((0,1), \lambda, \mathbb{R}) \to D(\Delta),$$

respectively. Recall the definition of the measures μ_1^Φ and μ^Φ and denote by $B_U = B_V = (d_k)_{k\in\mathbb{N}} = (e_k)_{k\in\mathbb{N}} = (\sqrt{2}\sin(k\pi\cdot))_{k\in\mathbb{N}}$ the orthonormal basis of $L^2((0,1), \lambda, \mathbb{R})$ diagonalizing Q_1 and Q_2 with corresponding eigenvalues

$(\lambda_k)_{k\in\mathbb{N}} = (\nu_k)_{k\in\mathbb{N}} = (\frac{1}{k^2\pi^2})_{k\in\mathbb{N}}$. Additionally we fix a continuous differentiable (convex) function $\phi : \mathbb{R} \to \mathbb{R}$, which is bounded from below. Assume that there are constants $C_1, C_2 \in [0,\infty)$, $p_1 \in [2,\infty)$ and $p_2 \in [1,\infty)$ such that

$$
\begin{aligned}
|\phi(t)| &\leq C_1(1+|t|^{p_1}), \quad t \in \mathbb{R},\\
|\phi'(t)| &\leq C_2(1+|t|^{p_2}), \quad t \in \mathbb{R}.
\end{aligned}
$$

I.e. ϕ and its derivative have at most polynomial growth. For such ϕ we consider potentials $\Phi : L^2((0,1),\lambda,\mathbb{R}) \to (-\infty,\infty]$ defined by

$$
\Phi(u) = \begin{cases} \int_{(0,1)} \phi \circ u \,\mathrm{d}\lambda & u \in L^{p_1}((0,1),\lambda,\mathbb{R}) \\ \infty & u \notin L^{p_1}((0,1),\lambda,\mathbb{R}) \end{cases}.
$$

Remark 5.1. Note that potentials as defined above are lower semicontinuous by Fatou's lemma, bounded from below and in $L^p(U,\mu_1,\mathbb{R})$ for all $p \in [1,\infty)$. If ϕ is convex the same holds true for Φ. Using [7, Proposition 5.2] we know that Φ is bounded from below, lower semicontinuous and in $W^{1,2}(U,\mu_1,\mathbb{R})$ with $D\Phi(u) = \phi' \circ u$ for a.e. $u \in L^2((0,1),\lambda,\mathbb{R})$ (namely, for all $u \in L^{2p_2}((0,1),\lambda,\mathbb{R})$).

Choose a continuous differentiable (convex) ϕ, which is bounded from below with at most polynomial growth. Further assume that the derivative of ϕ is bounded. Now define the potential Φ in terms of ϕ. In Hypothesis 4.3 and 4.10 all items are valid, were by construction and Remark 5.1 we have

$$
\|D\Phi\|_{L^\infty(\mu_1)} = \sup_{t\in\mathbb{R}}|\phi'(t)| < \infty.
$$

Therefore Theorem 4.11 is applicable and we obtain essential m-dissipativity in $L^2(L^2((0,1),\lambda,\mathbb{R}) \times L^2((0,1),\lambda,\mathbb{R}),\mu^\Phi,\mathbb{R})$ of $(L_\Phi, \mathcal{F}C_b^\infty(B_W))$, where for $f \in \mathcal{F}C_b^\infty(B_W)$ we have

$$
\begin{aligned}
L_\Phi f =& \operatorname{tr}[K_{22}D_2^2 f] + (v, K_{22}\Delta D_2 f)_{L^2(\lambda)}\\
&+ (u, K_{21}\Delta D_2 f)_{L^2(\lambda)} - (\phi' \circ u, K_{21}D_2 f)_{L^2(\lambda)} - (v, K_{12}\Delta D_1 f)_{L^2(\lambda)}.
\end{aligned}
$$

This is the starting point to construct a martingale and even a weak solution to the non-linear infinite-dimensional stochastic differential equation given by

$$
\begin{aligned}
\mathrm{d}U_t &= -K_{21}\Delta V_t \mathrm{d}t\\
\mathrm{d}V_t &= (K_{22}\Delta V_t + K_{12}\Delta U_t - K_{12}\phi'(U_t))\mathrm{d}t + \sqrt{2K_{22}}\mathrm{d}W_t.
\end{aligned} \tag{5.1}
$$

In the equation above $(W_t)_{t\geq 0}$ is a cylindrical Brownian motion on $(V, \mathcal{B}(V))$. A heuristically application of the Itô-formula suggest that this stochastic differential equation corresponds to $(L_\Phi, D(L_\Phi))$. In order to make this correspondence rigorous and to construct weak solutions we want to apply general resolvent methods described in [1].

I.e. we plan to construct a μ^Φ-standard right process (see [1, Appendix B.])

$$
X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (U_t, V_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (\mathbb{P}^w)_{w\in W}),
$$

providing a martingale and even a weak solution, with infinite lifetime $\mathbb{P}_{\mu^\Phi}$-a.e. and $\mathcal{T}$-continuous paths, $\mathbb{P}_{\mu^\Phi}$-a.e.. Here $\mathcal{T}$ denotes the weak topology on the Hilbert space W and $\mathbb{P}_{\mu^\Phi}$ the probability measure on $(\Omega, \mathcal{F})$ defined for $A \in \mathcal{F}$ by

$$\mathbb{P}_{\mu^\Phi}(A) = \int_W \mathbb{P}^w(A) \mathrm{d}\mu^\Phi(w) \quad A \in \mathcal{F}.$$

Moreover, the transition semigroup corresponding to the process X and the strongly continuous semigroup $(T_t)_{t\geq 0}$ generated by $(L_\Phi, D(L_\Phi))$ coincide in $L^2(\mu^\Phi)$. Using this correspondence we like to study the long-time behavior of the Process X via the long-time behavior of the semigroup $(T_t)_{t\geq 0}$. I.e. we plan to apply the abstract Hilbert space hypocoercivity method from [15], to show exponential convergence to equilibrium of the semigroup.

As announced in the introduction the results derived in Sect. 3 are naturally needed while applying the abstract Hilbert space hypocoercivity method. A rigorous application in our infinite-dimensional setting goes beyond the aim of this article. At this point we illustrate the rough idea and explain how to achieve such an exponential convergence result and how Ornstein-Uhlenbeck operators perturbed by the gradient of a potential with unbounded diffusions $(C, D(C))$ as coefficients appear during the application process. As in the article [13], of Dolbeault, Mouhot and Schmeiser we define the modified entropy functional $H_\varepsilon : L^2(\mu^\Phi) \to \mathbb{R}$, $\varepsilon \in [0, 1)$ by

$$H_\varepsilon[f] = \frac{1}{2}\|f\|^2_{L^2(\mu^\Phi)} + \varepsilon(Bf, f)_{L^2(\mu^\Phi)}, \quad f \in L^2(\mu^\Phi).$$

The bounded linear operator B, specified below is chosen such that H_ε defines a norm which is equivalent to the norm $\|\cdot\|_{L^2(\mu^\Phi)}$, compare [15]. To construct B, define $P_S : L^2(\mu^\Phi) \to L^2(\mu^\Phi)$ by

$$P_S f = \int_V f \mathrm{d}\mu_2,$$

where the integration is understood w.r.t. the second variable. Using that μ^Φ is a probability measure one can check that the map $P : L^2(\mu^\Phi) \to L^2(\mu^\Phi)$ given as

$$Pf = P_S f - (f, 1)_{L^2(\mu^\Phi)}, \quad f \in L^2(\mu^\Phi),$$

is an orthogonal projection with

$$Pf \in L^2(\mu_1^\Phi) \quad \text{and} \quad \|Pf\|_{L^2(\mu_1^\Phi)} = \|Pf\|_{L^2(\mu^\Phi)}, \quad f \in L^2(\mu^\Phi),$$

where we canonically embed $L^2(\mu_1^\Phi)$ into $L^2(\mu^\Phi)$. Now the bounded linear operator B on $L^2(\mu^\Phi)$ is defined as the unique extension of $(B, D((A_\Phi P)^*))$ to a continuous linear operator on $L^2(\mu^\Phi)$ where

$$B = (Id + (A_\Phi P)^* A_\Phi P)^{-1}(A_\Phi P)^* \quad \text{on} \quad D((A_\Phi P)^*).$$

Here $(A_\Phi P, D(A_\Phi P))$ is the linear operator $A_\Phi P$ with domain

$$D(A_\Phi P) = \{f \in L^2(\mu^\Phi) \mid Pf \in D(A_\Phi)\}.$$

and $((A_\Phi P)^*, D((A_\Phi P)^*))$ denotes its adjoint on $L^2(\mu^\Phi)$. Note that by the Neumann-Series theorem, the operator

$$Id + (A_\Phi P)^* A_\Phi P : D((A_\Phi P)^* A_\Phi P) \to L^2(\mu^\Phi)$$

with domain $D((A_\Phi P)^* A_\Phi P) = \{f \in D(A_\Phi P) \mid A_\Phi P f \in D((A_\Phi P)^*)\}$ is bijective and admits a bounded inverse. Hence B is indeed well-defined on $D((A_\Phi P)^*)$. For the fact that B extends to a bounded linear operator on $L^2(\mu^\Phi)$, see [22, Theo. 5.1.9]. Now one can calculate

$$\begin{aligned}&\frac{d}{dt} H_\varepsilon[T_t f]\\ &= (L_\Phi T_t f, T_t f)_{L^2(\mu^\Phi)} + \varepsilon (B L_\Phi T_t f, T_t f)_{L^2(\mu^\Phi)} + \varepsilon (B T_t f, L_\Phi T_t f)_{L^2(\mu^\Phi)}.\end{aligned}$$

Define the possible unbounded operator $(C, D(C))$ in U by

$$C = K_{21} Q_2^{-1} K_{12} \quad \text{with} \quad D(C) = \{u \in U | K_{12} u \in D(Q_2^{-1})\}.$$

One can show that for all $f \in \mathcal{F}C_b^\infty(B_W)$, the operator $(PA_\Phi^2 P, \mathcal{F}C_b^\infty(B_W))$ is given by the formula

$$PA_\Phi^2 P f = \mathrm{tr}[CD^2 P_S f] - (u, Q_1^{-1} CDP_S f)_U - (D\Phi(u), CDP_S f)_U.$$

The essential m-dissipativity of $(PA_\Phi^2 P, \mathcal{F}C_b^\infty(B_W))$ in $L^2(\mu^\Phi)$ and corresponding regularity estimates, which are applicable in view of Theorem 3.6 and Theorem 3.11 derived in Sect. 3, are fundamental to show [15, Corollary 2.13] and [15, Proposition 2.15], i.e. to derive

$$\frac{d}{dt} H_\varepsilon[T_t f] \le -\kappa H_\varepsilon[T_t f],$$

for an appropriate chosen $\varepsilon \in [0,1)$ and a positive constant $\kappa \in (0,\infty)$, compare [15, Theorem 2.18]. Applying Grönwall's lemma yields the desired exponential convergence to equilibrium.

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References

[1] Beznea, L., Boboc, N., Röckner, M.: Markov processes associated with L^p-resolvents and applications to stochastic differential equations on Hilbert spaces. J. Evol. Equs. **6**(4), 745–772 (2006)

[2] Bignamini, D.A., Ferrari, S.: On generators of transition semigroups associated to semilinear stochastic partial differential equations, J. Math. Anal. App. **508**(1), (2022)

[3] Conrad, F., Grothaus, M.: Construction of N-Particle Langevin Dynamics for $H^{1,\infty}$-Potentials via Generalized Dirichlet Forms. Potential Anal. **28**(3), 261–282 (2008)

[4] Conrad, F., Grothaus, M.: Construction, ergodicity and rate of convergence of N-particle Langevin dynamics with singular potentials. J. Evolut. Equs. **10**(3), 623–662 (2010)

[5] Conrad, F.: Construction and analysis of Langevin dynamics in continuous particle systems, PhD thesis, Mathematics Department, TU Kaiserslautern, Published by Verlag Dr. Hut, München, (2011)

[6] Da Prato, G., Lunardi, A.: On a class of degenerate elliptic operators in L^1 spaces with respect to invariant measures. Math. Z. **256**(3), 509–520 (2007)

[7] Da Prato, G., Lunardi, A.: Sobolev regularity for a class of second order elliptic PDE's in infinite dimension. Ann. Prob. **42**(5), 2113–2160 (2014)

[8] Da Prato, G., Tubaro, L.: Selfadjointness of some infinite-dimensional elliptic operators and application to stochastic quantization. Prob. Theory Relat Fields **118**, 131–145 (2000)

[9] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, 2nd edn. Cambridge University Press, Cambridge, Encyclopedia of Mathematics and its Applications (2014)

[10] Da Prato, G.: Introd. Infin-Dimens Anal. Springer-Verlag, Berlin Heidelberg (2006)

[11] Da Prato, G., Zabczyk, J.: Second Order Partial Differential Equations in Hilbert Spaces, Cambridge Univ. Press, London Mathematical Society Lecture Notes **293** (2002)

[12] Da Prato, G.: Applications croissantes et équations d'évolution dans les espaces Banach, Academic Press, 1976

[13] Dolbeault, J., Mouhot, C., Schmeiser, C.: Hypocoercivity for Linear Kinetic Equations Conserving Mass. Comptes Rendus Math. **347**(9), 511–516 (2009)

[14] Engel, K. J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, Graduate Texts in Mathematics **19** (2000)

[15] Grothaus, M., Stilgenbauer, P.: Hypocoercivity for Kolmogorov backward evolution equations and applications. J. Funct. Anal. **267**(10), 3515–3556 (2014)

[16] Grothaus, M., Wang, F.-Y.: Weak Poincaré inequalities for Convergence Rate of Degenerate Diffusion Processes. Ann. Prob. **47**(5), 2930–2952 (2019)

[17] Herzog, D.P., Mattingly, J.C.: Ergodicity and Lyapunov Functions for Langevin Dynamics with Singular Potentials. Comm. Pure Appl. Math. **72**(10), 2231–2255 (2019)

[18] Isserlis, L.: On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. Biometrika. **12**, 134–139 (1918)

[19] Kallenberg, O.: Foundations of Modern Probability. Springer-Verlag, New York (2002)

[20] Mertin, M., Grothaus, M.: Hypocoercivity of Langevin-type dynamics on abstract smooth manifolds. Stoch. Process. Their Appl. **146**, 22–59 (2022)

[21] Metafune, G., Pallara, D., Priola, E.: Spectrum of Ornstein-Uhlenbeck operators in Lp spaces with respect to invariant measures. J. Funct. Anal. **196**, 40–60 (2002)

[22] Pedersen, Gert K.: Analysis Now, Springer-Verlag, New York, Graduate Texts in Mathematics **118** (1989)

[23] Schwabl, F.: Statistical Mechanics, 2nd edn. Springer-Verlag, Berlin (2004)

[24] Vakhania, N.: The Topological Support of Gaussian Measure in Banach Space. Nagoya Mathematical Journal **57**, 59–63 (1975)

[25] Villani, C.L Hypocoercivity, Mem. Amer. Math. Soc. **202**(950), (2009) iv+141

[26] Zabczyk, J.: Mathematical Control Theory: An Introduction. Birkhaüser, Basel (1992)

Benedikt Eisenhuth (✉) and Martin Grothaus
Department of Mathematics
Technical University of Kaiserslautern
P.O. Box 3049
67653 Kaiserslautern
Germany
e-mail: `eisenhuth@mathematik.uni-kl.de`

Martin Grothaus
e-mail: `grothaus@mathematik.uni-kl.de`

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