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Integral Equations and Operator Theory

Eigenvalues of K-invariant Toeplitz Operators on Bounded Symmetric Domains

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Abstract. We determine the eigenvalues of certain “fundamental” K-invariant Toeplitz type operators on weighted Bergman spaces over bounded symmetric domains $D = G/K$, for the irreducible K-types indexed by all partitions of length $r = \text{rank}(D)$.

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1. Introduction

The well-known Toeplitz–Berezin calculus, acting on the Bergman space $H^2(D)$ of a bounded domain $D \subset \mathbf{C}^d$, is covariant under the biholomorphic automorphism group G of D. Actually, Berezin [3] considered two kinds of symbolic calculus (contravariant and covariant symbols) which are related by the Berezin transform. For a bounded symmetric domain $D = G/K$ of rank r, where G acts transitively on D and K is a maximal compact subgroup of G, one has a more general covariant Toeplitz–Berezin calculus acting on the weighted Bergman spaces $H^2_\nu(D)$ over D. Here ν is a scalar parameter for the (scalar) holomorphic discrete series of G and its analytic continuation. Since G acts irreducibly on $H^2_\nu(D)$, there are no non-trivial G-invariant operators in the C^*-algebra generated by Toeplitz operators. On the other hand, there exist interesting K-invariant Toeplitz type operators, which have been studied in relation to complex and harmonic analysis [2,6]. These operators are uniquely determined by a sequence of eigenvalues indexed over all partitions of length r. In this paper, we determine the eigenvalues of certain “fundamental” K-invariant Toeplitz type operators, both for the covariant and contravarient symbol. While the covariant symbol is treated as a direct generalization of [2], the contravariant symbol eigenvalue formula requires

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more effort. Here a crucial ingredient is the dimension formula for the irreducible K-types.

2. K-invariant Toeplitz Operators

In the following we use the Jordan theoretical description of bounded symmetric domains. For more details, see [1,5,10,11,16]. Each irreducible bounded symmetric domain D of rank r and dimension d can be realized as the (spectral) open unit ball of a hermitian Jordan triple $Z \approx \mathbf{C}^d$. Let G be the identity component of the biholomorphic automorphism group of D, and let K be the stabilizer subgroup at $0 \in D$. Then K is a compact linear group consisting of Jordan triple automorphisms of Z. The Shilov boundary S of D consists of all tripotents in Z of maximal rank r. Let $(z|w)$ denote the unique K-invariant inner product on Z such that $(e|e) = r$ for any $e \in S$. For any maximal tripotent $e \in Z$ the so-called Peirce 2-space $Z_2(e)$ [10] is a hermitian Jordan triple of **tube type**. Put

$$d_e := \dim Z_2(e).$$

Let $e_1, \ldots, e_r$ be a frame of orthogonal minimal tripotents, and put $e = e_1 + \ldots + e_r$. The joint Peirce decomposition [10] gives rise to two numerical invariants a, b such that

$$\frac{d_e}{r} = 1 + \frac{a}{2}(r-1),$$
$$\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b = \frac{d_e}{r} + b.$$

The tube type case is characterized by $d_e = d$ or, equivalently, $b = 0$. The triple (r, a, b) characterizes D up to isomorphism. As an important special case, the **spin factor** Z of dimension $d \geqslant 3$ has the invariants $r = 2,\ a = d-2$ and $b = 0$. Here the normalized inner product on $Z \approx \mathbf{C}^d$ is $(z|w) = 2z \cdot \overline{w}$, and the unit element $e = (1, 0, \ldots, 0)$.

Let $\mathcal{P}(Z)$ denote the algebra of all (holomorphic) polynomials on Z. The natural action

$$(k \cdot f)(z) := f(k^{-1}z) \tag{2.1}$$

of K on functions f on Z (or D), induces a **Peter–Weyl decomposition** [12]

$$\mathcal{P}(Z) = \sum_{\boldsymbol{m}} \mathcal{P}^{\boldsymbol{m}}(Z) \tag{2.2}$$

ranging over all **integer partitions** $\boldsymbol{m} = (m_1 \geqslant \ldots \geqslant m_r \geqslant 0)$ of length r. Here $\mathcal{P}^{\boldsymbol{m}}(Z)$ denotes the irreducible K-module of all polynomials on Z of type $\boldsymbol{m}$ [14]. By irreducibility, any two K-invariant inner products are proportional on each submodule $\mathcal{P}^{\boldsymbol{m}}(Z)$.

Let $H^2(Z)$ denote the **Fock space** of all entire functions on Z, with Fischer-Fock inner product

$$(\phi|\psi) = \int_Z \frac{dz}{\pi^d}\, e^{-(z|z)}\, \overline{\phi(z)}\psi(z) \tag{2.3}$$

and reproducing kernel

$$\mathcal{K}(z,w) = e^{-(z|w)}.$$

Note that for function spaces we use inner products which are conjugate-linear in the first variable.

Consider the classical Pochhammer symbol

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$$

which equals $\lambda(\lambda+1)\cdots(\lambda+k-1)$ if $k \in \mathbf{N}$. For any scalar parameter ν and r-tuple $\mu = (\mu_1, \ldots, \mu_r)$ we define the **multi-variable Pochhammer symbol**

$$(\nu)_\mu := \prod_{j=1}^{r} (\nu - \tfrac{a}{2}(j-1))_{\mu_j}. \tag{2.4}$$

The numerical invariant $p := \frac{1}{r}(d_e + d) = 2 + a(r-1) + b$ is called the **genus** of D. For a scalar parameter $\nu > p-1$ the **weighted Bergman space** $H^2_\nu(D)$ consists of all holomorphic functions ϕ on D which are square-integrable for the inner product

$$(\phi|\psi)_\nu = c_\nu \int_D \frac{dz}{\pi^d}\ \Delta(z,z)^{\nu-p}\ \overline{\phi(z)}\psi(z). \tag{2.5}$$

Here $\Delta(z,w)$ denotes the **Jordan triple determinant**, which is a sesqui-polynomial uniquely determined by the property

$$\Delta(z,z) = \prod_{i=1}^{r} (1 - t_i^2)$$

for all $z = \sum_{i=1}^{r} t_i e_i \in Z$, where $e_1, \ldots, e_r$ is any frame of minimal orthogonal tripotents and $t_1, \ldots, t_r$ are the (non-negative) "singular values" of z. For a Lie theoretic definition see [5, p. 262] (for tube type domains) and [4, (3.4)-(3.6)] (general case). The normalizing constant c_ν, giving rise to a probability measure, is

$$c_\nu = (\nu - \tfrac{d}{r})_{(d/r,\ldots,d/r)} = \prod_{j=1}^{r} \frac{\Gamma(\nu - \frac{a}{2}(j-1))}{\Gamma(\nu - \frac{d}{r} - \frac{a}{2}(j-1))}.$$

The reproducing kernel is

$$\mathcal{K}^\nu(z,w) = \Delta(z,w)^{-\nu}. \tag{2.6}$$

For the continuous part of the so-called Wallach set, explicitly given by the condition

$$\nu > \tfrac{a}{2}(r-1)$$

[4,8,9], the kernel (2.6) is strictly positive on $D \times D$, and the associated reproducing kernel Hilbert space, still denoted by $H^2_\nu(D)$, contains $\mathcal{P}(Z)$ as a dense subspace. As a special case, the parameter $\nu = \frac{d}{r} = 1 + b + \frac{a}{2}(r-1)$ corresponds to the **Hardy space**

$$H^2_{d/r}(D) = H^2(S)$$

on the Shilov boundary S of D.

Under the Fischer-Fock inner product (2.3) each finite-dimensional Hilbert space $\mathcal{P}^{\boldsymbol{m}}(Z)$ has a reproducing kernel

$$E^{\boldsymbol{m}}(z,w) = \sum_\alpha \psi_\alpha^{\boldsymbol{m}}(z)\,\overline{\psi_\alpha^{\boldsymbol{m}}(w)}$$

for any orthonormal basis $\psi_\alpha^{\boldsymbol{m}} \in \mathcal{P}^{\boldsymbol{m}}(Z)$. The Hilbert spaces $H^2(Z)$ and $H_\nu^2(D)$ (in the continuous Wallach set) are invariant under the action (2.1) of K. The **Faraut–Korányi binomial formula** [4]

$$\mathcal{K}^\nu(z,w) = \Delta(z,w)^{-\nu} = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\; E^{\boldsymbol{m}}(z,w) \tag{2.7}$$

implies that the inner product (2.5) on $H_\nu^2(D)$ and the Fischer-Fock inner product (2.3) are related by

$$(p|q)_\nu = \frac{(p|q)}{(\nu)_{\boldsymbol{m}}} \tag{2.8}$$

for each partition $\boldsymbol{m}$ and all $p,q \in \mathcal{P}^{\boldsymbol{m}}(Z)$, using the multi-variable Pochhammer symbol (2.4).

We now introduce Toeplitz operators. Let $F(z,w)$ be a sequi-holomorphic symbol function, written as a sum (or series)

$$F(z,w) = \sum_i \phi_i(z)\,\overline{\psi_i(w)}$$

for holomorphic functions ϕ_i, ψ_i. We are mainly concerned with sesqui-polynomials, where $\phi_i, \psi_i \in \mathcal{P}(Z)$ and the sum is finite. We write $F_w(z) := F(z,w)$ for fixed w.

Let M_ϕ (resp., M_ϕ^ν) denote the multiplication operator by a polynomial ϕ acting on $\mathcal{P}(Z)$ or $H_\nu^2(D)$, respectively. The ν-th Toeplitz operator T_F^ν on $H_\nu^2(D)$ with symbol function $F_|$ (where $F_|$ denotes the restriction of F to the diagonal) has the form

$$T_F^\nu = \sum_i M_{\psi_i}^{\nu *}\; M_{\phi_i}^\nu.$$

We define similarly

$$T_F = \sum_i M_{\psi_i}^*\; M_{\phi_i}$$

acting on $\mathcal{P}(Z)$, or as a densely defined unbounded operator on the Fock space $H^2(Z)$. Here M_ψ^* is the constant coefficient differential operator associated with a polynomial ψ via the normalized inner product. Thus

$$M_\psi^*\; e^{(z|w)} = \overline{\psi(w)}\; e^{(z|w)}$$

for all $z,w \in Z$. For $p,q,\psi \in \mathcal{P}(Z)$ we have

$$(p|q) = M_p^* q(0)$$

and

$$(p|M_\psi^* q) = (M_\psi p|q) = (\psi \cdot p|q).$$

Note that M_ψ^* depends in a conjugate-linear way on ψ.

The Toeplitz calculus is sometimes called the "anti-Wick" calculus. On the other hand, the "Wick" functional calculus (normal ordering, where annihilation operators are moved to the right) yields the operator

$$F_{T^\nu} := \sum_i M^\nu_{\phi_i}\ M^{\nu *}_{\psi_i}$$

on $H^2_\nu(D)$, and, similarly,

$$F_T := \sum_i M_{\phi_i}\ M^*_{\psi_i},$$

acting on $\mathcal{P}(Z)$ or as a densely defined unbounded operator on $H^2(Z)$.

If F is K-invariant in the sense that

$$F(kz, kw) = F(z, w)$$

for all $k \in K$, then the operators T_F, T^ν_F and F_T, F_{T^ν} commute with the K-action (2.1). Since the decomposition (2.2) is multiplicity free, it follows that K-invariant operators form a commutative algebra, and every such operator T is a block-diagonal operator uniquely determined by its sequence of eigenvalues $T(\boldsymbol{m})$ defined by

$$Tp = T(\boldsymbol{m})\ p$$

for all $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$. For T^ν_F, with F K-invariant, we obtain

$$T^\nu_F\ p =: T^\nu_F(\boldsymbol{m})\ p$$

for all $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$, with eigenvalues given by

$$\begin{aligned} T^\nu_F(\boldsymbol{m})\ (p|p)_\nu &= (p|T^\nu_F(\boldsymbol{m})p)_\nu = (p|T^\nu_F p)_\nu = (p|F_|\ p)_\nu \\ &= c_\nu \int\limits_D \frac{dz}{\pi^d}\ \overline{p(z)}\ \Delta(z,z)^{\nu-p}\ F(z,z)\ p(z). \end{aligned}$$

The Jordan triple determinant $\Delta(z, w) = \Delta_w(z)$ is K-invariant, and for its powers we obtain

Lemma 1.

$$T^\nu_{\Delta^\beta}(\boldsymbol{m}) = \frac{c_\nu}{c_{\nu+\beta}} \frac{(\nu)_{\boldsymbol{m}}}{(\nu+\beta)_{\boldsymbol{m}}}$$

Proof. Let $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$. Then (2.8) implies

$$\begin{aligned} T^\nu_{\Delta^\beta}(\boldsymbol{m})\ (p|p)_\nu &= c_\nu \int\limits_D \frac{dz}{\pi^d}\ \overline{p(z)}\ \Delta(z,z)^{\nu-p}\ \Delta(z,z)^\beta\ p(z) \\ &= \frac{c_\nu}{c_{\nu+\beta}} c_{\nu+\beta} \int\limits_D \frac{dz}{\pi^d}\ \overline{p(z)}\ \Delta(z,z)^{\nu+\beta-p}\ p(z) \\ &= \frac{c_\nu}{c_{\nu+\beta}} (p|p)_{\nu+\beta} = \frac{c_\nu}{c_{\nu+\beta}} \frac{(\nu)_{\boldsymbol{m}}}{(\nu+\beta)_{\boldsymbol{m}}} (p|p)_\nu \end{aligned}$$

□

Proposition 2. *For a K-invariant sesqui-holomorphic function F, let $F^{\vDash}$ denote the ν-th* **Berezin transform** *(defined via the Berezin symbol of operators on $H^2_\nu(D)$). Then*

$$T^\nu_F(\boldsymbol{m}) = F^{\Vdash}_{T^\nu}(\boldsymbol{m}).$$

Proof. Using Einstein summation convention, the identity

$$F(z,w)\ \mathcal{K}^\nu(z,w) = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ F_{T^\nu}(\boldsymbol{m})\ E^{\boldsymbol{m}}(z,w) \tag{2.9}$$

follows from the computation

$$\begin{aligned} F(z,w)\mathcal{K}^\nu(z,w) &= \phi_i(z)\overline{\psi_i(w)}(\mathcal{K}^\nu_z|\mathcal{K}^\nu_w)_\nu = (\overline{\phi_i(z)}\mathcal{K}^\nu_z|\overline{\psi_i(w)}\mathcal{K}^\nu_w)_\nu \\ &= (M^{\nu *}_{\phi_i}\mathcal{K}^\nu_z|M^{\nu *}_{\psi_i}\mathcal{K}^\nu_w)_\nu = (\mathcal{K}^\nu_z|M^\nu_{\phi_i}M^{\nu *}_{\psi_i}\mathcal{K}^\nu_w)_\nu = (\mathcal{K}^\nu_z|F_{T^\nu}\mathcal{K}^\nu_w)_\nu \\ &= \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}(\mathcal{K}^\nu_z|F_{T^\nu}E^{\boldsymbol{m}}_w)_\nu = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}(\mathcal{K}^\nu_z|F_{T^\nu}(\boldsymbol{m})E^{\boldsymbol{m}}_w)_\nu \\ &= \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ F_{T^\nu}(\boldsymbol{m})\ (\mathcal{K}^\nu_z|E^{\boldsymbol{m}}_w)_\nu = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ F_{T^\nu}(\boldsymbol{m})\ E^{\boldsymbol{m}}_w(z) \\ &= \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ F_{T^\nu}(\boldsymbol{m})\ E^{\boldsymbol{m}}(z,w). \end{aligned}$$

Similarly, the identity

$$F^{\vDash}(z,w)\ \mathcal{K}^\nu(z,w) = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ T^\nu_F(\boldsymbol{m})\ E^{\boldsymbol{m}}(z,w) \tag{2.10}$$

follows from the computation

$$\begin{aligned} F^{\vDash}(z,w)\ \mathcal{K}^\nu(z,w) &= (\mathcal{K}^\nu_z|T^\nu_F\mathcal{K}^\nu_w)_\nu = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}(\mathcal{K}^\nu_z|T^\nu_F E^{\boldsymbol{m}}_w)_\nu \\ &= \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}(\mathcal{K}^\nu_z|T^\nu_F(\boldsymbol{m})E^{\boldsymbol{m}}_w)_\nu = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ T^\nu_F(\boldsymbol{m})\ (\mathcal{K}^\nu_z|E^{\boldsymbol{m}}_w)_\nu \\ &= \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ T^\nu_F(\boldsymbol{m})\ E^{\boldsymbol{m}}_w(z) = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ T^\nu_F(\boldsymbol{m})\ E^{\boldsymbol{m}}(z,w). \end{aligned}$$

Comparing coefficients in (2.9) and (2.10), the assertion follows. □

In general, the Berezin transform of a K-invariant function is difficult to compute. For powers of Δ we obtain with Lemma 1

Corollary 3.

$$(\Delta^\beta)^{\vDash}(z,w)\ \mathcal{K}^\nu(z,w) = \frac{c_\nu}{c_{\nu+\beta}} \sum_{\boldsymbol{m}} \frac{(\nu)^2_{\boldsymbol{m}}}{(\nu+\beta)_{\boldsymbol{m}}}\ E^{\boldsymbol{m}}(z,w)$$

3. The First Eigenvalue Formula

The Jordan triple determinant has a decomposition

$$\Delta(z,w) = \sum_{\ell=0}^{r} (-1)^\ell\ \Delta^{(\ell)}(z,w)$$

into sesqui-polynomials $\Delta^{(\ell)}$ which are homogeneous of bi-degree (ℓ,ℓ). For $\ell=1$ we obtain the normalized K-invariant inner product

$$\Delta^{(1)}(z,w)=(z|w).$$

If Z is of tube type with unit element e, then

$$\Delta^{(r)}(z,w)=N(z)\ \overline{N(w)}, \tag{3.1}$$

where N is the Jordan algebra determinant normalized by $N(e)=1$. The first eigenvalue formula gives the eigenvalues of the K-invariant operators $\Delta^{(\ell)}_{T^\nu}$ and $\Delta^{(\ell)}_T$ for $0\leqslant\ell\leqslant r$. This generalizes the approach in [2] for $\ell=1$.

Consider the **fundamental partitions**

$$(\ell):=(1,\ldots,1,0,\ldots,0) \tag{3.2}$$

with ℓ ones. Let $\psi^{(\ell)}_\alpha$ be an orthonormal basis of $\mathcal{P}^{(\ell)}(Z)$ and consider the Fischer-Fock reproducing kernel

$$E^{(\ell)}(z,w)=\sum_\alpha \psi^{(\ell)}_\alpha(z)\ \overline{\psi^{(\ell)}_\alpha(w)} \tag{3.3}$$

of $\mathcal{P}^{(\ell)}(Z)$. Then $\mathcal{P}^{(0)}(Z)=\mathbf{C}$ consists of constant functions and $\Delta^{(0)}=E^{(0)}=1$. In the first interesting case $\ell=1$ we obtain the dual space

$$\mathcal{P}^{(1)}(Z)=Z^*$$

of all linear forms on Z, and

$$\Delta^{(1)}(z,w)=E^{(1)}(z,w)=(z|w)=\sum_i (z|u_i)(u_i|w)$$

for any orthonormal basis $u_i\in Z$.

Lemma 4.

$$\Delta^{(\ell)}(z,w)=E^{(\ell)}(z,w)\ \prod_{j=1}^{\ell}(1+\tfrac{a}{2}(j-1)).$$

Proof. The Faraut–Korányi formula (2.7) applied to the parameter -1 yields

$$\Delta^{(\ell)}(z,w)=(-1)^\ell\ (-1)_{(\ell)}\ E^{(\ell)}(z,w)$$

with (positive) constant given by

$$\begin{aligned}(-1)^\ell\ (-1)_{(\ell)}&=(-1)^\ell\prod_{j=1}^{r}(-1-\tfrac{a}{2}(j-1))_{(\ell)_j}\\ &=(-1)^\ell\prod_{j=1}^{\ell}(-1-\tfrac{a}{2}(j-1))=\prod_{j=1}^{\ell}(1+\tfrac{a}{2}(j-1)).\end{aligned}$$

□

Lemma 4 implies that

$$\Delta^{(\ell)}_{T^\nu}=\prod_{j=1}^{\ell}(1+\tfrac{a}{2}(j-1))\ E^{(\ell)}_{T^\nu}=\prod_{j=1}^{\ell}(1+\tfrac{a}{2}(j-1))\sum_\alpha M^\nu_{\psi^{(\ell)}_\alpha}M^{\nu*}_{\psi^{(\ell)}_\alpha}$$

and

$$\Delta_T^{(\ell)} = \prod_{j=1}^{\ell}(1+\tfrac{a}{2}(j-1))\ E_T^{(\ell)} = \prod_{j=1}^{\ell}(1+\tfrac{a}{2}(j-1))\sum_\alpha M_{\psi_\alpha^{(\ell)}} M^*_{\psi_\alpha^{(\ell)}}.$$

In particular, for $\ell = 1$,

$$\Delta_{T^\nu}^{(1)} = E_{T^\nu}^{(1)} = \sum_i M^\nu_{(z|u_i)} M^{\nu *}_{(z|u_i)}$$

and

$$\Delta_T^{(1)} = E_T^{(1)} = \sum_i M_{(z|u_i)} M^*_{(z|u_i)}.$$

For any partition $\boldsymbol{m}$ and $1 \leqslant k \leqslant r$ we put

$$m'_k := m_k - \tfrac{a}{2}(k-1). \tag{3.4}$$

This notation (unrelated to the 'dual' partition sometimes also denoted by $\boldsymbol{m}'$) will be used throughout the paper. For any ℓ-element subset $L \subset \{1, \ldots, r\}$, with characteristic function

$$\chi_i^L := \begin{cases} 1 & i \in L \\ 0 & i \notin L \end{cases},$$

we define

$$\alpha^L_{\boldsymbol{m}} := \prod_{k \in L \not\ni h} \left(1 + \frac{a/2}{m'_k - m'_h}\right) \tag{3.5}$$

whenever $\boldsymbol{m} + \chi^L$ is also a partition, and put $\alpha^L_{\boldsymbol{m}} := 0$ otherwise. Similarly, we define

$$\beta^L_{\boldsymbol{m}} := \prod_{k \in L \not\ni h} \left(1 - \frac{a/2}{m'_k - m'_h}\right) \tag{3.6}$$

whenever $\boldsymbol{m} - \chi^L$ is also a partition, and put $\beta^L_{\boldsymbol{m}} := 0$ otherwise.

Proposition 5. *Assume that Z is of tube type, with unit element e. Then the spherical polynomial $\Phi^{\boldsymbol{m}}$ of type $\boldsymbol{m}$ satisfies*

$$\Delta_e^{(\ell)}\ \Phi^{\boldsymbol{m}} = \sum_{|L|=\ell} \Phi^{\boldsymbol{m}+\chi^L}\ \alpha^L_{\boldsymbol{m}} \tag{3.7}$$

for $0 \leqslant \ell \leqslant r$, with summation over all ℓ-element subsets $L \subset \{1, \ldots, r\}$ such that $\boldsymbol{m} + \chi^L$ is a partition.

Proof. Choose a frame $e_1, \ldots, e_r$ with $e = e_1 + \ldots + e_r$. For $t = (t_1, \ldots, t_r) \in \mathbf{C}^r$ we put $t \cdot e := t_1 e_1 + \ldots + t_r e_r$. Then

$$\Delta_e(t \cdot e) = \Delta(t \cdot e, e) = \prod_{i=1}^r (1 - t_i).$$

It follows that

$$\Delta_e^{(\ell)}(t \cdot e) = \sigma_\ell(t) \tag{3.8}$$

is the ℓ-th elementary symmetric polynomial. Now consider the **'Selberg-Jack' symmetric functions** $s_{\boldsymbol{m}}^{a/2}$, for any partition $\boldsymbol{m}$ and parameter $k = a/2$, as defined in [7] (cf. also [13]). Putting

$$f_r^{a/2}(\boldsymbol{m}) := \prod_{i<j}(m_i' - m_j')_{a/2}$$

as in [7, (1.2)], it is shown in [7, identity (S) on p. 69] that

$$s_{\boldsymbol{m}}^{a/2}(1^r) = \frac{f_r^{a/2}(\boldsymbol{m})}{f_r^{a/2}((0))}.$$

Therefore

$$\Phi^{\boldsymbol{m}}(t\cdot e) = \frac{1}{s_{\boldsymbol{m}}^{a/2}(1^r)}\ s_{\boldsymbol{m}}^{a/2}(t) = \frac{f_r^{a/2}((0))}{f_r^{a/2}(\boldsymbol{m})}\ s_{\boldsymbol{m}}^{a/2}(t). \tag{3.9}$$

By [7, identity (U) on p. 44 and (5.2)] we have a **Pieri formula**

$$\sigma_\ell(t)\ s_{\boldsymbol{m}}^{a/2}(t) = \sum_{|L|=\ell} U^{a/2}(\boldsymbol{m}+\chi^L/\boldsymbol{m})\ s_{\boldsymbol{m}+\chi^L}^{a/2}(t)$$

where, according to [7, (3.15)], the coefficients are given by

$$U^{a/2}(\boldsymbol{m}+\chi^L/\boldsymbol{m}) = \frac{f_r^{a/2}(\boldsymbol{m})}{f_r^{a/2}(\boldsymbol{m}+\chi^L)}\ \prod_{i<j}\Big(1+\tfrac{a}{2}\ \frac{\chi_i^L-\chi_j^L}{m_i'-m_j'}\Big).$$

In view of (3.9) this implies

$$\Delta^{(\ell)}(t\cdot e)\ \Phi^{\boldsymbol{m}}(t\cdot e) = \sum_{|L|=\ell}\Phi^{\boldsymbol{m}+\chi^L}(t\cdot e)\prod_{i<j}\Big(1+\tfrac{a}{2}\ \frac{\chi_i^L-\chi_j^L}{m_i'-m_j'}\Big).$$

For any $i \neq j$ we have $\chi_i^L - \chi_j^L = 0$ whenever both i, j belong to L or belong to its complement. On the other hand

$$\chi_i^L - \chi_j^L = \begin{cases} 1 & i\in L,\ j\notin L \\ -1 & i\notin L,\ j\in L \end{cases}.$$

It follows that

$$\prod_{i<j}\Big(1+\tfrac{a}{2}\ \frac{\chi_i^L-\chi_j^L}{m_i'-m_j'}\Big) = \prod_{L\ni i<j\notin L}\Big(1+\frac{a/2}{m_i'-m_j'}\Big)\prod_{L\not\ni i<j\in L}\Big(1-\frac{a/2}{m_i'-m_j'}\Big)$$

$$= \prod_{k\in L\not\ni h}\Big(1+\frac{a/2}{m_k'-m_h'}\Big) = \alpha_{\boldsymbol{m}}^L. \tag{3.10}$$

Thus (3.7) holds for all $t\cdot e \in Z$. Since the spherical polynomials are uniquely determined by their values on the "diagonal" $t\cdot e$, the assertion follows. □

For $\ell = 1$ we use singletons $L = \{k\}$ and obtain

$$(z|e)\ \Phi^{\boldsymbol{m}}(z) = \sum_{k=1}^r \Phi^{\boldsymbol{m}+\chi^{\{k\}}}(z)\ \alpha_{\boldsymbol{m}}^{\{k\}} = \sum_{k=1}^r \Phi^{\boldsymbol{m}+\chi^{\{k\}}}(z)\ \prod_{h\neq k}\frac{m_k'-m_h'+\frac{a}{2}}{m_k'-m_h'}$$

in accordance with [2, Lemma 4.2].

Example 6. For the spin factor of rank $r=2$ and $m\in\mathbf{N},\ m\geqslant 1$, we obtain

$$(z|e)\ \Phi^{(m,0)}(z)=\Phi^{(m+1,0)}(z)\ \frac{m+a}{m+\frac{a}{2}}+\Phi^{(m,1)}(z)\ \frac{m}{m+\frac{a}{2}}.$$

Remark 7. Evaluating (3.7) at e yields the non-obvious identity

$$\binom{r}{\ell}=\sum_{|L|=\ell}\alpha_{\boldsymbol{m}}^L \tag{3.11}$$

valid for any partition $\boldsymbol{m}$ and any parameter a. This is explicitly stated (for $k=\frac{a}{2}$) in [7, (6.23)] in the form

$$\binom{r}{\ell}\prod_{1\leqslant i<j\leqslant r}(m_i-m_j+(j-i)\tfrac{a}{2})=\sum_{|L|=\ell}\prod_{1\leqslant i<j\leqslant r}(m_i+\tfrac{a}{2}\chi_i^L-m_j-\tfrac{a}{2}\chi_j^L+(j-i)\tfrac{a}{2}).$$

With (3.4) this yields

$$\binom{r}{\ell}=\sum_{|L|=\ell}\prod_{1\leqslant i<j\leqslant r}\frac{m_i'+\frac{a}{2}\chi_i^L-m_j'-\frac{a}{2}\chi_j^L}{m_i'-m_j'}$$

which, by (3.10), implies (3.11).

For any parameter ν define

$$(\nu)_{\boldsymbol{m}}^L:=\prod_{k\in L}(\nu+m_k'). \tag{3.12}$$

Lemma 8.

$$\frac{(\nu)_{\boldsymbol{m}+\chi^L}}{(\nu)_{\boldsymbol{m}}}=(\nu)_{\boldsymbol{m}}^L.$$

In particular, for a singleton $L=\{k\}$

$$\frac{(\nu)_{\boldsymbol{m}+\chi^{\{k\}}}}{(\nu)_{\boldsymbol{m}}}=(\nu)_{\boldsymbol{m}}^{\{k\}}=\nu+m_k'$$

Proof.

$$\begin{aligned}\frac{(\nu)_{\boldsymbol{m}+\chi^L}}{(\nu)_{\boldsymbol{m}}}&=\prod_{k=1}^r\frac{(\nu-\frac{a}{2}(k-1))_{m_k+\chi_k^L}}{(\nu-\frac{a}{2}(k-1))_{m_k}}\\&=\prod_{k\in L}\frac{(\nu-\frac{a}{2}(k-1))_{m_k+1}}{(\nu-\frac{a}{2}(k-1))_{m_k}}\\&=\prod_{k\in L}(\nu-\tfrac{a}{2}(k-1)+m_k)=\prod_{k\in L}(\nu+m_k')\end{aligned}$$

□

The **dimension** $d_{\boldsymbol{m}}:=\dim\mathcal{P}^{\boldsymbol{m}}(Z)$ has been computed in [15, Lemma 2.6] (for tube domains) and [15, Lemma 2.7] (general case). It satisfies

$$C\ \frac{(d_e/r)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}}\ d_{\boldsymbol{m}}=\prod_{i<j}(m_i'-m_j')\ (m_i'-m_j'+1-\tfrac{a}{2})_{a-1}$$

$$= \prod_{i<j}(m_i' - m_j') \ \frac{\Gamma(m_i' - m_j' + \frac{a}{2})}{\Gamma(m_i' - m_j' + 1 - \frac{a}{2})}$$

$$= \prod_{i<j} \frac{m_i' - m_j'}{m_i' - m_j' - \frac{a}{2}} \ \frac{\Gamma(m_i' - m_j' + \frac{a}{2})}{\Gamma(m_i' - m_j' - \frac{a}{2})}. \tag{3.13}$$

Here the constant

$$C = \prod_{i<j}(0_i' - 0_j') \ (0_i' - 0_j' + 1 - \tfrac{a}{2})_{a-1}$$

$$= \prod_{i<j}(\tfrac{a}{2}(j-i)) \ (1 + \tfrac{a}{2}(j-i-1))_{a-1} \tag{3.14}$$

is determined by the condition $d_{(0)} = 1$, corresponding to $\mathcal{P}^{(0)}(Z) = \mathbf{C}$.

Lemma 9. *For the partitions (ℓ) the dimension is given by*

$$d_{(\ell)} = \binom{r}{\ell} \frac{(d/r)_{(\ell)}}{\prod\limits_{j=1}^{\ell}(1 + \frac{a}{2}(j-1))} = \binom{r}{\ell} \prod_{j=1}^{\ell} \frac{1 + b + \frac{a}{2}(r-j)}{1 + \frac{a}{2}(j-1)}$$

Proof. This follows, with (3.13) and (3.14), from the computation

$$\prod_{i<j} \frac{(\ell)_i' - (\ell)_j'}{\frac{a}{2}(j-i)} \ \frac{((\ell)_i' - (\ell)_j' + 1 - \frac{a}{2})_{a-1}}{(1 + \frac{a}{2}(j-i-1))_{a-1}}$$

$$= \prod_{i<j} \frac{(\ell)_i - (\ell)_j + \frac{a}{2}(j-i)}{\frac{a}{2}(j-i)} \ \frac{((\ell)_i - (\ell)_j + 1 + \frac{a}{2}(j-i-1))_{a-1}}{(1 + \frac{a}{2}(j-i-1))_{a-1}}$$

$$= \prod_{i \leqslant \ell < j} \frac{1 + \frac{a}{2}(j-i)}{\frac{a}{2}(j-i)} \ \frac{(2 + \frac{a}{2}(j-i-1))_{a-1}}{(1 + \frac{a}{2}(j-i-1))_{a-1}}$$

$$= \prod_{i \leqslant \ell < j} \frac{1 + \frac{a}{2}(j-i)}{\frac{a}{2}(j-i)} \ \frac{\frac{a}{2}(j-i+1)}{1 + \frac{a}{2}(j-i-1)}$$

$$= \prod_{i=1}^{\ell} \prod_{j=\ell+1}^{r} \frac{1 + \frac{a}{2}(j-i)}{1 + \frac{a}{2}(j-i-1)} \ \frac{\frac{a}{2}(j-i+1)}{\frac{a}{2}(j-i)}$$

$$= \prod_{i=1}^{\ell} \frac{1 + \frac{a}{2}(r-i)}{1 + \frac{a}{2}(\ell-i)} \ \frac{\frac{a}{2}(r+1-i)}{\frac{a}{2}(\ell+1-i)}$$

$$= \frac{(d_e/r)_{(\ell)}}{\prod\limits_{k=1}^{\ell}(1 + \frac{a}{2}(k-1))} \binom{r}{\ell}$$

using

$$\prod_{i=1}^{\ell} \frac{\frac{a}{2}(r+1-i)}{\frac{a}{2}(\ell+1-i)} = \prod_{i=1}^{\ell} \frac{r+1-i}{\ell+1-i} = \frac{r(r-1)\cdots(r-\ell)}{\ell!} = \binom{r}{\ell}$$

and

$$\prod_{i=1}^{\ell}(1+\tfrac{a}{2}(r-i)) = \prod_{i=1}^{\ell}(1+\tfrac{a}{2}(r-1)-\tfrac{a}{2}(i-1)) = (d_e/r)_{(\ell)}.$$

□

For $\ell = 1$ we obtain

$$d_{(1,0\ldots,0)} = d_{(1)} = \binom{r}{1}\,(d/r)_{(1)} = r\,\tfrac{d}{r} = d$$

for $\mathcal{P}^{(1)}(Z) = Z^*$. Here the above computation (say, for tube type domains) simplifies to

$$\begin{aligned} d_{(1,0\ldots,0)} &= \prod_{j=2}^{r} \frac{1+\frac{a}{2}(j-1)}{\frac{a}{2}(j-1)} \cdot \frac{\frac{a}{2}j}{1+\frac{a}{2}(j-2)} = \prod_{j=2}^{r} \frac{1+\frac{a}{2}(j-1)}{1+\frac{a}{2}(j-2)} \cdot \frac{\frac{a}{2}j}{\frac{a}{2}(j-1)} \\ &= \frac{1+\frac{a}{2}(r-1)}{1} \cdot \frac{\frac{a}{2}r}{\frac{a}{2}} = (1+\tfrac{a}{2}(r-1)) \cdot r = d. \end{aligned}$$

Example 10. For the spin factor Z and $m \in \mathbf{N}$, $\mathcal{P}^{(m,0)}(Z)$ is the space of all m-homogeneous harmonic polynomials in d variables. Since $a = d-2$ and $b = 0$ in this case, we obtain the well-known dimension formula

$$\begin{aligned} d_{(m,0)} &= \frac{m+\frac{a}{2}}{\frac{a}{2}} \cdot \frac{\Gamma(m+a)}{\Gamma(m+1)} \cdot \frac{\Gamma(1)}{\Gamma(a)} \\ &= \frac{2m+a}{a} \cdot \frac{(m+a-1)!}{m!(a-1)!} = \frac{(2m+d-2)(m+d-3)!}{m!(d-2)!} \end{aligned}$$

Proposition 11. *Suppose* $\boldsymbol{m}$ *and* $\boldsymbol{m}+\chi^L$ *are partitions. Then*

$$\frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}+\chi^L}} = \frac{(d_e/r)^L_{\boldsymbol{m}}}{(d/r)^L_{\boldsymbol{m}}}\;\frac{\beta^L_{\boldsymbol{m}+\chi^L}}{\alpha^L_{\boldsymbol{m}}}.$$

Proof. For any $i \neq j$ we have $(\boldsymbol{m}+\chi^L)_i - (\boldsymbol{m}+\chi^L)_j = m_i - m_j$ whenever both i,j belong to L or belong to its complement. On the other hand

$$(\boldsymbol{m}+\chi^L)_i - (\boldsymbol{m}+\chi^L)_j = \begin{cases} m_i + 1 - m_j & i \in L,\ j \notin L \\ m_i - m_j - 1 & i \notin L,\ j \in L \end{cases}.$$

By (3.13) we have

$$\begin{aligned} \frac{(d/r)^L_{\boldsymbol{m}}}{(d_e/r)^L_{\boldsymbol{m}}}\,\frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}+\chi^L}} &= \frac{(d_e/r)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}}\;\frac{(d/r)_{\boldsymbol{m}+\chi^L}}{(d_e/r)_{\boldsymbol{m}+\chi^L}}\;\frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}+\chi_k}} \\ &= \frac{(d/r)_{\boldsymbol{m}+\chi^L}}{(d/r)_{\boldsymbol{m}}}\;\frac{(d_e/r)_{\boldsymbol{m}}}{(d_e/r)_{\boldsymbol{m}+\chi^L}}\;\frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}+\chi^L}} \\ &= \prod_{L\ni i<j\notin L} \frac{m_i'-m_j'}{m_i'-m_j'+1} \cdot \frac{(m_i'-m_j'+1-\frac{a}{2})_{a-1}}{(m_i'-m_j'+2-\frac{a}{2})_{a-1}} \cdot \\ &\quad \cdot \prod_{L\not\ni i<j\in L} \frac{m_i'-m_j'}{m_i'-m_j'-1} \cdot \frac{(m_i'-m_j'+1-\frac{a}{2})_{a-1}}{(m_i'-m_j'-\frac{a}{2})_{a-1}} \end{aligned}$$

$$= \prod_{L\ni i<j\notin L} \frac{m_i'-m_j'}{m_i'-m_j'+1} \cdot \frac{m_i'-m_j'+1-\frac{a}{2}}{m_i'-m_j'+\frac{a}{2}} \cdot$$
$$\cdot \prod_{L\not\ni i<j\in L} \frac{m_i'-m_j'}{m_i'-m_j'-1} \cdot \frac{m_i'-m_j'-1+\frac{a}{2}}{m_i'-m_j'-\frac{a}{2}}$$
$$= \prod_{L\ni i<j\notin L} \frac{m_i'-m_j'}{m_i'-m_j'+\frac{a}{2}} \; \frac{m_i'-m_j'+1-\frac{a}{2}}{m_i'-m_j'+1} \cdot$$
$$\cdot \prod_{L\not\ni i<j\in L} \frac{m_i'-m_j'}{m_i'-m_j'-\frac{a}{2}} \; \frac{m_i'-m_j'-1+\frac{a}{2}}{m_i'-m_j'-1}$$
$$= \prod_{k\in L\not\ni h} \frac{m_k'-m_h'}{m_k'-m_h'+\frac{a}{2}} \; \frac{m_k'-m_h'+1-\frac{a}{2}}{m_k'-m_h'+1}$$
$$= \prod_{k\in L\not\ni h} \frac{1-\frac{a/2}{m_k'-m_h'+1}}{1+\frac{a/2}{m_k'-m_h'}} = \frac{\beta^L_{\boldsymbol{m}+\chi^L}}{\alpha^L_{\boldsymbol{m}}}$$

□

Proposition 12. *Let* $0 \leqslant \ell \leqslant r$ *and* $w \in Z$. *Then*

$$\Delta_w^{(\ell)}\; E_w^{\boldsymbol{m}} = \sum_{|L|=\ell} (d_e/r)^L_{\boldsymbol{m}}\; \beta^L_{\boldsymbol{m}+\chi^L}\; E_w^{\boldsymbol{m}+\chi^L}. \tag{3.15}$$

Proof. We first show that (3.15) holds for any maximal tripotent $w = e \in Z$. Assume first that Z is of tube type. Since

$$\Delta_e^{(\ell)} = \binom{r}{\ell} \Phi^{(\ell)}$$

by (3.8) and

$$E_e^{\boldsymbol{m}} = \frac{d_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}} \Phi^{\boldsymbol{m}}$$

as a consequence of Schur orthogonality [5], it follows that

$$\Delta_e^{(\ell)}\; E_e^{\boldsymbol{m}} = \frac{d_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}}\; \Delta_e^{(\ell)}\; \Phi^{\boldsymbol{m}} = \frac{d_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}} \sum_{|L|=\ell} \alpha^L_{\boldsymbol{m}}\; \Phi^{\boldsymbol{m}+\chi^L}$$
$$= \frac{d_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}}} \sum_{|L|=\ell} \frac{(d/r)_{\boldsymbol{m}+\chi^L}}{d_{\boldsymbol{m}+\chi^L}}\; \alpha^L_{\boldsymbol{m}}\; E_e^{\boldsymbol{m}+\chi^L}$$
$$= \sum_{|L|=\ell} \frac{(d/r)_{\boldsymbol{m}+\chi^L}}{(d/r)_{\boldsymbol{m}}}\; \frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}+\chi^L}}\; \alpha^L_{\boldsymbol{m}}\; E_e^{\boldsymbol{m}+\chi^L}$$
$$= \sum_{|L|=\ell} (d/r)^L_{\boldsymbol{m}} \cdot \frac{(d_e/r)^L_{\boldsymbol{m}}}{(d/r)^L_{\boldsymbol{m}}}\; \frac{\beta^L_{\boldsymbol{m}+\chi^L}}{\alpha^L_{\boldsymbol{m}}} \cdot \alpha^L_{\boldsymbol{m}}\; E_e^{\boldsymbol{m}+\chi^L}$$
$$= \sum_{|L|=\ell} (d_e/r)^L_{\boldsymbol{m}}\; \beta^L_{\boldsymbol{m}+\chi^L}\; E_e^{\boldsymbol{m}+\chi^L}.$$

If Z is not of tube type, we have $\Delta_e^{(\ell)}(z) = \Lambda_e^{(\ell)}(Pz)$ and $E_e^{\boldsymbol{m}}(z) = E_e^{\boldsymbol{m}}(Pz)$, where $P : Z \to Z_2(e)$ is the Peirce 2-projection. Thus (3.15) for $w = e$ holds for Z. Since both sides of (3.15) are K-invariant and the orbit $S = K \cdot e$ is a set of uniqueness for (anti)-holomorphic functions, the assertion follows for all $w \in Z$. □

Lemma 13. *For $0 \leqslant \ell \leqslant r$ we have*

$$\Delta^{(\ell)} = \frac{\binom{r}{\ell}\ (d/r)_{(\ell)}}{d_{(\ell)}}\ E^{(\ell)}.$$

Proof. This follows from $\Delta_{e,e}^{(\ell)} = \binom{r}{\ell}$ and $E_{e,e}^{(\ell)} = \frac{d_{(\ell)}}{(d/r)_{(\ell)}}$. As a double check, the same result is obtained by combining Lemmas 9 and 4. □

For any polynomial $p \in \mathcal{P}(Z)$ we denote by $p_{\boldsymbol{m}} \in \mathcal{P}^{\boldsymbol{m}}(Z)$ its $\boldsymbol{m}$-th component under the Peter–Weyl decomposition (2.2).

Proposition 14. *Let $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$. Then for each ℓ-element subset L such that $\boldsymbol{m} - \chi^L$ is a partition, we have (on the Fock space)*

$$\sum_\alpha \psi_\alpha^{(\ell)} (M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L} = p\ \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}}\ (d_e/r)^L_{\boldsymbol{m}-\chi^L}\ \beta^L_{\boldsymbol{m}}$$

and

$$\sum_\alpha \|(M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L}\|^2 = \|p\|^2\ \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}}\ (d_e/r)^L_{\boldsymbol{m}-\chi^L}\ \beta^L_{\boldsymbol{m}}$$

Proof. We follow the argument, for $\ell = 1$, contained in [2]. For fixed w we have

$$E_w^{(\ell)} = \sum_\alpha \psi_\alpha^{(\ell)}\ \overline{\psi_\alpha^{(\ell)}(w)}$$

by (3.3). With Lemma 13 and Proposition 12 it follows that

$$\begin{aligned}
\sum_\alpha \psi_\alpha^{(\ell)}(w)(M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L}(w) &= \psi_\alpha^{(\ell)}(w)(E_w^{\boldsymbol{m}-\chi^L} | M^*_{\psi_\alpha^{(\ell)}} p) \\
&= \sum_\alpha \psi_\alpha^{(\ell)}(w)(\psi_\alpha^{(\ell)} E_w^{\boldsymbol{m}-\chi^L} | p) \\
&= \sum_\alpha \left(\overline{\psi_\alpha^{(\ell)}(w)}\psi_\alpha^{(\ell)} E_w^{\boldsymbol{m}-\chi^L} | p\right) \\
&= (E_w^{(\ell)} E_w^{\boldsymbol{m}-\chi^L} | p) = \left((E_w^{(\ell)} E_w^{\boldsymbol{m}-\chi^L})_{\boldsymbol{m}} | p\right) \\
&= \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}}\ (d_e/r)^L_{\boldsymbol{m}-\chi^L}\ \beta^L_{\boldsymbol{m}}\ (E_w^{\boldsymbol{m}} | p) \\
&= \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}}\ (d_e/r)^L_{\boldsymbol{m}-\chi^L}\ \beta^L_{\boldsymbol{m}}\ p(w).
\end{aligned}$$

For the second assertion,

$$\sum_\alpha \|(M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L}\|^2 = \sum_\alpha ((M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L} | (M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L})$$

$$
\begin{aligned}
&= \sum_\alpha (M^*_{\psi_\alpha^{(\ell)}} p | (M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L}) \\
&= \sum_\alpha (p | \psi_\alpha^{(\ell)} (M^*_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}-\chi^L}) \\
&= \frac{d_{(\ell)}}{\binom{r}{\ell} (d/r)_{(\ell)}} \; (d_e/r)^L_{\boldsymbol{m}-\chi^L} \; \beta^L_{\boldsymbol{m}} \; \|p\|^2
\end{aligned}
$$

□

For the weighted Bergman spaces we obtain

Lemma 15. *Let $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$ and $\phi \in \mathcal{P}^{(\ell)}(Z)$. Then*

$$
(M^{\nu *}_\phi p)_{\boldsymbol{m}-\chi^L} = \frac{(\nu)_{\boldsymbol{m}-\chi^L}}{(\nu)_{\boldsymbol{m}}} \; (M^*_\phi p)_{\boldsymbol{m}-\chi^L} = \frac{1}{(\nu)^L_{\boldsymbol{m}-\chi^L}} \; (M^*_\phi p)_{\boldsymbol{m}-\chi^L}.
$$

Proof. Let $q \in \mathcal{P}^{\boldsymbol{m}-\chi^L}(Z)$. Then, with (2.8),

$$
\begin{aligned}
\Big((M^{\nu *}_\phi p)_{\boldsymbol{m}-\chi^L} | q \Big)_\nu &= (M^{\nu *}_\phi p | q)_\nu = (p | \phi q)_\nu = \frac{1}{(\nu)_{\boldsymbol{m}}} (p|\phi q) = \frac{1}{(\nu)_{\boldsymbol{m}}} (M^*_\phi p | q) \\
&= \frac{(\nu)_{\boldsymbol{m}-\chi^L}}{(\nu)_{\boldsymbol{m}}} (M^*_\phi p|q)_\nu = \frac{(\nu)_{\boldsymbol{m}-\chi^L}}{(\nu)_{\boldsymbol{m}}} ((M^*_\phi p)_{\boldsymbol{m}-\chi^L} | q)_\nu.
\end{aligned}
$$

Since q is arbitrary, the assertion follows. □

The **first eigenvalue formula** is the following:

Theorem 16. *For $0 \leqslant \ell \leqslant r$ the K-invariant operators $\Delta^{(\ell)}_T$ and $\Delta^{(\ell)}_{T^\nu}$ have the eigenvalues*

$$
\Delta^{(\ell)}_T(\boldsymbol{m}) = \sum_{|L|=\ell} (d_e/r)^L_{\boldsymbol{m}-\chi^L} \; \beta^L_{\boldsymbol{m}}
$$

and

$$
\Delta^{(\ell)}_{T^\nu}(\boldsymbol{m}) = \sum_{|L|=\ell} \frac{(d_e/r)^L_{\boldsymbol{m}-\chi^L}}{(\nu)^L_{\boldsymbol{m}-\chi^L}} \; \beta^L_{\boldsymbol{m}},
$$

with $\beta^L_{\boldsymbol{m}}$ defined in (3.6).

Proof. If $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$ then

$$
E^{(\ell)}_T p = \sum_\alpha \psi^{(\ell)}_\alpha (M^*_{\psi^{(\ell)}_\alpha} p) = \sum_{|L|=\ell} \sum_\alpha \psi^{(\ell)}_\alpha (M^*_{\psi^{(\ell)}_\alpha} p)_{\boldsymbol{m}-\chi^L}.
$$

In view of Lemma 13, the first assertion for the Fock space follows from Proposition 14. For the second assertion, we use Lemma 15 and obtain, for each subset L such that $\boldsymbol{m} - \chi^L$ is a partition

$$
\begin{aligned}
\sum_\alpha M^\nu_{\psi^{(\ell)}_\alpha} (M^{\nu *}_{\psi^{(\ell)}_\alpha} p)_{\boldsymbol{m}-\chi^L} &= \sum_\alpha \psi^{(\ell)}_\alpha (M^{\nu *}_{\psi^{(\ell)}_\alpha} p)_{\boldsymbol{m}-\chi^L} \\
&= \frac{1}{(\nu)^L_{\boldsymbol{m}-\chi^L}} \; \sum_\alpha \psi^{(\ell)}_\alpha (M^*_{\psi^{(\ell)}_\alpha} p)_{\boldsymbol{m}-\chi^L} \\
&= \frac{1}{(\nu)^L_{\boldsymbol{m}-\chi^L}} \; \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} \; (d_e/r)^L_{\boldsymbol{m}-\chi^L} \; \beta^L_{\boldsymbol{m}} \; p.
\end{aligned}
$$

Now the assertion follows from

$$\Delta_{T^\nu}^{(\ell)} = \binom{r}{\ell}\frac{(d/r)_{(\ell)}}{d_{(\ell)}}\ E_{T^\nu}^{(\ell)} = \binom{r}{\ell}\frac{(d/r)_{(\ell)}}{d_{(\ell)}}\ \sum_\alpha M^\nu_{\psi_\alpha^{(\ell)}} M^{\nu *}_{\psi_\alpha^{(\ell)}}.$$

□

Example 17. For $\ell = 1$ we obtain the formula

$$\begin{aligned}\Delta_{T^\nu}^{(1)}(\boldsymbol{m}) &= \Big(\sum_i M^\nu_{(z|u_i)} M^{\nu *}_{(z|u_i)}\Big)(\boldsymbol{m}) = \sum_{k=1}^r \frac{m_k' + \frac{d_e}{r} - 1}{m_k' + \nu - 1}\ \beta_{\boldsymbol{m}}^{\{k\}} \\ &= \sum_{k=1}^r \frac{m_k' + \frac{d_e}{r} - 1}{m_k' + \nu - 1}\ \prod_{h \neq k} \frac{m_k' - m_h' - \frac{a}{2}}{m_k' - m_h'}\end{aligned}$$

previously obtained in [2, Proposition 4.4].

Example 18. For $\ell = r,\ L = \{1, \ldots, r\}$ we have $\beta_{\boldsymbol{m}}^L = 1$ (empty product). If Z is of tube type then $d_e = d$. Using (3.1) Theorem 16 simplifies to

$$\begin{aligned}(M_N M_N^*)(\boldsymbol{m}) &= \Delta_T^{(r)}(\boldsymbol{m}) = \frac{(d/r)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}-1}}, \\ (M_N^\nu M_N^{\nu *})(\boldsymbol{m}) &= \Delta_{T^\nu}^{(r)}(\boldsymbol{m}) = \frac{(d/r)_{\boldsymbol{m}}}{(d/r)_{\boldsymbol{m}-1}} \frac{(\nu)_{\boldsymbol{m}-1}}{(\nu)_{\boldsymbol{m}}}.\end{aligned}$$

4. The Second Eigenvalue Formula

The second eigenvalue formula gives the eigenvalues of the K-invariant operators $T^\nu_{\Delta^{(\ell)}}$ and $T_{\Delta^{(\ell)}}$, for $0 \leqslant \ell \leqslant r$. In this case the previous arguments, based on reproducing kernel identities, do not apply immediately.

Lemma 19. *Under the K-action (2.1) on polynomials, we have*

$$M_p^*(k^{-1} \cdot q) = k^{-1} \cdot (M_{k \cdot p}^* q).$$

Proof. It suffices to check for linear polynomials $p(z) = (z|u)$, where $u \in Z$. We have

$$\begin{aligned}(M^*_{(z|u)}(k^{-1} \cdot q))(z) &= (k^{-1} \cdot q)'(z)u = (q \circ k)'(z)u \\ &= q'(kz)ku = (M^*_{(z|ku)} q)(kz) = k^{-1} \cdot (M^*_{(z|ku)} q)(z).\end{aligned}$$

Since $(k \cdot p)(z) = p(k^{-1}z) = (k^{-1}z|u) = (z|ku)$, the assertion follows. □

Lemma 20. *Let ϕ_α and ψ_β be orthonormal bases of $\mathcal{P}^{(\ell)}(Z)$. Then for any sesqui-linear form $\langle \phi | \psi \rangle$ on $\mathcal{P}^{(\ell)}(Z)$ we have*

$$\sum_\alpha \langle \phi_\alpha | \phi_\alpha \rangle = \sum_\beta \langle \psi_\beta | \psi_\beta \rangle$$

Proof. Using Einstein summation convention to simplify notation, we have $\phi_\alpha = \Lambda_\alpha^\beta \psi_\beta$ for a unitary 'matrix' Λ. Then

$$\begin{aligned}\langle \phi_\alpha | \phi_\alpha \rangle &= \langle \Lambda_\alpha^\sigma \psi_\sigma | \Lambda_\alpha^\tau \psi_\tau \rangle = \overline{\Lambda_\alpha^\sigma}\ \langle \psi_\sigma | \psi_\tau \rangle\ \Lambda_\alpha^\tau \\ &= \langle \psi_\sigma | \psi_\tau \rangle\ (\Lambda^* \Lambda)_\sigma^\tau = \langle \psi_\sigma | \psi_\tau \rangle\ \delta_\sigma^\tau = \langle \psi_\sigma | \psi_\sigma \rangle.\end{aligned}$$

□

For any $p, q \in \mathcal{P}(Z)$ the map $(\phi, \psi) \mapsto \langle \phi|\psi\rangle := (p|M_\phi^* q)(M_\psi^* q|p)$ is sesqui-linear. Hence Lemma 20 implies for each $k \in K$

$$\begin{aligned}\sum_\alpha (p|M^*_{k\cdot\psi_\alpha^{(\ell)}} q)(M^*_{k\cdot\psi_\alpha^{(\ell)}} q|p) &= \sum_\alpha \langle k\cdot\psi_\alpha^{(\ell)}|k\cdot\psi_\alpha^{(\ell)}\rangle \\ &= \sum_\alpha \langle \psi_\alpha^{(\ell)}|\psi_\alpha^{(\ell)}\rangle \\ &= \sum_\alpha (p|M^*_{\psi_\alpha^{(\ell)}} q)(M^*_{\psi_\alpha^{(\ell)}} q|p), \qquad (4.1)\end{aligned}$$

since $k\cdot\psi_\alpha^{(\ell)}$ is also an orthonormal basis.

Proposition 21. *Let $\boldsymbol{m}$ and $\boldsymbol{m}+\chi^L$ be partitions. Then we have (on the Fock space)*

$$\sum_\alpha \|(\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L}\|^2 = \|p\|^2 \ \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} \ (d/r)^L_{\boldsymbol{m}} \ \alpha^L_{\boldsymbol{m}}$$

and

$$\sum_\alpha M^*_{\psi_\alpha^{(\ell)}} (\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L} = p \ \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} \ (d/r)^L_{\boldsymbol{m}} \ \alpha^L_{\boldsymbol{m}}$$

for all $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$.

Proof. Let $q \in \mathcal{P}^{\boldsymbol{m}+\chi^L}(Z)$. Schur orthogonality applied to $\mathcal{P}^{\boldsymbol{m}+\chi^L}(Z)$ yields for each α

$$\begin{aligned}&\frac{\|q\|^2}{d_{\boldsymbol{m}+\chi^L}} \ \|(\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L}\|^2 \\ &\quad = \int_K dk \ \left((\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L}|k^{-1}\cdot q\right) \ \left(k^{-1}\cdot q|(\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L}\right) \\ &\quad = \int_K dk \ \left(\psi_\alpha^{(\ell)} p|k^{-1}\cdot q\right) \ \left(k^{-1}\cdot q|\psi_\alpha^{(\ell)} p\right) \\ &\quad = \int_K dk \ \left(p|M^*_{\psi_\alpha^{(\ell)}}(k^{-1}\cdot q)\right) \ \left(M^*_{\psi_\alpha^{(\ell)}}(k^{-1}\cdot q)|p\right) \\ &\quad = \int_K dk \ \left(p|k^{-1}\cdot (M^*_{k\cdot\psi_\alpha^{(\ell)}} q)\right) \ \left(k^{-1}\cdot (M^*_{k\cdot\psi_\alpha^{(\ell)}} q)|p\right) \\ &\quad = \int_K dk \ \left(k\cdot p|M^*_{k\cdot\psi_\alpha^{(\ell)}} q\right) \ \left(M^*_{k\cdot\psi_\alpha^{(\ell)}} q|k\cdot p\right).\end{aligned}$$

With (4.1) and Schur orthogonality applied to $\mathcal{P}^{\boldsymbol{m}}(Z)$ we obtain

$$\begin{aligned}&\frac{\|q\|^2}{d_{\boldsymbol{m}+\chi^L}} \ \sum_\alpha \|(\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L}\|^2 \\ &\quad = \int_K dk \ \sum_\alpha \left(k\cdot p|M^*_{k\cdot\psi_\alpha^{(\ell)}} q\right) \ \left(M^*_{k\cdot\psi_\alpha^{(\ell)}} q|k\cdot p\right)\end{aligned}$$

$$= \int_K dk \sum_\alpha \left(k \cdot p | M^*_{\psi^{(\ell)}_\alpha} q\right) \left(M^*_{\psi^{(\ell)}_\alpha} q | k \cdot p\right)$$
$$= \sum_\alpha \int_K dk \left(k \cdot p | (M^*_{\psi^{(\ell)}_\alpha} q)_{\boldsymbol{m}}\right) \left((M^*_{\psi^{(\ell)}_\alpha} q)_{\boldsymbol{m}} | k \cdot p\right)$$
$$= \frac{\|p\|^2}{d_{\boldsymbol{m}}} \sum_\alpha \|(M^*_{\psi^{(\ell)}_\alpha} q)_{\boldsymbol{m}}\|^2$$
$$= \frac{\|p\|^2}{d_{\boldsymbol{m}}} \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} (d_e/r)^L_{\boldsymbol{m}} \ \beta^L_{\boldsymbol{m}+\chi^L} \ \|q\|^2,$$

where in the last step we use Proposition 14. It follows that

$$\begin{aligned}\sum_\alpha \|(\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L}\|^2 &= \|p\|^2 \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} \frac{d_{\boldsymbol{m}+\chi^L}}{d_{\boldsymbol{m}}} (d_e/r)^L_{\boldsymbol{m}} \ \beta^L_{\boldsymbol{m}+\chi^L} \\ &= \|p\|^2 \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} \frac{(d/r)^L_{\boldsymbol{m}}}{(d_e/r)^L_{\boldsymbol{m}}} \frac{\alpha^L_{\boldsymbol{m}}}{\beta^L_{\boldsymbol{m}+\chi^L}} (d_e/r)^L_{\boldsymbol{m}} \ \beta^L_{\boldsymbol{m}+\chi^L} \\ &= \|p\|^2 \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}} (d/r)^L_{\boldsymbol{m}} \ \alpha^L_{\boldsymbol{m}}.\end{aligned}$$

This proves the first assertion. The second assertion follows, since $\sum_\alpha M^*_{\psi^{(\ell)}_\alpha}(\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L}$ is a multiple of p and

$$\begin{aligned}\sum_\alpha \left(p | M^*_{\psi^{(\ell)}_\alpha}(\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L}\right) &= \sum_\alpha (\psi^{(\ell)}_\alpha p | (\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L}) \\ &= \sum_\alpha ((\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L} | (\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L}) \\ &= \sum_\alpha \|(\psi^{(\ell)}_\alpha p)_{\boldsymbol{m}+\chi^L}\|^2.\end{aligned}$$

□

The **second eigenvalue formula** is the following:

Theorem 22. *For $0 \leqslant \ell \leqslant r$ the K-invariant operators $T_{\Delta^{(\ell)}}$ and $T^\nu_{\Delta^{(\ell)}}$ have the eigenvalues*

$$T_{\Delta^{(\ell)}}(\boldsymbol{m}) = \sum_{|L|=\ell} (d/r)^L_{\boldsymbol{m}} \ \alpha^L_{\boldsymbol{m}}$$

and

$$T^\nu_{\Delta^{(\ell)}}(\boldsymbol{m}) = \sum_{|L|=\ell} \frac{(d/r)^L_{\boldsymbol{m}}}{(\nu)^L_{\boldsymbol{m}}} \ \alpha^L_{\boldsymbol{m}}$$

with $\alpha^L_{\boldsymbol{m}}$ defined in (3.5).

Proof. Let $p \in \mathcal{P}^{\boldsymbol{m}}(Z)$. By Lemma 15 applied to $\boldsymbol{m} + \chi^L$ we have

$$\begin{aligned}\sum_\alpha M^{\nu *}_{\psi_\alpha^{(\ell)}}(M^\nu_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}+\chi^L} &= \sum_\alpha M^{\nu *}_{\psi_\alpha^{(\ell)}}(\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L} \\ &= \frac{1}{(\nu)^L_{\boldsymbol{m}}} \sum_\alpha M^{*}_{\psi_\alpha^{(\ell)}}(\psi_\alpha^{(\ell)} p)_{\boldsymbol{m}+\chi^L} \\ &= p\, \frac{d_{(\ell)}}{\binom{r}{\ell}(d/r)_{(\ell)}}\, \frac{(d/r)^L_{\boldsymbol{m}}}{(\nu)^L_{\boldsymbol{m}}}\, \alpha^L_{\boldsymbol{m}}.\end{aligned}$$

Since

$$T^\nu_{\Delta^{(\ell)}} p = \frac{\binom{r}{\ell}(d/r)_{(\ell)}}{d_{(\ell)}}\, T^\nu_{E^{(\ell)}} p = \frac{\binom{r}{\ell}(d/r)_{(\ell)}}{d_{(\ell)}} \sum_{|L|=\ell} \sum_\alpha M^{\nu *}_{\psi_\alpha^{(\ell)}}(M^\nu_{\psi_\alpha^{(\ell)}} p)_{\boldsymbol{m}+\chi^L}$$

the assertion follows by summing over all ℓ-element subsets L such that $\boldsymbol{m} + \chi^L$ is a partition. The proof for the Fock space is similar. □

For the Hardy space $H^2(S)$ over the Shilov boundary S of D, corresponding to $\nu = \frac{d}{r}$, the above formula combined with (3.11) simplifies to

$$T^{d/r}_{\Delta^{(\ell)}}(\boldsymbol{m}) = \sum_{|L|=\ell} \alpha^L_{\boldsymbol{m}} = \binom{r}{\ell}.$$

This, however, is trivial since $\Delta^{(\ell)}(z, z) = \binom{r}{\ell}$ is constant on S.

Example 23. For $\ell = 1$, with $\Delta^{(1)}(z, z) = (z|z)$, we obtain as a special case

$$T^\nu_{(z|z)}(\boldsymbol{m}) = \sum_{k=1}^r \frac{(d/r)^{\{k\}}_{\boldsymbol{m}}}{(\nu)^{\{k\}}_{\boldsymbol{m}}}\, \alpha^{\{k\}}_{\boldsymbol{m}} = \sum_{k=1}^r \frac{m'_k + \frac{d}{r}}{m'_k + \nu} \prod_{h \neq k} \frac{m'_k - m'_h + \frac{a}{2}}{m'_k - m'_h}$$

for all partitions $\boldsymbol{m}$. This formula was conjectured in [6] and proved there, by a different argument, for all bounded symmetric domains of rank $r = 2$.

Besides the spin factors, which correspond to the rank 2 domains of tube type, there exist three types of Jordan triples of rank 2 which are not of tube type: (i) the space of all complex $(2 \times N)$-matrices with $N > 2$, where $d = 2N$ and $a = 2$, (ii) the space of all complex anti-symmetric (5×5)-matrices, where $d = 10$ and $a = 4$, and (iii) the exceptional domain of dimension $d = 16$, where $a = 6$.

Example 24. For $\ell = r,\ L = \{1, \ldots, r\}$ we have $\alpha^L_{\boldsymbol{m}} = 1$ (empty product). If Z is of tube type, then $d_e = d$. Using (3.1) Theorem 22 simplifies to

$$\begin{aligned}(M_N^* M_N)(\boldsymbol{m}) &= T_{\Delta^{(r)}}(\boldsymbol{m}) = \frac{(d/r)_{\boldsymbol{m}+1}}{(d/r)_{\boldsymbol{m}}}, \\ (M_N^{\nu *} M_N^\nu)(\boldsymbol{m}) &= T^\nu_{\Delta^{(r)}}(\boldsymbol{m}) = \frac{(d/r)_{\boldsymbol{m}+1}}{(d/r)_{\boldsymbol{m}}} \frac{(\nu)_{\boldsymbol{m}}}{(\nu)_{\boldsymbol{m}+1}}.\end{aligned}$$

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