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Integral Equations and Operator Theory

Maximal Regularity for Non-autonomous Evolutionary Equations

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Abstract. We discuss the issue of maximal regularity for evolutionary equations with non-autonomous coefficients. Here evolutionary equations are abstract partial-differential algebraic equations considered in Hilbert spaces. The catch is to consider time-dependent partial differential equations in an exponentially weighted Hilbert space. In passing, one establishes the time derivative as a continuously invertible, normal operator admitting a functional calculus with the Fourier–Laplace transformation providing the spectral representation. Here, the main result is then a regularity result for well-posed evolutionary equations solely based on an assumed parabolic-type structure of the equation and estimates of the commutator of the coefficients with the square root of the time derivative. We thus simultaneously generalise available results in the literature for non-smooth domains. Examples for equations in divergence form, integro-differential equations, perturbations with non-autonomous and rough coefficients as well as non-autonomous equations of eddy current type are considered.

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1. Introduction

If one considers partial differential equations depending on time as an equation in space-time the following problem of maximal regularity arises naturally. For the sake of the argument, let $\mathcal{H}$ be a Hilbert space modelling space-time and let D and A be two closed, densely defined (unbounded) operators, where the former contains the temporal and the latter the spatial derivative(s). Abstractly spoken, the PDE in question then may look like as

follows:

$$Du + Au = f$$

for some right-hand side $f \in \mathcal{H}$. In particular, when hyperbolic type problems are concerned (think of the transport equation or the wave equation), one cannot expect that for any $f \in \mathcal{H}$ (usually an L_2-type space) the solution u to belong to both $\operatorname{dom}(D)$ and $\operatorname{dom}(A)$. In general one can only hope for $u \in \operatorname{dom}(\overline{D + A})$, thus deeming the above equation to be true only in some *generalised* sense. A first example, where it is possible to show that u belongs to the individual domains given any $f \in \mathcal{H}$ is when $D = \partial_t$ and $A = -\Delta_D$ (Laplacian with Dirichlet boundary conditions on some open $\Omega \subseteq \mathbb{R}^n$) and $\mathcal{H} = L_2(0, T; L_2(\Omega))$ and u is assumed to satisfy homogeneous initial conditions. Then, the solution u indeed belongs to $H^1(0, T; L_2(\Omega)) \cap L_2(0, T; \operatorname{dom}(\Delta_D))$, see e.g. [10] or below. Traditionally, the method of choice to derive such a regularity result first establishes well-posedness of the equation at hand via bilinear forms and afterwards analysing the problem in terms of the associated generator. Quite naturally and generalising the above situation considerably, Lions raised the following problem (see [10, p. 68]):

Problem 1.1. Let V and H be Hilbert spaces such that $V \hookrightarrow H$ continuously and densely. Let $a\colon [0, T] \times V \times V \to \mathbb{C}$ be such that $a(t, \cdot, \cdot)$ is sesquilinear, satisfying suitable boundedness, coercivity and measurability conditions thus defining $\mathfrak{A}(t)$ via $\langle \mathfrak{A}(t)x, y\rangle_{V,V^*} := a(t, x, y)$, $t \in [0, T]$. Let $f \in L_2(0, T; H)$ be given. Then there exists a unique solution $u \in H^1(0, T; V^*) \cap L_2(0, T; V)$ of

$$u'(t) + \mathfrak{A}(t)u(t) = f(t) \quad u(0) = 0.$$

The question now is under which conditions on a, do we actually have $u \in H^1(0, T; H)$?

The latter problem indeed fits into the above abstract perspective for $D = \partial_t$ with domain $H^1(0, T; H)$ with Dirichlet boundary conditions at 0 and $\widetilde{A} = \mathfrak{A}\colon \operatorname{dom}(\mathfrak{A}) \subseteq L_2(0, T; H) \to L_2(0, T; H), u \mapsto (t \mapsto \mathfrak{A}(t)u(t))$ with maximal domain. Problem 1.1 has a long history and has rather recently gained some renewed attention. For the latest developments, we refer to the survey article in [2], to [1] and its introduction. We recall here that Hölder continuity for a (and particularly the Hölder exponent 1/2) with respect to time in a suitable sense plays a crucial role, see e.g. [9,13] for a positive and a negative result, respectively.

The available results in the literature up to this point consider explicit Cauchy problems similar to the one in Problem 1.1. Thus, in any case, the complexity of the problem is contained in the form a (or in the operator $\mathfrak{A}$).

In this article we set a different focus and try to keep the operator containing the spatial derivatives (i.e. $\mathfrak{A}$) as simple as possible and move the complexity over to the time derivative. The rationale behind this is the notion of so-called evolutionary equations, invented in [14] and rather self-contained discussed in [16,22]. More precisely, we consider equations of the form

$$(\partial_t \mathcal{M} + \mathcal{N} + A)\, U = F,$$

where f belongs to an exponentially weighted L_2-space, A is an unbounded skew-selfadjoint operator solely acting with respect to the spatial variables and $\mathcal{M}$ and $\mathcal{N}$ are suitable bounded linear operators in space-time. The solution theory developed in [27,28] asserts that under suitable positive definiteness conditions imposed on $\partial_t\mathcal{M}+\mathcal{N}$, one has that $\overline{(\partial_t\mathcal{M}+\mathcal{N}+A)}$ is continuously invertible. In the framework of evolutionary equations, the maximal regularity problem then reads as follows.

Problem 1.2. Given $\partial_t\mathcal{M}+\mathcal{N}$ satisfies the appropriate positive definiteness conditions and A skew-selfadjoint in order that $\overline{(\partial_t\mathcal{M}+\mathcal{N}+A)}$ is continuously invertible, what are the additional conditions on $\mathcal{M}$ and $\mathcal{N}$ (and the right-hand side F) such that

$$\overline{(\partial_t\mathcal{M}+\mathcal{N}+A)}^{-1}F=(\partial_t\mathcal{M}+\mathcal{N}+A)^{-1}F\ ?$$

We emphasise that even in the time-independent case Problems 1.1 and 1.2 are rather different types of questions. In fact, since A is skew-selfadjoint in Problem 1.2, the choices $\mathcal{M}=1$ and $\mathcal{N}=0$ for $F\notin H^1$ with respect to time do not lead to $\overline{(\partial_t+A)}^{-1}F=(\partial_t+A)^{-1}F$, if A is unbounded. As it will be obvious in the next example for a solution of Problem 1.2 one is particularly interested in cases where F belongs to spaces not as smooth as H^1.

In the autonomous case, Problem 1.2 has been addressed in [19]. The conditions derived describe a parabolic type evolutionary equation in an abstract manner. Indeed, one assume that there exists a densely defined closed linear operator C (acting in the spatial variables only) such that

$$A=\begin{pmatrix}0 & -C^*\\ C & 0\end{pmatrix}.$$

Moreover, one has that

$$\mathcal{M}=\begin{pmatrix}\mathcal{M}_{00} & 0\\ 0 & 0\end{pmatrix}\text{ and }\mathcal{N}=\begin{pmatrix}\mathcal{N}_{00} & \mathcal{N}_{01}\\ \mathcal{N}_{10} & \mathcal{N}_{11}\end{pmatrix}$$

with $\mathcal{N}_{11}$ satisfying an additional positive definiteness condition. The standard case of the heat equation $\partial_t u-\Delta_D u=f$ mentioned above is then recovered by putting $q=-\operatorname{grad}_0 u$ (gradient subject to homogeneous Dirichlet boundary conditions) and considering

$$\left(\partial_t\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad}_0 & 0\end{pmatrix}\right)\begin{pmatrix}u\\ q\end{pmatrix}=\begin{pmatrix}f\\ 0\end{pmatrix}.$$

Then, indeed, by the main result of [19], one has

$$\begin{aligned}&\overline{\left(\partial_t\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad}_0 & 0\end{pmatrix}\right)}^{-1}\begin{pmatrix}f\\ 0\end{pmatrix}\\ &=\left(\partial_t\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad}_0 & 0\end{pmatrix}\right)^{-1}\begin{pmatrix}f\\ 0\end{pmatrix}\end{aligned}$$

leading to the maximal regularity result mentioned at the beginning for $F=(f,0)$ with $f\in L_2(0,T;L_2(\Omega))$ only.

Even though the class of equations treated in [19] particularly contains integro-differential equations rendering rather different equations to enjoy maximal regularity, the case of the heat equation with non-symmetric but time-independent coefficients could not be treated with the methods developed there.

In this article we shall enlarge the class of coefficients $\mathcal{M}$ and $\mathcal{N}$ considerably leading to the equality highlighted in Problem 1.2. In particular, this class will involve non-symmetric conductivities in the case of the heat equation. What is more, we shall show that the conditions might be weaker than the conditions derived in both [3,8] if applied to divergence form problems. Since we do not consider bilinear forms as our central object of study, we do not invoke the Kato square root property explicitly, which proved instrumental in the main result in [1]. In particular, our methods also apply irrespective of the regularity of the considered underlying domains of the exemplarily considered divergence form problems. Another key difference to the results for non-autonomous maximal regularity available in the literature is the possibility of the variable operator coefficient $\mathcal{M}$, which allows us to consider integro-differential equations with the same approach as classical Cauchy problems in divergence form. Moreover, the operator coefficient $\mathcal{N}$ permits the introduction of rough (in time) lower order terms. Before we present a plan of our paper, we shortly describe the two main results and instrumental techniques used in the present article.

Theorem 4.1, our first main result on maximal regularity of evolutionary equations, in rough terms can be described as follows: Well-posedness in L_2 and $H^{1/2}$ together with a parabolic structure of $\mathcal{M},\mathcal{N}$ and A imply maximal regularity in the sense of Problem 1.2 for $F = (f, g) \in L_2 \times H^{1/2}$, which in the standard heat equation case is satisfied as $g = 0$ anyway. For a proof of Theorem 4.1, the framework of evolutionary equations is particularly helpful since ∂_t is continuously invertible and normal yielding a handy description of $H^{1/2}$ by the functional calculus for ∂_t. The functional calculus is provided with the help of the Fourier–Laplace transformation. Note that the application of this functional calculus naturally leads to the fractional Riemann–Liouville derivative, see also [17**?**]. In applications, the conditions on the parabolic structure and the well-posedness in L_2 are rather easy to show. The assumed positive definiteness in $H^{1/2}$ leading to the respective well-posedness result might be rather difficult to obtain, though. We emphasise, however, that in addition to the various positive definiteness estimates, we only need to assume that the involved coefficient operators $\mathcal{M}$ and $\mathcal{N}$ are bounded linear operators in $H^{1/2}$ thus leaving this space invariant. In particular, *no* bounded commutator assumptions need to be imposed suggesting room for improvement along the lines of the low regularity assumed for the coefficients in [1]. We shall not follow up on this but rather assume stronger commutator assumptions on $\mathcal{M}$ and $\mathcal{N}$ with ∂_t^{-1} and $\partial_t^{1/2}$, respectively, confirming the particular role of commutator estimates for maximal regularity already observed in [3,8]. Our second main theorem on maximal regularity

of evolutionary equations (Theorem 5.1) imposes the same parabolic structure assumption and well-posedness-in-L_2-requirement as Theorem 4.1. The conditions on the commutators then lead to the asked for well-posedness in $H^{1/2}$ of Theorem 4.1 via a perturbation argument.

For divergence form problems, the assumptions in Theorem 5.1 are implied by the fractional Sobolev (or BMO)-regularity properties imposed in [3,8]. This provides a way of classifying the a priori not comparable conditions in [3,8]. Furthermore, we recover an analogous regularity phenomenon first observed in [8] and confirmed in [3] of the solution belonging to $H^{1/2}$ -time regularity taking values in the form domain.

In the next section, we recall the framework of evolutionary equations and highlight the main ingredients of the non-autonomous solution theory in L_2 as well as some facts of the (time) derivative established in vector-valued exponentially weighted L_2-spaces. This particularly includes the spectral representation and the accompanying functional calculus. In Sect. 3, we provide a necessary new technical result, Theorem 3.4, which contains a solution theory for evolutionary equations in $H^{1/2}$. Our first main result is presented and proved in Sect. 4. The corresponding perturbation result with the mentioned commutator assumptions is presented in Sect. 5. Also, with a focus on operator-valued multiplication operators, we analyse the commutator condition imposed in Theorem 5.1 a bit more closely. We provide a proof of the results in [3,8] for divergence form problems with our methods in Subsection 6.1. An example for integro-differential equations being a non-autonomous variant of some equations considered in [23] is presented in Sect. 6.2. The last application of our abstract findings is concerned with the (non-autonomous) eddy current approximation for Maxwell's equations in Sect. 6.3. We provide a small conclusion in Sect. 7.

2. The Framework

We recall the framework of evolutionary equations. For more details and the proofs we refer to [15,22,28]. We start with the underlying Hilbert space setting and the definition of the time derivative operator.

Definition. For $\rho \geq 0$ we define the space

$$L_{2,\rho}(\mathbb{R};H) := \{f : \mathbb{R} \to H \,;\, f \text{ Bochner-measurable}, \int_{\mathbb{R}} \|f(t)\|^2 \mathrm{e}^{-2\rho t}\,\mathrm{d}t < \infty\},$$

where we as usual identify functions which are equal almost everywhere. This space is clearly a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\rho,0} := \int_{\mathbb{R}} \langle f(t), g(t)\rangle_H \mathrm{e}^{-2\rho t}\,\mathrm{d}t \quad (f, g \in L_{2,\rho}(\mathbb{R};H)).$$

Moreover, we define the operator $\partial_{t,\rho}$: $\mathrm{dom}(\partial_{t,\rho}) \subseteq L_{2,\rho}(\mathbb{R};H) \to L_{2,\rho}(\mathbb{R};H)$ as the closure of the operator

$$C_c^\infty(\mathbb{R};H) \subseteq L_{2,\rho}(\mathbb{R};H) \to L_{2,\rho}(\mathbb{R};H), \quad \phi \mapsto \phi',$$

where $C_c^\infty(\mathbb{R};H)$ denotes the space of arbitrarily differentiable functions having compact support attaining values in H. Finally, we define the *Fourier–Laplace transformation* $\mathcal{L}_\rho\colon L_{2,\rho}(\mathbb{R};H)\to L_2(\mathbb{R};H)$ as the continuous extension of the mapping

$$C_c^\infty(\mathbb{R};H)\subseteq L_{2,\rho}(\mathbb{R};H)\to L_2(\mathbb{R};H),\quad \phi\mapsto\left(t\mapsto\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{-(\mathrm{i}t+\rho)s}\phi(s)\,\mathrm{d}s\right).$$

We collect some properties of the so introduced operators.

Proposition 2.1. *Let $\rho\geq 0$.*

(a) *The operator $\partial_{t,\rho}$ is normal with $\operatorname{Re}\partial_{t,\rho}=\rho$. Thus, in particular $\partial_{t,\rho}^{-1}\in L(L_{2,\rho}(\mathbb{R};H))$ with $\|\partial_{t,\rho}^{-1}\|\leq\frac{1}{\rho}$ if $\rho\neq 0$. Moreover, for $\rho\neq 0$*

$$\left(\partial_{t,\rho}^{-1}f\right)(t)=\int_{-\infty}^{t}f(s)\,\mathrm{d}s\quad(t\in\mathbb{R},f\in L_{2,\rho}(\mathbb{R};H)).$$

(b) *The operator $\mathcal{L}_\rho$ is unitary and*

$$\mathcal{L}_\rho\partial_{t,\rho}=(\mathrm{i}\,\mathrm{m}+\rho)\mathcal{L}_\rho,$$

where $\mathrm{m}\colon\operatorname{dom}(\mathrm{m})\subseteq L_2(\mathbb{R};H)\to L_2(\mathbb{R};H)$ is given by $(\mathrm{m}\,f)(t)=tf(t)$ for $t\in\mathbb{R}$ and $f\in\operatorname{dom}(\mathrm{m})$ with maximal domain; that is,

$$\operatorname{dom}(\mathrm{m})=\{f\in L_2(\mathbb{R};H)\,;\,(t\mapsto tf(t))\in L_2(\mathbb{R};H)\}.$$

In particular $\sigma(\partial_{t,\rho})=\{\mathrm{i}t+\rho\,;\,t\in\mathbb{R}\}$.

(c) *As operators in $L_{2,\rho}(\mathbb{R};H)$ we have*

$$\partial_{t,\rho}^*=-\partial_{t,\rho}+2\rho.$$

With the help of the unitary equivalence of the operators $\partial_{t,\rho}$ and $\mathrm{i}\,\mathrm{m}+\rho$ we can also define derivatives of fractional order (see [17,22**?**]).

Proposition 2.2. *Let $\rho>0$ and $\alpha\in\mathbb{R}$ and set*

$$\partial_{t,\rho}^\alpha:=\mathcal{L}_\rho^*(\mathrm{i}\,\mathrm{m}+\rho)^\alpha\mathcal{L}_\rho.$$

Then $\partial_{t,\rho}^\alpha$ is densely defined and closed on $L_{2,\rho}(\mathbb{R};H)$ and if $\alpha\leq 0$, it is bounded with $\|\partial_{t,\rho}^\alpha\|\leq\frac{1}{\rho^\alpha}$. Moreover, for $\alpha>0$ we have $\operatorname{Re}\partial_{t,\rho}^\alpha\geq\rho^\alpha$ and the operator $\partial_{t,\rho}^{-\alpha}$ is given by

$$\left(\partial_{t,\rho}^{-\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}f(s)\,\mathrm{d}s\quad(t\in\mathbb{R},f\in L_{2,\rho}(\mathbb{R};H)).$$

With the help of these operators, we can define the fractional Sobolev spaces with respect to the exponentially weighted Lebesgue-measure.

Definition. Let $\rho>0$ and $\alpha\geq 0$. Then we set

$$H_\rho^\alpha(\mathbb{R};H):=\operatorname{dom}(\partial_{t,\rho}^\alpha)$$

and equip it with the norm (note that $\partial_{t,\rho}^\alpha$ is injective)

$$\|u\|_{\rho,\alpha}:=\|\partial_{t,\rho}^\alpha u\|_{\rho,0}\quad(u\in H_\rho^\alpha(\mathbb{R};H)).$$

The following proposition is an immediate consequence of the definitions above.

Proposition 2.3. *Let $\rho > 0$ and $\alpha \geq 0$. Then the following statements hold.*

(a) *For each $0 \leq \beta \leq \alpha$ the operator $\partial_{t,\rho}^{\beta} : H_\rho^\alpha(\mathbb{R}; H) \to H_\rho^{\alpha-\beta}(\mathbb{R}; H)$ is unitary.*

(b) *The operator $\mathcal{L}_\rho : H_\rho^\alpha(\mathbb{R}; H) \to H^\alpha(\mathrm{i}\,\mathrm{m} + \rho)$ is unitary. Here,*

$$H^\alpha(\mathrm{i}\,\mathrm{m} + \rho) = \{u \in L_2(\mathbb{R}; H)\,;\, (t \mapsto (\mathrm{i}t + \rho)^\alpha u(t)) \in L_2(\mathbb{R}; H)\}$$

equipped with the norm $\|u\|_{H^\alpha(\mathrm{i}\,\mathrm{m}+\rho)} := \|(\mathrm{i}\,\mathrm{m} + \rho)^\alpha u\|_{L_2(\mathbb{R};H)}$.

Remark 2.4. (a) For $\theta \in]0, 1[$ the space $H_\rho^\theta(\mathbb{R}; H)$ can also be obtained by complex interpolation. More precisely, we have

$$H_\rho^\theta(\mathbb{R}; H) = (L_{2,\rho}(\mathbb{R}; H), H_\rho^1(\mathbb{R}; H))_\theta \tag{2.1}$$

isometrically. For the theory of interpolation spaces we refer to [4,11]. We sketch the proof for (2.1), only. The details are provided in [25, Remark 2.4 (a)]. First, consider the unitarily transformed space $H^\theta(\mathrm{i}\,\mathrm{m} + \rho)$. We show

$$H^\theta(\mathrm{i}\,\mathrm{m} + \rho) = (L_2(\mathbb{R}; H), H^1(\mathrm{i}\,\mathrm{m} + \rho))_\theta$$

isometrically. For the left-hand side being contained in the right-hand side, for $u \in H^\theta(\mathrm{i}\,\mathrm{m} + \rho)$ consider

$$f_u : S \to L_2(\mathbb{R}; H), \quad z \mapsto (t \mapsto u(t)|\mathrm{i}t + \rho|^{\theta - z}),$$

where $S := \{z \in \mathbb{C}\,;\, \mathrm{Re}\, z \in [0, 1]\}$. Then f_u is well-defined, bounded, holomorphic in $\mathring{S}$, $f_u(\mathrm{i}\xi) \in L_2(\mathbb{R}; H)$, $f_u(\mathrm{i}\xi + 1) \in H^1(\mathrm{i}\,\mathrm{m} + \rho)$ and $f_u(\theta) = u$. Since $\|f_u(\mathrm{i}\xi)\|_{L_2(\mathbb{R};H)} = \|u\|_{H^\theta(\mathrm{i}\,\mathrm{m}+\rho)} = \|f_u(\mathrm{i}\xi + 1)\|_{H^1(\mathrm{i}\,\mathrm{m}+\rho)}$ the first inclusion and norm estimate is shown. For the converse inclusion and inequality, let $u \in (L_2(\mathbb{R}; H), H_\rho^1(\mathrm{i}\,\mathrm{m} + \rho))_\theta$ and $g : S \to L_2(\mathbb{R}; H) + H^1(\mathrm{i}\,\mathrm{m} + \rho)$ continuous, holomorphic in $\mathring{S}$, bounded with $g(\mathrm{i}\xi) \in L_2(\mathbb{R}; H)$ and $g(\mathrm{i}\xi + 1) \in H^1(\mathrm{i}\,\mathrm{m} + \rho)$ such that $g(\theta) = u$. In order that $u \in H^\theta(\mathrm{i}\,\mathrm{m} + \rho)$ we show that

$$H_c^\theta(\mathrm{i}\,\mathrm{m} + \rho) \ni v \mapsto \int_{\mathbb{R}} \langle u(t), v(t)\rangle_H |\mathrm{i}t + \rho|^{2\theta}\, \mathrm{d}t$$

defines a bounded functional on $H^\theta(\mathrm{i}\,\mathrm{m} + \rho)$, where $H_c^\theta(\mathrm{i}\,\mathrm{m} + \rho)$ denotes the elements in $H^\theta(\mathrm{i}\,\mathrm{m} + \rho)$ having compact support. For this, let $v \in H_c^\theta(\mathrm{i}\,\mathrm{m} + \rho)$ and set $f_v : S \to L_2(\mathbb{R}; H)$ as above and consider the function

$$F : S \to \mathbb{C}, z \mapsto \int_{\mathbb{R}} \langle g(z)(t), f_v(z)(t)\rangle_H |\mathrm{i}t + \rho|^{2z}\, \mathrm{d}t.$$

Then F is well-defined, continuous, holomorphic in the interior of S and bounded. The maximum principle and estimates on ∂S yield

$$\begin{aligned}\left|\int_{\mathbb{R}} \langle u(t), v(t)\rangle_H |\mathrm{i}t + \rho|^{2\theta}\, \mathrm{d}t\right| &= |F(\theta)| \\ &\leq \sup_{\xi \in \mathbb{R}} \{\|g(\mathrm{i}\xi)\|_{L_2(\mathbb{R};H)}, \|g(\mathrm{i}\xi + 1))\|_{H^1(\mathrm{i}\,\mathrm{m}+\rho)}\} \|v\|_{H^\theta(\mathrm{i}\,\mathrm{m}+\rho)}.\end{aligned}$$

Infimum taken over all appropriate g yields

$$\left|\int_{\mathbb{R}} \langle u(t), v(t)\rangle_H |\mathrm{i}t + \rho|^{2\theta}\, \mathrm{d}t\right| \leq \|u\|_{(L_2(\mathbb{R};H), H^1(\mathrm{i}\,\mathrm{m}+\rho))_\theta} \|v\|_{H^\theta(\mathrm{i}\,\mathrm{m}+\rho)}.$$

Thus, $u \in H^\theta(\mathrm{i}\,\mathrm{m}+\rho)$ with $\|u\|_{H^\theta(\mathrm{i}\,\mathrm{m}+\rho)} \leq \|u\|_{(L_2(\mathbb{R};H),H^1(\mathrm{i}\,\mathrm{m}+\rho))_\theta}$. Finally, using that $\mathcal{L}_\rho \colon H^\theta_\rho(\mathbb{R};H) \to H^\theta(\mathrm{i}\,\mathrm{m}+\rho)$ is unitary, we obtain the assertion.

(b) Let $\mathcal{M} \in L(L_{2,\rho}(\mathbb{R};H)) \cap L(H^1_\rho(\mathbb{R};H))$. Then, for all $\theta \in [0,1]$, by (a), $\mathcal{M} \in L(H^\theta_\rho(\mathbb{R};H))$; see also [11, Theorem 2.6].

Remark 2.5. (a) For the next result, we recall that by complex interpolation and Plancherel's theorem, we have that the Fourier transformation extends to be a continuous operator

$$\mathcal{F} \colon L_{p'}(\mathbb{R};H) \to L_p(\mathbb{R};H),$$

where $\frac{1}{p'} + \frac{1}{p} = 1$ with $1 < p' < 2 < p < \infty$. This applies verbatim to $\mathcal{F}^*$.

(b) Recall the following version of Hölder's inequality: If $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R};X)$, X a Banach space, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for some $p,q,r \in [1,\infty]$, then $t \mapsto f(t)g(t) \in L_r(\mathbb{R};X)$ and $\|fg\|_{L_r} \leq \|f\|_{L_p}\|g\|_{L_q}$.

Lemma 2.6. *Let $\rho > 0$ and $\alpha \in]0,1/2]$. Then for each $p \in [2, \frac{2}{1-2\alpha}[$ (where we set $\frac{2}{0} := \infty$) $H^\alpha_\rho(\mathbb{R};H) \hookrightarrow L_{p,\rho}(\mathbb{R};H)$, where*

$$L_{p,\rho}(\mathbb{R};H) := \{f \colon \mathbb{R} \to H \,;\, f \text{ measurable}, \int_{\mathbb{R}} \|f(t)\|^p \mathrm{e}^{-p\rho t}\,\mathrm{d}t < \infty\}$$

equipped with the obvious norm, denoted by $\|\cdot\|_{L_{p,\rho}}$.

Proof. For $p = 2$ there is nothing to show. Let $p \in]2, \frac{2}{1-2\alpha}[$ and $p' \in]\frac{2}{1+2\alpha}, 2[$ denote the conjugate exponent to p; i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. From Remark 2.5, we know that $\mathcal{F}^* \colon L_{p'} \to L_p$ is continuous. Hence, for $u \in C^\infty_c(\mathbb{R};H)$ we estimate

$$\|u\|_{L_{p,\rho}} = \|\mathrm{e}^{-\rho\cdot}u\|_{L_p} = \|\mathcal{F}^*\mathcal{L}_\rho u\|_{L_p} \lesssim \|\mathcal{L}_\rho u\|_{L_{p'}}.$$

Next, let $q := \frac{2p'}{2-p'}$. Then by Hölder's inequality

$$\begin{aligned}\|\mathcal{L}_\rho u\|_{L_{p'}} &= \|(\mathrm{i}\,\mathrm{m}+\rho)^{-\alpha}(\mathrm{i}\,\mathrm{m}+\rho)^\alpha \mathcal{L}_\rho u\|_{L_{p'}} \\ &\leq \|(\mathrm{i}\,\mathrm{m}+\rho)^{-\alpha}\|_{L_q}\|(\mathrm{i}\,\mathrm{m}+\rho)^\alpha \mathcal{L}_\rho u\|_{L_2}.\end{aligned}$$

Note that

$$\|(\mathrm{i}\,\mathrm{m}+\rho)^{-\alpha}\|_q = \left(\int_{\mathbb{R}} \frac{1}{(t^2+\rho^2)^{-\frac{\alpha q}{2}}}\,\mathrm{d}t\right)^{\frac{1}{q}} < \infty,$$

since $q = \frac{2p'}{2-p'} = \frac{2}{(2/p')-1} > \frac{1}{\alpha}$ and hence, the claim follows. □

The next statement contains an approximation result, which has been employed in [19,27] for the particular case $\alpha = 0$. To have a corresponding result for the case when $\alpha > 0$ (and particularly when $\alpha = 1/2$), will turn out to be useful in the next section, where we provide a well-posedness result for evolutionary equations in $H^{1/2}_\rho(\mathbb{R};H)$.

Lemma 2.7. *Let $\rho > 0$ and $\alpha \geq 0$. We consider the time derivative operator on $H^\alpha_\rho(\mathbb{R};H)$; that is,*

$$\partial_{t,\rho} \colon H^{\alpha+1}_\rho(\mathbb{R};H) \subseteq H^\alpha_\rho(\mathbb{R};H) \to H^\alpha_\rho(\mathbb{R};H).$$

Then for each $\varepsilon > 0$ the operator $1 + \varepsilon\partial_{t,\rho}$ is continuously invertible on $H^\alpha_\rho(\mathbb{R};H)$ and $(1+\varepsilon\partial_{t,\rho})^{-1} \to 1$ strongly in $H^\alpha_\rho(\mathbb{R};H)$ as $\varepsilon \to 0$. □

Proof. For $u \in H^{\alpha+1}_\rho(\mathbb{R};H)$ we have that

$$\operatorname{Re}\langle(1+\varepsilon\partial_{t,\rho})u,u\rangle_{\rho,\alpha} = \|u\|^2_{\rho,\alpha} + \varepsilon \operatorname{Re}\langle\partial_{t,\rho}\partial^\alpha_{t,\rho}u, \partial^\alpha_{t,\rho}u\rangle_{\rho,0} \geq \|u\|^2_{\rho,\alpha}.$$

Thus, $(1+\varepsilon\partial_{t,\rho})$ is injective, posseses a closed range and its inverse (defined on the range) is continuous with operator norm bounded by 1. Thus, to prove the continuous invertibility, we have to show that $\operatorname{ran}(1+\varepsilon\partial_{t,\rho})$ is dense in $H^\alpha_\rho(\mathbb{R};H)$. For doing so, we first compute the adjoint of $\partial_{t,\rho}$. For elements $v,w \in H^\alpha_\rho(\mathbb{R};H)$ we have that $v \in \operatorname{dom}(\partial^*_{t,\rho})$ with $\partial^*_{t,\rho}v = w$ if and only if

$$\langle\partial_{t,\rho}u, v\rangle_{\rho,\alpha} = \langle u, w\rangle_{\rho,\alpha} \quad (u \in H^{\alpha+1}_\rho(\mathbb{R};H)).$$

The latter is equivalent to

$$\langle\partial_{t,\rho}x, \partial^\alpha_{t,\rho}v\rangle_{\rho,0} = \langle x, \partial^\alpha_{t,\rho}w\rangle_{\rho,0} \quad (x \in H^1_\rho(\mathbb{R};H)),$$

which in turn is equivalent to $\partial^\alpha_{t,\rho}v \in H^1_\rho(\mathbb{R};H)$ and $\partial^\alpha_{t,\rho}w = (-\partial_{t,\rho}+2\rho)\partial^\alpha_{t,\rho}v$, where we used $\partial^*_{t,\rho} = -\partial_{t,\rho}+2\rho$ in $L_{2,\rho}(\mathbb{R};H)$, see Proposition 2.1. Thus,

$$\partial^*_{t,\rho} = -\partial_{t,\rho} + 2\rho,$$

where both operators are considered as operators on $H^\alpha_\rho(\mathbb{R};H)$. Thus,

$$\operatorname{Re}\langle(1+\varepsilon\partial^*_{t,\rho})u,u\rangle_{\rho,\alpha} = \|u\|^2_{\rho,\alpha} + \varepsilon\operatorname{Re}\langle u, \partial_{t,\rho}u\rangle_{\rho,\alpha} \geq \|u\|^2_{\rho,\alpha}$$

for all $u \in H^{\alpha+1}_\rho(\mathbb{R};H) = \operatorname{dom}(\partial^*_{t,\rho})$ and hence, $1+\varepsilon\partial^*_{t,\rho}$ is injective, which shows the density of $\operatorname{ran}(1+\varepsilon\partial_{t,\rho})$ in $H^{\alpha+1}_\rho(\mathbb{R};H)$. To prove the strong convergence, it suffices to show the convergence for elements in $H^{\alpha+1}_\rho(\mathbb{R};H)$, since

$$\sup_{\varepsilon>0}\|(1+\varepsilon\partial_{t,\rho})^{-1}\|_{L(H^\alpha_\rho(\mathbb{R};H))} \leq 1$$

by what we have shown above. For $u \in H^{\alpha+1}_\rho(\mathbb{R};H)$ we compute

$$\begin{aligned}\|(1+\varepsilon\partial_{t,\rho})^{-1}u - u\|_{\rho,\alpha} &= \|(1+\varepsilon\partial_{t,\rho})^{-1}(u-(u+\varepsilon\partial_{t,\rho}u))\|_{\rho,\alpha}\\ &\leq \varepsilon\|u\|_{\rho,\alpha+1} \to 0 \quad (\varepsilon\to 0).\end{aligned}$$

We conclude this section, by citing the main result of [27].

Theorem 2.8. ([27, Theorem 3.4]) *Let $\rho > 0$ and $\mathcal{M},\mathcal{N} \in L(L_{2,\rho}(\mathbb{R};H))$. Moreover, assume there exists $\mathcal{M}' \in L(L_{2,\rho}(\mathbb{R};H))$ such that*

$$\mathcal{M}\partial_{t,\rho} \subseteq \partial_{t,\rho}\mathcal{M} - \mathcal{M}'.$$

Let A: $\operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. Furthermore, assume there exists $c > 0$ such that

$$\operatorname{Re}\langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N})u,u\rangle_{\rho,0} \geq c\|u\|^2_{\rho,0} \quad (u \in H^1_\rho(\mathbb{R};H)).$$

Then the operator $\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A$ is closable, and its closure is continuously invertible. Here, A is identified with its canonical extension to a skew-selfadjoint operator on $L_{2,\rho}(\mathbb{R};H)$ with domain $L_{2,\rho}(\mathbb{R};\operatorname{dom}(A))$.

Remark 2.9. (a) In [27, Theorem 3.4] the assumptions are slightly weaker, but for our purposes, this version of the theorem is sufficient; for a comprehensive discussion, see [28, Theorem 3.3.2] for the version above and [28, Theorem 3.4.6] for the corresponding variant in [27].

(b) Theorem 2.8 provides a unified solution theory for a broad class of non-autonomous problems. Due to the flexibility of the choice of the operators $\mathcal{M}$ and $\mathcal{N}$, which act in space-time, the problem class comprises many different types of differential equations, like delay equations, fractional differential equations, integro-differential equations and coupled problems thereof (see e.g. [18,22] for some survey in the autonomous case and [20,24,28] for some non-autonomous and/or nonlinear examples).

3. The Solution Theory in $H_\rho^{1/2}(\mathbb{R};H)$

In this section, we have a closer look at the solution theory for evolutionary equations in $H_\rho^{1/2}(\mathbb{R};H)$; that is, we prove an analogous statement to Theorem 2.8, but now the equation is considered as an equation on $H_\rho^{1/2}(\mathbb{R};H)$. The basic setting is the following. Let $\mathcal{M},\mathcal{N},\mathcal{M}' \in L(L_{2,\rho}(\mathbb{R};H))$ with the following properties:

$$\operatorname{Re}\langle (\partial_{t,\rho}\mathcal{M}+\mathcal{N})\,\phi,\phi\rangle_{\rho,1/2} \geq c\langle \phi,\phi\rangle_{\rho,1/2}$$

for all $\phi \in H_\rho^{3/2}(\mathbb{R};H)$ some $\rho>0$ and $c>0$. Moreover, we assume that

$$\mathcal{M}\partial_{t,\rho} \subseteq \partial_{t,\rho}\mathcal{M}-\mathcal{M}'$$

and $\mathcal{N}|_{H_\rho^{1/2}(\mathbb{R};H)}, \mathcal{M}'|_{H_\rho^{1/2}(\mathbb{R};H)} \in L(H_\rho^{1/2}(\mathbb{R};H))$.

We discuss operators on $H_\rho^{1/2}(\mathbb{R};H)$ in more detail next. For this we recall the notation of the commutator of two operators S and T on some Hilbert space H,

$$[S,T]:=ST-TS, \quad \text{with } \operatorname{dom}([S,T])=\operatorname{dom}(ST)\cap\operatorname{dom}(TS).$$

In the case that $[S,T]$ is densely defined in H and extends to a bounded linear operator on H, we omit the closure bar and just write $[S,T]\in L(H)$. Consequently, we also use $[S,T]$ to denote the (then continuous operator) $\overline{[S,T]}$.

Lemma 3.1. (a) *Let $\mathcal{C}\in L(L_{2,\rho}(\mathbb{R};H))$. Then $\mathcal{C}|_{H_\rho^{1/2}} \in L(H_\rho^{1/2}(\mathbb{R};H))$ if and only if $\partial_{t,\rho}^{1/2}\mathcal{C}\partial_{t,\rho}^{-1/2}\in L(L_{2,\rho}(\mathbb{R};H))$ in either case, we have*

$$\|\mathcal{C}|_{H_\rho^{1/2}}\|_{L(H_\rho^{1/2}(\mathbb{R};H))} = \|\partial_{t,\rho}^{1/2}\mathcal{C}|_{H_\rho^{1/2}}\partial_{t,\rho}^{-1/2}\|_{L(L_{2,\rho}(\mathbb{R};H))}.$$

Either of the alternative conditions is satisfied if $[\mathcal{C},\partial_{t,\rho}^{1/2}]\in L(L_{2,\rho}(\mathbb{R};H))$. Moreover, in this case we have

$$\|\mathcal{C}|_{H_\rho^{1/2}}\|_{L(H_\rho^{1/2}(\mathbb{R};H))} \leq \|\mathcal{C}\|_{L(L_{2,\rho}(\mathbb{R};H))} + \frac{1}{\sqrt{\rho}}\|[\mathcal{C},\partial_{t,\rho}^{1/2}]\|_{L(L_{2,\rho}(\mathbb{R};H))}.$$

(b) $[\mathcal{N},(1+\varepsilon\partial_{t,\rho})^{-1}]\to 0$ *strongly in* $L(H_\rho^{1/2}(\mathbb{R};H))$ *as* $\varepsilon\to 0+$.

(c) $\left[\partial_{t,\rho}\mathcal{M},(1+\varepsilon\partial_{t,\rho})^{-1}\right]\in L(H^{1/2}_\rho(\mathbb{R};H))$ *and* $\left[\partial_{t,\rho}\mathcal{M},(1+\varepsilon\partial_{t,\rho})^{-1}\right]\to 0$ *strongly in* $L(H^{1/2}_\rho(\mathbb{R};H))$ *as* $\varepsilon\to 0+$.

Proof. (a) Let $\phi\in C_c^\infty(\mathbb{R};H)$. Assume that $\partial^{1/2}_{t,\rho}\mathcal{C}\partial^{-1/2}_{t,\rho}\in L(L_{2,\rho}(\mathbb{R};H))$. Then we compute

$$\|\mathcal{C}\phi\|_{\rho,\frac12}=\|\partial^{1/2}_{t,\rho}\mathcal{C}\partial^{-1/2}_{t,\rho}\partial^{1/2}_{t,\rho}\phi\|_{\rho,0}\le\|\partial^{1/2}_{t,\rho}\mathcal{C}\partial^{-1/2}_{t,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H))}\|\phi\|_{\rho,\frac12}.$$

If, on the other hand, $\mathcal{C}|_{H^{1/2}_\rho}\in L(H^{1/2}_\rho(\mathbb{R};H))$. Then $\partial^{1/2}_{t,\rho}\mathcal{C}|_{H^{1/2}_\rho}\partial^{-1/2}_{t,\rho}\in L(L_{2,\rho}(\mathbb{R};H))$ since $\partial^{-1/2}_{t,\rho}\in L(L_{2,\rho}(\mathbb{R};H),H^{1/2}_\rho(\mathbb{R};H))$ and $\partial^{1/2}_{t,\rho}\in L(H^{1/2}_\rho(\mathbb{R};H),L_{2,\rho}(\mathbb{R};H))$ are unitary. Assume now that $[\mathcal{C},\partial^{1/2}_{t,\rho}]\in L(L_{2,\rho}(\mathbb{R};H))$. Then

$$\partial^{1/2}_{t,\rho}\mathcal{C}\partial^{-1/2}_{t,\rho}=[\partial^{1/2}_{t,\rho},\mathcal{C}]\partial^{-1/2}_{t,\rho}+\mathcal{C}\in L(L_{2,\rho}(\mathbb{R};H))$$

and, using Proposition 2.2, we get

$$\begin{aligned}\|\mathcal{C}|_{H^{1/2}_\rho}\|_{L(H^{1/2}_\rho(\mathbb{R};H))}&=\|\partial^{1/2}_{t,\rho}\mathcal{C}|_{H^{1/2}_\rho}\partial^{-1/2}_{t,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H))}\\&\le\|\mathcal{C}\|_{L(L_{2,\rho}(\mathbb{R};H)}+\|[\mathcal{C},\partial^{1/2}_{t,\rho}]\partial^{-1/2}_{t,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H))}\\&\le\|\mathcal{C}\|_{L(L_{2,\rho}(\mathbb{R};H)}+\frac{1}{\sqrt{\rho}}\|[\mathcal{C},\partial^{1/2}_{t,\rho}]\|_{L(L_{2,\rho}(\mathbb{R};H))}.\end{aligned}$$

(b) Let $\varepsilon>0$. By (a) together with the proved inequality, it suffices to show that $\partial^{1/2}_{t,\rho}[\mathcal{N},(1+\varepsilon\partial_{t,\rho})^{-1}]\partial^{-1/2}_{t,\rho}\to 0$ strongly in $L(L_{2,\rho}(\mathbb{R};H))$. For this, we compute

$$\begin{aligned}&\partial^{1/2}_{t,\rho}[\mathcal{N},(1+\varepsilon\partial_{t,\rho})^{-1}]\partial^{-1/2}_{t,\rho}\\&\quad=\partial^{1/2}_{t,\rho}\mathcal{N}(1+\varepsilon\partial_{t,\rho})^{-1}\partial^{-1/2}_{t,\rho}-\partial^{1/2}_{t,\rho}(1+\varepsilon\partial_{t,\rho})^{-1}\mathcal{N}\partial^{-1/2}_{t,\rho}\\&\quad=\left[\partial^{1/2}_{t,\rho}\mathcal{N}\partial^{-1/2}_{t,\rho},(1+\varepsilon\partial_{t,\rho})^{-1}\right],\end{aligned}$$

the latter tends to 0 since $\partial^{1/2}_{t,\rho}\mathcal{N}\partial^{-1/2}_{t,\rho}\in L(L_{2,\rho}(\mathbb{R};H))$, by part (a) and $(1+\varepsilon\partial_{t,\rho})^{-1}\to 1$ strongly as $\varepsilon\to 0$ by Lemma 2.7.

(c) Let $\varepsilon>0$. Then we compute using $\mathcal{M}'=[\partial_{t,\rho},\mathcal{M}]$

$$\begin{aligned}\left[\partial_{t,\rho}\mathcal{M},(1+\varepsilon\partial_{t,\rho})^{-1}\right]&=\left(\partial_{t,\rho}\mathcal{M}(1+\varepsilon\partial_{t,\rho})^{-1}-(1+\varepsilon\partial_{t,\rho})^{-1}\partial_{t,\rho}\mathcal{M}\right)\\&=\partial_{t,\rho}\left(\mathcal{M}(1+\varepsilon\partial_{t,\rho})^{-1}-(1+\varepsilon\partial_{t,\rho})^{-1}\mathcal{M}\right)\\&=\partial_{t,\rho}(1+\varepsilon\partial_{t,\rho})^{-1}\left((1+\varepsilon\partial_{t,\rho})\mathcal{M}-\mathcal{M}(1+\varepsilon\partial_{t,\rho})\right)(1+\varepsilon\partial_{t,\rho})^{-1}\\&=\varepsilon\partial_{t,\rho}(1+\varepsilon\partial_{t,\rho})^{-1}\mathcal{M}'(1+\varepsilon\partial_{t,\rho})^{-1}.\end{aligned}$$

Hence, we deduce

$$\begin{aligned}&\partial^{1/2}_{t,\rho}\left[\partial_{t,\rho}\mathcal{M},(1+\varepsilon\partial_{t,\rho})^{-1}\right]\partial^{-1/2}_{t,\rho}\\&\quad=\varepsilon\partial_{t,\rho}(1+\varepsilon\partial_{t,\rho})^{-1}\partial^{1/2}_{t,\rho}\mathcal{M}'\partial^{-1/2}_{t,\rho}(1+\varepsilon\partial_{t,\rho})^{-1}\in L(L_{2,\rho}(\mathbb{R};H))\end{aligned}$$

and thus, $\left[\partial_{t,\rho}\mathcal{M}, (1+\varepsilon\partial_{t,\rho})^{-1}\right] \in L(H_\rho^{1/2}(\mathbb{R};H))$ by (a). Moreover, by Lemma 2.7

$$\begin{aligned}&\partial_{t,\rho}^{1/2}\left[\partial_{t,\rho}\mathcal{M}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]\partial_{t,\rho}^{-1/2}\\ &\quad= \varepsilon\partial_{t,\rho}(1+\varepsilon\partial_{t,\rho})^{-1}\partial_{t,\rho}^{1/2}\mathcal{M}'\partial_{t,\rho}^{-1/2}(1+\varepsilon\partial_{t,\rho})^{-1}\\ &\quad= (1-(1+\varepsilon\partial_{t,\rho})^{-1})\partial_{t,\rho}^{1/2}\mathcal{M}'\partial_{t,\rho}^{-1/2}(1+\varepsilon\partial_{t,\rho})^{-1}\to 0 \quad (\varepsilon\to 0)\end{aligned}$$

strongly in $L(L_{2,\rho}(\mathbb{R};H))$, which yields the asserted convergence again by part (a). □

Proposition 3.2. *We have* $\partial_{t,\rho}\mathcal{M}[H_\rho^{3/2}(\mathbb{R};H)] \subseteq H_\rho^{1/2}(\mathbb{R};H)$.

Proof. Let $\phi \in \operatorname{dom}(\partial_{t,\rho}^{3/2})$. Then $\partial_{t,\rho}\mathcal{M}\phi = \mathcal{M}'\phi + \mathcal{M}\partial_{t,\rho}\phi$. Since $\mathcal{M}' \in L(H_\rho^{1/2}(\mathbb{R};H))$, we obtain $\mathcal{M}'\phi \in H_\rho^{1/2}(\mathbb{R};H)$. Furthermore, since $\mathcal{M} \in L(L_{2,\rho}(\mathbb{R};H)) \cap L(H_\rho^1(\mathbb{R};H))$, we deduce $\mathcal{M} \in L(H_\rho^{1/2}(\mathbb{R};H))$ by complex interpolation (see Remark 2.4 (b)). Hence, we also have $\mathcal{M}\partial_{t,\rho}\phi \in H_\rho^{1/2}(\mathbb{R};H)$, which shows the assertion. □

Lemma 3.3. *For* $\varepsilon > 0$ *we set* $R_\varepsilon := (1+\varepsilon\partial_{t,\rho})^{-1}$. *Let* $u \in \operatorname{dom}(\partial_{t,\rho}\mathcal{M}) \subseteq H_\rho^{1/2}(\mathbb{R};H)$. *Then for each* $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *we have* $R_\varepsilon^k u \in \operatorname{dom}(\partial_{t,\rho}\mathcal{M})$ *and*

$$\partial_{t,\rho}\mathcal{M}R_\varepsilon^k u \to \partial_{t,\rho}\mathcal{M}u \quad (\varepsilon\to 0+).$$

In particular $H_\rho^{k+1/2}(\mathbb{R};H)$ *is a core for* $\partial_{t,\rho}\mathcal{M}$ *for each* $k \in \mathbb{N}$.

Proof. The proof follows by induction on k. For $k=0$ there is nothing to show. Assume now that the assertion holds for $k-1$. Then we compute, using Lemma 3.1 (c) and Lemma 2.7 □

$$\partial_{t,\rho}\mathcal{M}R_\varepsilon^k u = [\partial_{t,\rho}\mathcal{M}, R_\varepsilon]R_\varepsilon^{k-1}u + R_\varepsilon\partial_{t,\rho}\mathcal{M}R_\varepsilon^{k-1}u \to \partial_{t,\rho}\mathcal{M}u \quad (\varepsilon\to 0+).$$

Theorem 3.4. *The operator* $\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A$ *considered as an operator on* $H_\rho^{1/2}(\mathbb{R};H)$ *is closable and its closure is continuously invertible.*

Proof. Recall that all operators are now considered as operators acting on $H_\rho^{1/2}(\mathbb{R};H)$. Since $\partial_{t,\rho}\mathcal{M}+\mathcal{N}$ is strictly positive definite in the Hilbert space $H_\rho^{1/2}(\mathbb{R};H)$ with domain $H_\rho^{3/2}(\mathbb{R};H)$ and A is skew-selfadjoint, we derive that $\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A$ is strictly positive definite in the Hilbert space $H_\rho^{1/2}(\mathbb{R};H)$ with domain $H_\rho^{3/2}(\mathbb{R};\operatorname{dom}(A))$. By Lemma 3.3 this positive definiteness extends to all elements in $\operatorname{dom}(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)$ and thus, $\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A$ is one-to-one and has a continuous inverse defined on the range of $\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A$. Since $H_\rho^{3/2}(\mathbb{R};\operatorname{dom}(A))$ is dense in $H_\rho^{1/2}(\mathbb{R};H)$, the latter implies that $\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A$ is closable (see e.g. [28, Proposition 2.3.14] or [5, Theorem 4.2.5]). Moreover, it is a standard argument to show that $\overline{\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A}$ is continuously invertible on its range, which is closed. Hence, for showing that $\overline{\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A}$ is onto, it suffices to compute the adjoint and confirm that this adjoint is one-to-one, which in turn would imply the density of the range of $\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A$. For doing so, let $\varepsilon > 0$, $u \in \operatorname{dom}(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)$

and $f \in \operatorname{dom}\left((\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*\right)$. We put $f_\varepsilon := (1+\varepsilon\partial_{t,\rho}^*)^{-1} f \in \operatorname{dom}(\partial_{t,\rho}^{3/2})$. Then we compute

$$\begin{aligned}\langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)\,u, f_\varepsilon\rangle_{\rho,1/2} &= \langle(1+\varepsilon\partial_{t,\rho})^{-1}\,(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)\,u, f\rangle_{\rho,1/2}\\ &= \langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)\,(1+\varepsilon\partial_{t,\rho})^{-1}u -\\ &\quad - \left[\partial_{t,\rho}\mathcal{M}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]u - \left[\mathcal{N}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]u, f\rangle_{\rho,1/2}.\end{aligned}$$

Thus, invoking Lemma 3.1 (c), we obtain that $f_\varepsilon \in \operatorname{dom}(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*$ and

$$\begin{aligned}&(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^* f_\varepsilon\\ &\quad= (1+\varepsilon\partial_{t,\rho}^*)^{-1}\,(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^* f - \left[\partial_{t,\rho}\mathcal{M}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]^* f\\ &\qquad - \left[\mathcal{N}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]^* f.\end{aligned}$$

Since $\left[\partial_{t,\rho}\mathcal{M}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]^* f + \left[\mathcal{N}, (1+\varepsilon\partial_{t,\rho})^{-1}\right]^* f \to 0$ weakly in $H_\rho^{1/2}(\mathbb{R};H)$ as $\varepsilon \to 0+$ by Lemma 3.1 (b) and (c), we infer that

$$H_\rho^{3/2}(\mathbb{R};H) \cap \operatorname{dom}\left((\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*\right)$$

is a core for $(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*$ (recall $f_\varepsilon \to f$ in $H_\rho^{1/2}(\mathbb{R};H)$ by Lemma 2.7).

Next, we show $H_\rho^{3/2}(\mathbb{R};H)\cap\operatorname{dom}\left((\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*\right) \subseteq H_\rho^{1/2}(\mathbb{R};\operatorname{dom}(A))$: Let $f \in H_\rho^{3/2}(\mathbb{R};H) \cap \operatorname{dom}\left((\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*\right)$ and $\psi \in H_\rho^{3/2}(\mathbb{R};\operatorname{dom}(A))$. Then $\psi \in \operatorname{dom}(\partial_{t,\rho}\mathcal{M})$ by Proposition 3.2 and

$$\begin{aligned}\langle A\psi, f\rangle_{\rho,1/2} &= \langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)\psi, f\rangle_{\rho,1/2} - \langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N})\psi, f\rangle_{\rho,1/2}\\ &= \langle\psi, (\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^* f\rangle_{\rho,1/2} - \langle\psi, (\mathcal{M}^*\partial_{t,\rho}^*+\mathcal{N}^*)f\rangle_{\rho,1/2},\end{aligned}$$

where we used $\mathcal{M} \in L(H_\rho^{1/2}(\mathbb{R};H))$, (Remark 2.4 (b)). As $H_\rho^{3/2}(\mathbb{R};\operatorname{dom}(A))$ is dense in $H_\rho^{1/2}(\mathbb{R};\operatorname{dom}(A))$, $H_\rho^{3/2}(\mathbb{R};\operatorname{dom}(A))$ is a core for A. Thus, $f \in H_\rho^{1/2}(\mathbb{R};\operatorname{dom}(A))$ with

$$-Af = A^* f = (\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^* f - (\mathcal{M}^*\partial_{t,\rho}^*+\mathcal{N}^*)f.$$

Hence, for $f \in H_\rho^{3/2}(\mathbb{R};H) \cap \operatorname{dom}\left((\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*\right)$ we compute

$$\begin{aligned}\operatorname{Re}\langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^* f, f\rangle_{\rho,\frac{1}{2}} &= \operatorname{Re}\langle\mathcal{M}^*\partial_{t,\rho}^* f + \mathcal{N}^* f - Af, f\rangle_{\rho,\frac{1}{2}}\\ &= \operatorname{Re}\langle f, (\partial_{t,\rho}\mathcal{M}+\mathcal{N})\,f\rangle_{\rho,\frac{1}{2}} \geq c\langle f, f\rangle_{\rho,\frac{1}{2}}\end{aligned}$$

and since the set $H_\rho^{3/2}(\mathbb{R};H)\cap\operatorname{dom}\left((\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*\right)$ is a core for $(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*$, the latter implies that $(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)^*$ is one-to-one. □

Corollary 3.5. *For $k \geq 1$, $(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)\,[H_\rho^{k+1/2}(\mathbb{R};\operatorname{dom}(A))]$ is dense in $H_\rho^{1/2}(\mathbb{R};H)$.*

Proof. Let $f \in H_\rho^{1/2}(\mathbb{R};H)$ and set $u := \left(\overline{\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A}\right)^{-1} f \in H_\rho^{1/2}(\mathbb{R};H)$. Such a u exists by Theorem 3.4. Hence, we find a sequence $(u_n)_{n\in\mathbb{N}}$ in $\operatorname{dom}(\partial_{t,\rho}\mathcal{M})\cap\operatorname{dom}(A)$ with $u_n \to u$ and $(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)\,u_n \to f$ as $n \to \infty$ in $H_\rho^{1/2}(\mathbb{R};H)$. We now define $v_{n,\varepsilon} := (1+\varepsilon\partial_{t,\rho})^{-k} u_n \in H_\rho^{k+1/2}(\mathbb{R};\operatorname{dom}(A))$ for $n \in \mathbb{N}$. By Lemma 2.7, $Av_{n,\varepsilon} \to Au_n$ as well as $\mathcal{N}v_{n,\varepsilon} \to \mathcal{N}u_n$ as $\varepsilon \to 0$

and thus, it suffices to show $\partial_{t,\rho}\mathcal{M}v_{n,\varepsilon} \to \partial_{t,\rho}\mathcal{M}u_n$ as $\varepsilon \to 0$. This, however, follows from Lemma 3.3 and thus, the assertion follows. □

Corollary 3.6. *Let $H = H_0 \oplus H_1$ for each Hilbert spaces H_0 and H_1. Then for $k \geq 1$, $(\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)\,[H_\rho^{k+1/2}(\mathbb{R}; \mathrm{dom}(A))]$ is dense in the space $L_{2,\rho}(\mathbb{R}; H_0) \oplus H_\rho^{1/2}(\mathbb{R}; H_1)$.*

Proof. By Corollary 3.5, $(\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)\,[H_\rho^{k+1/2}(\mathbb{R}; \mathrm{dom}(A))]$ is dense in $H_\rho^{1/2}(\mathbb{R}; H)$. This space continuously and densely embeds into $L_{2,\rho}(\mathbb{R}; H_0) \oplus H_\rho^{1/2}(\mathbb{R}; H_1)$. □

4. Maximal Regularity for Evolutionary Equations

In the following we provide our main result: a criterion for maximal regularity for evolutionary equations. In a nutshell this criterion reads:

$$\mathrm{Well-posedness\ in\ both}\ L_{2,\rho}(\mathbb{R}; H)\ \mathrm{and}\ H_\rho^{1/2}(\mathbb{R}; H)\ \mathrm{together\ with\ a}$$
$$\mathrm{parabolic-like\ structure\ implies\ maximal\ regularity.}$$

Throughout, let H_0 and H_1 be two complex Hilbert spaces and set $H := H_0 \oplus H_1$. Moreover, let C: $\mathrm{dom}(C) \subseteq H_0 \to H_1$ densely defined closed and linear and set

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \tag{4.1}$$

(which easily can be verified to be skew-selfadjoint in H). Finally, we assume that $\mathcal{M}$ and $\mathcal{N}$ have the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{00} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.2}$$

as well as

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_{00} & \mathcal{N}_{01} \\ \mathcal{N}_{10} & \mathcal{N}_{11} \end{pmatrix} \tag{4.3}$$

with appropriate linear operators in $L(L_{2,\rho}(\mathbb{R}; H_j), L_{2,\rho}(\mathbb{R}; H_i))$, $i, j \in \{0, 1\}$.

Theorem 4.1. *Let $A, \mathcal{M}$ and $\mathcal{N}$ be as in (4.1)–(4.3) and assume there is $\mathcal{M}' \in L(L_{2,\rho}(\mathbb{R}; H))$ with*

$$\mathcal{M}\partial_{t,\rho} \subseteq \partial_{t,\rho}\mathcal{M} - \mathcal{M}'. \tag{4.4}$$

Assume, in addition, that

$$\mathcal{M}', \mathcal{N} \in L(H_\rho^{1/2}(\mathbb{R}; H)).$$

We shall assume the positive definiteness conditions

$$\mathrm{Re}\langle \mathcal{M}_{00}\phi, \phi\rangle_{\rho,0} \geq c\langle \phi, \phi\rangle_{\rho,0}, \tag{4.5}$$
$$\mathrm{Re}\langle (\partial_{t,\rho}\mathcal{M} + \mathcal{N})\,\phi, \phi\rangle_{\rho,0} \geq c\langle \phi, \phi\rangle_{\rho,0}, \tag{4.6}$$

and

$$\mathrm{Re}\langle (\partial_{t,\rho}\mathcal{M} + \mathcal{N})\,\phi, \phi\rangle_{\rho,\frac{1}{2}} \geq c\langle \phi, \phi\rangle_{\rho,\frac{1}{2}} \tag{4.7}$$

for some $c > 0$ *and all* $\phi \in H_\rho^{3/2}(\mathbb{R}; H)$. *Let* $\mathcal{S}_\rho := \left(\overline{\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A}\right)^{-1} \in L(L_{2,\rho}(\mathbb{R}; H)) \cap L(H_\rho^{1/2}(\mathbb{R}; H))$ (*cp. Theorems* 2.8 *and* 3.4). *Then*

$$\begin{aligned}&\mathcal{S}_\rho[L_{2,\rho}(\mathbb{R}; H_0) \times H_\rho^{1/2}(\mathbb{R}; H_1)]\\ &\quad \subseteq \left(H_\rho^1(\mathbb{R}; H_0) \cap H_\rho(\mathbb{R}; \mathrm{dom}(C))\right) \times L_{2,\rho}(\mathbb{R}; \mathrm{dom}(C^*));\end{aligned}$$

that is, for $f \in L_{2,\rho}(\mathbb{R}; H_0)$ *and* $g \in H_\rho^{1/2}(\mathbb{R}; H_1)$ *and* $(u, v) \in L_{2,\rho}(\mathbb{R}; H)$ *satisfying*

$$\overline{(\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

we have $u \in H_\rho^1(\mathbb{R}; H_0) \cap H_\rho(\mathbb{R}; \mathrm{dom}(C))$, $v \in L_{2,\rho}(\mathbb{R}; \mathrm{dom}(C^*))$.

Remark 4.2. As we shall see in the examples section, the above nutshell description of the Theorem 4.1 is visible as follows:

- Well-posedness in $L_{2,\rho}(\mathbb{R}; H)$ is guaranteed by assumption (4.6); see Theorem 2.8.
- Well-posedness in $H_\rho^{1/2}(\mathbb{R}; H)$ is guaranteed by assumption (4.7); see Theorem 3.4.
- The parabolic-like structure is visible in the block matrix structure (4.1), (4.2) and the positive definiteness condition (4.5).

Remark 4.3. (a) As in [3,8], we recover the same additional regularity $u \in H_\rho^{1/2}(\mathbb{R}; \mathrm{dom}(C))$. In fact, as the proof of Theorem 4.1 will show an estimate for $\|Cu\|_{\rho,\frac{1}{2}}$ is key for obtaining the main result.

(b) Note that $u \in H_\rho^{1/2}(\mathbb{R}; \mathrm{dom}(C))$ also has a consequence on the time-regularity of v. Indeed, taking (f, g) as right-hand sides as in Theorem 4.1, we see that (u, v) satisfies

$$\begin{aligned}\partial_{t,\rho}\mathcal{M}_{00}u + \mathcal{N}_{00}u + \mathcal{N}_{01}v - C^*v &= f\\ \mathcal{N}_{10}u + \mathcal{N}_{11}v + Cu &= g.\end{aligned}$$

Multiplying the second line by $\mathcal{N}_{11}^{-1}$, we get $v = \mathcal{N}_{11}^{-1}(g - Cu - \mathcal{N}_{11}u)$. Next, since $u \in H_\rho^1(\mathbb{R}; H_0) \subseteq H_\rho^{1/2}(\mathbb{R}; H_0)$, $Cu, g \in H_\rho^{1/2}(\mathbb{R}; H_1)$ and both $\mathcal{N}_{01}$ and $\mathcal{N}_{11}^{-1}$ leave $H_\rho^{1/2}$-regular mappings invariant (see also Lemma 4.6 below), we infer $v \in H_\rho^{1/2}(\mathbb{R}; H_1)$. We summarise all the regularity results in the next statement.

Corollary 4.4. *Under the assumptions of Theorem* 4.1, *let* $f \in L_{2,\rho}(\mathbb{R}; H_0)$, $g \in H_\rho^{1/2}(\mathbb{R}; H_1)$ *and* $(u, v) \in L_{2,\rho}(\mathbb{R}; H)$ *satisfying*

$$\overline{(\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Then $u \in H_\rho^1(\mathbb{R}; H_0) \cap H_\rho^{1/2}(\mathbb{R}; \mathrm{dom}(C))$, $v \in L_{2,\rho}(\mathbb{R}; \mathrm{dom}(C^*)) \cap H_\rho^{1/2}(\mathbb{R}; H_1)$.

Remark 4.5. (a) Note that the regularity statement in the latter result is also accompanied with the corresponding continuity statement; that is, there

exists a constant $\kappa \geq 0$ such that for all $f \in L_{2,\rho}(\mathbb{R}; H_0), g \in H_\rho^{1/2}(\mathbb{R}; H_1)$ with $\mathcal{S}_\rho(f,g) = (u,v)$ we have

$$\|u\|_{\rho,1} + \|Cu\|_{\rho,\frac{1}{2}} + \|C^*v\|_{\rho,0} + \|v\|_{\rho,\frac{1}{2}} \leq \kappa \left(\|f\|_{\rho,0} + \|g\|_{\rho,\frac{1}{2}} \right).$$

Moreover, note that, as a consequence, the closure bar in the formulation of the evolutionary equation can be omitted so that, indeed,

$$\overline{(\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)}^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = (\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)^{-1} \begin{pmatrix} f \\ g \end{pmatrix},$$

addressing Problem 1.2.

(b) We note here that Corollary 4.4 is in fact a result on maximal regularity for the special case of $g = 0$ and if the evolutionary equation is viewed as a 'scalar' equation in the following sense. Assume $g = 0$ and $f \in L_{2,\rho}(\mathbb{R}; H_0)$, then as in Remark 4.3(b), we have seen that then

$$\begin{aligned} \partial_{t,\rho}\mathcal{M}_{00}u + \mathcal{N}_{00}u + \mathcal{N}_{01}v - C^*v &= f \\ \mathcal{N}_{10}u + \mathcal{N}_{11}v + Cu &= 0. \end{aligned}$$

Rearranging the second equality, we infer $v = -\mathcal{N}_{11}^{-1}(Cu + \mathcal{N}_{10}u) \in \operatorname{dom}(C^*)$ and thus the first equation reads

$$\partial_{t,\rho}\mathcal{M}_{00}u + \mathcal{N}_{00}u - \mathcal{N}_{01}\mathcal{N}_{11}^{-1}(Cu + \mathcal{N}_{10}u) + C^*\mathcal{N}_{11}^{-1}(C + \mathcal{N}_{10})u = f.$$

By (4.5) and (4.4), it is not difficult to see that $\operatorname{dom}(\partial_{t,\rho}\mathcal{M}_{00}) = \operatorname{dom}(\partial_{t,\rho})$ and so u really admits the maximal regularity hoped for when $f \in L_{2,\rho}(\mathbb{R}; H)$.

In order to prove our main theorem, we need some prerequisites.

Lemma 4.6. *Assume the conditions imposed on $\mathcal{M}$ and $\mathcal{N}$ in Theorem 4.1. Then $\mathcal{N}_{11}^{-1} \in L(L_{2,\rho}(\mathbb{R}; H_1)) \cap L(H_\rho^{1/2}(\mathbb{R}; H_1))$. Moreover,*

$$\operatorname{Re}\langle \mathcal{N}_{11}^{-1}\phi_1, \phi_1\rangle_{\rho,1/2} \geq \frac{c}{\|\mathcal{N}_{11}\|^2_{L(H_\rho^{1/2}(\mathbb{R};H_1))}} \langle \phi_1, \phi_1\rangle_{\rho,1/2} \quad (\phi_1 \in H_\rho^{1/2}(\mathbb{R}; H_1)).$$

Proof. The conditions in (4.6) and (4.7) applied for $\phi = (0, \phi_1)$ with $\phi_1 \in H_\rho^{3/2}(\mathbb{R}; H_1)$ implies for $\alpha \in \{0, \frac{1}{2}\}$, $\operatorname{Re}\langle \mathcal{N}_{11}\phi_1, \phi_1\rangle_{\rho,\alpha} \geq c\langle \phi_1, \phi_1\rangle_{\rho,\alpha}$. By density, this extends to all $\phi_1 \in H_\rho^\alpha(\mathbb{R}; H_1)$. Since $\mathcal{N}_{11} \in L(L_{2,\rho}(\mathbb{R}; H_1)) \cap L(H_\rho^{1/2}(\mathbb{R}; H_1))$, we deduce the first statement. A standard argument also reveals that (see [22, Proposition 6.2.3 (b)]) □

$$\operatorname{Re}\langle \mathcal{N}_{11}^{-1}\phi_1, \phi_1\rangle_{\rho,\alpha} \geq \frac{c}{\|\mathcal{N}_{11}\|^2_{L(H_\rho^\alpha(\mathbb{R};H_1))}} \langle \phi_1, \phi_1\rangle_{\rho,\alpha}.$$

Lemma 4.7. *Assume the conditions imposed on $\mathcal{M}$ and $\mathcal{N}$ in Theorem 4.1. Then, for all $\phi_0 \in H_\rho^1(\mathbb{R}; H_0)$,*

$$\operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}\phi_0, |\partial_{t,\rho}|\phi_0\rangle_{\rho,0} \geq \left(c - \|\mathcal{N}_{00}\|_{L(H_\rho^{1/2}(\mathbb{R};H_0))} \right) \|\phi_0\|^2_{\rho,1/2}.$$

Proof. We apply the condition (4.7) to $\phi = (\phi_0, 0) \in H^{3/2}_\rho(\mathbb{R}; H)$ and obtain

$$\operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}\phi_0 + \mathcal{N}_{00}\phi_0, \phi_0\rangle_{\rho,1/2} \geq c\|\phi_0\|^2_{\rho,1/2}.$$

Rearranging terms, we obtain

$$\begin{aligned}\operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}\phi_0, |\partial_{t,\rho}|\phi_0\rangle_{\rho,0} &= \operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}\phi_0, \phi_0\rangle_{\rho,1/2}\\ &\geq c\|\phi_0\|^2_{1/2} - \operatorname{Re}\langle \mathcal{N}_{00}\phi_0, \phi_0\rangle_{\rho,1/2}\\ &\geq \left(c - \|\mathcal{N}_{00}\|_{L(H^{1/2}_\rho(\mathbb{R};H))}\right)\|\phi_0\|^2_{\rho,1/2}.\end{aligned}$$

Since $H^{3/2}_\rho(\mathbb{R}; H_0)$ is dense in $H^1_\rho(\mathbb{R}; H_0)$ the assertion follows. □

Lemma 4.8. *Let $P\colon \mathbb{R} \to \mathbb{R}$ be a polyomial of degree $k \in \mathbb{N}$; that is, $P(x) = \sum_{i=0}^k a_i x^i$ for some $a_i \in \mathbb{R}$ with $a_k \neq 0$. Let $x_0 \in \mathbb{R}_{\geq 0}$ with $x_0^\ell \leq P(x_0)$ for some $\ell > k$. Then*

$$x_0 \leq \max\left\{\sum_{i=0}^k |a_i|, \left(\sum_{i=0}^k |a_i|\right)^{\frac{1}{\ell}}\right\}.$$

Proof. Consider the polynomial $Q(x) := x^\ell - P(x)$. Then $Q(x_0) \leq 0$ and $Q(x) \to \infty$ as $x \to \infty$. Thus, there exists some $x_1 \geq x_0$ with $Q(x_1) = 0$. We estimate $x_1^\ell = P(x_1) = \sum_{i=0}^k a_i x_1^i \leq \sum_{i=0}^k |a_i| x_1^i$. We consider the cases $x_1 \leq 1$ and $x_1 > 1$ separately. Assume first that $x_1 \leq 1$. Then $x_1^\ell \leq \sum_{i=0}^k |a_i|$ and hence, $x_0 \leq x_1 \leq \left(\sum_{i=0}^k |a_i|\right)^{\frac{1}{\ell}}$. In the case that $x_1 > 1$ we can estimate $x_1^\ell \leq \sum_{i=0}^k |a_i| x_1^i \leq x_1^{\ell-1} \sum_{i=0}^k |a_i|$, which yields $x_0 \leq x_1 \leq \sum_{i=0}^k |a_i|$. □

Proof of Theorem 4.1. First, let $(f, g) \in (\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)[H^2_\rho(\mathbb{R}; \operatorname{dom}(A))]$ and $(u, v) := \mathcal{S}_\rho(f, g)$. Hence, by the uniqueness of the solution in $L_{2,\rho}(\mathbb{R}; H)$, we obtain $(u, v) \in H^2_\rho(\mathbb{R}; \operatorname{dom}(A))$. Then we can read off the equations line by line and obtain

$$\begin{aligned}\partial_{t,\rho}\mathcal{M}_{00}u + \mathcal{N}_{00}u + \mathcal{N}_{01}v - C^*v &= f,\\ \mathcal{N}_{10}u + \mathcal{N}_{11}v + Cu &= g.\end{aligned}$$

Since $\mathcal{N}_{01}$ and $\mathcal{N}_{11}$ are bounded linear operators mapping $H^{1/2}_\rho$ into $H^{1/2}_\rho$, we deduce that the second line particularily implies $g \in H^{1/2}_\rho(\mathbb{R}; H_1)$. We estimate using (4.5)

$$\begin{aligned}c\|u\|^2_{\rho,1} &\leq \operatorname{Re}\langle \mathcal{M}_{00}\partial_{t,\rho}u, \partial_{t,\rho}u\rangle_{\rho,0}\\ &= \operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}u - \mathcal{M}'_{00}u, \partial_{t,\rho}u\rangle_{\rho,0}\\ &\leq \operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}u, \partial_{t,\rho}u\rangle_{\rho,0} + \|\mathcal{M}'_{00}\|_{L(L_{2,\rho}(\mathbb{R};H_0))}\|u\|_{\rho,0}\|u\|_{\rho,1},\end{aligned}$$

where $\mathcal{M}'_{00} = [\partial_{t,\rho}, \mathcal{M}_{00}]$, which is bounded in $L(L_{2,\rho}(\mathbb{R}; H_0))$ by (4.2) and (4.4). Moreover,

$$\begin{aligned}&\operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}u, \partial_{t,\rho}u\rangle_{\rho,0}\\ &= \operatorname{Re}\langle f - \mathcal{N}_{00}u - \mathcal{N}_{01}v + C^*v, \partial_{t,\rho}u\rangle_{\rho,0}\\ &\leq (\|f\|_{\rho,0} + \|\mathcal{N}_{00}\|_{L(L_{2,\rho}(\mathbb{R};H_0))}\|u\|_{\rho,0} + \|\mathcal{N}_{01}\|_{L(L_{2,\rho}(\mathbb{R};H_0))}\|v\|_{\rho,0})\|u\|_{\rho,1}\\ &\quad + \operatorname{Re}\langle C^*v, \partial_{t,\rho}u\rangle_{\rho,0}.\end{aligned}$$

The last term can further be estimated by (using Lemma 4.6 and Lemma 3.1)

$$\begin{aligned}
&\operatorname{Re}\langle C^*v, \partial_{t,\rho}u\rangle_{\rho,0} = \operatorname{Re}\langle v, \partial_{t,\rho}Cu\rangle_{\rho,0}\\
&\quad= \operatorname{Re}\langle \mathcal{N}_{11}^{-1}(g - Cu - \mathcal{N}_{10}u), \partial_{t,\rho}Cu\rangle_{\rho,0}\\
&\quad= \operatorname{Re}\langle \left(\partial_{t,\rho}^*\right)^{1/2}\mathcal{N}_{11}^{-1}(g - Cu - \mathcal{N}_{10}u), \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}\\
&\quad= \operatorname{Re}\langle \left(\partial_{t,\rho}^*\right)^{1/2}\partial_{t,\rho}^{-1/2}\partial_{t,\rho}^{1/2}\mathcal{N}_{11}^{-1}\partial_{t,\rho}^{-1/2}\partial_{t,\rho}^{1/2}(g - Cu - \mathcal{N}_{10}u), \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}\\
&\quad\lesssim \left(\|g\|_{\rho,1/2} + \|u\|_{\rho,1/2} + \|Cu\|_{\rho,1/2}\right)\|Cu\|_{\rho,1/2},
\end{aligned}$$

where $\lesssim$ means an estimate including constants depending on the operators $\mathcal{N}$ and $\mathcal{M}$ and the positive definiteness constant $c > 0$; we also used

$$\|\left(\partial_{t,\rho}^*\right)^{1/2}\partial_{t,\rho}^{-1/2}\|_{L(L_{2,\rho}(\mathbb{R};H))} \leq 1,$$

which is immediately verified with the help of the Fourier–Laplace transformation, see Proposition 2.2. Thus, summarising we have shown

$$\begin{aligned}
\|u\|_{\rho,1}^2 &\lesssim \left(\|u\|_{\rho,0} + \|f\|_{\rho,0} + \|v\|_{\rho,0}\right)\|u\|_{\rho,1}\\
&\quad + \left(\|g\|_{\rho,1/2} + \|u\|_{\rho,1/2} + \|Cu\|_{\rho,1/2}\right)\|Cu\|_{\rho,1/2}.
\end{aligned}$$

Using $(u, v) = \mathcal{S}_\rho(f, g)$ we obtain $\|(u, v)\|_{\rho,0} \lesssim \|(f, g)\|_{\rho,0} \lesssim \|(f, g)\|_{\rho,(0,1/2)}$, where the last norm means that we take the $L_{2,\rho}$ norm of f and the $H_\rho^{1/2}$ norm of g. Hence, we can estimate further

$$\begin{aligned}
\|u\|_{\rho,1}^2 &\lesssim \|(f, g)\|_{\rho,(0,1/2)}(\|u\|_{\rho,1} + \|Cu\|_{\rho,1/2})\\
&\quad + \|u\|_{\rho,1/2}\|Cu\|_{\rho,1/2} + \|Cu\|_{\rho,1/2}^2. \qquad (4.8)
\end{aligned}$$

Next, we estimate the norm $\|Cu\|_{\rho,1/2}$. First, we compute using the positive definiteness estimate for $\mathcal{N}_{11}^{-1}$ in $H_\rho^{1/2}(\mathbb{R}; H_1)$ from Lemma 4.6

$$\begin{aligned}
\|Cu\|_{1/2}^2 &\lesssim \operatorname{Re}\langle \mathcal{N}_{11}^{-1}Cu, Cu\rangle_{\rho,1/2} = \operatorname{Re}\langle \partial_{t,\rho}^{1/2}\mathcal{N}_{11}^{-1}Cu, \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}\\
&= \operatorname{Re}\langle \partial_{t,\rho}^{1/2}\mathcal{N}_{11}^{-1}g - \partial_{t,\rho}^{1/2}\mathcal{N}_{11}^{-1}\mathcal{N}_{10}u - \partial_{t,\rho}^{1/2}v, \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}\\
&= \operatorname{Re}\langle \partial_{t,\rho}^{1/2}\mathcal{N}_{11}^{-1}\partial_{t,\rho}^{-1/2}\partial_{t,\rho}^{1/2}g - \partial_{t,\rho}^{1/2}\mathcal{N}_{11}^{-1}\mathcal{N}_{10}\partial_{t,\rho}^{-1/2}\partial_{t,\rho}^{1/2}u, \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}\\
&\quad - \operatorname{Re}\langle \partial_{t,\rho}^{1/2}v, \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}\\
&\lesssim (\|g\|_{\rho,1/2} + \|u\|_{\rho,1/2})\|Cu\|_{\rho,1/2} - \operatorname{Re}\langle \partial_{t,\rho}^{1/2}v, \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0}.
\end{aligned}$$

Moreover, we compute using Lemma 4.7

$$\begin{aligned}
&- \operatorname{Re}\langle \partial_{t,\rho}^{1/2}v, \partial_{t,\rho}^{1/2}Cu\rangle_{\rho,0} = \operatorname{Re}\langle -\partial_{t,\rho}^{1/2}C^*v, \partial_{t,\rho}^{1/2}u\rangle_{\rho,0}\\
&\quad= \operatorname{Re}\langle f - \partial_{t,\rho}\mathcal{M}_{00}u - \mathcal{N}_{00}u - \mathcal{N}_{01}v, |\partial_{t,\rho}|u\rangle_{\rho,0}\\
&\quad\lesssim (\|f\|_{\rho,0} + \|u\|_{\rho,0} + \|v\|_{\rho,0})\|u\|_{\rho,1} - \operatorname{Re}\langle \partial_{t,\rho}\mathcal{M}_{00}u, |\partial_{t,\rho}|u\rangle_{\rho,0}\\
&\quad\lesssim (\|f\|_{\rho,0} + \|u\|_{\rho,0} + \|v\|_{\rho,0})\|u\|_{\rho,1} + \|u\|_{\rho,1/2}^2.
\end{aligned}$$

Summarising, we obtain

$$\begin{aligned}
\|Cu\|_{\rho,1/2}^2 &\lesssim (\|g\|_{\rho,1/2} + \|u\|_{\rho,1/2})\|Cu\|_{\rho,1/2}\\
&\quad + (\|f\|_{\rho,0} + \|u\|_{\rho,0} + \|v\|_{\rho,0})\|u\|_{\rho,1} + \|u\|_{\rho,1/2}^2\\
&\lesssim (\|(f, g)\|_{\rho,(0,1/2)} + \|u\|_{\rho,1/2})\|Cu\|_{\rho,1/2} + \|(f, g)\|_{\rho,(0,1/2)}\|u\|_{\rho,1} + \|u\|_{\rho,1/2}^2,
\end{aligned}$$

which is a quadratic inequality in $\|Cu\|_{\rho,1/2} \geq 0$ and yields

$$\begin{aligned}\|Cu\|_{\rho,1/2} &\lesssim \|(f,g)\|_{\rho,(0,1/2)} + \|u\|_{\rho,1/2} + \sqrt{\|(f,g)\|_{\rho,(0,1/2)}\|u\|_{\rho,1} + \|u\|^2_{\rho,1/2}} \\ &\lesssim \|(f,g)\|_{\rho,(0,1/2)} + \|u\|_{\rho,1/2} + \sqrt{\|(f,g)\|_{0,1/2}}\sqrt{\|u\|_{\rho,1}}. \qquad (4.9)\end{aligned}$$

Next, from $\|u\|^2_{\rho,1/2} = \langle \partial^{1/2}_{t,\rho} u, \partial^{1/2}_{t,\rho} u\rangle_{\rho,0} = \langle |\partial_{t,\rho}| u, u\rangle_{\rho,0}$, we deduce $\|u\|^2_{\rho,1/2} \leq \|u\|_{\rho,1}\|u\|_{\rho,0} \lesssim \|u\|_{\rho,1}\|(f,g)\|_{\rho,(0,1/2)}$. This inequality, (4.8), and (4.9) yield

$$\begin{aligned}\|u\|^2_{\rho,1} &\lesssim \|(f,g)\|_{\rho,(0,1/2)}\|u\|_{\rho,1} \\ &\quad + \|(f,g)\|_{\rho,(0,1/2)} \\ &\qquad \cdot \left(\|(f,g)\|_{\rho,(0,1/2)} + \|u\|_{\rho,1/2} + \sqrt{\|(f,g)\|_{\rho,(0,1/2)}}\sqrt{\|u\|_{\rho,1}}\right) + \\ &\quad + \|u\|_{\rho,1/2}\left(\|(f,g)\|_{\rho,(0,1/2)} + \|u\|_{\rho,1/2} + \sqrt{\|(f,g)\|_{\rho,(0,1/2)}}\sqrt{\|u\|_{\rho,1}}\right) \\ &\quad + \|(f,g)\|^2_{\rho,(0,1/2)} + \|u\|^2_{\rho,1/2} + \|(f,g)\|_{\rho,(0,1/2)}\|u\|_{\rho,1} \\ &\lesssim \|(f,g)\|_{\rho,(0,1/2)}\|u\|_{\rho,1} + \|(f,g)\|^{3/2}_{\rho,(0,1/2)}\|u\|^{1/2}_{\rho,1} + \|(f,g)\|^2_{\rho,(0,1/2)}.\end{aligned}$$

Lemma 4.8 applied to $x_0 = \|u\|^{1/2}_{\rho,1}$ leads to $\|u\|_{\rho,1} \leq F(\|(f,g)\|_{\rho,(0,1/2)})$, where $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous with $F(0) = 0$. This proves that

$$\begin{aligned}\mathcal{S}_\rho : (\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A)[H^2_\rho(\mathbb{R}; \mathrm{dom}(A))] &\subseteq L_{2,\rho}(\mathbb{R}; H_0) \times H^{1/2}_\rho(\mathbb{R}; H_1) \\ &\to H^1_\rho(\mathbb{R}; H_0) \times L_{2,\rho}(\mathbb{R}; H_1)\end{aligned}$$

is continuous at 0 and hence, bounded. Since $(\partial_{t,\rho}\mathcal{M}+\mathcal{N}+A)[H^2_\rho(\mathbb{R}; \mathrm{dom}(A))]$ is dense in $L_{2,\rho}(\mathbb{R}; H_0) \times H^{1/2}_\rho(\mathbb{R}; H_1)$ by Corollary 3.6, the first regularity statement holds. The additional regularity $Cu \in H^{1/2}_\rho$ then follows from estimate (4.9). □

5. Applications

5.1. Maximal Regularity and Bounded Commutators

In this section, we will apply our main result Theorem 4.1 to prove maximal regularity for a broad class of evolutionary equations. Note that the second main theorem of the present manuscript is concerned with the case, where well-posedness in $H^{1/2}_\rho$ is obtained by a bounded commutator assumption involving $\mathcal{N}$ and by restricting $\mathcal{M}$ to the case commuting with $\partial^{-1}_{t,\rho}$. It turns out that this situation is closer to the applications as we shall outline below.

As above, we assume that H_0 and H_1 are two complex Hilbert spaces and we set $H{:=}H_0 \oplus H_1$. Moreover, $A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}$ for some densely defined closed linear operator $C\colon \mathrm{dom}(C) \subseteq H_0 \to H_1$ and $\mathcal{M}$ and $\mathcal{N}$ have the form $\mathcal{M} = \begin{pmatrix} \mathcal{M}_{00} & 0 \\ 0 & 0 \end{pmatrix}$ as well as $\mathcal{N} = \begin{pmatrix} \mathcal{N}_{00} & \mathcal{N}_{01} \\ \mathcal{N}_{10} & \mathcal{N}_{11} \end{pmatrix}$ with appropriate linear operators in $L(L_{2,\rho}(\mathbb{R}; H_j), L_{2,\rho}(\mathbb{R}; H_i))$, $i, j \in \{0, 1\}$.

Theorem 5.1. *Let $A, \mathcal{M}$ and $\mathcal{N}$ be as in (4.1)–(4.3). Assume, in addition, that $\partial_{t,\rho}^{-1}\mathcal{M} = \mathcal{M}\partial_{t,\rho}^{-1}$. Moreover, we assume that there is $c > 0$ such that*

$$\operatorname{Re}\langle \mathcal{M}_{00}\phi_0, \phi_0\rangle_{\rho,0} \geq c\langle \phi_0, \phi_0\rangle_{\rho,0} \quad (\phi \in L_{2,\rho}(\mathbb{R}; H_0))$$

and

$$\operatorname{Re}\langle (\partial_{t,\rho}\mathcal{M} + \mathcal{N})\,\phi, \phi\rangle_{\rho,0} \geq c\langle \phi, \phi\rangle_{\rho,0},$$

for all $\phi \in H_\rho^{3/2}(\mathbb{R}; H)$. Finally, we assume that there is $0 \leq \widetilde{c} < c$ and $d > 0$ such that

$$\|[\partial_{t,\rho}^{1/2}, \mathcal{N}]\phi\|_{\rho,0} \leq \widetilde{c}\|\phi\|_{\rho,1/2} + d\|\phi\|_{\rho,0} \quad (\phi \in H_\rho^{1/2}(\mathbb{R}; H)).$$

Then

$$\mathcal{S}_\rho[L_{2,\rho}(\mathbb{R}; H_0) \times H_\rho^{1/2}(\mathbb{R}; H_1)] \subseteq \left(H_\rho^1(\mathbb{R}; H_0) \cap H_\rho^{1/2}(\mathbb{R}; \operatorname{dom}(C))\right) \times \left(L_{2,\rho}(\mathbb{R}; \operatorname{dom}(C^*)) \cap H_\rho^{1/2}(\mathbb{R}; H_1)\right),$$

where $\mathcal{S}_\rho := \overline{\left(\partial_{t,\rho}\mathcal{M} + \mathcal{N} + A\right)}^{-1} \in L(L_{2,\rho}(\mathbb{R}; H))$.

Remark 5.2. The condition $\partial_{t,\rho}^{-1}\mathcal{M} = \mathcal{M}\partial_{t,\rho}^{-1}$ implies $\partial_{t,\rho}^{-1/2}\mathcal{M} = \mathcal{M}\partial_{t,\rho}^{-1/2}$. Indeed, to start off with, the fact that $\mathcal{M}$ commutes with $\partial_{t,\rho}^{-1}$ yields $\mathcal{M}\partial_{t,\rho} \subseteq \partial_{t,\rho}\mathcal{M}$. Hence, $\mathcal{M}\,(\partial_{t,\rho} - 2\rho) \subseteq (\partial_{t,\rho} - 2\rho)\,\mathcal{M}$ and, thus, by Proposition 2.1 (c), we infer $-\left(\partial_{t,\rho}^*\right)^{-1}\mathcal{M} = (\partial_{t,\rho} - 2\rho)^{-1}\,\mathcal{M} \subseteq \mathcal{M}\,(\partial_{t,\rho} - 2\rho)^{-1} = -\mathcal{M}\left(\partial_{t,\rho}^*\right)^{-1}$. Since $(\partial_{t,\rho}^*)^{-1}\mathcal{M}$ is defined everywhere, this inclusion is an equality. Next, by the approximation theorem of Weierstraß, polynomials in z and z^* as continuous functions on $C(V)$ are dense in $C(V)$ endowed with the sup-norm, where $V := \overline{B_{\mathbb{C}}(1/(2\rho), 1/(2\rho))}$. Hence, we find a sequence of polynomials $(z \mapsto p_n(z, z^*))_n$ in z and z^* such that $p_n \to (z \mapsto \sqrt{z})$ uniformly on V as $n \to \infty$. In consequence, using the Fourier–Laplace transformation, we obtain that $p_n\left(\partial_{t,\rho}^{-1}, \left(\partial_{t,\rho}^*\right)^{-1}\right) \to \partial_{t,\rho}^{-1/2}$ in $L(L_{2,\rho}(\mathbb{R}; H))$. Thus, we infer using the commutator properties of $\mathcal{M}$ shown above

$$\partial_{t,\rho}^{-1/2}\mathcal{M} = \lim_{n\to\infty} p_n(\partial_{t,\rho}^{-1}, (\partial_{t,\rho}^*)^{-1})\mathcal{M} = \lim_{n\to\infty} \mathcal{M}p_n(\partial_{t,\rho}^{-1}, (\partial_{t,\rho}^*)^{-1}) = \mathcal{M}\partial_{t,\rho}^{-1/2}.$$

Proof of Theorem 5.1. Let $f \in L_{2,\rho}(\mathbb{R}; H_0)$ and $g \in H_\rho^{1/2}(\mathbb{R}; H_1)$. Moreover, let $(u, v) = \mathcal{S}_\rho(f, g) \in L_{2,\rho}(\mathbb{R}; H)$. We choose $0 < \varepsilon < c - \widetilde{c}$ and $\delta \geq \frac{d^2}{4\varepsilon\sqrt{\rho}}$ and consider the operator $\widetilde{\mathcal{N}} := \mathcal{N} + \delta\partial_{t,\rho}^{-1/2}$. It is clear that

$$\overline{\left(\partial_{t,\rho}\mathcal{M} + \widetilde{\mathcal{N}} + A\right)}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + \delta\partial_{t,\rho}^{-1/2}\begin{pmatrix} u \\ v \end{pmatrix} \in L_{2,\rho}(\mathbb{R}; H_0) \times H_\rho^{1/2}(\mathbb{R}; H)$$

and hence, to show the claim, it suffices to prove that the operators $\mathcal{M}$ and $\widetilde{\mathcal{N}}$ satisfy the assumptions of Theorem 4.1. We first note that $\mathcal{M}' = 0$ and that (4.5) holds by assumption and (4.6) follows from the inequality assumed for $\partial_{t,\rho}\mathcal{M} + \mathcal{N}$ and the fact that $\operatorname{Re}\partial_{t,\rho}^{-1/2} \geq 0$. Thus, it remains to show (4.7).

For doing so, let $\phi \in H_\rho^{3/2}(\mathbb{R};H)$. Since $\mathcal{M}$ commutes with $\partial_{t,\rho}^{-1}$, it follows that it also commutes with $\partial_{t,\rho}^{-1/2}$ (see Remark 5.2) and thus

$$\partial_{t,\rho}^{3/2}\mathcal{M}\phi = \partial_{t,\rho}^{3/2}\mathcal{M}\partial_{t,\rho}^{-1/2}\partial_{t,\rho}^{1/2}\phi = \partial_{t,\rho}\mathcal{M}\partial_{t,\rho}^{1/2}\phi.$$

Hence, we can compute

$$\begin{aligned}
\operatorname{Re}\langle\Big(\partial_{t,\rho}\mathcal{M}+\widetilde{\mathcal{N}}\Big)\phi,\phi\rangle_{\rho,1/2} &= \operatorname{Re}\langle\partial_{t,\rho}^{3/2}\mathcal{M}\phi+\partial_{t,\rho}^{1/2}\widetilde{\mathcal{N}}\phi,\partial_{t,\rho}^{1/2}\phi\rangle_{\rho,0}\\
&= \operatorname{Re}\langle\partial_{t,\rho}\mathcal{M}\partial_{t,\rho}^{1/2}\phi+\partial_{t,\rho}^{1/2}\mathcal{N}\phi,\partial_{t,\rho}^{1/2}\phi\rangle_{\rho,0}+\delta\operatorname{Re}\langle\phi,\partial_{t,\rho}^{1/2}\phi\rangle_{\rho,0}\\
&\geq \operatorname{Re}\langle(\partial_{t,\rho}\mathcal{M}+\mathcal{N})\partial_{t,\rho}^{1/2}\phi,\partial_{t,\rho}^{1/2}\phi\rangle_{\rho,0}+\operatorname{Re}\langle[\partial_{t,\rho}^{1/2},\mathcal{N}]\phi,\partial_{t,\rho}^{1/2}\phi\rangle_{\rho,0}+\sqrt{\rho}\delta\|\phi\|_{\rho,0}^2\\
&\geq c\|\phi\|_{\rho,1/2}^2-\|[\partial_{t,\rho},\mathcal{N}]\phi\|_{\rho,0}\|\phi\|_{\rho,1/2}+\sqrt{\rho}\delta\|\phi\|_{\rho,0}^2\\
&\geq (c-\widetilde{c})\,\|\phi\|_{\rho,1/2}^2-d\|\phi\|_{\rho,0}\|\phi\|_{\rho,1/2}+\sqrt{\rho}\delta\|\phi\|_{\rho,0}^2\\
&\geq (c-\widetilde{c}-\varepsilon)\|\phi\|_{\rho,1/2}^2+(\sqrt{\rho}\delta-\frac{d^2}{4\varepsilon})\|\phi\|_{\rho,0}^2\geq (c-\widetilde{c}-\varepsilon)\|\phi\|_{\rho,1/2}^2,
\end{aligned}$$

which shows (4.7). □

If the coefficient operators, $\mathcal{M}$ and $\mathcal{N}$, act in a 'physically meaningful manner'; that is, if they are causal (see definition below), then the latter result (as well as the other main result Theorem 4.1) also imply a maximal regularity result locally in time. For this we need a closer look into the well-posedness result Theorem 2.8, which in turn prerequisites the following notion. We define

$$S_c(\mathbb{R};H) := \operatorname{lin}\{f\colon \mathbb{R}\to H;\, f \text{ simple function with compact support}\}.$$

Definition 5.3. Let K_0, K_1 be Hilbert spaces, $\rho_0\in\mathbb{R}$, and

$$\mathcal{C}\colon\ S_c(\mathbb{R};K_0)\to\bigcap_{\rho\geq\rho_0}L_{2,\rho}(\mathbb{R};K_1)$$

linear. Then we call $\mathcal{C}$ *evolutionary (at ρ_0)*, if, for all $\rho\geq\rho_0$, $\mathcal{C}$ admits a continuous extension $\mathcal{C}_\rho\in L(L_{2,\rho}(\mathbb{R};K_0),L_{2,\rho}(\mathbb{R};K_1))$ satisfying

$$\sup_{\rho\geq\rho_0}\|\mathcal{C}_\rho\|_{L(L_{2,\rho}(\mathbb{R};K_0),L_{2,\rho}(\mathbb{R};K_1))}<\infty.$$

We gather two results important for evolutionary mappings of the type discussed in the latter definition. For the intricacies of the interplay of causality and closure of operators, we refer to [26].

Proposition 5.4. ([28, Remark 2.1.5] or [22, Lemma 4.2.5 (a)]) *Let K_0, K_1 be Hilbert spaces, $\rho_0\in\mathbb{R}$, and $\mathcal{C}\colon\ S_c(\mathbb{R};K_0)\to\bigcap_{\rho\geq\rho_0}L_{2,\rho}(\mathbb{R};K_1)$ linear and $\mathcal{C}$ evolutionary at ρ_0. Then $\mathcal{C}_\rho$ is* causal *for all $\rho\geq\rho_0$; that is, for all $t\in\mathbb{R}$ and $f\in L_{2,\rho}(\mathbb{R};K_0)$ we have*

$$\operatorname{spt} f\subseteq[t,\infty)\Rightarrow\operatorname{spt}\mathcal{C}_\rho f\subseteq[t,\infty).$$

Theorem 5.5. ([28, Theorem 3.4.6] or [27, Theorem 3.4]) *In addition to the assumptions in Theorem 2.8, assume that $\mathcal{M},\mathcal{M}'$ and $\mathcal{N}$ are evolutionary. Then*

$$\mathcal{S}_\rho=\overline{\big(\partial_{t,\rho}\mathcal{M}_\rho+\mathcal{N}_\rho+A\big)}^{-1}\in L(L_{2,\rho}(\mathbb{R};H))$$

is causal.

Having presented the remaining technical ingredients for the localisation on bounded time-intervals, we can present the local maximal regularity statement next.

Corollary 5.6. *In addition to the assumptions in Theorem 5.1, assume that $\mathcal{M}$ and $\mathcal{N}$ are evolutionary. Let $T \in]0,\infty[$. Then there exists $\kappa \geq 0$ such that for all $f \in L_{2,\rho}(\mathbb{R};H_0)$ and $g \in H^{1/2}_\rho(\mathbb{R};H_1)$ with $\operatorname{spt} f, \operatorname{spt} g \subseteq [0,T[$, $(u,v) = \mathcal{S}_\rho(f,g)$ are supported in $[0,\infty)$ only and satisfy*

$$\begin{aligned}&\|u\|_{H^1[0,T)} + \|Cu\|_{H^{1/2}[0,T)} + \|C^*v\|_{L_2[0,T)} + \|v\|_{H^{1/2}[0,T)}\\ &\quad\leq \kappa\left(\|f\|_{L_2[0,T)} + \|g\|_{H^{1/2}[0,T)}\right).\end{aligned}$$

Proof. Firstly, observe that $H^{1/2}_\rho(\mathbb{R};H) \ni u \mapsto u|_{[0,T)} \in H^{1/2}([0,T);H)$ continuously, by complex interpolation, see also Remark 2.4. Next, let $\phi \in C_c^\infty(\mathbb{R})$ with $0 \leq \phi \leq 1$ and

$$\phi(t) = \begin{cases} 0, & t \leq -T/2,\\ 1 & 0 \leq t \leq T,\\ 0 & t \geq 3T/2; \end{cases} \text{ define } \hat{h} := \phi(\cdot) \begin{cases} h(-\cdot), & \text{on }]-\infty,0],\\ h(\cdot), & \text{on }]0,T[,\\ h(T-\cdot) & \text{on } [T,\infty[. \end{cases}$$

for $h \in L_2(]0,T[;H)$. Then it is not difficult to see that $E\colon h \mapsto \hat{h}$ is continuous as a mapping from $L_2(]0,T[;H)$ to $L_2(\mathbb{R};H)$ and as a mapping from $H^1(]0,T[;H)$ to $H^1_\rho(\mathbb{R};H)$. Thus, by interpolation, we infer continuity as a mapping from $H^{1/2}(]0,T[;H)$ to $H^{1/2}_\rho(\mathbb{R};H)$. Thus, there exists $\kappa \geq 0$ such that for all $g \in H^{1/2}_\rho(\mathbb{R};H)$ with $\operatorname{spt} g \subseteq [0,T[$ we have $\|g\|_{\rho,\frac{1}{2}} \leq \kappa \|g\|_{H^{1/2}[0,T[}$. This estimate together with Theorem 5.1 then implies the assertion. □

Remark 5.7. Note that a prototype of evolutionary operators are operators defined as multiplication by a function, see also [28, Example 2.1.1]. This prototype will be discussed next.

5.2. Commutators for Multiplication Operators

In this subsection we inspect the conditions on the operator $\mathcal{N}$ assumed in Theorem 5.1 for the concrete case of $\mathcal{N}$ being a multiplication operator. More precisely, we assume the following: Let $N\colon \mathbb{R} \to L(H)$ be a strongly measurable bounded mapping. Then N induces an evolutionary operator

$$\mathcal{N}\colon S_c(\mathbb{R};H) \to \bigcap_{\rho\geq 0} L_{2,\rho}(\mathbb{R};H), \quad f \mapsto (t \mapsto N(t)f(t))$$

with $\|\mathcal{N}_\rho\|_{L(L_{2,\rho}(\mathbb{R};H))} = \|N\|_\infty$ for all $\rho \geq 0$. Note that all continuous extensions $\mathcal{N}_\rho$, $\rho \geq 0$, act as multiplication by N. We provide a formula for $\partial^{1/2}_{t,\rho}\phi$ for regular ϕ first.

Lemma 5.8. *Let $\rho > 0$ and $\phi \in H^1_\rho(\mathbb{R};H)$. Then*

$$\left(\partial^{1/2}_{t,\rho}\phi\right)(t) = \frac{1}{2\Gamma(1/2)} \int_{-\infty}^t (t-s)^{-3/2}(\phi(t)-\phi(s))\,\mathrm{d}s \quad (t \in \mathbb{R} \text{ a.e.}).$$

Proof. We have $\partial_{t,\rho}^{1/2}\phi = \partial_{t,\rho}^{-1/2}\partial_{t,\rho}\phi = \partial_{t,\rho}^{-1/2}\phi'$ (see also Proposition 2.2) and thus, by Proposition 2.2

$$\begin{aligned}
\left(\partial_{t,\rho}^{1/2}\phi\right)(t) &= \frac{1}{\Gamma(1/2)}\int_{-\infty}^{t}(t-s)^{-1/2}\phi'(s)\,\mathrm{d}s\\
&= \frac{1}{\Gamma(1/2)}\int_0^\infty s^{-1/2}\phi'(t-s)\,\mathrm{d}s = \frac{1}{2\Gamma(1/2)}\int_0^\infty\int_s^\infty r^{-3/2}\,\mathrm{d}r\,\phi'(t-s)\,\mathrm{d}s\\
&= \frac{1}{2\Gamma(1/2)}\int_0^\infty r^{-3/2}\int_0^r \phi'(t-s)\,\mathrm{d}s\,\mathrm{d}r\\
&= \frac{1}{2\Gamma(1/2)}\int_0^\infty r^{-3/2}(\phi(t)-\phi(t-r))\,\mathrm{d}r\\
&= \frac{1}{2\Gamma(1/2)}\int_{-\infty}^{t}(t-s)^{-3/2}(\phi(t)-\phi(s))\,\mathrm{d}s \quad (t\in\mathbb{R}\text{ a.e.}).
\end{aligned}$$

□

Using this expression, we can prove our first result on commutators with the fractional derivative.

Proposition 5.9. *Let $\rho_0 > 0$ and assume that $[\mathcal{N}_{\rho_0}, \partial_{t,\rho_0}^{1/2}]$ is bounded as an operator on $L_{2,\rho_0}(\mathbb{R};H)$. Then for each $\rho \geq \rho_0$ we have $\mathcal{N}_\rho[H_\rho^{1/2}(\mathbb{R};H)] \subseteq H_\rho^{1/2}(\mathbb{R};H)$ and*

$$\|[\mathcal{N}_\rho, \partial_{t,\rho}^{1/2}]\|_{L(L_{2,\rho}(\mathbb{R};H))} \leq \|[\mathcal{N}_{\rho_0}, \partial_{t,\rho_0}^{1/2}]\|_{L(L_{2,\rho_0}(\mathbb{R};H))} + 2\|N\|_\infty\left(\sqrt{\rho}-\sqrt{\rho_0}\right).$$

Proof. Let $\rho \geq \rho_0$. To start off with, we prove the following statement:

$$\phi \in H_\rho^{1/2}(\mathbb{R};H) \;\Leftrightarrow\; \mathrm{e}^{(\rho_0-\rho)\cdot}\phi \in H_{\rho_0}^{1/2}(\mathbb{R};H).$$

For this, let $\phi \in H_\rho^{1/2}(\mathbb{R};H)$; that is, $(\mathrm{i}\,\mathrm{m}+\rho)^{\frac{1}{2}}\mathcal{L}_\rho\phi \in L_2(\mathbb{R};H)$. Note that $\mathcal{L}_{\rho_0}\mathrm{e}^{(\rho_0-\rho)\cdot}\phi = \mathcal{L}_\rho\phi$ and hence, it suffices to show that

$$(\mathrm{i}\,\mathrm{m}+\rho_0)^{\frac{1}{2}}\mathcal{L}_\rho\phi \in L_2(\mathbb{R};H).$$

The latter however is clear, since $\left(t\mapsto(\mathrm{i}t+\rho_0)^{1/2}(\mathrm{i}t+\rho)^{-1/2}\right)\in L_\infty(\mathbb{R};H)$. Since this argument is completely symmetric in ρ and ρ_0, the asserted equivalence holds.
If now $\phi \in H_\rho^{1/2}(\mathbb{R};H)$, we infer that

$$\mathrm{e}^{(\rho_0-\rho)\cdot}\mathcal{N}_\rho\phi = \mathcal{N}_{\rho_0}\mathrm{e}^{(\rho_0-\rho)\cdot}\phi \in H_{\rho_0}^{1/2}(\mathbb{R};H)$$

and thus, $\mathcal{N}_\rho[H_\rho^{1/2}(\mathbb{R};H)] \subseteq H_\rho^{1/2}(\mathbb{R};H)$. To compute the norm of $[\mathcal{N}_\rho, \partial_{t,\rho}^{1/2}]$, let $\phi \in C_c^\infty(\mathbb{R};H)$. Then we estimate

$$\begin{aligned}
\|[\mathcal{N}_\rho, \partial_{t,\rho}^{1/2}]\phi\|_{\rho,0} &= \|\mathrm{e}^{(\rho_0-\rho)\cdot}\left(\mathcal{N}_\rho\partial_{t,\rho}^{1/2} - \partial_{t,\rho}^{1/2}\mathcal{N}_\rho\right)\phi\|_{\rho_0,0}\\
&= \|\mathcal{N}_{\rho_0}\mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2}\phi - \mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2}\mathcal{N}_\rho\phi\|_{\rho_0,0}\\
&\leq \|[\mathcal{N}_{\rho_0}, \partial_{t,\rho_0}^{1/2}]\mathrm{e}^{(\rho_0-\rho)\cdot}\phi\|_{\rho_0,0} + \|\mathcal{N}_{\rho_0}\left(\mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2} - \partial_{t,\rho_0}^{1/2}\mathrm{e}^{(\rho_0-\rho)\cdot}\right)\phi\|_{\rho_0,0} +\\
&\quad + \|\left(\mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2} - \partial_{t,\rho_0}^{1/2}\mathrm{e}^{(\rho_0-\rho)\cdot}\right)\mathcal{N}_\rho\phi\|_{\rho_0,0}.
\end{aligned}$$

Hence, we need to estimate the norm of the operator $\mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2}-\partial_{t,\rho_0}^{1/2}\mathrm{e}^{(\rho_0-\rho)\cdot}$. For this, let $\psi\in C_c^\infty(\mathbb{R};H)$ and compute, using Lemma 5.8

$$\begin{aligned}
&(\mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2}\psi-\partial_{t,\rho_0}^{1/2}\mathrm{e}^{(\rho_0-\rho)\cdot}\psi)(t)\\
&\quad=\mathrm{e}^{(\rho_0-\rho)t}\frac{1}{2\Gamma(1/2)}\int_{-\infty}^{t}(t-s)^{-3/2}(\psi(t)-\psi(s))\,\mathrm{d}s-\\
&\qquad-\frac{1}{2\Gamma(1/2)}\int_{-\infty}^{t}(t-s)^{-3/2}(\mathrm{e}^{(\rho_0-\rho)t}\psi(t)-\mathrm{e}^{(\rho_0-\rho)s}\psi(s))\,\mathrm{d}s\\
&\quad=\frac{1}{2\Gamma(1/2)}\int_{-\infty}^{t}(t-s)^{-3/2}\left(\mathrm{e}^{(\rho_0-\rho)s}-\mathrm{e}^{(\rho_0-\rho)t}\right)\psi(s)\,\mathrm{d}s\\
&\quad=\frac{1}{2\Gamma(1/2)}\int_{-\infty}^{t}(t-s)^{-3/2}\left(1-\mathrm{e}^{(\rho_0-\rho)(t-s)}\right)\mathrm{e}^{(\rho_0-\rho)s}\psi(s)\,\mathrm{d}s\\
&\quad=(k_{\rho_0-\rho}*\mathrm{e}^{(\rho_0-\rho)\cdot}\psi)(t),
\end{aligned}$$

where $k_\mu(t):=\frac{1}{2\Gamma(1/2)}\chi_{\mathbb{R}_{\geq 0}}(t)t^{-3/2}(1-\mathrm{e}^{\mu t})$ for $t,\mu\in\mathbb{R}$. Note that for $\mu=(\rho_0-\rho)\leq 0$ the function $k_{\rho_0-\rho}$ is positive and hence, using the convolution theorem, we obtain

$$\|k_{\rho_0-\rho}*\|_{L(L_{2,\rho_0}(\mathbb{R};H))}=\int_{\mathbb{R}}k_{\rho_0-\rho}(t)\mathrm{e}^{-\rho_0 t}\,\mathrm{d}t.$$

We compute using the integral representation of the Γ-function

$$\begin{aligned}
\int_{\mathbb{R}}k_{\rho_0-\rho}(t)\mathrm{e}^{-\rho_0 t}\,\mathrm{d}t&=\frac{1}{2\Gamma(1/2)}\int_0^\infty t^{-3/2}(1-\mathrm{e}^{(\rho_0-\rho)t})\mathrm{e}^{-\rho_0 t}\,\mathrm{d}t\\
&=\frac{1}{\Gamma(1/2)}\int_0^\infty t^{-1/2}\left(-\rho_0\mathrm{e}^{-\rho_0 t}+\rho\mathrm{e}^{-\rho t}\right)\,\mathrm{d}t=\sqrt{\rho}-\sqrt{\rho_0}
\end{aligned}$$

and thus,

$$\begin{aligned}
&\|(\mathrm{e}^{(\rho_0-\rho)\cdot}\partial_{t,\rho}^{1/2}-\partial_{t,\rho_0}^{1/2}\mathrm{e}^{(\rho_0-\rho)\cdot})\phi\|_{\rho_0,0}\leq(\sqrt{\rho}-\sqrt{\rho_0})\,\|\mathrm{e}^{(\rho_0-\rho)\cdot}\phi\|_{\rho_0,0}\\
&\quad=(\sqrt{\rho}-\sqrt{\rho_0})\,\|\phi\|_{\rho,0}.
\end{aligned}\tag{5.1}$$

Summarising, we obtain the estimate

$$\|[\mathcal{N}_\rho,\partial_{t,\rho}^{1/2}]\phi\|_{\rho,0}\leq\left(\|[\mathcal{N}_{\rho_0},\partial_{t,\rho_0}^{1/2}]\|_{L(L_{2,\rho_0}(\mathbb{R};H))}+2\|N\|_\infty(\sqrt{\rho}-\sqrt{\rho_0})\right)\|\phi\|_{\rho,0},$$

which shows the claim. □

The next proposition is devoted to the limit case $\rho_0=0$, which is the case usually treated in the literature.

Proposition 5.10. *Assume that $[\mathcal{N}_0,\partial_{t,0}^{1/2}]$ is bounded as an operator on $L_2(\mathbb{R};H)$. Then for each $\rho\geq 0$ we have $\mathcal{N}[H_\rho^{1/2}(\mathbb{R};H)]\subseteq H_\rho^{1/2}(\mathbb{R};H)$ and*

$$\|[\mathcal{N}_\rho,\partial_{t,\rho}^{1/2}]\|_{L(L_{2,\rho}(\mathbb{R};H))}\leq\|[\mathcal{N}_0,\partial_{t,0}^{1/2}]\|_{L(L_2(\mathbb{R};H))}+2\|N\|_\infty\sqrt{\rho}.$$

Proof. Similar to the proof of Proposition 5.9, at first we show

$$\phi\in H_\rho^{1/2}(\mathbb{R};H)\iff\mathrm{e}^{-\rho\cdot}\phi\in H^{1/2}(\mathbb{R};H).$$

For this, let $\phi \in H^{1/2}_\rho(\mathbb{R};H)$. Then $(\mathrm{i}\,\mathrm{m}+\rho)^{1/2}\mathcal{L}_\rho\phi = (\mathrm{i}\,\mathrm{m}+\rho)^{1/2}\mathcal{F}\mathrm{e}^{-\rho\cdot}\phi \in L_2(\mathbb{R};H)$. The latter implies $(\mathrm{i}\,\mathrm{m})^{1/2}\mathcal{F}\mathrm{e}^{-\rho\cdot}\phi \in L_2(\mathbb{R};H)$ and hence, $\mathrm{e}^{-\rho\cdot}\phi \in H^{1/2}(\mathbb{R};H)$. If, on the other hand, $\mathrm{e}^{-\rho\cdot}\phi \in H^{1/2}(\mathbb{R};H)$, then $\phi \in L_{2,\rho}(\mathbb{R};H)$ and $(\mathrm{i}\,\mathrm{m})^{1/2}\mathcal{L}_\rho\phi = (\mathrm{i}\,\mathrm{m})^{1/2}\mathcal{F}\mathrm{e}^{-\rho\cdot}\phi \in L_2(\mathbb{R};H)$ and hence,

$$\begin{aligned}
&\int_{\mathbb{R}} \|(\mathrm{i}t+\rho)^{1/2}\left(\mathcal{L}_\rho\phi\right)(t)\|_H^2\,\mathrm{d}t\\
&\quad= \int_{[-1,1]} \|(\mathrm{i}t+\rho)^{1/2}\left(\mathcal{L}_\rho\phi\right)(t)\|_H^2\,\mathrm{d}t + \int_{|t|>1} \|(\mathrm{i}t+\rho)^{1/2}\left(\mathcal{L}_\rho\phi\right)(t)\|_H^2\,\mathrm{d}t\\
&\quad\le (1+\rho^2)^{1/2}\|\phi\|^2_{L_{2,\rho}(\mathbb{R};H)} + (1+\rho^2)^{1/2}\int_{\mathbb{R}} \|(\mathrm{i}t)^{1/2}\left(\mathcal{L}_\rho\phi\right)(t)\|_H^2\,\mathrm{d}t < \infty
\end{aligned}$$

and thus, $\phi \in H^{1/2}_\rho(\mathbb{R};H)$.

By the same argumentation as in the proof of Proposition 5.9 we infer $\mathcal{N}[H^{1/2}_\rho(\mathbb{R};H)] \subseteq H^{1/2}_\rho(\mathbb{R};H)$. Following the lines of the proof of Proposition 5.9, we need to find an estimate for $\|\left(\mathrm{e}^{-\rho\cdot}\partial^{1/2}_{t,\rho} - \partial^{1/2}_{t,0}\mathrm{e}^{-\rho\cdot}\right)\phi\|_{0,0}$, for $\phi \in C^\infty_c(\mathbb{R};H)$. The main problem in proving such an estimate is that we do not have an explicit integral representation for $\partial^{1/2}_{t,0}$ thus far. However, we have

$$\begin{aligned}
\partial^{1/2}_{t,0}\psi &= \mathcal{F}^*(\mathrm{i}\,\mathrm{m})^{1/2}\mathcal{F}\psi = \lim_{\rho_0\to 0}\mathcal{F}^*(\mathrm{i}\,\mathrm{m}+\rho_0)^{1/2}\mathcal{F}\psi\\
&= \lim_{\rho_0\to 0}\mathrm{e}^{-\rho_0\cdot}\mathcal{L}^*_{\rho_0}(\mathrm{i}\,\mathrm{m}+\rho_0)^{1/2}\mathcal{L}_{\rho_0}\mathrm{e}^{\rho_0\cdot}\psi = \lim_{\rho_0\to 0}\mathrm{e}^{-\rho_0\cdot}\partial^{1/2}_{t,\rho_0}\mathrm{e}^{\rho_0\cdot}\psi
\end{aligned}$$

with convergence in $L_2(\mathbb{R};H)$ for each $\psi \in H^{1/2}(\mathbb{R};H)$, where we have used dominated convergence in the second line. Thus, for $\phi \in C^\infty_c(\mathbb{R};H)$ we have that

$$\begin{aligned}
\|\left(\mathrm{e}^{-\rho\cdot}\partial^{1/2}_{t,\rho} - \partial^{1/2}_{t,0}\mathrm{e}^{-\rho\cdot}\right)\phi\|_{0,0} &= \lim_{\rho_0\to 0}\|\left(\mathrm{e}^{-\rho\cdot}\partial^{1/2}_{t,\rho} - \mathrm{e}^{-\rho_0\cdot}\partial^{1/2}_{t,\rho_0}\mathrm{e}^{(\rho_0-\rho)\cdot}\right)\phi\|_{0,0}\\
&= \lim_{\rho_0\to 0}\|\left(\mathrm{e}^{(\rho_0-\rho)\cdot}\partial^{1/2}_{t,\rho} - \partial^{1/2}_{t,\rho_0}\mathrm{e}^{(\rho_0-\rho)\cdot}\right)\phi\|_{\rho_0,0}\\
&\le \lim_{\rho_0\to 0}\left(\sqrt{\rho}-\sqrt{\rho_0}\right)\|\phi\|_{\rho,0} = \sqrt{\rho}\|\phi\|_{\rho,0},
\end{aligned}$$

where we have used (5.1). Following the lines of the proof of Proposition 5.9 the assertion follows. □

Remark 5.11. Note that Theorem 5.1 in combination with Proposition 5.9 or Proposition 5.10 yields maximal regularity of the corresponding evolutionary equation, if $\mathcal{N}$ has a bounded commutator for some $\rho \ge 0$. In particular, this covers the case treated in [3] (see also Sect. 6.1 below).

Our next goal is to prove the following proposition.

Proposition 5.12. *Assume that*

$$C := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|N(t)-N(s)\|^2}{|t-s|^{2+\delta}}\,\mathrm{d}t\,\mathrm{d}s < \infty \tag{5.2}$$

for some $\delta > 0$. Then

$$\forall \varepsilon > 0\,\exists c > 0\,\forall \phi \in H^{1/2}_\rho(\mathbb{R};H)\colon \|[\partial^{1/2}_{t,\rho},\mathcal{N}_\rho]\phi\|_{\rho,0} \le \varepsilon\|\phi\|_{\rho,1/2} + c\|\phi\|_{\rho,0}.$$

Remark 5.13. Assumption (5.2) is the main assumption imposed in [8, Corollary 1.1].

In order to prove Proposition 5.12, we want to apply Lemma 5.8 to derive an integral expression for the commutator. Since Lemma 5.8 just holds for functions in $H^1_\rho(\mathbb{R};H)$, we need to regularise $\mathcal{N}$.

Lemma 5.14. *For $\varepsilon>0$ we define*

$$N_\varepsilon(t):=\frac{1}{\varepsilon}\int_t^{t+\varepsilon} N(s)\,\mathrm{d}s \quad (t\in\mathbb{R}),$$

where the integral is defined in the strong sense. We denote the associated multiplication operator by $\mathcal{N}_\varepsilon$. Then the following statements hold:

(a) *For each $\varepsilon>0$ we have $\|N_\varepsilon\|_\infty\leq\|N\|_\infty$ and $\mathcal{N}_{\varepsilon,\rho}[H^1_\rho(\mathbb{R};H)]\subseteq H^1_\rho(\mathbb{R};H)$,*
(b) *$\mathcal{N}_{\varepsilon,\rho}\to\mathcal{N}_\rho$ strongly in $L_{2,\rho}(\mathbb{R};H)$ as $\varepsilon\to 0$,*
(c) *If there exist $c_1,c_2\geq 0$ such that $\|[\partial_{t,\rho}^{1/2},\mathcal{N}_{\varepsilon,\rho}]\phi\|_{\rho,0}\leq c_1\|\phi\|_{\rho,1/2}+c_2\|\phi\|_{\rho,0}$ for all $\phi\in H^{1/2}_\rho(\mathbb{R};H)$ and $\varepsilon>0$, then $\mathcal{N}_\rho[H^{1/2}_\rho(\mathbb{R};H)]\subseteq H^{1/2}_\rho(\mathbb{R};H)$ with $\|[\partial_{t,\rho}^{1/2},\mathcal{N}_\rho]\phi\|_{\rho,0}\leq c_1\|\phi\|_{\rho,1/2}+c_2\|\phi\|_{\rho,0}$ for all $\phi\in H^{1/2}_\rho(\mathbb{R};H)$.*

Proof. (a) Let $\varepsilon>0$. The estimate $\|N_\varepsilon\|_\infty\leq\|N\|_\infty$ is obvious. Let $u\in H^1_\rho(\mathbb{R};H)$. In order to show $\mathcal{N}_{\varepsilon,\rho}u\in H^1_\rho(\mathbb{R};H)$, let $\phi\in C_c^\infty(\mathbb{R})$. Then we compute

$$\begin{aligned}
\int_{\mathbb{R}}(\mathcal{N}_{\varepsilon,\rho}u)(t)\phi'(t)\,\mathrm{d}t &= \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_t^{t+\varepsilon} N(s)u(t)\phi'(t)\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_0^{\varepsilon} N(s+t)u(t)\phi'(t)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{\varepsilon}\int_0^{\varepsilon}\int_{\mathbb{R}} N(s+t)u(t)\phi'(t)\,\mathrm{d}t\,\mathrm{d}s\\
&= \frac{1}{\varepsilon}\int_0^{\varepsilon}\int_{\mathbb{R}} N(r)u(r-s)\phi'(r-s)\,\mathrm{d}r\,\mathrm{d}s\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}} N(r)\int_0^{\varepsilon} u(r-s)\phi'(r-s)\,\mathrm{d}s\,\mathrm{d}r\\
&= -\frac{1}{\varepsilon}\int_{\mathbb{R}} N(r)\int_0^{\varepsilon}\partial_{t,\rho}u(r-s)\phi(r-s)\,\mathrm{d}s\,\mathrm{d}r\\
&\quad -\frac{1}{\varepsilon}\int_{\mathbb{R}} N(r)\left(u(r-\varepsilon)\phi(r-\varepsilon)-u(r)\phi(r)\right)\mathrm{d}r\\
&= -\int_{\mathbb{R}}(\mathcal{N}_{\varepsilon,\rho}\partial_{t,\rho}u)(t)\phi(t)\,\mathrm{d}t-\frac{1}{\varepsilon}\int_{\mathbb{R}}(N(t+\varepsilon)-N(t))u(t)\phi(t)\,\mathrm{d}t.
\end{aligned}$$

Since $(t\mapsto(\mathcal{N}_{\varepsilon,\rho}\partial_{t,\rho}u)(t)+(N(t+\varepsilon)-N(t))\,u(t))\in L_{2,\rho}(\mathbb{R};H)$, the claim follows from [22, Proposition 4.1.1].

(b) Let $\psi\in L_{2,\rho}(\mathbb{R})$ and $x\in H$. Then

$$N_\varepsilon(t)x\psi(t)-N(t)x\psi(t)=\frac{1}{\varepsilon}\int_t^{t+\varepsilon}(N(s)-N(t))x\,\mathrm{d}s\,\psi(t)\to 0\quad(\varepsilon\to 0)$$

for almost every $t\in\mathbb{R}$ by Lebesgue's differentiation theorem. Moreover,

$$\|N_\varepsilon(t)x\psi(t)-N(t)x\psi(t)\|_H\leq 2\|N\|_\infty\|\psi(t)x\|_H$$

and thus, $\mathcal{N}_\varepsilon(\psi x) \to \mathcal{N}(\psi x)$ in $L_{2,\rho}(\mathbb{R}; H)$ by dominated convergence. Since

$$\|\mathcal{N}_{\varepsilon,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H))} = \|N_\varepsilon\|_\infty \leq \|N\|_\infty$$

for each $\varepsilon > 0$, the strong convergence follows, since $\operatorname{lin}\{\psi x\,;\, \psi \in L_{2,\rho}(\mathbb{R}), x \in H\}$ lies dense in $L_{2,\rho}(\mathbb{R}; H)$.

(c) For $\phi \in H^{1/2}_\rho(\mathbb{R}; H)$ we estimate

$$\begin{aligned}\|\partial^{1/2}_{t,\rho}\mathcal{N}_{\varepsilon,\rho}\phi\|_{\rho,0} &\leq \|\mathcal{N}_{\varepsilon,\rho}\partial^{1/2}_{t,\rho}\phi\|_{\rho,0} + c_1\|\phi\|_{\rho,1/2} + c_2\|\phi\|_{\rho,0}\\ &\leq (\|N\|_\infty + c_1)\|\phi\|_{\rho,1/2} + c_2\|\phi\|_{\rho,0}.\end{aligned}$$

Hence, the family $(\mathcal{N}_{\varepsilon,\rho}\phi)_{\varepsilon>0}$ is bounded in $H^{1/2}_\rho(\mathbb{R}; H)$ and thus, w.l.o.g. it converges weakly in $H^{1/2}_\rho(\mathbb{R}; H)$ as $\varepsilon \to 0$. Since $\mathcal{N}_{\varepsilon,\rho}\phi \to \mathcal{N}_\rho\phi$ in $L_{2,\rho}(\mathbb{R}; H)$ as $\varepsilon \to 0$ by (b), we derive that $\mathcal{N}_\rho\phi \in H^{1/2}_\rho(\mathbb{R}; H)$ and $\partial^{1/2}_{t,\rho}\mathcal{N}_{\varepsilon,\rho}\phi \rightharpoonup \partial^{1/2}_{t,\rho}\mathcal{N}_\rho\phi$ in $L_{2,\rho}(\mathbb{R}; H)$. Hence,

$$\|[\partial^{1/2}_{t,\rho}, \mathcal{N}_\rho]\phi\|_{\rho,0} \leq \lim_{\varepsilon\to 0}\|[\partial^{1/2}_{t,\rho}, \mathcal{N}_{\varepsilon,\rho}]\phi\|_{\rho,0} \leq c_1\|\phi\|_{\rho,1/2} + c_2\|\phi\|_{\rho,0}.$$

Proof of Proposition 5.12. For the proof, we follow the rationale presented in [8, Lemma 5.3].Therefore, we rather sketch the arguments here and refer to [25, proof of Proposition 5.12] for the details. We first prove this assertion for the case $\mathcal{N}[H^1_\rho(\mathbb{R}; H)] \subseteq H^1_\rho(\mathbb{R}; H)$. Let $\phi \in H^1_\rho(\mathbb{R}; H)$. Then we get by Lemma 5.8

$$[\partial^{1/2}_{t,\rho}, \mathcal{N}_\rho]\phi(t) = \frac{1}{2\Gamma(1/2)}\int_{-\infty}^t (t-s)^{-3/2}(N(t) - N(s))\phi(s)\,\mathrm{d}s \quad (t \in \mathbb{R}).$$

Let now $0 < \varepsilon \leq 1$. Then

$$\begin{aligned}&\|[\partial^{1/2}_{t,\rho}, \mathcal{N}_\rho]\phi\|_{\rho,0}\\ &= \frac{1}{2\Gamma(1/2)}\left(\int_{\mathbb{R}}\left\|\int_{-\infty}^t (t-s)^{-3/2}(N(t) - N(s))\phi(s)\,\mathrm{d}s\right\|^2_H \mathrm{e}^{-2\rho t}\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2\Gamma(1/2)}\left(\int_{\mathbb{R}}\left\|\int_{t-\varepsilon}^t (t-s)^{-3/2}(N(t) - N(s))\phi(s)\,\mathrm{d}s\right\|^2_H \mathrm{e}^{-2\rho t}\,\mathrm{d}t\right)^{\frac{1}{2}} +\\ &\quad + \frac{1}{2\Gamma(1/2)}\left(\int_{\mathbb{R}}\left\|\int_{-\infty}^{t-\varepsilon} (t-s)^{-3/2}(N(t) - N(s))\phi(s)\,\mathrm{d}s\right\|^2_H \mathrm{e}^{-2\rho t}\,\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

Using Young's inequality as in [8, Lemma 5.3], the second integral can be estimated by

$$2\|N\|_\infty c\|\phi\|_{\rho,0}, \text{ where } c := \int_\varepsilon^\infty s^{-3/2}\mathrm{e}^{-\rho s}\,\mathrm{d}s \leq \varepsilon^{-3/2}\frac{1}{\rho}.$$

For estimating the first integral, let $\alpha \in]1-\delta, 1[$. Then

$$\left(\int_{\mathbb{R}} \left\| \int_{t-\varepsilon}^{t} (t-s)^{-3/2}(N(t)-N(s))\phi(s)\,\mathrm{d}s \right\|_H^2 \mathrm{e}^{-2\rho t}\,\mathrm{d}t\right)^{\frac{1}{2}} \leq$$
$$\left(\tfrac{1}{1-\alpha}\varepsilon^{1-\alpha}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \int_{t-\varepsilon}^{t} (t-s)^{-3+\alpha} \|N(t)-N(s)\|_{L(H)}^2 \|\phi(s)\|_H^2\,\mathrm{d}s\,\mathrm{e}^{-2\rho t}\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Similarly to [8, Lemma 5.3] choose $p' \in]2, 2\frac{2+\delta}{3-\alpha}]$ and $p \in]2,\infty[$ such that $\frac{1}{p}+\frac{1}{p'}=\frac{1}{2}$. Then Hölder's inequality (see Remark 2.5(b)) yields

$$\left(\int_{\mathbb{R}} \int_{t-\varepsilon}^{t} (t-s)^{-3+\alpha} \|N(t)-N(s)\|_{L(H)}^2 \|\phi(s)\|_H^2\,\mathrm{d}s\,\mathrm{e}^{-2\rho t}\,\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leq \varepsilon^{\frac{1}{p}} \left(\int_{\mathbb{R}} \int_{t-\varepsilon}^{t} (t-s)^{\frac{(-3+\alpha)}{2}p'} \|N(t)-N(s)\|_{L(H)}^{p'}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{p'}} \|\phi\|_{\rho,p}.$$

Since, $t-s \leq \varepsilon \leq 1$ for $s \in [t-\varepsilon, t]$, $\frac{(-3+\alpha)}{2}p' \geq -(2+\delta)$ and $\|N(t)-N(s)\|^{p'} \leq \|N(t)-N(s)\|^2 (2\|N\|_\infty)^{p'-2}$, we obtain

$$\left(\int_{\mathbb{R}} \int_{t-\varepsilon}^{t} (t-s)^{\frac{(-3+\alpha)}{2}p'} \|N(t)-N(s)\|_{L(H)}^{p'}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{p'}} \leq (2\|N\|_\infty)^{1-\frac{2}{p'}} C^{\frac{1}{p'}}.$$

Hence,

$$\|[\partial_{t,\rho}^{1/2}, \mathcal{N}]\phi\|_{\rho,0} \leq \frac{1}{2\Gamma(1/2)} \left(\frac{1}{1-\alpha}\varepsilon^{1-\alpha}\right)^{\frac{1}{2}} \varepsilon^{\frac{1}{p}} (2\|N\|_\infty)^{1-\frac{2}{p'}} C^{\frac{1}{p'}} \|\phi\|_{\rho,p}$$
$$+\frac{1}{2\Gamma(1/2)} 2\|N\|_\infty \varepsilon^{-3/2} \frac{1}{\rho} \|\phi\|_{\rho,0}.$$

Under the assumption that $\mathcal{N}$ leaves $H^1_\rho(\mathbb{R}; H)$ invariant, the assertion follows from Lemma 2.6 and the fact that $H^1_\rho(\mathbb{R}; H)$ is dense in $H^{1/2}_\rho(\mathbb{R}; H)$.
If $\mathcal{N}$ does not leave $H^1_\rho(\mathbb{R}; H)$ invariant, replace it by $\mathcal{N}_\varepsilon$ as defined in Lemma 5.14. For each $\varepsilon > 0$ we thus obtain

$$\|[\partial_{t,\rho}^{1/2}, \mathcal{N}_{\varepsilon,\rho}]\phi\|_{\rho,0} \leq \left(\frac{1}{1-\alpha}\varepsilon^{1-\alpha}\right)^{\frac{1}{2}} \varepsilon^{\frac{1}{p}} (2\|N_\varepsilon\|_\infty)^{1-\frac{2}{p'}} C_\varepsilon^{\frac{1}{p'}} \|\phi\|_{\rho,p}$$
$$+2\|N_\varepsilon\|_\infty \varepsilon^{-3/2} \frac{1}{\rho} \|\phi\|_{\rho,0},$$

where

$$\begin{aligned} C_\varepsilon &:= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|N_\varepsilon(t)-N_\varepsilon(s)\|^2_{L(H)}}{|t-s|^{2+\delta}}\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\frac{1}{\varepsilon^2}\|\int_0^\varepsilon N(r+t)\,\mathrm{d}r - \int_0^\varepsilon N(r+s)\,\mathrm{d}r\|^2_{L(H)}}{(t-s)^{2+\delta}}\,\mathrm{d}s\,\mathrm{d}t \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\varepsilon^2}\frac{\varepsilon\int_0^\varepsilon \|N(r+t)-N(r+s)\|^2_{L(H)}\,\mathrm{d}r}{(t-s)^{2+\delta}}\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\|N(r+t)-N(r+s)\|^2_{L(H)}}{(t-s)^{2+\delta}}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}r = C. \end{aligned}$$

Since furthermore $\|N_\varepsilon\|_\infty \leq \|N\|_\infty$ by Lemma 5.14 (a), we can apply Lemma 5.14 (c) and thus, the claim follows. □

6. Examples

6.1. Divergence Form Equations

In order to treat a first standard example, we consider heat type equations in this section and analyse the relationship to available results in the literature. For this, we need to introduce the following operators.

Definition 6.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. We define

$$\begin{aligned} \operatorname{grad}_{(0)} &: H^1_{(0)}(\Omega) \subseteq L_2(\Omega) \to L_2(\Omega)^n, \phi \mapsto \nabla\phi, \\ \operatorname{div}_{(0)} &: H_{(0)}(\operatorname{div},\Omega) \subseteq L_2(\Omega)^n \to L_2(\Omega), \psi \mapsto \nabla\cdot\psi, \end{aligned}$$

where $H^1(\Omega)$ is the standard Sobolev space of weakly differentiable $L_2(\Omega)$ functions, $H^1_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. Similarly, $H(\operatorname{div},\Omega)$ is the space of $L_2(\Omega)$-vector fields with distributional divergence in $L_2(\Omega)$ and $H_0(\operatorname{div},\Omega)$ is the closure of $C_c^\infty(\Omega)^n$ in $H(\operatorname{div},\Omega)$.

It is not difficult to see that $\operatorname{div}_0^* = -\operatorname{grad}$ and $\operatorname{grad}_0^* = -\operatorname{div}$, see [22, Chapter 6].

Next, we rephrase a sufficient condition from [3], yielding boundedness of $[\mathcal{N}_0, \partial_{t,0}^{1/2}]$. The result itself is a combination of the techniques used in [3], the BMO-characterisation by Strichartz and the commutator estimate by Murray [12].

Theorem 6.2. *Let $H = L_2(\Omega)^n$ for some open $\Omega \subseteq \mathbb{R}^n$, $N\colon \mathbb{R}\times\Omega \to \mathbb{C}^{n\times n}$ measurable and bounded. Assume there exists $C \geq 0$ such that we have for a.e. $x \in \Omega$ and for all intervals $I \subseteq \mathbb{R}$,*

$$\frac{1}{\ell(I)}\int_I\int_I \frac{\|N(t,x)-N(s,x)\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,\mathrm{d}s\,\mathrm{d}t \leq C. \tag{6.1}$$

Then $[\mathcal{N}_0, \partial_{t,0}^{1/2}]$ is bounded as an operator in $L_2(\mathbb{R};H)$.

Proof. A direct computation shows that N_ε defined in Lemma 5.14 satisfies the same condition imposed on N in the present theorem (with the same C). Thus, using Lemma 5.14 it suffices to treat the case of Lipschitz continuous

N. In this case, the arguments in [3, Corollary 7] show that both $[\mathcal{N}_0, |\partial_{t,0}|^{1/2}]$ and (using [12]) $[\mathcal{N}_0, \operatorname{sgn}(-i\partial_{t,0})\,|\partial_{t,0}|^{1/2}]$ are bounded. Since

$$\begin{aligned}\partial_{t,0}^{1/2} &= \chi_{[0,\infty)}(-i\partial_{t,0})\partial_{t,0}^{1/2} + \chi_{(-\infty,0)}(-i\partial_{t,0})\partial_{t,0}^{1/2}\\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi/4}\left(|\partial_{t,0}|^{1/2} + \operatorname{sgn}(-i\partial_{t,0})\,|\partial_{t,0}|^{1/2}\right)\\ &\quad + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi/4}\left(|\partial_{t,0}|^{1/2} - \operatorname{sgn}(-i\partial_{t,0})\,|\partial_{t,0}|^{1/2}\right)\end{aligned}$$

and due to linearity of the commutator in the second argument, we infer the assertion. □

Remark 6.3. Assume N is given as in Theorem 6.2. Moreover, assume that $\widetilde{N}\colon (t,x) \mapsto N(t,x)^{-1}$ is well-defined and bounded. Then $\widetilde{N}$ satisfies the same integral condition (6.1) as N does. Indeed, we compute for a non-empty interval $I \subseteq \mathbb{R}$

$$\begin{aligned}&\frac{1}{\ell(I)}\int_I\int_I \frac{\|\widetilde{N}(t,x) - \widetilde{N}(s,x)\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,\mathrm{d}s\,\mathrm{d}t\\ &\quad= \frac{1}{\ell(I)}\int_I\int_I \frac{\|N(t,x)^{-1} - N(s,x)^{-1}\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,\mathrm{d}s\,\mathrm{d}t\\ &\quad= \frac{1}{\ell(I)}\int_I\int_I \frac{\|N(t,x)^{-1}\,(N(s,x) - N(t,x))\,N(s,x)^{-1}\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,\mathrm{d}s\,\mathrm{d}t\\ &\quad\le \|\widetilde{N}\|_\infty^4 \frac{1}{\ell(I)}\int_I\int_I \frac{\|\,(N(s,x) - N(t,x))\,\|^2_{\mathbb{C}^{n\times n}}}{|t-s|^2}\,\mathrm{d}s\,\mathrm{d}t \le \|\widetilde{N}\|_\infty^4 C.\end{aligned}$$

At first we provide a proof of the main theorem in [3] with the present tools.

Theorem 6.4. ([3, Theorem 2]) *Let $\Omega \subseteq \mathbb{R}^n$ be open, $N\colon \mathbb{R}\times\Omega \to \mathbb{C}^{n\times n}$ measurable and bounded, satisfying (6.1). Furthermore assume that there exists $c > 0$ such that for a.e. $(t,x) \in \mathbb{R}\times\Omega$:*

$$\operatorname{Re}\langle \xi, N(t,x)\xi\rangle_{\mathbb{C}^n} \ge c\|\xi\|^2_{\mathbb{C}^n}.$$

Let $C\colon \operatorname{dom}(C) \subseteq L_2(\Omega) \to L_2(\Omega)^n$ be densely defined and closed such that $\operatorname{grad}_0 \subseteq C \subseteq \operatorname{grad}$, $\rho > 0$ and let $f \in L_{2,\rho}(\mathbb{R}; L_2(\Omega))$. Then the (unique) solution $u \in L_{2,\rho}(\mathbb{R}; L_2(\Omega))$ of

$$\partial_{t,\rho}u + C^*\mathcal{N}_\rho C u = f$$

admits maximal regularity, that is,

$$u \in H^1_\rho(\mathbb{R}; L_2(\Omega)) \cap H^{1/2}_\rho(\mathbb{R}; \operatorname{dom}(C)) \cap L_{2,\rho}(\mathbb{R}; \operatorname{dom}(C^*\mathcal{N}_\rho C)).$$

Moreover, the solution mapping $f \mapsto u$ is continuous as an operator from $L_{2,\rho}(\mathbb{R}; L_2(\Omega))$ into $H^1_\rho(\mathbb{R}; L_2(\Omega)) \cap H^{1/2}_\rho(\mathbb{R}; \operatorname{dom}(C)) \cap L_{2,\rho}(\mathbb{R}; \operatorname{dom}(C^*\mathcal{N}_\rho C))$.

Proof. In order to put ourselves into the framework of evolutionary equations, we introduce the variable $q := -\mathcal{N}_\rho C u$ and consider

$$\left(\partial_{t,\rho}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \mathcal{N}_\rho^{-1}\end{pmatrix} + \begin{pmatrix}0 & -C^*\\ C & 0\end{pmatrix}\right)\begin{pmatrix}u\\ q\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}. \quad (6.2)$$

The latter equation is equivalent to

$$\partial_{t,\rho} u + C^* \mathcal{N}_\rho C u = f \text{ and } q = -\mathcal{N}_\rho C u.$$

For a more detailed rationale on this we refer to [22, Chapter 6]. It is not difficult to see that (6.2) satisfies the well-posedness condition yielding unique existence of $(u, q) \in L_{2,\rho}(\mathbb{R}; L_2(\Omega)^{1+n})$, see e.g. [14,22]. Next, Theorem 6.2 and Remark 6.3 yield $\left[\mathcal{N}_0^{-1}, \partial_{t,0}^{1/2}\right] \in L(L_2(\mathbb{R}; L_2(\Omega)^n))$. Thus, Proposition 5.10 implies $\left[\mathcal{N}_\rho^{-1}, \partial_{t,0}^{1/2}\right] \in L(L_{2,\rho}(\mathbb{R}; L_2(\Omega)^n))$. Hence, Theorem 5.1 applies with $\widetilde{c} = 0$ to (6.2), which implies the assertion. □

Remark 6.5. (a) Using the extension result in [3, Lemma 11] and Corollary 5.6, we obtain the corresponding local-in-time result in [3, Theorem 2].

(b) The assumption of C to be sandwiched inbetween grad_0 and grad is the same assumption as in [3] asking for either Dirichlet, Neumann or mixed boundary conditions.

Next we provide our perspective on a main implication of the work in [8] for homogeneous initial values.

Theorem 6.6. ([8, Corollary 1.1]) *Let $\Omega \subseteq \mathbb{R}^n$ be open, $N\colon \mathbb{R} \to L(L_2(\Omega)^n)$ strongly measurable and bounded, satisfying (5.2). Furthermore assume that there exists $c > 0$ such that for a.e. $t \in \mathbb{R}$:*

$$\operatorname{Re}\langle \xi, N(t)\xi \rangle_{L_2(\Omega)^n} \geq c \|\xi\|^2_{L_2(\Omega)^n}.$$

Let $C\colon \operatorname{dom}(C) \subseteq L_2(\Omega) \to L_2(\Omega)^n$ be densely defined and closed such that $\operatorname{grad}_0 \subseteq C \subseteq \operatorname{grad}$, $\rho > 0$ and let $f \in L_{2,\rho}(\mathbb{R}; L_2(\Omega))$. Then the (unique) solution $u \in L_{2,\rho}(\mathbb{R}; L_2(\Omega))$ of

$$\partial_{t,\rho} u + C^* \mathcal{N}_\rho C u = f$$

admits maximal regularity, that is,

$$u \in H^1_\rho(\mathbb{R}; L_2(\Omega)) \cap H^{1/2}_\rho(\mathbb{R}; \operatorname{dom}(C)) \cap L_{2,\rho}(\mathbb{R}; \operatorname{dom}(C^* \mathcal{N}_\rho C)).$$

Moreover, the solution mapping $f \mapsto u$ is continuous as an operator from $L_{2,\rho}(\mathbb{R}; L_2(\Omega))$ into $H^1_\rho(\mathbb{R}; L_2(\Omega)) \cap H^{1/2}_\rho(\mathbb{R}; \operatorname{dom}(C)) \cap L_{2,\rho}(\mathbb{R}; \operatorname{dom}(C^* \mathcal{N}_\rho C))$.

Proof. Reformulating the equation in the variables u and $q = -\mathcal{N}_\rho C u$, we obtain

$$\left(\partial_{t,\rho} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{N}_\rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right) \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

With the same argument as in Remark 6.3, we infer that $\widetilde{N}\colon t \mapsto N(t)^{-1}$ satisfies (5.2). Thus, with Proposition 5.12, Theorem 5.1 is applicable, which yields the assertion. □

Remark 6.7. (a) Even though the conditions (5.2) and (6.1) do not compare (see [3, Introduction]), we have established that both of the results in [3,8] applied to standard divergence form equations can be obtained by the same overriding principle of suitably bounded commutators with

$\partial_{t,\rho}^{1/2}$. Note that (6.1) implies boundedness as an operator in L_2, whereas (5.2) yields infinitesimal boundedness relative to $\partial_{t,\rho}^{1/2}$ only.

(b) The condition on the regularity of the coefficient N leading to maximal regularity of the considered divergence form equation obtained in [1] seems to be weaker than the one of (infinitesimal) boundedness of the commutator with $\partial_{t,\rho}^{1/2}$. However, note that in order to apply the maximal regularity theorem in [1], one needs to assume Kato's square root property (potentially) resulting in undue regularity requirements of the boundary of Ω, which we do not want to impose here.

The above results (and the corresponding proofs) provide potential for the following maximal regularity result, which invokes both lower order terms and (time-)nonlocal effects in the time derivative term. This will be addressed next.

6.2. Maximal Regularity for Integro-Differential Equations

In this section, we consider equations of the following form (see [21,23]), which has applications for instance in visco-elasticity. We need the following notion.

Definition 6.8. Let G be a Hilbert space, $T \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(G))$ for some $\mu \geq 0$. We call T *admissible*, if the following conditions are met:

1. for all $t \geq 0$, $T(t)$ is selfadjoint,
2. there exists $d \geq 0$ and $\rho_0 \geq \mu$ such that for all $t \in \mathbb{R}$ we have

$$t \operatorname{Im} \hat{T}(t - \mathrm{i}\rho_0) \leq d,$$

where

$$\langle \hat{T}(t - \mathrm{i}\rho)\phi, \psi\rangle_G := \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\mathrm{i}ts}\mathrm{e}^{-\rho s}\langle T(s)\phi, \psi\rangle_G \,\mathrm{d}s \quad (\phi, \psi \in G).$$

Remark 6.9. As highlighted in [23, Remark 3.6]; T being admissible generalises the standard assumption for convolution kernels for the class of integro-differential equations considered in the literature.

Proposition 6.10. ([23, Proposition 3.9 (and its proof)]) *Let G be a Hilbert space, $T \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(G))$ for some $\mu \geq 0$. Assume that T is admissible. Then there exists $c_1, c_2 > 0$, $\rho_1 \geq \rho_0$ such that for all $\rho \geq \rho_1$ we have*

$$\operatorname{Re}\langle \partial_{t,\rho}(1 + T*)\phi, \phi\rangle_{\rho,0} \geq (c_1\rho - c_2)\,\langle \phi, \phi\rangle_{\rho,0} \quad (\phi \in H^1_\rho(\mathbb{R}; G)),$$

where $T* \in L(L_{2,\rho}(\mathbb{R}; G))$ is defined as the operator of convolving with T (extended by zero to $\mathbb{R}$).

The corresponding theorem for maximal regularity of parabolic-type non-autonomous integro-differential equations, now reads as follows.

Theorem 6.11. *Let H_0, H_1 be Hilbert spaces, $\mu \geq 0$, $T \in L_{1,\mu}(\mathbb{R}; L(H_0))$ admissible. Assume $\mathcal{N}_{ij}\colon S_c(\mathbb{R}; H_j) \to \bigcap_{\rho \geq \rho_0} L_{2,\rho}(\mathbb{R}; H_i)$ is evolutionary at ρ_0 for each pair $(i, j) \in \{(0,0), (0,1), (1,1)\}$ and some ρ_0 satisfying*

$$\operatorname{Re}\langle \mathcal{N}_{11}\phi_1, \phi_1\rangle_{\rho,0} \geq c\langle \phi_1, \phi_1\rangle_{\rho,0} \quad (\phi_1 \in S_c(\mathbb{R}; H_1), \rho \geq \rho_0)$$

for some $c > 0$. In addition, assume that for all $\rho \geq \rho_0$ and all $\varepsilon > 0$ there exists $d > 0$ such that

$$\| \left[\mathcal{N}_{11,\rho}, \partial_{t,\rho}^{1/2}\right] \phi_1 \|_{\rho,0} \leq \varepsilon \|\phi_1\|_{\rho,\frac{1}{2}} + d\|\phi_1\|_{\rho,0} \quad (\phi_1 \in H_\rho^{1/2}(\mathbb{R}; H_1)).$$

Furthermore, let $C\colon \operatorname{dom}(C) \subseteq H_0 \to H_1$ be densely defined and closed. Then we find $\rho_1 \geq \rho_0$ such that for all $\rho \geq \rho_1$ and $f \in L_{2,\rho}(\mathbb{R}; H_0)$ there exists a unique $u \in L_{2,\rho}(\mathbb{R}; H_0)$ with

$$\partial_{t,\rho}(1 + T*)u + \mathcal{N}_{00,\rho}u + \mathcal{N}_{01,\rho}\mathcal{N}_{11,\rho}^{-1}Cu + C^*\mathcal{N}_{11,\rho}^{-1}Cu = f.$$

Moreover, u satisfies the regularity

$$u \in H_\rho^1(\mathbb{R}; H_0) \cap H_\rho^{1/2}(\mathbb{R}; \operatorname{dom}(C)) \cap L_{2,\rho}(\mathbb{R}; \operatorname{dom}(C^*\mathcal{N}_{11,\rho}^{-1}C)).$$

Proof. Using the substitution $q := -\mathcal{N}_{11,\rho}^{-1}Cu$, we consider for $\rho \geq \rho_1$ for $\rho_1 \geq \rho_0$ to be fixed later

$$\left(\partial_{t,\rho}\begin{pmatrix}(1+T*) & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}\mathcal{N}_{00,\rho} & -\mathcal{N}_{01,\rho}\\ 0 & \mathcal{N}_{11,\rho}\end{pmatrix} + \begin{pmatrix}0 & -C^*\\ C & 0\end{pmatrix}\right)\begin{pmatrix}u\\ q\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}. \tag{6.3}$$

For $\mathcal{M} = \begin{pmatrix}(1+T*) & 0\\ 0 & 0\end{pmatrix}$ and $\mathcal{N} = \begin{pmatrix}\mathcal{N}_{00,\rho} & -\mathcal{N}_{01,\rho}\\ 0 & \mathcal{N}_{11,\rho}\end{pmatrix}$, we show that the positive definiteness condition in Theorem 2.8 is satisfied. For this, we use Proposition 6.10 and estimate for $\phi = (\phi_0, \phi_1) \in L_{2,\rho}(\mathbb{R}; H_0 \times H_1)$ and $\varepsilon > 0$

$$\begin{aligned}
&\operatorname{Re}\langle(\partial_{t,\rho}\mathcal{M} + \mathcal{N})\,\phi, \phi\rangle_{\rho,0}\\
&\quad \geq (c_1\rho - c_2)\,\langle\phi_0, \phi_0\rangle_{\rho,0} - \|\mathcal{N}_{00,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H_0))}\|\phi_0\|_{\rho,0}^2\\
&\qquad - \|\mathcal{N}_{01,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H_0))}\|\phi_0\|_{\rho,0}\|\phi_1\|_{\rho,0} + c\|\phi_1\|_{\rho,0}^2\\
&\quad \geq \left(c_1\rho - c_2 - \|\mathcal{N}_{00,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H_0))} - \frac{1}{2\varepsilon}\|\mathcal{N}_{01,\rho}\|_{L(L_{2,\rho}(\mathbb{R};H_0))}^2\right)\|\phi_0\|_{\rho,0}^2\\
&\qquad + (c - \frac{1}{2}\varepsilon)\|\phi_1\|_{\rho,0}^2.
\end{aligned}$$

Thus, choosing $\varepsilon > 0$ small enough, we find $\rho_1 \geq \rho_0$ such that for all $\rho \geq \rho_1$ we have

$$\operatorname{Re}\langle(\partial_{t,\rho}\mathcal{M} + \mathcal{N})\,\phi, \phi\rangle_{\rho,0} \geq \frac{c}{2}\|\phi\|_{\rho,0}^2.$$

Hence, $(u, q) \in L_{2,\rho}(\mathbb{R}; H_0 \times H_1)$ are uniquely determined by (6.3). Note that Theorem 2.8 asserts that actually

$$\overline{\left(\partial_{t,\rho}\begin{pmatrix}(1+T*) & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}\mathcal{N}_{00,\rho} & -\mathcal{N}_{01,\rho}\\ 0 & \mathcal{N}_{11,\rho}\end{pmatrix} + \begin{pmatrix}0 & -C^*\\ C & 0\end{pmatrix}\right)}\begin{pmatrix}u\\ q\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}.$$

Since both $\mathcal{N}_{00,\rho}$ and $\mathcal{N}_{01,\rho}$ are bounded linear operators, we obtain

$$\begin{aligned}&\overline{\left(\partial_{t,\rho}\begin{pmatrix}(1+T*) & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & \mathcal{N}_{11,\rho}\end{pmatrix}+\begin{pmatrix}0 & -C^*\\ C & 0\end{pmatrix}\right)}\begin{pmatrix}u\\ q\end{pmatrix}\\ &=\begin{pmatrix}f\\ 0\end{pmatrix}+\begin{pmatrix}-\mathcal{N}_{00,\rho} & \mathcal{N}_{01,\rho}\\ 0 & 0\end{pmatrix}\begin{pmatrix}u\\ q\end{pmatrix}\\ &=\begin{pmatrix}f-\mathcal{N}_{00,\rho}u+\mathcal{N}_{01,\rho}q\\ 0\end{pmatrix}.\end{aligned}$$

Next, as the convolution operator $(1+T*)$ commutes with $\partial_{t,0}^{-1}$, we infer with the help of Theorem 5.1 the desired regularity statement. □

Remark 6.12. (a) Note that the coefficients of the lower order terms $\mathcal{N}_{01}$ and $\mathcal{N}_{00}$ are not required to satisfy any regularity in time, which is in line with the concluding example in [1]. Moreover, in the theorem presented here the coefficient $\mathcal{N}_{11,\rho}$ may well depend suitably regular on time, i.e., $\mathcal{N}_{11,\rho}$ may be induced by a multiplication operator, which satisfies either (6.1) or (5.2).

(b) The results above directly apply to systems of divergence form equations, see [8, Parabolic systems] for examples concerning maximal regularity and [7, Proposition 3.8] for the corresponding formulation as evolutionary equation.

6.3. Maxwell's Equations

The concluding example is concerned with Maxwell's equations. For this, we introduce the necessary operator from vector analysis:

Definition 6.13. Let $\Omega\subseteq\mathbb{R}^3$ open. Then we define

$$\operatorname{curl}_{(0)}:\ H_{(0)}(\operatorname{curl},\Omega)\subseteq L_2(\Omega)^3\to L_2(\Omega)^3,\phi\mapsto\nabla\times\phi,$$

where $H(\operatorname{curl},\Omega)$ is the space of $L_2(\Omega)$-vector fields with distributional curl in $L_2(\Omega)^3$ and $H_0(\operatorname{curl},\Omega)$ is the closure of $C_c^\infty(\Omega)^3$ in $H(\operatorname{curl},\Omega)$. It is not difficult to see that $\operatorname{curl}_0^*=\operatorname{curl}$.

The result on maximal regularity for Maxwell's equations is concerned with the eddy current approximation, which is a parabolic variant of the original Maxwell's equations. The catch is that in electrically conducting materials like metals the dielectricity ε is negligible compared to the conductivity σ, which we assume to depend on time. This setting has applications to moving domains, see e.g. [6]. The result reads as follows.

Theorem 6.14. *Let $\Omega\subseteq\mathbb{R}^3$ open, $\rho>0$ and let $\mu=\mu^*\in L(L_2(\Omega)^3)$; assume $\mu\geq c$ for some $c>0$. Moreover, let $\sigma\in L(L_{2,\rho}(\mathbb{R};L_2(\Omega)^3)$ satisfy*

$$\operatorname{Re}\langle\sigma E,E\rangle_{\rho,0}\geq c\|E\|_{\rho,0}^2\quad(E\in L_{2,\rho}(\mathbb{R};L_2(\Omega)^3).$$

and for all $\varepsilon>0$ we find $d>0$ such that

$$\|\left[\sigma,\partial_{t,\rho}^{1/2}\right]\phi\|\leq\varepsilon\|\phi\|_{\rho,\frac{1}{2}}+d\|\phi\|_{\rho,0}\quad(\phi\in H_\rho^{1/2}(\mathbb{R};L_2(\Omega)^3)).$$

Then for all $\rho>0$ and for all $(J,K)\in L_{2,\rho}(\mathbb{R};L_2(\Omega)^6)$ the equation

$$\left(\partial_{t,\rho}\begin{pmatrix}\mu & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & \sigma\end{pmatrix}+\begin{pmatrix}0 & \operatorname{curl}_0\\ -\operatorname{curl} & 0\end{pmatrix}\right)\begin{pmatrix}H\\ E\end{pmatrix}=\begin{pmatrix}K\\ -J\end{pmatrix}$$

admits a unique solution $(E, H) \in L_{2,\rho}(\mathbb{R}; L_2(\Omega)^6)$. *If, in addition,* $J \in H_\rho^{1/2}(\mathbb{R}; L_2(\Omega)^3)$ *then* $E \in H_\rho^{1/2}(\mathbb{R}; L_2(\Omega)^3) \cap L_{2,\rho}(\mathbb{R}; H_0(\mathrm{curl}, \Omega))$ *and* $H \in H_\rho^1(\mathbb{R}; L_2(\Omega)^3) \cap H_\rho^{1/2}(\mathbb{R}; H(\mathrm{curl}, \Omega))$.

Proof. The proof is a direct application of Theorem 5.1. □

Remark 6.15. (a) The commutator condition imposed on σ is satisfied, if σ is a multiplication operator induced by a function satisfying (6.1) or (5.2).

(b) In applications, non-zero terms K can occur for inhomogeneous boundary values. A result corresponding to Theorem 6.14 is valid also for mixed boundary conditions or with homogeneous boundary conditions for H.

(c) There is no condition assumed on the regularity of the boundary of Ω.

7. Conclusion

We presented a maximal regularity theorem for evolutionary equations. The core assumptions abstractly describe a parabolic type evolutionary equation and lead to well-posedness on $L_{2,\rho}$ and $H_\rho^{1/2}$. For applications, the operator theoretic insight of the need of commutator estimates for the commutator with $\partial_t^{1/2}$ found in [3,8] showed to be decisive also for evolutionary equations. Moreover, we showed that both conditions on the coefficients imposed in [3,8], which are not comparable, imply the well-posedness in $H_\rho^{1/2}$ and hence, yield the maximal regularity of the problem under consideration within the presented framework. Naturally, the regularity phenomenon for the unknown to belong to $H^{1/2}$ with values in the form domain, observed in [3,8], resurfaced also in the framework of evolutionary equations. The conditions derived here are deliberately focussed on the coefficients rather than the whole space-time operator in order that it is possible to generate results independent of the regularity of the boundary of the underlying domain, which is needed in [1] in order to warrant some form of the square root property. Due to the view of the time derivative as a normal continuously invertible operator it is possible to use a straightforward functional calculus and to compute fractional powers of the time derivative and to work with them without the need of explicitly invoking the Hilbert transform or other technicalities. It remains to be seen, whether the commutator assumptions or the basic result Theorem 4.1 implying maximal regularity lead to slightly stronger statements also in the situation of divergence form equations.

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References

[1] Achache, M., Ouhabaz, E.M.: Lions' maximal regularity problem with $H^{\frac{1}{2}}$-regularity in time. J. Differ. Equ. **266**(6), 3654–3678 (2019)

[2] Arendt, W., Dier, D., Fackler, S.: J. L. Lions' problem on maximal regularity. Arch. Math. (Basel) **109**(1), 59–72 (2017)

[3] Auscher, P., Egert, M.: On non-autonomous maximal regularity for elliptic operators in divergence form. Arch. Math. (Basel) **107**(3), 271–284 (2016)

[4] Bergh, J., Löfström, J.: Interpolation Spaces. An introduction. Springer-Verlag, Berlin-New York. Grundlehren der Mathematischen Wissenschaften, No. 223 (1976)

[5] Beyer, H.: Beyond partial differential equations. In: On Linear and Quasi-Linear Abstract Hyperbolic Evolution Equations. Springer, Berlin (2007)

[6] Cooper, J., Strauss, W.: The initial boundary problem for the Maxwell equations in the presence of a moving body. SIAM J. Math. Anal. **16**(6), 1165–1179 (1985)

[7] Cooper, S., Waurick, M.: Fibre homogenisation. J. Funct. Anal. **276**(11), 3363–3405 (2019)

[8] Dier, D., Zacher, R.: Non-autonomous maximal regularity in Hilbert spaces. J. Evol. Equ. **17**(3), 883–907 (2017)

[9] Fackler, S.: J. L. Lions' problem concerning maximal regularity of equations governed by non-autonomous forms. Ann. Inst. H. Poincaré Anal. Non Linéaire **34**(3), 699–709 (2017)

[10] Lions, J.-L.: Équations différentielles opérationnelles et problèmes aux limites. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer, Berlin (1961)

[11] Lunardi, A.: Interpolation theory, volume 16 of Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, (2018). Third edition

[12] Murray, M.A.M.: Commutators with fractional differentiation and BMO Sobolev spaces. Indiana Univ. Math. J. **34**(1), 205–215 (1985)

[13] Ouhabaz, E.M., Spina, C.: Maximal regularity for non-autonomous Schrödinger type equations. J. Differ. Equ. **248**(7), 1668–1683 (2010)

[14] Picard, R.: A structural observation for linear material laws in classical mathematical physics. Math. Methods Appl. Sci. **32**(14), 1768–1803 (2009)

[15] Picard, R., McGhee, D.: Partial differential equations, volume 55 of De Gruyter Expositions in Mathematics. Walter de Gruyter GmbH & Co. KG, Berlin. A unified Hilbert space approach (2011)

[16] Picard, R., McGhee, D., Trostorff, S., Waurick, M.: A Primer for a Secret Shortcut to PDEs of Mathematical Physics. Frontiers in Mathematics. Birkhäuser. manuscript accepted; contract signed, approx 200 pp (2020)

[17] Picard, R., Trostorff, S., Waurick, M.: On evolutionary equations with material laws containing fractional integrals. Math. Methods Appl. Sci. **38**(15), 3141–3154 (2015)

[18] Picard, R., Trostorff, S., Waurick, M.: Well-posedness via monotonicity—an overview. In: Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics, volume 250 of Oper. Theory Adv. Appl., pp. 397–452. Birkhäuser/Springer, Cham (2015)

[19] Picard, R., Trostorff, S., Waurick, M.: On maximal regularity for a class of evolutionary equations. J. Math. Anal. Appl. **449**(2), 1368–1381 (2017)

[20] Picard, R., Trostorff, S., Waurick, M., Wehowski, M.: On non-autonomous evolutionary problems. J. Evol. Equ. **13**, 751–776 (2013)

[21] Prüss, J.: Evolutionary integral equations and applications. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel. [2012] reprint of the 1993 edition (1993)

[22] Seifert, C., Trostorff, S., Waurick, M.: Evolutionary equations. Lecture notes of the 23rd Internet seminar. arXiv: 2003.12403 (2020)

[23] Trostorff, S.: On integro-differential inclusions with operator-valued kernels. Math. Methods Appl. Sci. **38**(5), 834–850 (2015)

[24] Trostorff, S.: Well-posedness for a general class of differential inclusions. J. Differ. Equ. **268**(11), 6489–6516 (2020)

[25] Trostorff, S., Waurick, M.: Maximal Regularity for Non-Autonomous Evolutionary Equations arXiv:2006.16696; version of 30th of June (2020)

[26] Waurick, M.: A note on causality in Banach spaces. Indag. Math. **26**(2), 404–412 (2015)

[27] Waurick, M.: On non-autonomous integro-differential-algebraic evolutionary problems. Math. Methods Appl. Sci. **38**(4), 665–676 (2015)

[28] Waurick, M.: On the continuous dependence on the coefficients of evolutionary equations. Habilitation, TU Dresden, (2016). arXiv:1606.07731

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