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Integral Equations and Operator Theory

Hyperrigidity of C*-Correspondences

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Abstract. We show that hyperrigidity for a C*-correspondence (A, X) is equivalent to non-degeneracy of the left action of the Katsura ideal $\mathcal{J}_X$ on X. This extends the work of Kakariadis (Bull Lond Math Soc 45(6):1119–1130, 2013, Theorem 3.3) and Dor-On and Salomon (J Lond Math Soc 98(2):416–438, 2018, Theorem 3.5) who establish this equivalence for discrete graphs as well as the work of Katsoulis and Ramsey (The non-selfadjoint approach to the Hao–Ng isomorphism, arXiv preprint arXiv:1807.11425, 2018, Theorem 3.1), who establish one direction of this equivalence.

1. Introduction

An operator system is a *-closed unital subspace of a unital C*-algebra. The works of Kalantar–Kennedy, Kavruk, and Katsoulis–Ramsey demonstrate that properties of certain C*-algebras such as simplicity, nuclearity, and isomorphism problems can be tackled in new and fruitful ways by looking at an appropriate operator system associated to the C*-algebra [7,11,14]. Beyond this, the work of Webster–Winkler [20] demonstrate that the category of operator systems admit a duality to non-commutative analogues of convex sets, providing new convex theoretic tools for the study of operator systems and their generating C*-algebras [3,16]. A property of operator systems that we are concerned with in this paper is hyperrigidity:

Definition 1.1. Suppose that S is an operator system generating a C*-algebra A. We say that a representation $\pi : A \to B(H)$ has the unique extension property if whenever $\varphi : A \to B(H)$ is a unital and completely positive (ucp) map that agrees with π on S, φ agrees with π on A. We say that S is hyperrigid in A if every representation on A admits the unique extension property.

Heuristically, hyperrigidity tells us the representation theory of A is completely determined by the operator system S. Hyperrigidity is introduced by Arveson in [1]. In his paper, Arveson shows that to check whether an operator system S generating a C*-algebra A with countable spectrum is hyperrigid, it suffices to show that all *irreducible* representations π on A have the unique extension property. Arveson then poses the following problem:

Question 1. (Hyperrigidity Conjecture) Let S be a separable operator system generating a C*-algebra A. Suppose that all irreducible representations π on A have the unique extension property. Is it always the case that S is hyperrigid in A?

It is known by [4, Theorem 3.9] and [19, Corollary 4.7] that the hyperrigidity conjecture holds for certain *-closed subspaces associated to graph C*-algebras. Beyond Arveson's initial class of examples of operator systems and Dor-On and Salomon's class of examples, the hyperrigidity conjecture is wide open.

We say that a non-degenerate C*-correspondence (A, X) is hyperrigid if the operator space

$$S(A, X) := \operatorname{span}\{x + a + y^* : x, y \in X, a \in A\} \subset \mathcal{O}_X$$

has the following extension property: given a representation $\pi : \mathcal{O}_X \to B(H)$, if $\varphi : \mathcal{O}_X \to B(H)$ is a completely positive and completely contractive map which agrees with π on $\mathcal{S}(A, X)$ then φ must agree with π on $\mathcal{O}_X$. If A is unital, hyperrigidity in the above sense agrees with operator system hyperrigidity of $S(A, X)$. Please see [19] for further discussion of the case when A is non-unital. Our main Theorem is the following. Please see the discussion before Theorem 2.3 for the definition of the Katsura ideal $\mathcal{J}_X$.

Theorem 1.2. *Let (A, X) be a C*-correspondence. Let $\mathcal{J}_X \trianglelefteq A$ be the Katsura ideal of the C*-correspondence (A, X). The following are equivalent:*

1. *The C*-correspondence (A, X) is hyperrigid.*
2. *We have the identity $\mathcal{J}_X \cdot X = X$.*

This extends a result of Kakariadis [6, Theorem 3.3] and Dor-On and Salomon [4, Theorem 3.5] who establish the equivalence for C*-correspondences associated to discrete graphs and a result of Katsoulis and Ramsey who give a sufficient condition for hyperrigidity when X is countably generated over A [11, Theorem 3.1]. By establishing an exact characterization of hyperrigidity for this class of operator spaces, we provide a step forward in verifying the hyperrigidity conjecture for a larger class of operator systems. In particular, the techniques used to prove our main Theorem do not tell us whether non-degeneracy of $\mathcal{J}_X$ on X is equivalent to all irreducible representations admitting the unique extension property. Thus, establishing or finding counterexamples to this stronger fact would provide greater insight into whether the hyperrigidity conjecture should hold. At the end, we use our main Theorem to give an exact characterization for when the C*-correspondence associated to a topological graph is hyperrigid when the range map r is open.

2. Preliminaries

In this section, we give a brief overview of the various results on operator systems and Cuntz-Pimsner algebras that we will need for this paper.

2.1. Operator Systems

An operator system S is a closed subspace of a unital C*-algebra A for which $1_A \in S$ and $S^* = S$. The class of operator systems has an abstract axiomatization [2]. We will only say a word about the abstract characterization: to axiomatize operator systems it is enough to keep track of the involution *, the cone $M_n(S)_+$ of positive operators on $M_n(S) \subset M_n(A)$, and the unit $1_{M_n(A)} \in M_n(S)$. The appropriate morphisms for operator systems are unital completely positive (ucp) maps and the appropriate embeddings for operator systems are unital complete order embeddings. Given an operator system S, we say that a pair (C, ρ) is a C*-cover of S if C is a C*-algebra and ρ is a unital complete order embedding $\rho : S \hookrightarrow C$ for which $C^*(\rho(S)) = C$. Given an operator system S there is always a minimal C*-cover called the C*-envelope $(C_e^*(S), \iota)$. It is minimal in the following sense: if (C, ρ) is another C*-cover of S then there is a *-homomorphism $\pi : C \to C_e^*(S)$ for which the diagram

$$\begin{tikzcd} C \arrow[dr, "\pi"] & \\ S \arrow[u, "\rho"] \arrow[r, "\iota"] & C_e^*(S) \end{tikzcd}$$

commutes. There is also a universal C*-cover $(C_{\max}^*(S), \iota)$ which is maximal in the following sense: if (C, ρ) is another C*-cover of S then there is a *-homomorphism $\pi : C_{\max}^*(S) \to C$ for which the diagram

$$\begin{tikzcd} C_{\max}^*(S) \arrow[dr, "\pi"] & \\ S \arrow[u, "\iota"] \arrow[r, "\rho"] & C \end{tikzcd}$$

commutes.

An operator subsystem S of a C*-algebra A is said to be hyperrigid in A if we have the following unique extension property: whenever $\pi : C^*(S) \to B(H)$ is a *-homomorphism and whenever $\varphi : C^*(S) \to B(H)$ is a unital completely positive (ucp) map extending the ucp map $\pi|_S$ then we must have $\varphi = \pi$. Hyperrigid operator systems give us a strong relation between operator systems and their C*-envelope. For example, if S is hyperrigid in A then we must have $C^*(S) \simeq C_e^*(S)$. We say that S is hyperrigid if S is hyperrigid in $C_e^*(S)$. The above definition of hyperrigidity is not the original one. Indeed, the concept of hyperrigidity is introduced by Arveson in [1] to initiate a study of a non-commutative analogue of approximation theory. Consequently, in [1, Definition 1.1], a subspace $S \subset A$ is said to be hyperrigid if whenever we have a faithful embedding $A \subset B(H)$ and whenever $\varphi_n : B(H) \to B(H)$ is a sequence of completely contractive and completely positive maps, we have the implication

$$\lim_{n\to\infty} \|\varphi_n(x) - x\| = 0 \text{ for all } x \in S \text{ implies}$$
$$\lim_{n\to\infty} \|\varphi_n(a) - a\| = 0 \text{ for all } a \in A\,.$$

In [1, Theorem 2.1], Arveson proves that these two definitions are equivalent in the separable case. The density character of a topological space X is the smallest cardinal κ for which there is a subset $E \subset X$ of size κ that is dense in X. Arveson's proof will go through verbatim when we replace all instances of *separable* with *density character at most* κ for any infinite cardinal κ.

If S is *-closed but non-unital, so long as S contains an approximate unit of A, it follows from [19, Proposition 3.6] that S is hyperrigid in $C^*(S)$ if and only if $S^1 := S + \mathbb{C}1$ in the unitization $C^*(S)^1$ is hyperrigid.

A representation $\pi : C^*(S) \to B(H)$ is said to be boundary if π is irreducible and π is the unique ucp extension of $\pi|_S$ to $C^*(S)$. Suppose that S is furthermore unital. Arveson's hyperrigidity conjecture asserts that if all irreducible representations are boundary then the operator system S must be hyperrigid in A. For more information on operator systems, see [17]. See [1] for the formulation of the hyperrigidity conjecture and more details on the above results.

2.2. The Tensor Algebra $\mathcal{T}_X^+$

Let (A, X) be a C*-correspondence and let C be a C*-algebra. We say that a pair of maps $(\pi^0, \pi^1) : (A, X) \to C$ is a Toeplitz pair if

1. $\pi^0 : A \to C$ is a *-homomorphism,
2. $\pi^1 : X \to C$ is a linear map,
3. For any $a \in A$ and $x \in X$ we have $\pi^0(a)\pi^1(x) = \pi^1(a \cdot x)$, and
4. For any x and y in X we have $\pi^0(\langle x, y\rangle) = \pi^1(x)^*\pi^1(y)$.

Given a Toeplitz pair (π^0, π^1), we always have $\pi^1(x)\pi^0(a) = \pi^1(x \cdot a)$ for any $x \in X$ and $a \in A$. A Toeplitz pair can also be thought of as a morphism from the C*-correspondence (A, X) into the C*-correspondence (C, C) where left and right action is given by multiplication and the inner product is given by $\langle x, y\rangle = x^*y$. There is always a maximal C*-algebra associated to C*-correspondences called the Toeplitz-Pimsner algebra $\mathcal{T}_X$. This C*-algebra is maximal in the following sense: there is always a Toeplitz pair

$$\kappa^0 : A \to \mathcal{T}_X$$
$$\kappa^1 : X \to \mathcal{T}_X$$

into $\mathcal{T}_X$ and whenever $(\pi^0, \pi^1) : (A, X) \to C$ is a Toeplitz pair then there is a *-homomorphism

$$\pi^0 \times \pi^1 : \mathcal{T}_X \to C$$

for which the diagram

$$\begin{tikzcd} \mathcal{T}_X \arrow[r, "\pi^0\times\pi^1"] & C \\ (A, X) \arrow[u, "{(\kappa^0,\kappa^1)}"] \arrow[ur, "{(\pi^0,\pi^1)}"'] & \end{tikzcd}$$

commutes. The Toeplitz–Pimsner algebra always contains a canonical norm closed non-selfadjoint operator algebra $\mathcal{T}_X^+$ called the Tensor algebra. This algebra is described as the non-selfadjoint operator algebra generated by

$\kappa^0(A)$ and $\kappa^1(X)$ in $\mathcal{T}_X$. The following example shows us that an approximate identity of A will not necessarily translate to an approximate identity in $\mathcal{T}_X$.

Example 2.1. Consider the C*-correspondence $(\mathbb{C}, \mathbb{C}^2)$, where $\mathbb{C}^2$ is a Hilbert space, with scalar multiplication thought of as a right-$\mathbb{C}$-action. Fix an orthonormal basis $e, f \in \mathbb{C}^2$ and define the left action by $a \cdot x := ae \langle e, x\rangle$ for all $a \in \mathbb{C}$ and $x \in \mathbb{C}^2$. Consider the pair of maps $(\pi^0, \pi^1) : (\mathbb{C}, \mathbb{C}^2) \to M_2$, where $\pi^0(a) = E_{1,1}a$ and $\pi^1(x) = E_{1,1} \langle e, x\rangle + E_{2,1} \langle f, x\rangle$. We see that $\pi^0(a)\pi^1(x) = E_{1,1}a \langle e, x\rangle = \pi^1(a \cdot x)$ and $\pi^0(\langle x, y\rangle) = \pi^1(x)^*\pi^1(y)$. Hence (π^0, π^1) is a Toeplitz pair, but the unit of $\mathbb{C}$ maps to the corner $E_{1,1}$.

In order to guarantee that the approximate identity of A produces an approximate identity of $\mathcal{T}_X$, we will assume that A acts non-degenerately on X. Indeed, if A acts non-degenerately on X and if (e_i) is an approximate unit of A, we see that for all $a \in A$ and $x \in X$, $\kappa^0(e_i)\kappa^0(a) = \kappa^0(e_i a)$ converges to $\kappa^0(a)$ and $\kappa^0(e_i)\kappa^1(x) = \kappa^1(e_i \cdot x)$ converges to $\kappa^1(x)$. Thus, $\kappa^0(e_i)$ is an approximate unit for $\mathcal{T}_X$.

The Toeplitz-Pimsner algebra $\mathcal{T}_X$ always admits a canonical continuous $\mathbb{T}$-action γ called the gauge action. Using the universal property of $\mathcal{T}_X$, it is enough to define γ as an action on (A, X): for $z \in \mathbb{T}$,

$$\gamma_z^0 : A \to A : a \mapsto a$$
$$\gamma_z^1 : X \to X : x \mapsto z \cdot x$$

will give us the action.

Although the Toeplitz-Pimsner algebra $\mathcal{T}_X$ is a canonical algebra associated to (A, X), it is often too big for our purposes. The next example shows that is not the case that the C*-envelope of an operator system of the form $S(A, X)$ is $\mathcal{T}_X$.

Example 2.2. Consider $A = X = \mathbb{C}$ with left and right actions given by multiplication. The C*-algebra $\mathcal{T}_X$ is the universal C*-algebra generated by a single isometry v and our operator system $S(\mathbb{C}, \mathbb{C})$ is the operator system spanned by the set $\{1, v, v^*\}$. Let $\mathcal{T}_X \subset B(H)$ for some Hilbert space H. By [17, Theorem 1.1], there is some Hilbert space K and and isometry $V : H \hookrightarrow K$ such that for some unitary $U \in B(K)$, $v = V^*UV$. This means in particular that there is a unital and completely positive map from the operator system $\operatorname{span}\{1, U, U^*\}$ into $S(\mathbb{C}, \mathbb{C})$ given by $x \mapsto V^*xV$. Since v generates the universal C*-algebra of a single isometry, these two operator systems are in fact isomorphic. By [15, Lemma 5.5], the operator system $\operatorname{span}\{1, U, U^*\}$ is hyperrigid in $C^*(1, U)$. Thus, the C*-envelope of $S(\mathbb{C}, \mathbb{C})$ is $C^*(1, U)$.

Since hyperrigid operator systems necessarily generate a C*-envelope, we will need to take an appropriate quotient of $\mathcal{T}_X$ in order to identify the correct ambient C*-algebra in which to check the unique extension property. The remedy for this is to restrict our class of representations.

Fix a C*-correspondence (A, X). The space $\mathcal{K}(X)$ is the C*-subalgebra of the space $\mathcal{L}(X)$ of adjointable right-A-linear operators on X spanned by the operators $x\langle y,\cdot\rangle$ for $x,y\in X$. We think of the left action of A on X as the *-homomorphism $\lambda : A \to \mathcal{L}(X)$. Given a Toeplitz pair $(\pi^0,\pi^1):(A,X)\to C$, there is always a *-homomorphism

$$\varphi_\pi : \mathcal{K}(X) \to C : x\langle y,\cdot\rangle \mapsto \pi^1(x)\pi^1(y)^* \ .$$

The Katsura ideal $\mathcal{J}_X$ associated to (A,X), introduced in [13], consists of elements $a\in A$ for which $\lambda(a)\in\mathcal{K}(X)$ and for which $ab=0$ whenever b belongs to the kernel of λ. Note that $\mathcal{J}_X$ is the largest ideal in A for which λ is an injection into $\mathcal{K}(X)$. A Toeplitz pair $(\pi^0,\pi^1):(A,X)\to C$ is said to be covariant if for any element $a\in\mathcal{J}_X$, we have the identity

$$\pi^0(a) = \varphi_\pi(\lambda(a)) \ .$$

The appropriate choice of C*-algebra is the universal C*-algebra associated to covariant Toeplitz pairs. Observe that in Example 2.2, if x a unit vector in $X=\mathbb{C}$, covariance is imposing the relation $\pi_0(1)=1=\varphi_\pi(\lambda(1))=\varphi_\pi(x\langle x,\cdot\rangle)=\pi^1(x)\pi^1(x)^*$, which is exactly the condition we need to guarantee that our isometry is a unitary. This algebra is called the Cuntz-Pimsner algebra $\mathcal{O}_X$. We will let

$$\iota^0 : A \to \mathcal{O}_X$$
$$\iota^1 : X \to \mathcal{O}_X$$

be the canonical covariant Toeplitz pair. The $\mathbb{T}$-action on $\mathcal{T}_X$ induces a $\mathbb{T}$-action on $\mathcal{O}_X$ since the pair $(\gamma_z^0,\gamma_z^1):(A,X)\to\mathcal{T}_X$ is covariant for all $z\in\mathbb{T}$. Since $\mathcal{O}_X$ is the universal C*-algebra associated to a restricted class of Toeplitz pairs, there is a canonical quotient map $\mathcal{T}_X\to\mathcal{O}_X$. We also have the gauge invariant uniqueness Theorem [13, Theorem 6.4].

Theorem 2.3. *(Gauge-invariant uniqueness theorem) Suppose that there is a covariant Toeplitz pair $(\pi^0,\pi^1):(A,X)\to C$ with π^0 injective and suppose that there is a gauge action $\mathbb{T}\curvearrowright C^*(\pi^0,\pi^1)$ for which the Toeplitz pair (π^0,π^1) is $\mathbb{T}$-equivariant. The *-homomorphism*

$$\pi^0\times\pi^1 : \mathcal{O}_X \to C$$

is necessarily injective.

A result of Katoulis and Kribs shows that the tensor algebra $\mathcal{T}_X^+$ always sits completely isometrically as a subset of $\mathcal{O}_X$ [8, Lemma 3.5]. Moreover, they show that $\mathcal{O}_X$ is the C*-envelope of $\mathcal{T}_X^+$ [8, Theorem 3.7].

Definition 2.4. Let (A,X) be a C*-correspondence. We define the operator space $S(A,X)$ as the *-closed operator subspace of $\mathcal{T}_X$ generated by X and A.

One can similarly define a *-closed operator subspace of $\mathcal{O}_X$ generated by A and X. The following Proposition tells us that the two operator spaces are isomorphic.

Proposition 2.5. *Let (A,X) be a C*-correspondence. Suppose that $\rho:\mathcal{T}_X\to\mathcal{O}_X$ is the canonical quotient map. The restriction $\rho|_{S(A,X)}$ is a completely isometric map.*

Proof. Consider the unital *-homomorphism $\rho^1 : \mathcal{T}_X^1 \to \mathcal{O}_X^1$ induced by ρ. Let $\mathcal{M}$ and $\mathcal{N}$ denote the unital operator spaces generated by A and X in $\mathcal{T}_X^1$ and $\mathcal{O}_X^1$ respectively. That is, $\mathcal{M} + \mathcal{M}^* = S(A,X)^1$ and our goal is to show that $\rho^1|_{S(A,X)^1} : S(A,X)^1 \to \mathcal{N} + \mathcal{N}^*$ is an isomorphism of operator systems. To get a unital and completely positive inverse map, first observe that as $\mathcal{N}$ is a subspace of $\mathcal{T}_X^+$, by [8, Lemma 3.5] there is a completely contractive inverse map $\varphi := (\rho^1|_{\mathcal{N}})^{-1}$. By [17, Proposition 3.4], the map $\tilde{\varphi} : \mathcal{N} + \mathcal{N}^* \to \mathcal{T}_X : a + b^* \mapsto \varphi(a) + \varphi(b)^*$ for $a, b \in \mathcal{N}$ is a well-defined unital and completely positive map. Since $\tilde{\varphi} = (\rho^1|_{S(A,X)^1})^{-1}$, this completes the proof. □

Because of the above Proposition, we will denote by $S(A,X)$ the *-closed subspace generated by A and X in $\mathcal{O}_X$ as well. Next we will show that the C*-envelope of $S(A,X)$ is $\mathcal{O}_X$. If A is non-unital, by the C*-envelope $C_e^*(S(A,X))$ of $S(A,X)$, we mean the C*-subalgebra of the C*-envelope of $S(A,X)^1$ generated by $S(A,X)$. This C*-algebra has the universal property that whenever D is a C*-algebra generated by $S(A,X)$ in the sense that D^1 is a C*-cover of $S(A,X)^1$, then there is a *-homomorphism $D \to C_e^*(S(A,X))$ preserving $S(A,X)$.

Proposition 2.6. *Let (A,X) be a C*-correspondence. The C*-envelope of $S(A,X)$ is $\mathcal{O}_X$.*

Proof. Suppose that D is the C*-envelope of $S(A,X)$ and let $\theta : S(A,X) \hookrightarrow D$ denote the associated embedding. There is always a quotient map $\rho : \mathcal{O}_X \to D$ preserving $S(A,X)$. The pair $(\rho^0, \rho^1) := (\rho \circ \iota^0, \rho \circ \iota^1)$ is necessarily a covariant pair since it arises from a *-homomorphism on $\mathcal{O}_X$. As well, $\rho \circ \iota^0$ is necessarily isometric since θ is isometric. Thus, to show that ρ is isometric, it suffices to show that D admits a gauge action. For each $z \in \mathbb{T}$, let $\gamma_z : \mathcal{O}_X \to \mathcal{O}_X$ denote the canonical automorphism. Since $S(A,X)$ is invariant under γ_z, we may define an embedding $\theta \circ (\gamma_z|_{S(A,X)})$ of $S(A,X)$ into D. By the universal property of the C*-envelope, there is always a *-homomorphism $\alpha_z : D \to D$ such that the diagram

$$\begin{tikzcd} D \arrow[rr, "\alpha_z"] & & D \\ & S(A,X) \arrow[ul, hook, "\theta"] \arrow[ur, hook, "\theta\circ(\gamma_z|_{S(A,X)})"'] & \end{tikzcd}$$

commutes. The map α_z has inverse map $\alpha_{\bar{z}}$, so α_z is in fact an isomorphism. By construction, $\alpha : \mathbb{T} \to \mathrm{Aut}(D)$ makes the map ρ $\mathbb{T}$-equivariant. By the gauge-invariant uniqueness theorem, $\rho : \mathcal{O}_X \to D$ is an isomorphism. □

3. Hyperrigidity of Operator Spaces $S(A,X)$

In [11, Theorem 3.1], Katsoulis and Ramsey show that to achieve hyperrigidity of a C*-correspondence X that is countably generated over A, it is sufficient for the left action of $\mathcal{J}_X$ to act non-degenerately X . We show that not only does this condition hold without any assumption on X but that this condition is also necessary. The following two definitions are in [18].

Definition 3.1. Let (A, X) be a Hilbert A-module. We treat the multiplier algebra $M(A)$ as the C*-algebra $\mathcal{L}(A)$. The Hilbert $M(A)$-module $M(X)$ is defined as follows: As a linear space, $M(X) = \mathcal{L}(A, X)$. The right action is given by composition and the inner product is given by $\langle x, y\rangle := x^* \circ y$.

If $x \in X$ and $y \in M(X)$ then $\langle y, x\rangle \in A$ and if $a \in A$ then $y \cdot a \in X$. If (A, X) is a C*-correspondence and $a \in A$ is such that $\lambda(a) \in K(X)$ then for any $x \in M(X)$, we have $a \cdot x \in X$. In particular, if $a \in \mathcal{J}_X$ then $a \cdot x \in X$.

Definition 3.2. Let (A, X) be a Hilbert A-module. We say that X is countably generated over A if there exists a sequence $(x_n)_{n\geq 1}$ in $M(X)$ for which the A-linear span of $(x_n)_n$ is dense in X. A standard normalized frame for (A, X) is a sequence $(x_n)_{n\geq 1}$ in $M(X)$ for which for every $x \in X$ we have the identity

$$\langle x, x\rangle = \sum_{n\geq 1} \langle x, x_n\rangle \langle x_n, x\rangle \ .$$

By [18, Corollary 3.3], whenever X is countably generated over A, a standard normalized frame for X exists.

The reconstruction formula [18, Theorem 3.4] states that a sequence $(x_n)_{n\geq 1}$ is a standard normalized frame if and only if we have the identity

$$x = \sum_{n\geq 1} x_n \langle x_n, x\rangle$$

for every $x \in X$.

Lemma 3.3. *Suppose that (A, X) is a C*-correspondence. Let $\mathcal{M}$ denote the space of all countably generated right A-submodules of X. For each $Y \in \mathcal{M}$, let $(x_n(Y))_{n\geq 1}$ denote a standard normalized frame for Y. Let*

$$e_n(Y) := \sum_{k=1}^{n} x_k(Y) \langle x_k(Y), \cdot\rangle \ .$$

The set $(e_n(Y))_{(n,Y)\in\mathbb{N}\times\mathcal{M}}$ is an approximate unit for $\mathcal{K}(X)$ in the following sense: if $T \in \mathcal{K}(X)$ then we have the identity

$$\lim_{Y\to\infty} \lim_{n\to\infty} e_n(Y) \cdot T = T \ .$$

Proof. Let $T \in \mathcal{K}(X)$. Let $\epsilon > 0$. Suppose that $y_1, \ldots, y_n, z_1, \ldots, z_n \in X$ is such that

$$\|T - \sum_{k=1}^{n} y_k \langle z_k, \cdot\rangle \| < \epsilon \ .$$

Let $S = \sum_k y_k \langle z_k, \cdot\rangle$. Consider any $Y \in \mathcal{M}$ for which y_k, z_k belong to Y for all k. For any $x \in X$,

$$\sum_k y_k \langle z_k, x\rangle \in Y \ .$$

By the reconstruction formula, we know that

$$y_k = \sum_{n \geq 1} x_n(Y) \langle x_n(Y), y_k \rangle = \lim_{n \to \infty} e_n(Y)(y_k) \, .$$

for all k. This means in particular, that for n large enough,

$$\|e_n(Y)S - S\| < \epsilon \, .$$

Therefore,

$$\|T - e_n(Y)T\| \leq 2\|T - S\| + \|S - e_n(Y)S\| < 3\epsilon \, .$$

This proves that $e_n(Y)$ is an approximate unit for $\mathcal{K}(X)$. $\square$

The following Lemma provides a quantitative variant of [17, Theorem 3.18].

Lemma 3.4. *Let A be a C^*-algebra. Fix $m, n \geq 1$. Suppose that $\varphi : A \to B(H)$ is a completely positive and contractive map for which for some $\epsilon > 0$ and $a \in M_{m,n}(A)$, we have the bound*

$$\|\varphi(aa^*) - \varphi(a)\varphi(a)^*\| < \epsilon \, .$$

It is then the case that for any $b \in M_{m,n}(A)$, we have the estimate

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \sqrt{\epsilon}\|b\| \, .$$

Proof. For a positive $p \in M_{2m}(A)$, let

$$P(p) := \left[\begin{array}{c|cc} I_n & a^* & b^* \\ \hline a & \multicolumn{2}{c}{\multirow{2}{*}{p}} \\ b & & \end{array}\right] \, .$$

The same argument as in [17, Lemma 3.1] shows that the matrix $P(p)$ is positive if and only if we have the bound

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a^* & b^* \end{bmatrix} \leq p \, .$$

Taking

$$p = \begin{bmatrix} aa^* & ab^* \\ ba^* & bb^* \end{bmatrix} \, ,$$

we can conclude $P(p)$ is positive in this case. Since φ is contractive and completely positive, applying the $(2n+2)$-amplification of the unitization of φ onto $P(p)$, we get the bound

$$\begin{bmatrix} \varphi(a) \\ \varphi(b) \end{bmatrix} \begin{bmatrix} \varphi(a)^* & \varphi(b)^* \end{bmatrix} \leq \varphi(p) \, .$$

That is, the matrix

$$\begin{bmatrix} \varphi(aa^*) - \varphi(a)\varphi(a)^* & \varphi(ab^*) - \varphi(a)\varphi(b^*) \\ \varphi(ba^*) - \varphi(b)\varphi(a)^* & \varphi(bb^*) - \varphi(b)\varphi(b)^* \end{bmatrix}$$

is positive. Since the $(1,1)$ corner of this matrix is at most ϵ, we get positivity of the matrix

$$\begin{bmatrix} \epsilon I_2 & \varphi(ab^*) - \varphi(a)\varphi(b^*) \\ \varphi(ba^*) - \varphi(b)\varphi(a)^* & \varphi(bb^*) - \varphi(b)\varphi(b)^* \end{bmatrix} .$$

In particular, we have the bound

$$\begin{aligned} \|\varphi(ab^*) - \varphi(a)\varphi(b)^*\|^2 &\leq \epsilon\|\varphi(bb^*) - \varphi(b)\varphi(b)^*\| \\ &\leq \epsilon\|bb^*\| , \end{aligned} \tag{1}$$

where the final inequality follows from the fact that

$$0 \leq \varphi(bb^*) - \varphi(b)\varphi(b)^* \leq \varphi(bb^*) \leq \|bb^*\|1 .$$

□

Theorem 3.5. *Let (A, X) be a C*-correspondence. The following are equivalent:*

1. *The left action of $\mathcal{J}_X$ on X is non-degenerate.*
2. *$S(A, X)$ is hyperrigid.*

Proof. First assume that $\mathcal{J}_X$ acts on X non-degenerately. We denote by (i^0, i^1) the canonical covariant pair

$$(i^0, i^1) : (A, X) \to \mathcal{O}_X .$$

Fix any *-homomorphism $\pi : \mathcal{O}_X \to B(H)$ and suppose that $\varphi : \mathcal{O}_X \to B(H)$ is any cpcc-extension of $\pi|_{S(A,X)}$. We claim that for any $a \in \mathcal{J}_X$ and $x \in X$, we have

$$\varphi(\iota^1(a \cdot x)\iota^1(a \cdot x)^*) = \varphi(\iota^1(a \cdot x))\varphi(\iota^1(a \cdot x))^* .$$

Suppose first that we have the above claim. By [17, Theorem 3.18] the set

$$\mathrm{mult}(\varphi) := \{a \in \mathcal{O}_X : \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a^*)\}$$

is a C*-algebra that agrees with the set

$$\{a \in \mathcal{O}_X : \varphi(ab) = \varphi(a)\varphi(b) \text{ and } \varphi(ba) = \varphi(b)\varphi(a) \text{ for all } b \in \mathcal{O}_X\} .$$

Because $\varphi \circ \iota^0$ and ι^0 are *-homomorphisms, for any $a \in A$, $a \in \mathrm{mult}(\varphi)$. As well, for any $x \in X$ and $a \in \mathcal{J}_X$,

$$\begin{aligned} \varphi(\iota^1(a \cdot x)^*\iota^1(a \cdot x)) &= \varphi(\iota^0(\langle a \cdot x, a \cdot x\rangle)) = \pi(\iota^0(\langle a \cdot x, a \cdot x\rangle)) \\ &= \pi(\iota^1(a \cdot x))^*\pi(\iota^1(a \cdot x)) = \varphi(\iota^1(a \cdot x))^*\varphi(\iota^1(a \cdot x)) . \end{aligned}$$

Thus, with our claim, we may also conclude that $\iota^1(a \cdot x) \in \mathrm{mult}(\varphi)$. By non-degeneracy of $\mathcal{J}_X$ acting on X, it follows that $\iota^1(x) \in \mathrm{mult}(\varphi)$. Since $\iota^0(A)$ and $\iota^1(X)$ generate $\mathcal{O}_X$, it follows that φ is multiplicative, and hence agrees with π.

Let $\mathcal{M}$ and $x_n(Y), e_n(Y)$ be as in Lemma 3.3. Let

$$\phi_\iota : \mathcal{K}(X) \to \mathcal{O}_X : x\,\langle y, \cdot\rangle \mapsto \iota^1(x)\iota^1(y)^* .$$

For any $a \in \mathcal{J}_X$, since $\lambda(a)$ is a compact operator, we have

$$\begin{aligned}\iota^0(aa^*) &= \phi_\iota\left(\lim_{Y\to\infty}\lim_{n\to\infty}\lambda(a)\cdot e_n(Y)\lambda(a)^*\right)\\ &= \lim_{Y\to\infty}\lim_{n\to\infty}\sum_{k<n}\iota^1(a\cdot x_k(Y))\iota^1(a\cdot x_k(Y))^*\ .\end{aligned}$$

By the Schwarz inequality,

$$\begin{aligned}\varphi(\iota^0(aa^*)) &= \lim_Y \lim_n \sum_{k<n}\varphi(\iota^1(a\cdot x_k(Y))\iota^1(a\cdot x_k(Y))^*)\\ &\geq \lim_Y \lim_n \sum_{k<n}\varphi(\iota^1(a\cdot x_k(Y)))\varphi(\iota^1(a\cdot x_k(Y)))^*\\ &= \lim_Y \lim_n \sum_{k<n}\pi(\iota^1(a\cdot x_k(Y)))\pi(\iota^1(a\cdot x_k(Y)))^*\\ &= \pi(\iota^0(aa^*)) = \varphi(\iota^0(aa^*))\ .\end{aligned}$$

From this, we have the identity

$$\begin{aligned}&\lim_Y \lim_n \sum_{k<n}\varphi(\iota^1(a\cdot x_k(Y))\iota^1(a\cdot x_k(Y))^*)\\ &= \lim_Y \lim_n \sum_{k<n}\pi(\iota^1(a\cdot x_k(Y))\iota^1(a\cdot x_k(Y))^*)\ .\end{aligned}$$

By the reconstruction formula, for any $x \in X$ and for any $Y \in \mathcal{M}$ with $x \in Y$, we have for all $a \in \mathcal{J}_X$,

$$a\cdot x = \sum_{n\geq 1} a\cdot x_n(Y)\langle x_n(Y), x\rangle\ .$$

Let $\epsilon > 0$. Fix any $Y \in \mathcal{M}$ for which we have the bound

$$0 \leq \sum_{n\geq 1}\varphi(\iota^1(a\cdot x_n(Y))\iota^1(a\cdot x_n(Y))^*) - \varphi(\iota^1(a\cdot x_n(Y)))\varphi(\iota^1(a\cdot x_n(Y)))^* \leq \epsilon 1\ .$$

Let $\alpha_n = \iota^1(a\cdot x_n(Y))$ and let $\beta_n = \iota^1(a\cdot x\langle x, x_n(Y)\rangle)$. Observe that

$$\iota^1(a\cdot x)\iota^1(a\cdot x)^* = \sum_{n\geq 1}\alpha_n\beta_n^*\ .$$

Consider for fixed $n \geq 1$ the $1\times n$-matrices $A_n = (\alpha_1,\ldots,\alpha_n)$ and $B_n = (\beta_1,\ldots,\beta_n)$. A calculation shows that

$$\begin{aligned}\|B_nB_n^*\| &= \left\|\sum_{k\leq n}\iota^1(a\cdot x)\iota^0(\langle x, x_k(Y)\rangle\langle x_k(Y), x\rangle)\iota^1(a\cdot x)^*\right\|\\ &= \left\|\iota^1(a\cdot x)\ \iota^0\left(\sum_{k\leq n}\langle x, x_k(Y)\rangle\langle x_k(Y), x\rangle\right)\iota^1(a\cdot x)^*\right\|\\ &\leq \|a\cdot x\|^2\left\|\sum_{k\leq n}\langle x, x_k(Y)\rangle\langle x_k(Y), x\rangle\right\|\ .\end{aligned}$$

Since the sequence $x_n(Y)$ is a standard normalized frame, we have the inequality

$$\|B_n B_n^*\| \leq \|a \cdot x\|^2 \|x\|^2$$

for any n. As well, a calculation shows that

$$\begin{aligned}\lim_{n\to\infty} \varphi(A_n B_n^*) - \varphi(A_n)\varphi(B_n)^* &= \lim_{n\to\infty} \sum_{k\leq n} \varphi(\iota^1(a \cdot x_n(Y) \langle x_n(Y), x\rangle)\iota^1(a\cdot x)^*) \\ &\quad - \varphi(\iota^1(a \cdot x_n(Y) \langle x_n(Y), x\rangle))\varphi(\iota^1(a\cdot x))^* \\ &= \varphi(\iota^1(a\cdot x)\iota^1(a\cdot x)^*) - \varphi(\iota^1(a\cdot x))\varphi(\iota^1(a\cdot x))^* \ .\end{aligned}$$

The above calculation with Lemma 3.4 give us the bound

$$\begin{aligned}&\|\varphi(\iota^1(a\cdot x)\iota^1(a\cdot x)^*) - \varphi(\iota^1(a\cdot x))\varphi(\iota^1(a\cdot x))^*\|^2 \\ &= \lim_{n\to\infty} \|\varphi(A_n B_n^*) - \varphi(A_n)\varphi(B_n)^*\|^2 \leq \epsilon \|a\cdot x\|^2 \|x\|^2 \ .\end{aligned}$$

Since this identity is independent of the choice of Y and ϵ, we may conclude that for any $a \in \mathcal{J}_X$ and for any $x \in X$, the element $\iota^1(a\cdot x)$ belongs to $\mathrm{mult}(\varphi)$, showing hyperrigidity.

For the converse, assume that $\mathcal{J}_X$ does not act on X non-degenerately. Fix a faithful covariant representation $(\pi^0, \pi^1) : (A, X) \to B(H)$. Let $N \subset \mathcal{J}_{X,+}$ form a contractive approximate unit for $\mathcal{J}_X$ under the ordering induced by the positive operators. Define operators $P = \lim_{a\in N} \pi^0(a)$ and $Q = 1 - P$ where the limit is taken in the strong operator topology on $B(H)$. For any isometry $V \in B(\mathcal{K})$, let $(\tau^0, \tau^1_V) : (A, X) \to B(H \otimes \mathcal{K})$ be the following pair of maps

$$\begin{aligned}&\tau^0 : A \to B(H\otimes\mathcal{K}) : a \mapsto \pi^0(a) \otimes I \\ &\tau^1_V : X \to B(H\otimes\mathcal{K}) : x \mapsto P\pi^1(x) \otimes I + Q\pi^1(x) \otimes V \ .\end{aligned}$$

It is immediate that τ^0 is a *-homomorphism and that τ^1_V is linear. For any $a \in A$ and $x \in X$, first observe that since P is the projection which generates the ideal $\pi^0(\mathcal{J}_X)$ in $\pi^0(A)$, that P commutes with $\pi^0(a)$. Thus,

$$\begin{aligned}\tau^0(a)\tau^1_V(x) &= (\pi^0(a)\otimes I)(P\pi^1(x)\otimes I + Q\pi^1(x)\otimes V) \\ &= P(\pi^0(a)\pi^1(x)) \otimes I + Q\pi^0(a)\pi^1(x) \otimes V \\ &= P\pi^1(a\cdot x)\otimes I + Q\pi^1(a\cdot x)\otimes V = \tau^1(a\cdot x) \ .\end{aligned}$$

As well, for $x, y \in X$, we have

$$\begin{aligned}\tau^1_V(x)^*\tau^1_V(x) &= (P\pi^1(x)\otimes I + Q\pi^1(x)\otimes V)^*(P\pi^1(y)\otimes I + Q\pi^1(y)\otimes V) \\ &= \pi^1(x)^* P\pi^1(y)\otimes I + \pi^1(x)^* Q\pi^1(y)\otimes I \\ &= \pi^1(x)^*(P+Q)\pi^1(y)\otimes I = \pi^0(\langle x, y\rangle)\otimes I \\ &= \tau^0(\langle x, y\rangle) \ .\end{aligned}$$

This is therefore a Toeplitz representation for (A, X). To see that this representation is covariant, let $a \in \mathcal{J}_X$. Since $\lambda(a) \in \mathcal{K}(X)$, for $\epsilon > 0$, let

$x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that for any contraction $z \in X$, we have

$$\left\| a \cdot z - \sum_{k=1}^{n} x_k \langle y_k, z \rangle \right\| < \epsilon \ .$$

For any $b \in N$, we have

$$\left\| bab \cdot z - \sum_{k=1}^{n} b \cdot x_k \langle b \cdot y_k, z \rangle \right\| < \epsilon \ .$$

In particular, $\lambda(bab)$ is within ϵ of the compact operator $\sum_k bx_k \langle by_k, \cdot \rangle$. Let

$$\varphi_V : \mathcal{K}(X) \to B(H) : x_i \langle y_i, \cdot \rangle \mapsto \tau_V^1(x_i)\tau_V^1(y_i)^* \ .$$

A calculation shows

$$\begin{aligned}
&\varphi_V \left(\sum\nolimits_k bx_k \langle by_k, \cdot \rangle \right) = \sum\nolimits_k \tau^1(bx_k)\tau^1(by_k)^* \\
&= \sum\nolimits_k (P\pi^1(bx_k) \otimes I + Q\pi^1(bx_k) \otimes V)(P\pi^1(by_k) \otimes I + Q\pi^1(by_k) \otimes V)^* \\
&= \sum\nolimits_k (P\pi^1(bx_k) \otimes I)(P\pi^1(by_k) \otimes I)^* \\
&= \sum\nolimits_k (\pi^1(bx_k) \otimes I)(\pi^1(by_k) \otimes I)^* \\
&= \left(\sum\nolimits_k \pi^1(bx_k)\pi^1(by_k)^* \right) \otimes I \ .
\end{aligned}$$

For any $b \in N$,

$$\begin{aligned}
\left\| \varphi_V(\lambda(bab)) - \pi^0(bab) \otimes I \right\| \leq & \left\| \varphi_V(\lambda(bab)) - \varphi_V \left(\sum\nolimits_k bx_k \langle by_k, \cdot \rangle \right) \right\| \\
& + \left\| \varphi_V \left(\sum\nolimits_k bx_k \langle by_k, \cdot \rangle \right) - \pi^0(bab) \otimes I \right\| \\
< & 2\epsilon \ .
\end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, we conclude that $\varphi_V(\lambda(bab)) = \tau^0(bab)$ for all $b \in N$. Since N is an approximate unit for $\mathcal{J}_X$ and $a \in \mathcal{J}_X$, we have $\varphi_V(\lambda(a)) = \tau^0(a)$.

Let us fix the unilateral shift $V \in B(\ell^2(\mathbb{Z}_+))$ and the bilateral shift $U \in B(\ell^2(\mathbb{Z}))$. Let $\Phi : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}_+))$ be the ucp map given by restriction. The diagram

$$\begin{tikzcd}
 & B(H \otimes \ell^2(\mathbb{Z})) \arrow[d, "\Phi"] \\
\mathcal{O}_X \arrow[ur, "\tau^0 \times \tau_U^1"] \arrow[r, "\tau^0 \times \tau_V^1"] & B(H \otimes \ell^2(\mathbb{Z}_+))
\end{tikzcd}$$

commutes. So long as we can show $Q\pi^1(X) \neq 0$, we are done, since $\Phi \circ (\tau^0 \times \tau_U^1) \neq \tau^0 \times \tau_V^1$ but agree on $S(A, X)$. Suppose that $Q\pi^1(X) = 0$ in order to derive a contradiction. Since $P + Q = I$, this means that $P\pi^1(x) = \pi^1(x)$ for every $x \in X$. If $\mathcal{J}_X$ acts on X degenerately, then by taking a subnet if necessary, there is some $\epsilon > 0$ and some $x \in X$ so that for every $b \in N$, there is some unit vector $h_b \in H$ for which we have the identity

$$\left\langle (\pi^1(x)^*\pi^1(x) - \pi^1(x)^*\pi^0(b)\pi^1(x))h_b, h_b \right\rangle \geq \epsilon \ .$$

If $a \geq b$ in N then we have the identity

$$\begin{aligned}&\left\langle(\pi^1(x)^*\pi^1(x) - \pi^1(x)^*\pi^0(b)\pi^1(x))h_a, h_a\right\rangle \\ &\geq \left\langle(\pi^1(x)^*\pi^1(x) - \pi^1(x)^*\pi^0(a)\pi^1(x))h_a, h_a\right\rangle \geq \epsilon\ .\end{aligned}$$

If we could replace the net $(h_b)_{b\in N}$ with a fixed vector $h_b = h \in H$ for all b then we may conclude from the above inequality that $P\pi^1(x) \neq \pi^1(x)$ and we would have our contradiction.

In order to guarantee that a vector $h \in H$ as above exists, we need to fix a specific faithful representation. Let $\mathcal{P}(N)$ denote the powerset of N. Take any non-principal ultrafilter $\mathcal{U} \subset \mathcal{P}(N)$ containing the set

$$S := \{\{a \in N : a \geq b\} : b \in N\}\ .$$

Such an ultrafilter exists since S has the finite intersection property. Consider the covariant pair

$$(\overline{\pi}^0, \overline{\pi}^1) : (A, X) \to B(H^{\mathcal{U}})$$

so that $\overline{\pi}^0(a)(\lim_{\mathcal{U}} k_b) = \lim_{\mathcal{U}} \pi^0(a)\cdot k_b$ and $\overline{\pi}^1(x)(\lim_{\mathcal{U}} k_b) = \lim_{\mathcal{U}} \pi^1(x)\cdot k_b$. Replacing (π^0, π^1) with $(\overline{\pi}^0, \overline{\pi}^1)$ and taking $h = \lim_{\mathcal{U}} h_b$ will do. $\square$

As an application, we look at hyperrigidity of *-closed operator spaces associated to topological graphs, as introduced by Katsura in [12]. A topological graph is a quadruple $E = (E^0, E^1, s, r)$ such that E^0 and E^1 are locally compact hausdorff spaces and $s, r : E^1 \to E^0$ are continuous functions. We furthermore assume that s is a local homeomorphism. We want to think of E^0 as vertices of a graph, while E^1 are thought of as edges such that the functions s and r assign the source and the range of the edge respectively. That s is a local homeomorphism means in particular that for every $x \in E^0$, the preimage $s^{-1}(x)$ is a discrete subspace of E^1. Given a topological graph E, define a C*-correspondence $X(E)$ over the C*-algebra $C_0(E^0)$ as the completion of $C_c(E^1)$ with right and left actions given by

$$\begin{aligned}f\cdot g &: e \mapsto f(e)g(s(e)) \text{ and}\\ g\cdot f &: e \mapsto g(r(e))f(e)\end{aligned}$$

for any $f \in C_c(E^1)$ and $g \in C_0(E^0)$ and with inner product given by

$$\langle f, h\rangle : x \in E^0 \mapsto \sum_{e\in E^1 : s(e)=x} \overline{f(e)}h(e)$$

for any $f, h \in C_c(E^1)$. The graph C*-algebra $C^*(E)$ is defined to be the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$.

Let us characterize the topological graphs with range map r open for which the associated space $S(C_0(E^0), X(E))$ is hyperrigid. Let $E^0_{\text{fin.}}$ be the open subset of E^0 for which we have the identity

$$C_0(E^0_{\text{fin.}}) = \lambda^{-1}(\mathcal{K}(X(E)))\ .$$

Theorem 3.6. *Let E be a topological graph and let r be open. The following are equivalent:*

1. *The space $S(C_0(E^0), X(E))$ is hyperrigid.*

2. *The set* $E^0_{\mathrm{fin.}}$ *is dense in* E^0.

Proof. The kernel of λ consists of those elements $f \in C_0(E^0)$ for which $f|_{r(E^1)} = 0$. Thus,

$$\ker \lambda = C_0(E^0 \setminus \overline{r(E^1)}) \ .$$

This implies that $\mathcal{J}_{X(E)} = C_0(E^0_{\mathrm{fin.}} \cap \mathrm{int}(\overline{r(E^1)}))$. Let $Y = \mathrm{int}(\overline{r(E^1)})$. Assume that $E^0_{\mathrm{fin.}} \cap Y$ is dense in Y. We claim that $\mathcal{J}_{X(E)} X(E) = X(E)$. Let φ_i be a monotonically increasing approximate unit for $C_0(E^0_{\mathrm{fin.}} \cap Y)$. For any $f \in C_c(E^1)$, we claim that $\varphi_i \cdot f$ converges to f. Consider the positive function $\mathcal{F}_i = \langle f - \varphi_i \cdot f, f - \varphi_i \cdot f\rangle$. Observe that as f is compactly supported that all $\mathcal{F}_i$ are supported on a compact set K. As well, $\mathcal{F}_i(x)$ is a decreasing net for all $x \in E^0$. By Dini's theorem and the uniform limit theorem, the function

$$\mathcal{F} : E^0 \to \mathbb{C} : x \mapsto \lim_{i \to \infty} \mathcal{F}_i(x)$$

is continuous and compactly supported. We need to show that $\mathcal{F} = 0$. If not, there is some open set $U \subset E^0$ for which $\mathcal{F}|_U > 0$. If $x \in U$ then for any $e \in s^{-1}(x)$, $r(e) \notin E^0_{\mathrm{fin.}}$. That is, if $x \in r(s^{-1}(U))$ then $x \notin E^0_{\mathrm{fin.}}$. Since r is open, $r(s^{-1}(U))$ is an open subset of Y. That $r(s^{-1}(U)) \cap E^0_{\mathrm{fin.}} = \varnothing$ is a contradiction on the density of $E^0_{\mathrm{fin.}} \cap Y$ in Y. Thus we have $\mathcal{J}_{X(E)} X(E) = X(E)$.

If $E^0_{\mathrm{fin.}} \cap Y$ is not dense in Y then there is some open subset U of Y so that $U \cap E^0_{\mathrm{fin.}} = \varnothing$. Consider any non-zero function $f \in C_c(E^1)$ supported on $r^{-1}(U)$. If $\mathcal{J}_{X(E)}$ acts non-degenerately on $X(E)$, then by Cohen's factorization theorem, there is some $x \in X(E)$ and some $g \in \mathcal{J}_{X(E)}$ for which $g \cdot x = f$. Say $f_i \in C_c(E^1)$ for which $\lim_i f_i = x$. For any point $e \in E^1$, if $f(e) \neq 0$ then $r(e) \in U$. This implies that $g(r(e)) = 0$. For any i,

$$\langle g \cdot f_i, f\rangle : x \mapsto \sum_{e \in E^1 : s(e) = x} \overline{g(r(e))} f(e) \overline{f_i(e)} = 0 \ .$$

Thus we have $\langle f, f\rangle = \lim_i \langle g \cdot f_i, f\rangle = 0$ – a contradiction.

Let $Y = \mathrm{int}(\overline{r(E^1)})$. By the above argument, hyperrigidity of the operator space $S(C_0(E^0), X(E))$ is equivalent to density of $E^0_{\mathrm{fin.}} \cap Y$ in Y. To finish the argument, suppose that $E^0_{\mathrm{fin.}} \cap Y$ is dense in Y. If x is a point in $E^0 \setminus \overline{r(E^1)}$ then there is a non-negative function f supported outside of $\overline{r(E^1)}$ for which $f(x) = 1$. Since $\lambda(f) = 0$, we must conclude that $x \in E^0_{\mathrm{fin.}}$. In particular, whenever U is an open set in E^0 for which $U \cap E^0_{\mathrm{fin.}} = \varnothing$ then we must have $U \subset Y$. By our assumption, we must have $U = \varnothing$. □

Example 3.7. Assume that E is a topological graph for which E^0 and E^1 are discrete spaces. By Theorem 3.6, the space $S(c_0(E^0), X(E))$ is hyperrigid if and only if we have the identity $E^0_{\mathrm{fin.}} = E^0$. In [12, Example 1], the set $E^0_{\mathrm{fin.}}$ for discrete graphs is described as those points $x \in E^0$ for which $r^{-1}(x)$ is a finite set. The graphs for which $r^{-1}(x)$ is finite for every $x \in E^0$ is known as a row-finite graph in the literature. Thus, our operator space is hyperrigid if and only if the graph E is row-finite. This recovers a result of Kakariadis [6, Theorem 3.3] and Dor-On and Salomon [4, Theorem 3.5] that characterize hyperrigid tensor algebras of discrete graphs as exactly those that are row-finite.

Example 3.8. Suppose Σ is a compact topological space and suppose that σ is an homeomorphism on Σ. We let $\sigma^* : f \mapsto f \circ \sigma$ denote the associated isomorphism on $C(\Sigma)$. We can define a topological graph $E_{\Sigma,\sigma}$ associated to Σ by setting $E^0_{\Sigma,\sigma} = E^1_{\Sigma,\sigma} = X$, $s = \mathrm{id}_X$, and $r = \sigma$. In this instance, for any $f \in C(E^0_{\Sigma,\sigma})$, $g \in C(E^1_{\Sigma,\sigma})$, and $x \in E^1_{\Sigma,\sigma}$,

$$\begin{aligned}\sigma^*(f)\langle 1, g\rangle(x) = f(\sigma(x))\langle 1, g\rangle(s(x)) &= f(x)\left(\sum_{e\in E^1 : s(e)=x} \overline{1}(e)g(e)\right)\\ &= f(\sigma(x))g(x) = f(r(x))g(x) = (f\cdot g)(x)\ .\end{aligned}$$

Thus, $\lambda(f) = \sigma^*(f)\langle 1, \cdot\rangle$, from which it follows that $\lambda^{-1}(\mathcal{K}(X(E_{\Sigma,\sigma}))) = C(\Sigma)$. In particular, $\Sigma = (E^1_{\Sigma,\sigma})_{\text{fin.}}$ and we conclude by Theorem 3.6 that all operator systems $S(C(E^0_{\Sigma,\sigma}), C(E^1_{\Sigma,\sigma}))$ are hyperrigid.

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