m -Isometric Composition Operators on Directed Graphs with One Circuit

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This paper is dedicated to Professor Jan Stochel on the occasion of his 70th birthday.

Abstract. The aim of this paper is to investigate m -isometric composition operators on directed graphs with one circuit. We establish a characterization of m -isometries and prove that complete hyperexpansivity coincides with 2-isometricity within this class. We discuss the m -isometric completion problem for unilateral weighted shifts and for composition operators on directed graphs with one circuit. The paper is concluded with an affirmative solution of the Cauchy dual subnormality problem in the subclass with circuit containing one element.

Mathematics Subject Classification. Primary 47B33; Secondary 47B37, 47B20.

Keywords. Composition operators, m -isometry, Complete hyperexpansivity, Completion problem.

1. Introduction

The notion of m -isometric operator was introduced by Agler in [4] and initially investigated by Agler and Stankus in [5]. Recently, there have been published many papers devoted to problems related to m -isometric operators (see [1, 6, 10–12, 18, 23]). In particular, much attention was paid to m -isometric unilateral and bilateral weighted shifts. Bermúdez et al. characterized m -isometric unilateral weighted shifts in [12, Theorem 3.4]. Later on, Abdullah and Le provided convenient characterizations of m -isometric unilateral and bilateral weighted shifts that bind their sequences of weights with certain real polynomials in one variable of degree at most $m - 1$ (see [1, Theorem 2.1, Theorem 5.2]).

Lately, more general classes of operators such as (m, p) -isometries (see [11, 18]) have been investigated. Both papers [11] and [18] deal with composition operators on ℓ_p for $p \geq 1$. Gu provided a characterization of (m, p) -isometric composition operators defined on ℓ_p that ties cardinality of preimages $\phi^{-n}(j)$ for $j \in \mathbb{N}$ with certain polynomials of degree at most $m - 1$

(see [18, Theorem 2.9]). The operators considered in the above papers can be seen as composition operators on discrete measure spaces with counting measures. As opposed to them, we are focused on composition operators on discrete measure spaces with arbitrary discrete measures (see Sect. 2).

The paper is devoted to investigation of classes of m -isometric operators which are more rich in examples than the classes of unilateral and bilateral weighted shifts. We focus on the class of composition operators on directed graphs with one circuit. It was used in [8, Section 3] in connection with the study of unbounded subnormality. Its weighted counterpart appeared much earlier in [14]. Moreover, these composition operators have been very popular lately. For example, Anand, Chavan and Trivedi utilized weighted composition operators of this type to provide an interesting example of analytic 3-isometric cyclic operator without the wandering subspace property (see [2, Example 3.1]). Although the operators belonging to this class can be, in general, unbounded and densely defined, we restrict ourselves to their subclass containing only bounded operators.

As shown in the paper, the class of m -isometric composition operators on directed graphs with one circuit is rich in strict m -isometries for $m \geq 2$. Surprisingly, it turns out that the only completely hyperexpansive composition operators in this class are 2-isometric (see Corollary 2.15). This phenomenon is a consequence of the fact that, in particular, 2-expansiveness implies 2-isometricity within this class. It is worth to mention that 3-isometries have been very popular recently (see e.g. [2, 10, 23]). Beyond the results already mentioned from [2], these operators were, for instance, utilized by Bermúdez et al. as examples to shed some light on relations of certain ergodic and dynamical properties of m -isometric operators (see [10] for more details). That is why we provide an explicit description of 3-isometric composition operators on directed graphs with one circuit that involves real polynomials in one variable (see Theorem 3.5).

The paper is organized as follows. Section 2 begins with setting up preliminary notation and definitions. Later on, discrete measure spaces and composition operators on directed graphs with one circuit are introduced (see the standing assumption (2.8)). Next several technical results required in the further proofs are given. Theorem 2.11, which is the main result of this section, characterizes m -isometric composition operators on directed graphs with one circuit. As a consequence, Corollary 2.12 provides a characterization of m -isometric operators in the subclass of these operators with the circuit containing only one element. Finally, Corollary 2.15 shows that the notion of complete hyperexpansivity coincides with the notion of 2-isometricity in this class.

In Sect. 3 we state and investigate the m -isometric completion problem for unilateral weighted shifts and composition operators on directed graphs with one circuit. At the beginning we provide a complete description of the existence of a solution of the m -isometric completion problem for unilateral weighted shifts (see Proposition 3.2). Proposition 3.4, whose proof essentially relies on the properties of circulant matrices, shows that, under some additional conditions on the length of the circuit, there exists a solution

of the m -isometric completion problem, if the measure of elements located on branches of ϕ is a priori given by polynomials of degree at most $m - 1$. Using this fact we establish a characterization of 2-isometric and 3-isometric composition operators on directed graphs with one circuit having arbitrary number of elements in the circuit (see Theorem 3.5). This result is used in the proof of Proposition 3.7 which characterizes the existence of a solution of another 2-isometric completion problem, in which we assume that measure of elements located in the circuit is given.

Section 4 is devoted to the study of subnormality of the Cauchy dual operators of 2-isometric composition operators on directed graphs with one circuit. We begin with a characterization of analyticity of composition operators on directed graphs with one circuit (see Proposition 4.2) and prove that m -isometric operators within this class are analytic (see Corollary 4.3). Theorem 4.6 gives the affirmative answer to the Cauchy dual subnormality problem for 2-isometric operators with circuit having one element. We conclude this section with two results related to 2-isometric composition operators on directed graphs with one circuit, namely, we state a characterization of Δ_{C_ϕ} -regularity of these operators (see Proposition 4.7) and show that they never satisfy the kernel condition (see Theorem 4.9).

2. Characterization of m -isometries

In this paper we use the following notation. Symbols $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$ and $\mathbb{R}_+$ stand for the sets of positive integers, nonnegative integers, integers and nonnegative reals, respectively. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. For $i, j \in \mathbb{Z} \cup \{\infty\}$ set $J_{[i,j]} = \{k \in \mathbb{Z} : i \leq k \leq j\}$. We adhere to the conventions that $\sum_{k \in \emptyset} \alpha_k = 0$ and $\prod_{k \in \emptyset} \alpha_k = 1$.

Define a linear transformation $\Delta : \mathbb{R}^{\mathbb{Z}_+} \rightarrow \mathbb{R}^{\mathbb{Z}_+}$ by

$$(\Delta \gamma)_n = \gamma_{n+1} - \gamma_n, \quad n \in \mathbb{Z}_+, \quad \gamma = \{\gamma_n\}_{n=0}^\infty \in \mathbb{R}^{\mathbb{Z}_+}.$$

Relations $\Delta^0 \gamma = \gamma$ and $\Delta^{j+1} \gamma = \Delta \Delta^j \gamma$ for $j \in \mathbb{Z}_+$ inductively define Δ^j for all $j \in \mathbb{Z}_+$. Using Newton's binomial formula, we can easily prove that

$$(\Delta^m \gamma)_n = (-1)^m \sum_{k=0}^m (-1)^k \binom{m}{k} \gamma_{n+k}, \quad m, n \in \mathbb{Z}_+, \quad \gamma = \{\gamma_n\}_{n=0}^\infty \in \mathbb{R}^{\mathbb{Z}_+}. \quad (2.1)$$

Let $\{\gamma_n\}_{n=0}^\infty \in \mathbb{R}^{\mathbb{Z}_+}$. We say that γ_n is a *polynomial in n (of degree k)* if there exists a polynomial p in one indeterminate x with real coefficients (of degree k) such that $p(n) = \gamma_n$ for all $n \in \mathbb{Z}_+$. As a consequence of the Fundamental Theorem of Algebra we see that, if exists, such p is unique.

The following well-known lemma plays a key role in our considerations (for more details see [21, Section 2]).

Lemma 2.1. *If $m \in \mathbb{N}$ and $\gamma = \{\gamma_n\}_{n=0}^\infty \in \mathbb{R}^{\mathbb{Z}_+}$, then the following conditions are equivalent:*

- (i) $\Delta^m \gamma = 0$,
- (ii) $\sum_{k=0}^m (-1)^k \binom{m}{k} \gamma_{n+k} = 0$ for $n \in \mathbb{Z}_+$,

(iii) γ_n is a polynomial in n of degree at most $m - 1$.

Corollary 2.2. *If $m \in \mathbb{N}$, p is a polynomial in one indeterminate x with coefficients in $\mathbb{R}$ of the form $p(x) = a_{m-1}x^{m-1} + \dots + a_0$ and $\gamma = \{p(n)\}_{n=0}^\infty \in \mathbb{R}^{\mathbb{Z}_+}$, then $(\Delta^{m-1}\gamma)_n = (m-1)!a_{m-1}$ for all $n \in \mathbb{Z}_+$. In particular, if $p(n) \geq 0$ for all $n \in \mathbb{Z}_+$, then $\Delta^{m-1}\gamma \subseteq \mathbb{R}_+$.*

Let $\mathcal{H}$ be a nonzero complex Hilbert space. A linear map $T: \mathcal{D}(T) \rightarrow \mathcal{H}$, where $\mathcal{D}(T)$ is a subspace of $\mathcal{H}$, is called an *operator* on $\mathcal{H}$. By $\mathbf{B}(\mathcal{H})$ we denote the $\mathcal{C}^*$ -algebra of all bounded linear operators defined on $\mathcal{H}$ and by I the identity operator on $\mathcal{H}$. For $T \in \mathbf{B}(\mathcal{H})$ let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ stand for the kernel and the range of operator T , respectively. If T is positive, then by $T^{\frac{1}{2}}$ we denote its positive square root. As usual, we write T^* for the adjoint of T . Recall that an operator $T \in \mathbf{B}(\mathcal{H})$ is *analytic* if $\bigcap_{n=1}^\infty T^n(\mathcal{H}) = \{0\}$.

For $T \in \mathbf{B}(\mathcal{H})$ we define

$$\mathcal{B}_m(T) = \sum_{k=0}^m (-1)^k \binom{m}{k} T^{*k} T^k, \quad m \in \mathbb{Z}_+.$$

Given $m \in \mathbb{N}$, we say that T is an *m -isometry* if $\mathcal{B}_m(T) = 0$. Following [13], we say that an m -isometry T is *strict* if T is an isometry or $m \geq 2$ and T is not an $(m-1)$ -isometry. An operator T is said to be *m -expansive* if $\mathcal{B}_m(T) \leq 0$ and *completely hyperexpansive*, if $\mathcal{B}_n(T) \leq 0$ for every $n \in \mathbb{N}$.

Let us recall an important property of $\mathcal{B}_m(T)$ (cf. [25, Lemma 1(a)]).

Proposition 2.3. [17, Theorem 2.5] *If $T \in \mathbf{B}(\mathcal{H})$, $m \geq 2$ is an integer and $(-1)^m \mathcal{B}_m(T) \leq 0$, then $(-1)^m \mathcal{B}_{m-1}(T) \leq 0$.*

For an operator $T \in \mathbf{B}(\mathcal{H})$ and $f \in \mathcal{H}$ we define the sequence $\gamma_{T,f} = \{(\gamma_{T,f})_n\}_{n=0}^\infty$ by

$$(\gamma_{T,f})_n = \|T^n f\|^2, \quad n \in \mathbb{Z}_+. \quad (2.2)$$

Observe that

$$\langle \mathcal{B}_m(T)f, f \rangle = \sum_{k=0}^m (-1)^k \binom{m}{k} (\gamma_{T,f})_k, \quad f \in \mathcal{H}, \quad m \in \mathbb{Z}_+. \quad (2.3)$$

The following fact is a consequence of (2.3), (2.1) and Lemma 2.1.

Proposition 2.4. *If $T \in \mathbf{B}(\mathcal{H})$ and $m \in \mathbb{N}$, then the following are equivalent:*

- (i) T is m -isometric,
- (ii) $\sum_{k=0}^m (-1)^k \binom{m}{k} (\gamma_{T,f})_k = 0$ for every $f \in \mathcal{H}$,
- (iii) $(\gamma_{T,f})_n$ is a polynomial in n of degree at most $m - 1$ for every $f \in \mathcal{H}$.

For the reader's convenience we recall the classification of all composition operators induced by self-maps of directed graphs having only one vertex with degree greater than one and all other vertices having degrees equal to one. It turns out that there are only two possible cases (see [8, Theorem 3.2.1]). The first case is related to the directed trees with one branching vertex and the second one is connected with the composition operators on directed graphs with one circuit (see Fig. 1 below). From now on we focus on the latter class of operators.

Let us now recall the definition and fundamental facts regarding composition operators. By a *discrete measure space* we mean a measure space $(X, \mathcal{A}, \mu)$, where X is a countably infinite set, $\mathcal{A}$ is the σ -algebra of all subsets of X and μ is a positive measure on $\mathcal{A}$ such that $\mu(x) := \mu(\{x\}) \in (0, \infty)$ for all $x \in X$. Suppose now that $(X, \mathcal{A}, \mu)$ is a measure space and a mapping $\phi: X \rightarrow X$ is measurable, i.e. $\phi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$. Denote by $\mu \circ \phi^{-1}$ the measure on $\mathcal{A}$ given by

$$\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta)), \quad \Delta \in \mathcal{A}.$$

We say that ϕ is *nonsingular* if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ . If ϕ is nonsingular, then the operator of the form $C_\phi f := f \circ \phi$ acting in $L^2(\mu) := L^2(X, \mathcal{A}, \mu)$ with the domain

$$\mathcal{D}(C_\phi) := \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}$$

is well-defined and it is called a *composition operator*. If $(X, \mathcal{A}, \mu)$ is a discrete measure space, then ϕ is automatically nonsingular. Suppose now that $(X, \mathcal{A}, \mu)$ is a discrete measure space. By the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$ -measurable function $h_\phi: X \rightarrow [0, \infty]$ such that

$$\mu \circ \phi^{-1}(\Delta) = \int_\Delta h_\phi d\mu, \quad \Delta \in \mathcal{A}.$$

Recall that

$$C_\phi \text{ is bounded if and only if } h_\phi \in L^\infty(\mu); \quad (2.4)$$

if this is the case, then $\|C_\phi\|^2 = \|h_\phi\|_{L^\infty(\mu)}$ (see [24, Theorem 1]). If $n \in \mathbb{N}$, then by ϕ^n we denote the n -fold composition of ϕ with itself; ϕ^0 is the identity mapping. It is easily seen that if ϕ is nonsingular and $n \in \mathbb{Z}_+$, then ϕ^n is also nonsingular, so h_{ϕ^n} makes sense; in particular $h_{\phi^0} \equiv 1$ and $h_{\phi^1} = h_\phi$. Note that if $C_\phi \in \mathcal{B}(L^2(\mu))$, then

$$\|C_\phi^n f\|^2 = \int_X |f \circ \phi^n|^2 d\mu = \int_X h_{\phi^n} |f|^2 d\mu, \quad f \in L^2(\mu), \quad n \in \mathbb{Z}_+. \quad (2.5)$$

Combining this, Proposition 2.3 and [20, Lemma 2.1] we get the following fact.

$$\begin{aligned} &\text{If } C_\phi \in \mathcal{B}(L^2(\mu)), \quad m \geq 2 \text{ is an integer and } (-1)^m \sum_{k=0}^m (-1)^k \binom{m}{k} h_{\phi^k} \leq 0, \\ &\text{then } (-1)^m \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} h_{\phi^k} \leq 0. \end{aligned} \quad (2.6)$$

The next result, which is a direct consequence of [20, Lemma 2.3(ii)], Lemma 2.1 and (2.5), provides a characterization of bounded m -isometric composition operators on discrete measure spaces.

Proposition 2.5. *If $m \in \mathbb{N}$, $(X, \mathcal{A}, \mu)$ is a discrete measure space and $C_\phi \in \mathcal{B}(L^2(\mu))$, then the following are equivalent:*

- (i) C_ϕ is an m -isometry,

FIGURE 1. The graph connected with a composition operator on a directed graph with one circuit

(ii) for all $x \in X$,

$$\sum_{k=0}^m (-1)^k \binom{m}{k} h_{\phi^k}(x) = 0, \quad (2.7)$$

- (iii) $\sum_{k=0}^m (-1)^k \binom{m}{k} h_{\phi^{n+k}}(x) = 0$ for all $x \in X$ and $n \in \mathbb{Z}_+$,
 (iv) $h_{\phi^n}(x)$ is a polynomial in n of degree at most $m-1$ for all $x \in X$.

Observe that, if $\text{card}(\phi^{-1}(x)) = 0$ for some $x \in X$, then C_{ϕ} is not an m -isometry, as condition (ii) from Proposition 2.5 is not satisfied.

Let us gather the following assumptions.

Suppose that $\kappa \in \mathbb{N}$, $\eta \in \mathbb{N} \cup \{\infty\}$ and

$$X = \{x_1, \dots, x_{\kappa}\} \cup \bigcup_{i=1}^{\eta} \{x_{i,j} : j \in \mathbb{N}\},$$

where $\{x_i\}_{i=1}^{\kappa}$ and $\{x_{i,j}\}_{i=1, j=1}^{\eta, \infty}$ are two disjoint systems of distinct points of X . Assume that $(X, \mathcal{A}, \mu)$ is a discrete measure space and a self-map ϕ of X is defined by

$$\phi(x) = \begin{cases} x_{i,j-1} & \text{if } x = x_{i,j} \text{ for some } i \in J_{[1,\eta]} \text{ and } j \in \mathbb{N} \setminus \{1\}, \\ x_{\kappa} & \text{if } x = x_{i,1} \text{ for some } i \in J_{[1,\eta]} \text{ or } x = x_1, \\ x_{i-1} & \text{if } x = x_i \text{ for some } i \in \{j \in \mathbb{N} : 2 \leq j \leq \kappa\}. \end{cases}$$

An operator C_{ϕ} defined on a discrete measure space $(X, \mathcal{A}, \mu)$ satisfying (2.8) is called a *composition operator on a directed graph with one circuit*.

For $\kappa \in \mathbb{N}$, denote by $\Phi_1: \mathbb{Z} \rightarrow \mathbb{Z}$ and $\Phi_2: \mathbb{Z} \rightarrow \{1, \dots, \kappa\}$ the functions uniquely determined by the following formula

$$n = \Phi_1(n)\kappa + \Phi_2(n), \quad n \in \mathbb{Z}.$$

It follows from the definition of functions Φ_1 and Φ_2 that

$$\begin{aligned} \Phi_1(l\kappa + 1) &= \Phi_1(l\kappa + r), \quad l \in \mathbb{Z}, r \in J_{[1,\kappa]}, \\ \Phi_2(l\kappa + r_1 + r_2) &= \Phi_2(l\kappa + r_1) + r_2, \quad l \in \mathbb{Z}, r_1 \in \mathbb{N}, r_2 \in \mathbb{Z}_+, r_1 + r_2 \leq \kappa. \end{aligned} \quad (2.9)$$

It is easily seen that the Radon-Nikodym derivative h_{ϕ} and $h_{\phi^n}(x)$ for $x \in \{x_{\kappa}\} \cup \{x_{i,j} : i \in J_{[1,\eta]}, j \in \mathbb{N}\}$ can be calculated as follows (cf. [8, (3.4.6)] &

(3.4.7))]:

$$h_\phi(x) = \begin{cases} \frac{\mu(x_1) + \sum_{i=1}^{\eta} \mu(x_{i,1})}{\mu(x_\kappa)} & \text{if } x = x_\kappa, \\ \frac{\mu(x_{r+1})}{\mu(x_r)} & \text{if } x = x_r \text{ for some } r \in J_{[1, \kappa-1]}, \\ \frac{\mu(x_{i,j+1})}{\mu(x_{i,j})} & \text{if } x = x_{i,j} \text{ for some } i \in J_{[1, \eta]}, j \in \mathbb{N}, \end{cases} \quad (2.10)$$

$$h_{\phi^n}(x_\kappa) = \frac{\mu(x_{\Phi_2(n)})}{\mu(x_\kappa)} + \sum_{i=1}^{\eta} \sum_{l=0}^{\Phi_1(n)} \frac{\mu(x_{i, l\kappa + \Phi_2(n)})}{\mu(x_k)}, \quad n \in \mathbb{Z}_+, \quad (2.11)$$

$$h_{\phi^n}(x_{i,j}) = \frac{\mu(x_{i, n+j})}{\mu(x_{i,j})}, \quad j \in \mathbb{N}, i \in J_{[1, \eta]}, n \in \mathbb{Z}_+. \quad (2.12)$$

Observe also that (cf. [8, (3.4.7)])

$$h_{\phi^n}(x_r) = \frac{\mu(x_1)}{\mu(x_r)} h_{\phi^{n+r-1}}(x_1), \quad r \in J_{[1, \kappa]}, n \in \mathbb{Z}_+. \quad (2.13)$$

Using (2.11) and (2.13) we can prove that

$$h_{\phi^n}(x_1) = \frac{\mu(x_{\Phi_2(n+1)})}{\mu(x_1)} + \sum_{i=1}^{\eta} \sum_{l=0}^{\Phi_1(n+1)-1} \frac{\mu(x_{i, l\kappa + \Phi_2(n+1)})}{\mu(x_1)}, \quad n \in \mathbb{Z}_+. \quad (2.14)$$

Now we collect some necessary facts needed in the proof of Theorem 2.10. First recall that (cf., [20, p. 524])

$$\text{if } m \in \mathbb{N} \text{ and } j \in J_{[1, m]}, \text{ then } \sum_{p=j}^m (-1)^p \binom{m}{p} = (-1)^j \binom{m-1}{j-1}. \quad (2.15)$$

The following result is used in the proof of Lemma 2.7.

Lemma 2.6. *If $\kappa, p \in \mathbb{N}$ and $\{a_n\}_{n=1}^p \subseteq \mathbb{R}$, then*

$$\sum_{r=1}^{\kappa} \sum_{l=0}^{\Phi_1(p+r)-1} a_{l\kappa + \Phi_2(p+r)} = \sum_{j=1}^p a_j. \quad (2.16)$$

Proof. Set

$$A = \{r \in \mathbb{Z}_+ : 1 \leq r \leq \kappa - \Phi_2(p)\},$$

$$B = \{r \in \mathbb{Z}_+ : \kappa - \Phi_2(p) + 1 \leq r \leq \kappa\},$$

and observe that (use equality $p = \Phi_1(p)\kappa + \Phi_2(p)$ and (2.9))

$$\begin{aligned} \Phi_1(p+r) &= \Phi_1(p), & \Phi_2(p+r) &= \Phi_2(p) + r, & r \in A, \\ \Phi_1(p+r) &= \Phi_1(p) + 1, & \Phi_2(p+r) &= \Phi_2(p) + r - \kappa, & r \in B, \end{aligned} \quad (2.17)$$

This implies that

$$\begin{aligned} \sum_{r \in B} \sum_{l=0}^{\Phi_1(p)} a_{l\kappa + \Phi_2(p+r)} &= \sum_{r \in B} \sum_{l=0}^{\Phi_1(p)} a_{(l-1)\kappa + \Phi_2(p)+r} \\ &= \sum_{r \in B} \sum_{l=0}^{\Phi_1(p)-1} a_{l\kappa + \Phi_2(p)+r} + \sum_{j=1}^{\Phi_2(p)} a_j, \end{aligned} \quad (2.18)$$

and consequently

$$\begin{aligned}
 & \sum_{r=1}^{\kappa} \sum_{l=0}^{\Phi_1(p+r)-1} a_{l\kappa+\Phi_2(p+r)} \\
 & \stackrel{(2.17)}{=} \sum_{r \in A} \sum_{l=0}^{\Phi_1(p)-1} a_{l\kappa+\Phi_2(p)+r} + \sum_{r \in B} \sum_{l=0}^{\Phi_1(p)} a_{l\kappa+\Phi_2(p+r)} \\
 & \stackrel{(2.18)}{=} \sum_{r=1}^{\kappa} \sum_{l=0}^{\Phi_1(p)-1} a_{l\kappa+\Phi_2(p)+r} + \sum_{j=1}^{\Phi_2(p)} a_j \\
 & = \sum_{j=1}^p a_j.
 \end{aligned}$$

This completes the proof. $\square$

Now we prove a technical lemma which is vital for later results.

Lemma 2.7. *Suppose that $m \in \mathbb{N}$, (2.8) holds and $\sum_{i=1}^{\eta} \sum_{j=1}^m \mu(x_{i,j}) < \infty$. Then*

$$\begin{aligned}
 \sum_{r=1}^{\kappa} \mu(x_r) \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r) &= - \sum_{i=1}^{\eta} \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} \mu(x_{i,p+1}) \\
 &= - \sum_{i=1}^{\eta} \mu(x_{i,1}) \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,1}).
 \end{aligned}$$

Proof. First observe that by our assumptions $h_{\phi^p}(x_r) \in \mathbb{R}_+$ for all $p \in J_{[1,m]}$ and $r \in J_{[1,\kappa]}$, and for $r \in J_{[1,\kappa]}$,

$$\begin{aligned}
 & \mu(x_r) \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r) \\
 & \stackrel{(2.13)}{=} \sum_{p=0}^m (-1)^p \mu(x_1) \binom{m}{p} h_{\phi^{p+r-1}}(x_1) \\
 & \stackrel{(2.14)}{=} \sum_{p=0}^m (-1)^p \binom{m}{p} \left(\mu(x_{\Phi_2(p+r)}) + \sum_{i=1}^{\eta} \sum_{l=0}^{\Phi_1(p+r)-1} \mu(x_{i,l\kappa+\Phi_2(p+r)}) \right).
 \end{aligned} \tag{2.19}$$

The latter, combined with the fact that

$$\sum_{r=1}^{\kappa} \mu(x_{\Phi_2(p+r)}) = \sum_{r=1}^{\kappa} \mu(x_r), \quad p \in J_{[0,m]},$$

yields

$$\sum_{r=1}^{\kappa} \mu(x_r) \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r)$$

$$\begin{aligned}
&= \sum_{i=1}^{\eta} \sum_{p=0}^m (-1)^p \binom{m}{p} \sum_{r=1}^{\kappa} \sum_{l=0}^{\Phi_1(p+r)-1} \mu(x_{i,l\kappa+\Phi_2(p+r)}) \\
&\stackrel{(2.16)}{=} \sum_{i=1}^{\eta} \sum_{p=1}^m (-1)^p \binom{m}{p} \sum_{j=1}^p \mu(x_{i,j}) \\
&= \sum_{i=1}^{\eta} \sum_{j=1}^m \sum_{p=j}^m (-1)^p \binom{m}{p} \mu(x_{i,j}) \\
&\stackrel{(2.15)}{=} \sum_{i=1}^{\eta} \sum_{j=1}^m (-1)^j \binom{m-1}{j-1} \mu(x_{i,j}) \\
&= - \sum_{i=1}^{\eta} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \mu(x_{i,j+1}) \\
&\stackrel{(2.12)}{=} - \sum_{i=1}^{\eta} \mu(x_{i,1}) \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} h_{\phi^j}(x_{i,1}).
\end{aligned}$$

This completes the proof. $\square$

Remark 2.8. Suppose that (2.8) holds. Observe that, if the composition operator C_ϕ is bounded, then $\sum_{i=1}^{\eta} \sum_{j=1}^k \mu(x_{i,j}) < \infty$ for any $k \in \mathbb{N}$. This is a consequence of (2.4), (2.11) and the fact that $C_\phi^k = C_{\phi^k}$ for any $k \in \mathbb{N}$.

The below proposition is the main tool used in the proof of Theorem 2.10.

Proposition 2.9. *Suppose $m \geq 2$, (2.8) holds, $\mu(x_{i,j+1})$ is a polynomial in j of degree at most $m-1$ for all $i \in J_{[1,\eta]}$ and $\sum_{i=1}^{\eta} \sum_{j=1}^m \mu(x_{i,j}) < \infty$. Then $\mu(x_{i,j+1})$ is a polynomial in j of degree at most $m-2$ for all $i \in J_{[1,\eta]}$ if and only if*

$$\sum_{r=1}^{\kappa} \mu(x_r) \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r) = 0. \quad (2.20)$$

Proof. Set $\gamma_i = \{\mu(x_{i,j+1})\}_{j=0}^{\infty}$ for $i \in J_{[1,\eta]}$. Applying Lemma 2.1 we deduce that $\Delta^m \gamma_i = 0$ for $i \in J_{[1,\eta]}$. Hence, by Corollary 2.2, there exists a system $\{a_i : i \in J_{[1,\eta]}\} \subseteq \mathbb{R}_+$ such that

$$(\Delta^{m-1} \gamma_i)_j = a_i, \quad j \in \mathbb{N}, \quad i \in J_{[1,\eta]}. \quad (2.21)$$

Using Lemma 2.7 we get

$$\begin{aligned}
\sum_{r=1}^{\kappa} \mu(x_r) \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r) &= - \sum_{i=1}^{\eta} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \mu(x_{i,j+1}) \\
&\stackrel{(2.1)}{=} (-1)^m \sum_{i=1}^{\eta} (\Delta^{m-1} \gamma_i)_0 = (-1)^m \sum_{i=1}^{\eta} a_i.
\end{aligned}$$

This combined with the fact that $a_i \geq 0$ for $i \in J_{[1,\eta]}$, (2.21) and Lemma 2.1 proves the required equivalence and completes the proof. $\square$

The next theorem states that if C_ϕ is a bounded m -isometric composition operator on a directed graph with one circuit, then measure on branches form, in fact, polynomials of degree at most $m - 2$.

Theorem 2.10. *Suppose that (2.8) holds, m is an integer such that $m \geq 2$ and $C_\phi \in \mathcal{B}(L^2(\mu))$ is an m -isometry. Then $\mu(x_{i,j+1})$ is a polynomial in j of degree at most $m - 2$ for all $i \in J_{[1,\eta]}$.*

Proof. Set $\gamma_i = \{\mu(x_{i,j+1})\}_{j=0}^\infty$ for $i \in J_{[1,\eta]}$. Applying Proposition 2.5, substituting $x = x_{i,j}$, $i \in J_{[1,\eta]}$, $j \in \mathbb{N}$, into (2.7) and using (2.12) and (2.1) we deduce that $\Delta^m \gamma_i = 0$ for all $i \in J_{[1,\eta]}$. Thus, by Lemma 2.1, $\mu(x_{i,j+1})$ is a polynomial in j of degree at most $m - 1$ for all $i \in J_{[1,\eta]}$. It follows from Proposition 2.5 and Remark 2.8 that (2.20) holds, so by Proposition 2.9 the proof is completed. $\square$

The following theorem, which is the main result of this section, establishes a characterization of m -isometric composition operators on directed graphs with one circuit.

Theorem 2.11. *Suppose (2.8) holds, m is an integer such that $m \geq 2$ and $C_\phi \in \mathcal{B}(L^2(\mu))$. Then the following conditions are equivalent:*

- (i) C_ϕ is an m -isometry,
- (ii) $\mu(x_{i,j+1})$ is a polynomial in j of degree at most $m - 2$ for all $i \in J_{[1,\eta]}$ and

$$\sum_{p=0}^m (-1)^p \binom{m}{p} \left(\mu(x_{\Phi_2(p+r)}) + \sum_{i=1}^{\eta} \sum_{l=0}^{\Phi_1(p+r)-1} \mu(x_{i,l\kappa+\Phi_2(p+r)}) \right) = 0, \quad r \in J_{[1,\kappa]}. \quad (2.22)$$

Proof. (i) $\Rightarrow$ (ii). Apply Theorem 2.10, Proposition 2.5 and (2.19).

(ii) $\Rightarrow$ (i). This can be proved by using (2.13), (2.19), (2.22), Proposition 2.5, Lemma 2.1 and the following observation: if a_j is a polynomial in j of degree at most k , then a_{j+n} is a polynomial in j of degree at most k for all $n \in \mathbb{Z}_+$. $\square$

Our next result is a consequence of Proposition 2.9 and Theorem 2.11.

Corollary 2.12. *Suppose that (2.8) holds with $\kappa = 1$, $C_\phi \in \mathcal{B}(L^2(\mu))$ and m is an integer such that $m \geq 2$. Then C_ϕ is an m -isometry if and only if $\mu(x_{i,j+1})$ is a polynomial in j of degree at most $m - 2$ for all $i \in J_{[1,\eta]}$. Moreover, if there exists $i_0 \in J_{[1,\eta]}$ such that $\mu(x_{i_0,j+1})$ is a polynomial in j of degree $m - 2$, then C_ϕ is a strict m -isometry.*

Note that, under the assumptions of Corollary 2.12, if $C_\phi \in \mathcal{B}(L^2(\mu))$ is m -isometric, then for every $t \in \mathbb{R}_+$, $C_\phi \in \mathcal{B}(L^2(\mu_t))$ is also m -isometric, where μ_t is the measure on $\mathcal{A}$ uniquely determined by

$$\mu_t(x) = \begin{cases} \mu(x) + t & \text{if } x = x_1, \\ \mu(x) & \text{otherwise.} \end{cases}$$

Below we show that bounded composition operators on directed graphs with one circuit separate the class of m -isometric operators for $m \geq 2$.

Example 2.13. Suppose (2.8) holds with $\kappa = 1$, $\eta \in J_{[1, \infty]}$ and $m \geq 2$ is an integer. Let q be a real polynomial of degree $m - 2$ such that all coefficients of q are nonnegative, and set $\mu(x_1) = 1$ and

$$\mu(x_{i,j}) = 2^{-i}q(j), \quad i \in J_{[1, \eta]}, j \in \mathbb{N}.$$

Observe that by (2.10),

$$h_\phi(x) = \begin{cases} 1 + q(1) \sum_{i=1}^{\eta} 2^{-i} & \text{if } x = x_1, \\ \frac{q(j+1)}{q(j)} & \text{if } x = x_{i,j} \text{ for some } i \in J_{[1, \eta]} \text{ and } j \in \mathbb{N}. \end{cases}$$

Since $\lim_{j \rightarrow \infty} \frac{q(j+1)}{q(j)} = 1$, this and (2.4) implies that $C_\phi \in \mathbf{B}(L^2(\mu))$. It follows from Corollary 2.12, that C_ϕ is strictly m -isometric.

The result below provides a sufficient condition for a bounded composition operator on a directed graph with one circuit to be an m -isometry.

Proposition 2.14. *Suppose (2.8) holds, $C_\phi \in \mathbf{B}(L^2(\mu))$ and $m \geq 2$ is an integer. If $(-1)^m \mathcal{B}_m(C_\phi) \leq 0$, then $\mathcal{B}_m(C_\phi) = 0$.*

Proof. Assume that $(-1)^m \mathcal{B}_m(C_\phi) \leq 0$. This, (2.5) and [20, Lemma 2.1] imply that

$$(-1)^m \sum_{k=0}^m (-1)^k \binom{m}{k} h_{\phi^k} \leq 0, \quad (2.23)$$

and hence, by (2.6),

$$(-1)^m \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} h_{\phi^k} \leq 0. \quad (2.24)$$

Using Lemma 2.7 we deduce that

$$\begin{aligned} 0 &\stackrel{(2.23)}{\geq} \sum_{r=1}^{\kappa} \mu(x_r) (-1)^m \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r) \\ &= - \sum_{i=1}^{\eta} \mu(x_{i,1}) (-1)^m \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,1}) \stackrel{(2.24)}{\geq} 0, \end{aligned}$$

and thus, it follows from (2.23) and (2.24) that

$$\sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_r) = 0, \quad r \in J_{[1, \kappa]}, \quad (2.25)$$

$$\sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,1}) = 0, \quad i \in J_{[1, \eta]}. \quad (2.26)$$

Observe now that

$$\begin{aligned}
 0 &\stackrel{(2.23)}{\geq} (-1)^m \sum_{p=0}^m (-1)^p \binom{m}{p} h_{\phi^p}(x_{i,j}) \\
 &= (-1)^m \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,j}) \\
 &\quad - (-1)^m \frac{\mu(x_{i,j+1})}{\mu(x_{i,j})} \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,j+1}), \quad i \in J_{[1,\eta]}, j \in \mathbb{N}.
 \end{aligned}$$

This and an induction argument on j implies that

$$\begin{aligned}
 0 &\stackrel{(2.24)}{\geq} (-1)^m \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,j+1}) \\
 &\geq (-1)^m \frac{\mu(x_{i,1})}{\mu(x_{i,j+1})} \sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,1}) \stackrel{(2.26)}{=} 0, \quad i \in J_{[1,\eta]}, j \in \mathbb{N},
 \end{aligned}$$

and consequently

$$\sum_{p=0}^{m-1} (-1)^p \binom{m-1}{p} h_{\phi^p}(x_{i,j+1}) = 0, \quad i \in J_{[1,\eta]}, j \in \mathbb{N}.$$

Combined with (2.25), (2.26) and Proposition 2.5, this completes the proof. $\square$

Since bounded 2-isometric operators are completely hyperexpansive and completely hyperexpansive operators are 2-expansive, the following corollary is an immediate consequence of Proposition 2.14.

Corollary 2.15. *If (2.8) holds and $C_\phi \in \mathbf{B}(L^2(\mu))$ then C_ϕ is completely hyperexpansive if and only if it is 2-isometric.*

Concerning Proposition 2.14, one may ask if $m \geq 2$ and $C_\phi \in \mathbf{B}(L^2(\mu))$ are such that $(-1)^m \mathcal{B}_m(C_\phi) \geq 0$ implies $\mathcal{B}_m(C_\phi) = 0$. The following example shows that this is not true.

Example 2.16. Assume (2.8) holds with $\kappa = 1$ and $\eta = 1$ and suppose that $m \geq 2$ is an integer. Let p be a polynomial of degree $m-1$ such that $p(j) > 0$ for all $j \in \mathbb{N}$. Set $\mu(x_{1,j}) = p(j)$ for $j \in \mathbb{N}$ and $\mu(x_1) = 1$. Then, by Corollary 2.12 the operator C_ϕ is strictly $(m+1)$ -isometric, in particular, it is not m -isometric. It follows from Proposition 2.3 that $(-1)^m \mathcal{B}_m(C_\phi) \geq 0$.

3. m -isometric completion problem

In this section we discuss some results related to the m -isometric completion problem. The counterparts of it for unilateral weighted shifts appeared in e.g. [1, 22]. In [22, Sections 4 and 5] authors studied completely hyperexpansive and 2-isometric completion problems for unilateral weighted shifts. Abdullah and Le proved that for each sequence $\{a_n\}_{n=1}^m \subset \mathbb{C} \setminus \{0\}$ there exists an

$(m+2)$ -isometric unilateral weighted shift with weight sequence starting with $\{a_n\}_{n=1}^m$ (see [1, Proposition 2.7]). In this section we focus also on the m -isometric completion problem for composition operators on directed graphs with one circuit.

For future reference we need a technical lemma.

Lemma 3.1. *Let $l \in \mathbb{Z}_+$ and $\{b_n\}_{n=0}^l \subseteq (0, \infty)$. Then there exists $c \in (0, \infty)$ such that for every $t \in [c, \infty)$ there exists a polynomial w_t in one indeterminate x with real coefficients of degree $l+1$ such that $w_t(n) = b_n$ for $n \in J_{[0,l]}$, $w_t(l+1) = t$ and $w_t(n) > 0$ for $n \in J_{[l+2,\infty]}$.*

Proof. We use induction on l . The case $l = 0$ is obvious. Assume that lemma holds for a fixed unspecified $l \in \mathbb{N}$ and let $\{b_n\}_{n=0}^{l+1} \subseteq (0, \infty)$. By the induction hypothesis there exists a polynomial v in one indeterminate x with real coefficients of degree $l+1$ such that $v(n) = b_n$ for $n \in J_{[0,l]}$, $v(l+1) \neq b_{l+1}$ and $v(n) > 0$ for $n \in J_{[l+1,\infty]}$. For $s \in [1, \infty)$, let

$$w_s(x) = v(x) + \left(s|\alpha|(x-l-1) + \alpha \right) \frac{\prod_{j=0}^l (x-j)}{\prod_{j=0}^l (l+1-j)}, \quad x \in \mathbb{R},$$

where $\alpha = b_{l+1} - v(l+1)$. It is easily seen that w_s is a polynomial in one indeterminate x with real coefficients of degree $l+2$ and for any $s \geq 1$, $w_s(n) = b_n$ for $n \in J_{[0,l+1]}$ and $w_s(n) > 0$ for $n \in J_{[l+2,\infty]}$. Since the function $\psi: [1, \infty) \ni s \rightarrow w_s(l+2) \in \mathbb{R}_+$ is continuous and $\lim_{s \rightarrow \infty} \psi(s) = +\infty$, the proof is completed. $\square$

For $k \in J_{[1,\infty]}$ define a transformation $\beta: \mathbb{R}^{J_{[1,k]}} \rightarrow \mathbb{R}^{J_{[0,k]}}$ by

$$(\beta\gamma)_n = \prod_{i=1}^n \gamma_i^2, \quad n \in J_{[0,k]}, \quad \gamma = \{\gamma_n\}_{n=1}^k \in \mathbb{R}^{J_{[1,k]}}.$$

Recall that an operator $S \in \mathcal{B}(\ell_2)$ is called a *unilateral weighted shift* with weights $\{\lambda_n\}_{n=1}^\infty \subseteq \mathbb{C} \setminus \{0\}$, if $Se_n = \lambda_{n+1}e_{n+1}$ for all $n \in \mathbb{Z}_+$, where $\{\lambda_n\}_{n=1}^\infty$ is a bounded sequence. It is a matter of a straightforward verification to see that these operators are bounded and injective. The reader should be aware that our notation for unilateral weighted shifts is different from the conventional one, i.e., $Se_n = \lambda_n e_{n+1}$.

Let $\mathcal{C}$ be a class of operators. For a classical weighted shift, the completion problem within the class $\mathcal{C}$ entails determining whether or not a given initial finite sequence of positive weights may be extended to the sequence of weights of an injective, bounded unilateral weighted shift which belongs to the class $\mathcal{C}$; such a shift is called a $\mathcal{C}$ class completion of the initial weight sequence.

Now we provide the solution of the completion problem within the class of m -isometries for the classical weighted shift (cf. [1, Proposition 2.7]). Since a unilateral weighted shift with weights $\{\lambda_n\}_{n=1}^\infty \subseteq \mathbb{C} \setminus \{0\}$ is unitarily equivalent to the shift with weight sequence $\{|\lambda_n|\}_{n=1}^\infty \subseteq (0, \infty)$, the following result can be generalized to unilateral weighted shifts with non-zero weights.

Proposition 3.2. *Let $m, k \in \mathbb{N}$ and let $\tilde{\lambda} := \{\lambda_n\}_{n=1}^k \subseteq (0, \infty)$. Then*

- (i) *if $k \leq m - 2$, then there exists $c > 0$ such that for every $r \in [c, \infty)$ the sequence $\{\lambda_n\}_{n=1}^{m-1}$ with an arbitrary sequence $\{\lambda_n\}_{n=k+1}^{m-2} \subseteq (0, \infty)$ and $\lambda_{m-1} = r$ admits a strict m -isometric completion,*
- (ii) *if $k > m - 2$, then $\tilde{\lambda}$ admits an m -isometric completion if and only if $\sum_{j=0}^m (-1)^j \binom{m}{j} (\beta \tilde{\lambda})_{n+j} = 0$ for $n \in J_{[0, k-m]}$ and the unique polynomial w of degree at most $m - 1$ which satisfies the conditions $w(n) = (\beta \tilde{\lambda})_n$ for $n \in J_{[0, m-1]}$, also satisfies $w(n) > 0$ for all $n \in J_{[m, \infty]}$. Moreover, this completion is strict if and only if the degree of w is equal to $m - 1$.*

Proof. (i). If $k < m - 2$, let us fix an arbitrary sequence $\{\lambda_n\}_{n=k+1}^{m-2} \subseteq (0, \infty)$. Let $\hat{\lambda} := \{\lambda_n\}_{n=1}^{m-2}$. It follows from Lemma 3.1 applied to the sequence $\{(\beta \hat{\lambda})_n\}_{n=0}^{m-2}$ that there exists $d \in (0, \infty)$ such that for every $t \in [d, \infty)$ there exists a polynomial w_t in one indeterminate x with real coefficients of degree $m - 1$ such that $w_t(n) = (\beta \hat{\lambda})_n$ for $n \in J_{[0, m-2]}$, $w_t(m-1) = t$ and $w_t(n) > 0$ for $n \in J_{[m, \infty]}$. Let $\lambda_n = \sqrt{\frac{w_t(n)}{w_t(n-1)}}$ for $n \in J_{[m-1, \infty]}$ and let S be the unilateral weighted shift with weights $\lambda := \{\lambda_n\}_{n=1}^\infty$. Since $\gamma_{S, e_0} = \{w_t(n)\}_{n=0}^\infty$ (see (2.2) for the definition of γ_{S, e_0}), it follows from [1, Theorem 2.1] that S is a strict m -isometric completion of the weight sequence $\{\lambda_n\}_{n=1}^k$. Now it is a routine matter to check that (i) holds.

(ii). In view of [1, Theorem 2.1] and the uniqueness part of the Lagrange interpolation formula (see [16, Section 2.5]), $\tilde{\lambda}$ admits an m -isometric completion if and only if there exists a polynomial w in one indeterminate x with real coefficients of degree at most $m - 1$ such that $w(n) = (\beta \tilde{\lambda})_n$ for $n \in J_{[0, m-1]}$ and $w(n) > 0$ for $n \in J_{[m, \infty]}$. Now, Lemma 2.1 completes the proof.

The moreover part follows directly from [1, Corollary 2.3]. $\square$

We present an example which shows that for each $m \geq 3$ there exists $\{\lambda_n\}_{n=1}^{m-1} \subseteq (0, \infty)$ such that the m -isometric completion problem does not have a solution (cf. [1, Remark 2.8]).

Example 3.3. First, assume that $m \geq 3$ is an odd number. Let

$$w_m(x) = -\frac{(x-m)^{m-2}(x+m-2)}{m^{m-2}(m-2)}.$$

Since $w_m(i) > 0$ for $i = 0, \dots, m-1$, the sequence $\{\lambda_n\}_{n=1}^{m-1}$ given by the formula

$$\lambda_i = \sqrt{\frac{w_m(i)}{w_m(i-1)}}, \quad i \in J_{[1, m-1]},$$

is well-defined and consists of positive real numbers. This and the fact that $w_m(0) = 1$ imply that

$$w_m(n) = \prod_{i=1}^n \lambda_i^2, \quad n \in J_{[0, m-1]}.$$

Combined with Proposition 3.2(ii) and $w_m(m) = 0$, this implies that there is no solution of the m -isometric completion problem for $\{\lambda_n\}_{n=1}^{m-1}$.

If m is even, then it is enough to consider

$$w_m(x) = -\frac{(x-m)^{m-1}}{m^{m-1}}.$$

We leave the details to the reader.

Now we concentrate on the completion problem for m -isometric composition operators on directed graphs with one circuit.

Proposition 3.4. *Suppose $m, \kappa \in \mathbb{Z}$ are such that $\kappa > m \geq 2$. Let $\eta, X, \mathcal{A}$ and ϕ be as in (2.8), $M \in (0, \infty)$ and $\{w_i\}_{i=1}^\eta$ be a system of polynomials of degree at most $m-2$ such that $w_i(j) > 0$ for all $i \in J_{[1, \eta]}$ and $j \in \mathbb{N}$ and*

$$\max \left\{ \sum_{i=1}^{\eta} w_i(1), \sup \left\{ \frac{w_i(j+1)}{w_i(j)} : i \in J_{[1, \eta]}, j \in \mathbb{N} \right\} \right\} \leq M. \quad (3.1)$$

Then there exists a measure μ on $\mathcal{A}$ such that $\mu(x_{i,j}) = w_i(j)$ for $i \in J_{[1, \eta]}$, $j \in \mathbb{N}$ and the associated $C_\phi \in \mathbf{B}(L^2(\mu))$ is an m -isometry. Moreover, there exists $t_0 \in \mathbb{R}$ such that all measures μ_t having the above properties can be parameterized by $t \in (t_0, \infty)$, where

$$\mu_t(x) := \begin{cases} \mu(x) + t & \text{if } x = x_j \text{ with } j \in J_{[1, \kappa]}, \\ \mu(x) & \text{otherwise.} \end{cases}$$

Proof. Define $\mu(x_{i,j}) = w_i(j)$ for $i \in J_{[1, \eta]}$ and $j \in \mathbb{N}$. In view of (2.4), (2.10) and Theorem 2.11 any solution $[\mu(x_1), \dots, \mu(x_\kappa)]^T$ of (2.22) fulfills our requirements. Observe that (2.22) can be written in the following matrix form

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{\kappa-2} & a_{\kappa-1} \\ a_{\kappa-1} & a_0 & a_1 & & a_{\kappa-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2 & & \ddots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{\kappa-1} & a_0 \end{bmatrix} \begin{bmatrix} \mu(x_1) \\ \mu(x_2) \\ \vdots \\ \vdots \\ \mu(x_\kappa) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_k \end{bmatrix}, \quad (3.2)$$

where $a_p = (-1)^p \binom{m}{p}$ for $p \in J_{[0, m]}$, $a_p = 0$ for $p > m$ and

$$b_r = \sum_{p=0}^m (-1)^{p+1} \binom{m}{p} \sum_{i=1}^{\eta} \sum_{l=0}^{\Phi_1(p+r)-1} \mu(x_{i, l\kappa + \Phi_2(p+r)}), \quad r \in J_{[1, \kappa]}.$$

Since the matrix in (3.2), call it A , is circulant, it follows from [19, Proposition 1.1] that the rank of A is equal to $\kappa-1$ because the associated polynomial of A is $(x-1)^m$ and the degree of the greatest common divisor of $1-x^\kappa$ and $(x-1)^m$ is equal to 1.

We claim that the rank of the augmented matrix of (3.2) is equal to $\kappa-1$. To prove this assume that $\mu(x_r) = 1$ for every $r \in J_{[1, \kappa]}$. It follows from (2.4) and (2.11) that C_ϕ is bounded. Thus, by Remark 2.8, the assumptions of Proposition 2.9 are satisfied. Hence (2.20) holds with $\mu(x_r) = 1$ for every

$r \in J_{[1,\kappa]}$. Now (2.19) implies that the sum of the rows of the augmented matrix of (3.2) is equal to 0. Thus the range of the augmented matrix is at most $\kappa - 1$. Combined with the fact that the rank of A is equal to $\kappa - 1$, this proves our claim.

By the Rouché-Capelli theorem there exists a solution of (3.2) and the solutions of the system (3.2) form a one-dimensional affine subspace of $\mathbb{R}^\kappa$. Note that the vector $v = [t, \dots, t]$ for $t \in \mathbb{R}$ satisfies the equality $Av = 0$. The proof is complete. $\square$

By using similar reasoning we prove the following useful characterization of 2-isometric and 3-isometric composition operators on directed graphs with one circuit, which provides an explicit description of the measure μ on X .

Theorem 3.5. *Suppose (2.8) holds with $\kappa > 1$. Then the following are equivalent:*

- (i) $C_\phi \in \mathcal{B}(L^2(\mu))$ is 3-isometric,
- (ii) *there exist $M, t \in (0, \infty)$, a system of polynomials $\{w_i\}_{i=1}^\eta$ of degree at most 1 and two systems $\{c_i\}_{i=1}^\eta \subseteq (0, \infty)$ and $\{d_i\}_{i=1}^\eta \subseteq \mathbb{R}_+$ such that (3.1) holds, $c := \sum_{i=1}^\eta c_i < \infty$, $d := \sum_{i=1}^\eta d_i < \infty$ and*

$$\begin{aligned}\mu(x_{i,j}) &= w_i(j) = c_i + d_i(j-1), \quad i \in J_{[1,\eta]}, j \in \mathbb{N}, \\ \mu(x_i) &= w_{c,d}^{\kappa,t}(i), \quad i \in J_{[1,\kappa]},\end{aligned}\tag{3.3}$$

where $w_{c,d}^{\kappa,t}$ is a polynomial of degree at most 2 given by

$$w_{c,d}^{\kappa,t}(x) = \frac{d}{2\kappa}x^2 + \left(\frac{c}{\kappa} - \frac{\kappa+2}{2\kappa}d\right)x - c + d + t, \quad x \in \mathbb{R}.\tag{3.4}$$

Moreover, C_ϕ is 2-isometric if and only if (ii) holds with a system of polynomials $\{w_i\}_{i=1}^\eta$ of degree equal to 0 and with $d_i = 0$ for all $i \in J_{[1,\eta]}$.

Proof. Let us first assume that $\kappa \geq 4$. In view of Proposition 3.4 and its proof it is enough to show that a vector $[\mu(x_i)]_{i=1}^\kappa$ given by (3.3) is a solution of (3.2). This can be proved by using the fact that $\Delta^3 w_{c,d}^{\kappa,t} = 0$. It is a matter of direct verification that our result is true for $\kappa = 2, 3$. $\square$

Using the above tools we are ready to prove some results regarding the m -isometric completion problem for composition operators on directed graphs with one circuit. One can think of many ways to state the m -isometric completion problem in the case of these composition operators, thus we limit our considerations to only a selected number of possibilities.

First we begin with the simplest situation, namely, when the circuit has only one element. In this situation we can prove a result, similar in its nature, to the analogous completion problem for unilateral weighted shifts.

Proposition 3.6. *Let $m \in \mathbb{N}$ and assume that $\{a_n\}_{n=1}^m \subseteq (0, \infty)$. Then there exist a discrete measure space $(X, \mathcal{A}, \mu)$ and a self-map $\phi: X \rightarrow X$ satisfying (2.8) with $\kappa = 1$ and $\eta = 1$ such that $\mu(x_{1,n}) = a_n$ for $n \in J_{[1,m]}$ and C_ϕ is an $(m+2)$ -isometry. Moreover, if $m = 1$, then $(X, \mathcal{A}, \mu)$ and $\phi: X \rightarrow X$ can be chosen so that C_ϕ is a 2-isometry.*

Proof. If $m \geq 2$, then we use Lemma 3.1 for $\{a_n\}_{n=1}^m$ and Corollary 2.12. If $m = 1$, then the result follows directly from Corollary 2.12. $\square$

Let us now prove another result regarding the 2-isometric completion problem in which we assume that measure of elements located in the circuit is given a priori.

Proposition 3.7. *Suppose that $\kappa > 1$, $\eta \in \mathbb{N} \cup \{\infty\}$ and let $a := \{a_n\}_{n=1}^\kappa \subseteq (0, \infty)$. Then the following are equivalent:*

- (i) *there exist a discrete measure space $(X, \mathcal{A}, \mu)$ and a self-map $\phi: X \rightarrow X$ satisfying (2.8) such that $\mu(x_n) = a_n$ for $n \in J_{[1, \kappa]}$ and $C_\phi \in \mathcal{B}(L^2(\mu))$ is a 2-isometry,*
- (ii) *$\{(\Delta a)_n\}_{n=1}^{\kappa-1}$ is a constant sequence such that $(\Delta a)_1 > 0$.*

Proof. (i) $\Rightarrow$ (ii). This implication is a straightforward application of Theorem 3.5. Indeed, if C_ϕ is a 2-isometry, then equation (3.4) implies that $\mu(x_i) = \frac{c}{\kappa}i - c + t$ for $i \in J_{[1, \kappa]}$, where $c = \sum_{i=1}^\eta \mu(x_{i,1})$. Thus $\{(\Delta a)_n\}_{n=1}^{\kappa-1}$ is a constant sequence and $(\Delta a)_1 = \mu(x_2) - \mu(x_1) = \frac{c}{\kappa} > 0$.

(ii) $\Rightarrow$ (i). Define X and $\mathcal{A}$ as in (2.8). Let μ be a discrete measure on $\mathcal{A}$ such that

$$\mu(x) = \begin{cases} a_n & \text{if } x = x_n \text{ for some } n \in J_{[1, \kappa]}, \\ r_i(\Delta a)_1 & \text{if } x = x_{i,j} \text{ for some } i \in J_{[1, \eta]} \text{ and } j \in \mathbb{N}, \end{cases} \quad (3.5)$$

where

$$r_i := \begin{cases} 2^{-i}\kappa & \text{if } \eta = \infty, \\ \frac{\kappa}{\eta} & \text{if } \eta < \infty. \end{cases} \quad (3.6)$$

Now, set $c = \kappa(\Delta a)_1$ and note that (3.5) combined with (3.6) imply that $c = \sum_{i=1}^\eta \mu(x_{i,1})$. Define $t = a_1 + c(1 - \frac{1}{\kappa})$ and $w_{c,d}^{\kappa,t}$ as in (3.4) with $d = 0$. Then, since $\{(\Delta a)_n\}_{n=1}^{\kappa-1}$ is a constant sequence,

$$\begin{aligned} \mu(x_i) &\stackrel{(3.5)}{=} a_i = a_1 + (\Delta a)_1(i-1) = a_1 + \frac{c}{\kappa}i - \frac{c}{\kappa} = t - c(1 - \frac{1}{\kappa}) + \frac{c}{\kappa}i - \frac{c}{\kappa} \\ &= \frac{c}{\kappa}i - c + t = w_{c,d}^{\kappa,t}(i), \quad i \in J_{[1, \kappa]}. \end{aligned}$$

Theorem 3.5 implies that C_ϕ is a 2-isometry. $\square$

Let us note that, if $\eta > 1$ in Proposition 3.7, then, if exists, the 2-isometric operator that solves the completion problem is not unique. Indeed, let $(X, \mathcal{A}, \mu)$ be as in Proposition 3.7(i). For $t \in (0, 1)$ define a measure $\nu_t: \mathcal{A} \rightarrow (0, \infty)$ in the following way:

$$\nu_t(x) = \begin{cases} \mu(x_{1,j}) + t\mu(x_{2,j}) & \text{if } x = x_{1,j} \text{ for some } j \in \mathbb{N}, \\ (1-t)\mu(x_{2,j}) & \text{if } x = x_{2,j} \text{ for some } j \in \mathbb{N}, \\ \mu(x) & \text{otherwise.} \end{cases}$$

It is an easy observation that C_ϕ satisfies Proposition 3.7(i) with ν_t in place of μ .

4. Subnormality of the Cauchy dual of a 2-isometry

We begin this section with characterization of analytic composition operators on directed graphs with one circuit. The equivalence (i) $\Leftrightarrow$ (ii) of the below lemma is a counterpart of a known result that describes $\mathcal{R}^\infty(C_\phi) := \bigcap_{n=1}^\infty C_\phi^n(L^2(\mu))$ for a composition operator C_ϕ (cf. [9, Remark 45] and [2, Lemma 2.3]).

Lemma 4.1. *Suppose (2.8) holds, $C_\phi \in \mathcal{B}(L^2(\mu))$ and $f \in L^2(\mu)$. Then the following are equivalent:*

- (i) $f \in \mathcal{R}^\infty(C_\phi)$,
- (ii) $f(x_r) = f(x_{i,l\kappa+r})$ for all $r \in J_{[1,\kappa]}$, $i \in J_{[1,\eta]}$ and $l \in \mathbb{Z}_+$,
- (iii) $C_\phi^\kappa f = f$.

Proof. First observe that

$$\phi^{r+n}(x_{i,n}) = x_{\kappa-r}, \quad n \in \mathbb{N}, \quad r \in J_{[0,\kappa-1]}, \quad i \in J_{[1,\eta]}, \quad (4.1)$$

$$\phi^{l\kappa}(x_r) = x_r, \quad l \in \mathbb{Z}_+, \quad r \in J_{[1,\kappa]}. \quad (4.2)$$

(i) $\Rightarrow$ (ii). Assume that $f \in \mathcal{R}^\infty(C_\phi)$ and fix $l \in \mathbb{Z}_+$. Then there exists $g \in L^2(\mu)$ such that

$$f = C_\phi^{\kappa(l+1)} g. \quad (4.3)$$

Thus, for $r \in J_{[1,\kappa]}$ and $i \in J_{[1,\eta]}$ we get that

$$\begin{aligned} f(x_{i,l\kappa+r}) &\stackrel{(4.3)}{=} (C_\phi^{\kappa(l+1)} g)(x_{i,l\kappa+r}) = g(\phi^\kappa(x_{i,r})) \\ &\stackrel{(4.1) \& (4.2)}{=} g(x_r) \stackrel{(4.2)}{=} g(\phi^{\kappa(l+1)}(x_r)) = (C_\phi^{\kappa(l+1)} g)(x_r) \stackrel{(4.3)}{=} f(x_r), \end{aligned}$$

which yields (ii).

(ii) $\Rightarrow$ (iii). This implication is a consequence of (4.2) with $l = 0$ and the following equality

$$\phi^\kappa(x_{i,l\kappa+r}) = \begin{cases} x_{i,\kappa(l-1)+r} & \text{if } l \in \mathbb{N} \text{ and } r \in J_{[1,\kappa]}, \\ x_r & \text{if } l = 0 \text{ and } r \in J_{[1,\kappa-1]}, \end{cases}$$

which holds for all $l \in \mathbb{Z}_+$ and $r \in J_{[1,\kappa]}$.

(iii) $\Rightarrow$ (i). It is obvious. $\square$

Note that in the case of finite measure spaces $C_\phi \in \mathcal{B}(L^2(\mu))$ associated with $(X, \mathcal{A}, \mu)$ and ϕ given by (2.8) is not analytic. Indeed, setting $f \equiv 1$ we deduce from Lemma 4.1 that $f \in L^2(\mu) \cap \mathcal{R}^\infty(C_\phi)$, which contradicts analyticity of C_ϕ . Using the above idea we can characterize analyticity of bounded composition operators on directed graphs with one circuit in the following way.

Proposition 4.2. *Suppose (2.8) holds and $C_\phi \in \mathcal{B}(L^2(\mu))$. Then the following are equivalent:*

- (i) C_ϕ is analytic,
- (ii) $\sum_{i=1}^\eta \sum_{l=0}^\infty \mu(x_{i,l\kappa+r})$ is divergent for all $r \in J_{[1,\kappa]}$.

Proof. (i) $\Rightarrow$ (ii). Suppose, to the contrary, that $\sum_{i=1}^{\eta} \sum_{l=0}^{\infty} \mu(x_{i,l\kappa+r}) < \infty$ for some $r \in J_{[1,\kappa]}$. Set

$$f_r(x) = \begin{cases} 1 & \text{if } x \in \{x_r\} \cup \{x_{i,l\kappa+r} : i \in J_{[1,\eta]}, l \in \mathbb{Z}_+\}, \\ 0 & \text{otherwise,} \end{cases} \quad x \in X. \quad (4.4)$$

Then, in view of Lemma 4.1, function $f \in L^2(\mu) \cap \mathcal{R}^\infty(C_\phi)$, which contradicts analyticity of C_ϕ .

(ii) $\Rightarrow$ (i). Assume that (ii) holds and $f \in \mathcal{R}^\infty(C_\phi)$. Using Lemma 4.1(ii) we can represent f as $f = \sum_{r=1}^{\kappa} f(x_r)f_r$, where f_r is given by (4.4) for every $r \in J_{[1,\kappa]}$. This, the facts that the supports of functions f_r , $r \in J_{[1,\kappa]}$ are disjoint and $f \in L^2(\mu)$ together with (ii) imply that $f \equiv 0$. $\square$

The following important result is a direct consequence of Theorem 2.10 and Proposition 4.2.

Corollary 4.3. *If (2.8) holds and $C_\phi \in \mathcal{B}(L^2(\mu))$ is an m -isometry for some $m \geq 2$, then C_ϕ is analytic.*

In what follows we investigate whether the Cauchy dual operator of a composition operator on a directed graph with on circuit is subnormal. The notion of the Cauchy dual operator was introduced by Shimorin in [26] on the occasion of a study of the Wold-type decomposition and the wandering subspace property. Recently, these topics have been very popular and authors established interesting results (see e.g. [2,3,6]). In particular, there is still an open problem of determining a characterization of the subnormality of the Cauchy dual operator of a 2-isometry (for a preliminary version of this problem, see [15]). Anand et al. provided two sufficient conditions under which 2-isometric operator satisfies the above property, namely, the kernel condition (see [3, Theorem 3.3]) and Δ_T -regularity (see [3, Theorem 4.5]). In what follows, we also characterize when composition operators on directed graphs with one circuit satisfy these properties.

Let us recall that for a left-invertible operator $T \in \mathcal{B}(\mathcal{H})$ the Cauchy dual operator T' of T is given by $T' = T(T^*T)^{-1}$. It is well-known (see [27]) that, if $C_\phi \in \mathcal{B}(L^2(\mu))$ is a composition operator, then

$$C_\phi^* C_\phi f = h_\phi f, \quad f \in L^2(\mu), \quad (4.5)$$

and consequently, if moreover C_ϕ is left-invertible, then C'_ϕ is a weighted composition operator with symbol ϕ and weight $w_\phi := \frac{1}{h_\phi \circ \phi}$, i.e.,

$$C'_\phi f = \frac{1}{h_\phi \circ \phi} \cdot C_\phi f, \quad f \in L^2(\mu). \quad (4.6)$$

In what follows we use the following notation. If $(X, \mathcal{A}, \mu)$ is a discrete measure space and $w: X \rightarrow (0, \infty)$ is a function, then by μ_w and $\{\hat{w}_n\}_{n=0}^\infty$ we denote a discrete measure on $\mathcal{A}$ and a sequence of functions uniquely determined by

$$\mu_w(x) = |w(x)|^2 \mu(x), \quad x \in X \quad (4.7)$$

and

$$\hat{w}_0 \equiv 1, \quad \hat{w}_{n+1} = \prod_{j=0}^n w \circ \phi^j, \quad n \in \mathbb{Z}_+. \quad (4.8)$$

By $h_{\phi, w}$ we denote the Radon-Nikodym derivative $\frac{d\mu_w \circ \phi^{-1}}{d\mu}$.

We gather below some necessary notation and properties of the Cauchy dual operator C'_ϕ of C_ϕ , which are used later.

Lemma 4.4. *If (2.8) holds with $\kappa = 1$, $C_\phi \in \mathcal{B}(L^2(\mu))$ is a 2-isometry and $w := w_\phi$, then*

$$\begin{aligned} \text{(i)} \quad \hat{w}_n(x) &= \begin{cases} \alpha^n & \text{if } x \in \{x_1\} \cup \{x_{i,1} : i \in J_{[1,\eta]}\}, \\ \alpha^{(n+1-j)} & \text{if } x \in \{x_{i,j} : i \in J_{[1,\eta]}, j \in J_{[2,n]}\}, x \in X, n \in \mathbb{Z}_+ \\ 1 & \text{otherwise,} \end{cases} \\ \text{(ii)} \quad h_{\phi^n, \hat{w}_n}(x) &= \begin{cases} \frac{\alpha^{2n} \mu(x_1) + c \sum_{j=1}^n \alpha^{2(n+1-j)}}{\mu(x_1)} & \text{if } x = x_1, \quad x \in X, n \in \mathbb{Z}_+, \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha = \frac{\mu(x_1)}{\mu(x_1)+c}$ and $c = \sum_{i=1}^\eta \mu(x_{i,1})$.

Proof. It follows from Corollary 2.12 that there exists a system $\{c_i\}_{i=1}^\eta \subseteq (0, \infty)$ such that

$$\mu(x_{i,j}) = c_i \quad i \in J_{[1,\eta]}, j \in \mathbb{N}. \quad (4.9)$$

(i) A direct computation shows that (i) holds for $n = 0, 1$. Using (4.9), for $k \in \mathbb{N}$ and $x \in X$ we get

$$w_\phi \circ \phi^k(x) = \begin{cases} \alpha & \text{if } x \in \{x_1\} \cup \{x_{i,j} : i \in J_{[1,\eta]}, j \in J_{[2,k+1]}\}, \\ 1 & \text{otherwise.} \end{cases}$$

This and induction argument show that (i) holds for $n \geq 2$.

(ii) Let us recall that, by [7, eq. (6.5)],

$$h_{\phi^n, \hat{w}_n}(x) = \frac{\mu_{\hat{w}_n}((\phi^n)^{-1}(\{x\}))}{\mu(x)}, \quad x \in X.$$

This implies that

$$\begin{aligned} h_{\phi^n, \hat{w}_n}(x_{i,j}) &= \frac{\mu_{\hat{w}_n}(x_{i,j+n})}{\mu(x_{i,j})} \\ &\stackrel{(4.7)}{=} \frac{\hat{w}_n(x_{i,j+n})^2 \mu(x_{i,j+n})}{\mu(x_{i,j})} \stackrel{\text{(i) \& (4.9)}}{=} 1, \quad i \in J_{[1,\eta]}, j \in \mathbb{N}. \end{aligned}$$

Similarly,

$$\begin{aligned}
 h_{\phi^n, \hat{w}_n}(x_1) &= \frac{\mu_{\hat{w}_n}(\{x_1\} \cup \{x_{i,j} : i \in J_{[1,n]}, j = 1, \dots, n\})}{\mu(x_1)} \\
 &\stackrel{(4.7)}{=} \frac{\hat{w}_n(x_1)^2 \mu(x_1) + \sum_{i=1}^n \sum_{j=1}^n \hat{w}_n(x_{i,j})^2 \mu(x_{i,j})}{\mu(x_1)} \\
 &\stackrel{(4.9) \& (i)}{=} \frac{\alpha^{2n} \mu(x_1) + \left(\sum_{i=1}^n c_i \right) \sum_{j=1}^n \alpha^{2(n+1-j)}}{\mu(x_1)} \\
 &= \frac{\alpha^{2n} \mu(x_1) + c \sum_{j=1}^n \alpha^{2(n+1-j)}}{\mu(x_1)}.
 \end{aligned}$$

□

Now we provide a characterization of subnormality of the Cauchy dual operator of a composition operator on a directed graph with one circuit.

Theorem 4.5. *If (2.8) holds and $C_\phi \in \mathcal{B}(L^2(\mu))$, then the following conditions are equivalent:*

- (i) C'_ϕ is subnormal,
- (ii) $\{h_{\phi^n, \hat{w}_n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for all $x \in X$, where $w = w_\phi$ and $\{\hat{w}_n\}_{n=0}^\infty$ is given by (4.8).

Proof. The proof follows directly from (4.6) and [7, Theorem 49]. □

The below theorem is the main result of this section. It answers affirmatively the question whether the Cauchy dual operator of 2-isometric composition operator on a directed graph with one circuit with $\kappa = 1$ is subnormal.

Theorem 4.6. *If (2.8) holds with $\kappa = 1$ and $C_\phi \in \mathcal{B}(L^2(\mu))$ is a 2-isometry, then C'_ϕ is subnormal.*

Proof. By Lemma 4.4 and Theorem 4.5, it remains to prove that sequence $\left\{ \frac{\alpha^{2n} \mu(x_1) + c \sum_{j=1}^n \alpha^{2(n+1-j)}}{\mu(x_1)} \right\}_{n=0}^\infty$ is a Stieltjes moment sequence, where α and c are as in Lemma 4.4. Observe that

$$\begin{aligned}
 \frac{\alpha^{2n} \mu(x_1) + c \sum_{j=1}^n \alpha^{2(n+1-j)}}{\mu(x_1)} &= \alpha^{2n} + \frac{c \alpha^2}{\mu(x_1)} \frac{\alpha^{2n} - 1}{\alpha^2 - 1} \\
 &= \frac{\mu(x_1) + c}{2\mu(x_1) + c} \alpha^{2n} + \frac{\mu(x_1)}{2\mu(x_1) + c}, \quad n \in \mathbb{Z}_+.
 \end{aligned}$$

Hence $\left\{ \frac{\alpha^{2n} \mu(x_1) + c \sum_{j=1}^n \alpha^{2(n+1-j)}}{\mu(x_1)} \right\}_{n=0}^\infty$ is a Stieltjes moment sequence with the representing measure $\frac{\mu(x_1) + c}{2\mu(x_1) + c} \delta_{\alpha^2} + \frac{\mu(x_1)}{2\mu(x_1) + c} \delta_1$, where δ_x stands for the Borel probability measure on $\mathbb{R}$ supported on $\{x\}$ for $x \in \mathbb{R}$. This completes the proof. □

Suppose (2.8) holds. In what follows we denote by χ_i the characteristic function of $\{x_i\}$ for $i \in J_{[1,\kappa]}$ and by $\chi_{i,j}$ the characteristic function of $\{x_{i,j}\}$

for $i \in J_{[1,\eta]}$ and $j \in \mathbb{N}$. If $A \subseteq X$, then χ_A stands for the characteristic function of A .

Let us recall that a 2-isometric operator $T \in \mathcal{B}(\mathcal{H})$ is said to be Δ_T -regular if $\Delta_T T = \Delta_T^{\frac{1}{2}} T \Delta_T^{\frac{1}{2}}$, where $\Delta_T = T^*T - I$ (see [3, 6]). By [25, Lemma 1(a)] operator Δ_T is positive, hence the above definition makes sense. Now we establish an equivalent condition for a 2-isometric composition operator on a directed graph with one circuit to be Δ_T -regular.

Proposition 4.7. *Assume (2.8) holds and C_ϕ is 2-isometric. Then C_ϕ is Δ_{C_ϕ} -regular if and only if $\kappa = 1$.*

Proof. Assume $\kappa = 1$. We want to show that $\Delta_{C_\phi} C_\phi = \Delta_{C_\phi}^{\frac{1}{2}} C_\phi \Delta_{C_\phi}^{\frac{1}{2}}$. By Corollary 2.12 there exists $\{c_i\}_{i=1}^\eta \subseteq (0, \infty)$ such that

$$\mu(x_{i,j}) = c_i \quad i \in J_{[1,\eta]}, j \in \mathbb{N}.$$

It follows from (4.5) that

$$\Delta_{C_\phi} f = (h_\phi - 1)f, \quad f \in L^2(\mu). \quad (4.10)$$

Moreover, by (2.10)

$$(h_\phi - 1)(x) = \begin{cases} \frac{c}{\mu(x_1)} & \text{if } x = x_1, \\ 0 & \text{otherwise,} \end{cases}$$

where $c = \sum_{i=1}^\eta c_i$. This implies that $\Delta_{C_\phi} f = \frac{c}{\mu(x_1)} f(x_1) \chi_1$ and consequently $\Delta_{C_\phi}^{\frac{1}{2}} f = \sqrt{\frac{c}{\mu(x_1)}} f(x_1) \chi_1$ for $f \in L^2(\mu)$. Hence $\sqrt{\frac{c}{\mu(x_1)}} \chi_1 = \Delta_{C_\phi}^{\frac{1}{2}} \chi_1 = \Delta_{C_\phi}^{\frac{1}{2}} (\chi_1 + \sum_{i=1}^\eta \chi_{i,1})$. These together yield

$$\begin{aligned} \Delta_{C_\phi} C_\phi f &= \frac{c}{\mu(x_1)} f(x_1) \chi_1 = \sqrt{\frac{c}{\mu(x_1)}} f(x_1) \Delta_{C_\phi}^{\frac{1}{2}} \left(\chi_1 + \sum_{i=1}^\eta \chi_{i,1} \right) \\ &= \sqrt{\frac{c}{\mu(x_1)}} f(x_1) \left(\Delta_{C_\phi}^{\frac{1}{2}} C_\phi \right) \chi_1 = \Delta_{C_\phi}^{\frac{1}{2}} C_\phi \Delta_{C_\phi}^{\frac{1}{2}} f, \quad f \in L^2(\mu). \end{aligned}$$

Therefore C_ϕ is Δ_{C_ϕ} -regular.

Now, suppose $\kappa > 2$. Assume to the contrary that C_ϕ is Δ_{C_ϕ} -regular. It is easily seen that

$$\Delta_{C_\phi} C_\phi \chi_1 = \Delta_{C_\phi} \chi_2 \stackrel{(4.10)}{=} (h_\phi(x_2) - 1) \chi_2 \quad (4.11)$$

and

$$\begin{aligned} \Delta_{C_\phi}^{\frac{1}{2}} C_\phi \Delta_{C_\phi}^{\frac{1}{2}} \chi_1 &= \Delta_{C_\phi}^{\frac{1}{2}} C_\phi (h_\phi(x_1) - 1)^{\frac{1}{2}} \chi_1 \\ &= (h_\phi(x_1) - 1)^{\frac{1}{2}} (h_\phi(x_2) - 1)^{\frac{1}{2}} \chi_2. \end{aligned} \quad (4.12)$$

Observe that, by (3.4) from Theorem 3.5, $h_\phi(x_2) - 1 = \frac{\frac{c}{\kappa}}{\frac{c}{\kappa}(2-\kappa)+t}$ and $h_\phi(x_1) - 1 = \frac{\frac{c}{\kappa}}{\frac{c}{\kappa}(1-\kappa)+t}$. Since C_ϕ is Δ_{C_ϕ} -regular, this, (4.11) and (4.12) imply that

$$h_\phi(x_2) - 1 = h_\phi(x_1) - 1$$

which yields a contradiction.

Finally, if $\kappa = 2$, then it is a matter of a similar verification as in the case when $\kappa > 2$ to prove that C_ϕ is not Δ_{C_ϕ} -regular (take χ_1 and use Δ_{C_ϕ} -regularity to get a contradiction by showing that $\mu(x_2) < 0$). $\square$

If (2.8) holds with $\kappa = 1$ and $C_\phi \in \mathcal{B}(L^2(\mu))$ is 2-isometric, then a combination of Proposition 4.7 with [3, Theorem 4.5] implies that C'_ϕ is subnormal. The above argument seems to be simpler than the proof of Theorem 4.6, however, this approach does not yield a direct form of the representing measure.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy the kernel condition if $T^*T\mathcal{N}(T^*) \subseteq \mathcal{N}(T^*)$ (see [3]). Observe also that if X is a set, $\phi: X \rightarrow X$ and $f: X \rightarrow \mathbb{C}$ is a function such that f is constant on $\phi^{-1}(\{x\})$ for all $x \in X$, then the function $f \circ \phi^{-1}: X \rightarrow \mathbb{C}$, where

$$(f \circ \phi^{-1})(x) := \begin{cases} f(y) & \text{if there exists } y \in X \text{ such that } x = \phi(y), \\ 0 & \text{otherwise.} \end{cases}$$

is well-defined and $(f \circ \phi^{-1}) \circ \phi = f$. This implies that f is constant on $\phi^{-1}(\{x\})$ for all $x \in X$ if and only if there exists a function $g: X \rightarrow \mathbb{C}$ such that $g \circ \phi = f$. The following proposition provides a characterization of left-invertible composition operators that satisfy the kernel condition.

Proposition 4.8. *Let $(X, \mathcal{A}, \mu)$ be a discrete measure space, ϕ be a nonsingular self-map of X and $C_\phi \in \mathcal{B}(L^2(\mu))$ be a composition operator. Then the following conditions hold:*

- (i) $f \in \mathcal{R}(C_\phi)$ if and only if $f: X \rightarrow \mathbb{C}$ is a function such that f is constant on $\phi^{-1}(\{x\})$ for all $x \in X$ and $f \circ \phi^{-1} \in L^2(\mu)$,
- (ii) if C_ϕ left-invertible, then C_ϕ satisfies the kernel condition if and only if h_ϕ is constant on preimages $\phi^{-1}(\{x\})$ for all $x \in X$.

Proof. (i) To prove the “if” part, assume that $f \in \mathcal{R}(C_\phi)$. Then there exists $g \in L^2(\mu)$ such that $f = g \circ \phi$ and hence f is constant on $\phi^{-1}(\{x\})$ for all $x \in X$. Since $g \in L^2(\mu)$, $g|_{\phi(X)} = (f \circ \phi^{-1})|_{\phi(X)}$ and $f \circ \phi^{-1}|_{X \setminus \phi(X)} = 0$, we deduce that $f \circ \phi^{-1} \in L^2(\mu)$. It is a routine matter to show that the reverse implication is also true.

(ii) Since C_ϕ is left-invertible, the range $\mathcal{R}(C_\phi)$ is closed. Hence, it is a direct consequence of the kernel-range decomposition that $\mathcal{N}(C_\phi^*)^\perp = \mathcal{R}(C_\phi)$. Therefore, C_ϕ satisfies the kernel condition if and only if

$$C_\phi^* C_\phi f \in \mathcal{R}(C_\phi), \quad f \in \mathcal{R}(C_\phi). \quad (4.13)$$

Now, suppose that C_ϕ satisfies the kernel condition and fix $x \in X$. Let $y, z \in X$ be such that $\phi(y) = \phi(z) = x$ and set $f = C_\phi \chi_{\{x\}} = \chi_{\phi^{-1}(\{x\})}$. It is easily seen that $f(y) = f(z) = 1$. This combined with the fact that $C_\phi^* C_\phi f = h_\phi f$, (4.13) and (i), imply that $h_\phi(y) = h_\phi(z)$. Thus h_ϕ is constant on preimages $\phi^{-1}(\{x\})$ for all $x \in X$.

To prove the reverse implication suppose that h_ϕ is constant on preimages $\phi^{-1}(\{x\})$ for all $x \in X$. Then the function $h_\phi \circ \phi^{-1}$ is well-defined and bounded. Thus, if $f \in \mathcal{R}(C_\phi)$, then, by (i), we get

$$C_\phi^* C_\phi f = h_\phi f = ((h_\phi \circ \phi^{-1})(f \circ \phi^{-1})) \circ \phi \in \mathcal{R}(C_\phi), \quad f \in \mathcal{R}(C_\phi). \quad \square$$

It is known that the Cauchy dual operators of 2-isometries that satisfy the kernel condition are subnormal (see [3, Theorem 3.3]). The following result states that if C_ϕ is a 2-isometric composition operator on a directed graph with one circuit, then C_ϕ does not satisfy the kernel condition.

Theorem 4.9. *Suppose (2.8) holds and $C_\phi \in \mathcal{B}(L^2(\mu))$ is a 2-isometry. Then C_ϕ does not satisfy the kernel condition.*

Proof. Suppose $\kappa = 1$. Since C_ϕ is a 2-isometry, it follows from Corollary 2.12 that there exists $\{c_i\}_{i=1}^\eta \subseteq (0, \infty)$ such that $\mu(x_{i,j}) = c_i$ for $i \in J_{[1,\eta]}$ and $j \in \mathbb{N}$. Define $c = \sum_{i=1}^\eta c_i$ and observe that $c < \infty$ due to boundedness of C_ϕ . Note that $h_\phi(x_1) = 1 + \frac{c}{\mu(x_1)}$ and $h_\phi(x_{i,1}) = 1$ for $i \in J_{[1,\eta]}$. It follows from Proposition 4.8 that C_ϕ satisfies the kernel condition if and only if $h_\phi(x_1) = h_\phi(x_{i,1})$ for $i \in J_{[1,\eta]}$. The latter implies that $c = 0$, which contradicts our assumptions.

Now assume $\kappa > 1$. It follows from Theorem 3.5 that there exist $t \in (0, \infty)$, $\{c_i\}_{i=1}^\eta \subseteq (0, \infty)$ and a polynomial w in one indeterminate x with real coefficients of degree at most 1 such that $\mu(x_{i,j}) = c_i$ for $i \in J_{[1,\eta]}$, $j \in \mathbb{N}$ and $\mu(x_i) = w(i)$ for $i \in J_{[1,\kappa]}$, where $w(x) = \frac{c}{\kappa}x - c + t$ and $c = \sum_{i=1}^\eta c_i < \infty$. Note that $h_\phi(x_1) = \frac{w(2)}{w(1)}$ and $h_\phi(x_{i,1}) = 1$ for $i \in J_{[1,\eta]}$. If C_ϕ satisfies the kernel condition, then it follows from Proposition 4.8 that $h_\phi(x_1) = h_\phi(x_{i,1})$, for $i \in J_{[1,\eta]}$. Again, the last equality implies that $c = 0$, which contradicts our assumptions. Hence, the proof is completed. $\square$

Acknowledgements

The authors would like to thank the anonymous reviewer for the insightful review of the paper and for suggestions that helped to improve its final version.

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Received: December 13, 2020.

Revised: June 7, 2021.