Conjugations in $L^2(\mathcal{H})$

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Abstract. Conjugations commuting with $\mathbf{M}_z$ and intertwining $\mathbf{M}_z$ and $\mathbf{M}_{\bar{z}}$ in $L^2(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, are characterized. We also investigate which of them leave invariant the whole Hardy space $H^2(\mathcal{H})$ or a model space $K_\Theta = H^2(\mathcal{H}) \ominus \Theta H^2(\mathcal{H})$, where Θ is a pure operator valued inner function.

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1. Introduction

The motivation to study conjugations (i.e., antilinear isometric involutions) has its roots in physics ([8]), in particular in non-hermitian quantum mechanics and spectral analysis of complex symmetric operators. There are many important examples of complex symmetric operators, that is C -symmetric operators with respect to some conjugation C , namely normal operators, Hankel operators, truncated Toeplitz operators (see for example [3–9, 11, 14]).

In [2, 3] all conjugations in the classical L^2 space on the unit circle commuting with M_z or intertwining the operators M_z and $M_{\bar{z}}$ (in other words, all conjugations C according to which the operator M_z is C -symmetric, see the definition below) were fully characterized. The behaviour of such conjugations was also studied in connection with an analytic part of the space L^2 and model spaces, in particular there were characterized all conjugations leaving the whole Hardy space and model spaces invariant. In what follows we study similar questions concerning conjugations in L^2 spaces with values in a

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certain Hilbert space $\mathcal{H}$. The investigation in this direction is important for its relation with Sz.-Nagy–Foiaş theory [12, Chap. 6] saying that C_0 contractions with finite defect indexes are unitarily equivalent to multiplication by the independent variable in a certain model space given by an operator valued inner function. In other words, the results from the paper can be moved by unitary equivalence to contractions on Hilbert spaces, keeping suitable assumptions.

Denote by $\mathcal{H}$ a complex Hilbert space, by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $LA(\mathcal{H})$ the space of all bounded antilinear operators on $\mathcal{H}$. A *conjugation* C in $\mathcal{H}$ is an antilinear isometric involution, i.e., $C^2 = I_{\mathcal{H}}$ and

$$\langle Cf, Cg \rangle = \langle g, f \rangle \quad \text{for all } f, g \in \mathcal{H}. \quad (1.1)$$

An operator $A \in L(\mathcal{H})$ is called C -symmetric if $CAC = A^*$. Recall that for $A \in LA(\mathcal{H})$ there exists a unique antilinear operator $A^\sharp$, called the *antilinear adjoint* of A , defined by the equality

$$\langle Af, g \rangle = \overline{\langle f, A^\sharp g \rangle}, \quad (1.2)$$

for all $f, g \in \mathcal{H}$. It is clear, see [3], that $C^\sharp = C$ for any conjugation C , $(AB)^\sharp = B^*A^\sharp$ and similarly $(BA)^\sharp = A^\sharp B^*$ for $A \in LA(\mathcal{H})$, $B \in L(\mathcal{H})$.

Let $L^2 = L^2(\mathbb{T}, m)$ and $L^\infty = L^\infty(\mathbb{T}, m)$ where $\mathbb{T}$ is the unit circle and m is the normalized Lebesgue measure and let H^2 denote the classical Hardy space on the unit disc $\mathbb{D}$. For an inner function θ (i.e., $\theta \in L^\infty \cap H^2$ and $|\theta| = 1$ a.e. on $\mathbb{T}$) one can define the *model space* $K_\theta = H^2 \ominus \theta H^2$.

The most natural conjugation $\tilde{J}$ in L^2 is defined as $\tilde{J}f = \bar{f}$, for $f \in L^2$. This conjugation has two natural properties: the operator M_z is $\tilde{J}$ -symmetric, i.e., $M_z\tilde{J} = \tilde{J}M_{\bar{z}}$, and $\tilde{J}$ maps an analytic function into a co-analytic one, i.e., $\tilde{J}H^2 = \overline{H^2}$. Another natural conjugation in L^2 is $J^*f = f^\#$, $f^\#(z) = \overline{f(\bar{z})}$. The conjugation J^* has a completely different behaviour: it commutes with multiplication by z ($M_zJ^* = J^*M_z$) and leaves analytic functions invariant, $J^*H^2 = H^2$.

In Sect. 3 we recall some basic properties of vector and operator valued functions. In Sect. 4, for any separable Hilbert space $\mathcal{H}$, we naturally extend the definitions of the conjugations $\tilde{J}$ and J^* on L^2 to the conjugations $\tilde{\mathbf{J}}$ and $\mathbf{J}^*$ on a vector valued space $L^2(\mathcal{H})$ keeping the same properties with respect to the multiplication by the independent variable, i.e., $\tilde{\mathbf{J}}\mathbf{M}_z = \mathbf{M}_{\bar{z}}\tilde{\mathbf{J}}$, $\mathbf{J}^*\mathbf{M}_z = \mathbf{M}_z\mathbf{J}^*$. However, one needs to fix some conjugation J on $\mathcal{H}$ (in case of L^2 the natural conjugation in $\mathbb{C}$, $z \mapsto \bar{z}$, plays this role). Theorems 4.8 and 4.3 characterize all $\mathbf{M}_z$ -conjugations $\mathbf{C}$ in $L^2(\mathcal{H})$, i.e.,

$$\mathbf{C}\mathbf{M}_z = \mathbf{M}_{\bar{z}}\mathbf{C} \quad (1.3)$$

and all $\mathbf{M}_z$ -commuting conjugations in $L^2(\mathcal{H})$, i.e.,

$$\mathbf{C}\mathbf{M}_z = \mathbf{M}_z\mathbf{C}. \quad (1.4)$$

In Sect. 5, Theorem 5.1 and Proposition 5.5, we describe all conjugations satisfying (1.3) or (1.4) and leaving the Hardy space $H^2(\mathcal{H})$ invariant. In Sect. 6 we study these conjugations for which vector valued model spaces are

invariant. We concentrate on the case when the dimension of the underlying Hilbert space $\mathcal{H}$ is finite and the vector valued inner function is pure. The last section is devoted to those conjugations which leave shift invariant subspaces invariant. Generally, the vector valued case is much more complicated than the scalar one, see for example Theorems 6.6, 7.3, 8.6 and Example 8.8. Section 2 is devoted to the special case $\mathcal{H} = \mathbb{C}^2$. This illustrates the general theorems and gives stronger results than in [11]. On the other hand, it gives more precise characterizations (Theorem 2.5), which are of independent interest.

2. Conjugations in $L^2 \oplus L^2$

In Theorem 2.4 and Proposition 2.6 [11] there are given the conditions for a 2×2 operator matrix to be a conjugation. We will, however, use equivalent conditions obtained by checking the antilinear selfadjointness and involutive property of a conjugation. Namely, let $\mathbf{C} = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$, where D_j are antilinear operators on $\mathcal{H}$ for $j = 1, 2, 3, 4$. Then $\mathbf{C}$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$ if and only if the following conditions hold:

$$D_1 = D_1^\sharp, D_4 = D_4^\sharp, D_3 = D_2^\sharp, \quad (2.1)$$

$$D_1 D_1^\sharp + D_2 D_2^\sharp = I, D_2^\sharp D_2 + D_4 D_4^\sharp = I, \quad (2.2)$$

$$D_2^\sharp D_1 + D_4 D_2^\sharp = 0. \quad (2.3)$$

Denote $\mathbf{M}_z = \begin{bmatrix} M_z & 0 \\ 0 & M_z \end{bmatrix}$. We will investigate the conditions for $\mathbf{C}$ to be an $\mathbf{M}_z$ -conjugation and for it to commute with $\mathbf{M}_z$.

In [2, Theorem 2.4] all conjugations in L^2 commuting with M_z were characterized. In particular it was shown that such a conjugation has to be of the form $M_\psi J^*$ for some unimodular function $\psi \in L^\infty$ which is *symmetric*, i.e., $\psi(z) = \psi(\bar{z})$ a.e. on $\mathbb{T}$. The following theorem gives a characterization of $\mathbf{M}_z$ -commuting conjugations in $L^2 \oplus L^2$.

Theorem 2.1. *Let $\mathbf{C}$ be an antilinear operator on $L^2 \oplus L^2$. Then $\mathbf{C} = \begin{bmatrix} D_1 & D_2 \\ D_2^\sharp & D_4 \end{bmatrix}$ is a conjugation such that $\mathbf{M}_z \mathbf{C} = \mathbf{C} \mathbf{M}_z$ if and only if there are functions $\psi_i \in L^\infty$, $i = 1, 2, 4$, such that $D_i = M_{\psi_i} J^*$ and*

$$\psi_1^\sharp = \bar{\psi}_1, \quad \psi_4^\sharp = \bar{\psi}_4, \quad (2.4)$$

$$|\psi_1|^2 = |\psi_4|^2 = 1 - |\psi_2|^2, \quad (2.5)$$

$$\psi_1^\sharp \psi_2 + \psi_2^\sharp \psi_4 = 0. \quad (2.6)$$

Proof. Easy calculations show that $\mathbf{M}_z \mathbf{C} = \mathbf{C} \mathbf{M}_z$ if and only if $M_z D_i = D_i M_z$ for $i = 1, 2, 4$ and $M_z D_2^\sharp = D_2^\sharp M_z$. Hence for $i = 1, 2, 4$ we have

$$M_z D_i J^* = D_i M_z J^* = D_i J^* M_z.$$

Thus the linear operators $D_i J^*$ commute with M_z , so they have to be of the form $D_i J^* = M_{\psi_i}$ for $\psi_i \in L^\infty$.

Condition (2.1) implies that for $i = 1, 4$, $M_{\psi_i} J^* = J^* M_{\overline{\psi}_i}$, i.e., $\psi_i^\# = \overline{\psi}_i$. This means that ψ_1 and ψ_4 are symmetric, i.e., $\psi_i(z) = \psi_i(\bar{z})$ for $i = 1, 4$ (a.e. on $\mathbb{T}$). By (2.2), for $i = 1, 4$, we get

$$(M_{\psi_i} J^*)^2 + M_{\psi_2} J^* J^* M_{\overline{\psi}_2} = I$$

which is equivalent to

$$\psi_i^\# \psi_i + |\psi_2|^2 = 1.$$

Finally, by (2.3) we get

$$J^* M_{\overline{\psi}_2} M_{\psi_1} J^* + J^* M_{\psi_4^\#} M_{\overline{\psi}_2^\#} J^* = 0 \quad (2.7)$$

which is equivalent to

$$\overline{\psi}_2 \psi_1 + \psi_4^\# \overline{\psi}_2^\# = 0. \quad (2.8)$$

Taking into consideration the fact that ψ_1 and ψ_4 are symmetric we obtain (2.6). $\square$

Remark 2.2. Note that if $D_2 = 0$, then conditions (2.4)–(2.6) imply that D_1 and D_4 are conjugations which commute with M_z . Hence by [2, Theorem 2.4] we get $D_1 = M_{\psi_1} J^*$, $D_4 = M_{\psi_4} J^*$. On the other hand, this is also a consequence of Theorem 2.1.

If now $\psi_1 = 0$ (which is equivalent to $\psi_4 = 0$), then (2.5) implies that $|\psi_2| = 1$.

Example 2.3. The following conjugations satisfy the conditions of Theorem 2.1:

$$J_1^* = \begin{bmatrix} J^* & 0 \\ 0 & J^* \end{bmatrix}, \quad \text{or} \quad J_2^* = \begin{bmatrix} 0 & J^* \\ J^* & 0 \end{bmatrix} \quad \text{or} \quad C = \frac{1}{\sqrt{2}} \begin{bmatrix} J^* & J^* \\ J^* & -J^* \end{bmatrix}.$$

Example 2.4. Let $\psi_1(z) = \psi_4(z) = \frac{1}{2}(\bar{z} + z)$ and $\psi_2(z) = \frac{1}{2i}(-\bar{z} + z)$. In other words, $\psi_1(e^{it}) = \psi_4(e^{it}) = \cos t$ and $\psi_2(e^{it}) = \sin t$. Then $\psi_2^\#(z) = \frac{1}{2i}(\bar{z} - z)$, i.e., $\psi_2^\#(e^{it}) = -\sin t$. Observe that such functions satisfy conditions (2.4)–(2.6), so $C = \begin{bmatrix} M_{\psi_1} J^* & M_{\psi_2} J^* \\ M_{\overline{\psi}_2^\#} J^* & M_{\psi_4} J^* \end{bmatrix}$ is a conjugation satisfying Theorem 2.1.

The characterization of all M_z -conjugations in L^2 was given in [3]. It was proved that such conjugations are of the form $M_\psi \tilde{J}$, where $\psi \in L^\infty$, $|\psi| = 1$ a.e. on $\mathbb{T}$. In the space $L^2(\mathbb{C}^2)$ the characterization is more complex.

Theorem 2.5. Let C be an antilinear operator on $L^2(\mathbb{C}^2)$. Then $C = \begin{bmatrix} D_1 & D_2 \\ D_2^\# & D_4 \end{bmatrix}$ is a conjugation such that $\mathbf{M}_z C = C \mathbf{M}_{\bar{z}}$ if and only if there are functions $\psi_i \in L^\infty$, $i = 1, 2, 4$, such that $D_i = M_{\psi_i} \tilde{J}$ and

$$|\psi_1|^2 = |\psi_4|^2 = 1 - |\psi_2|^2, \quad (2.9)$$

$$\overline{\psi}_1 \psi_2 + \overline{\psi}_2 \psi_4 = 0. \quad (2.10)$$

Proof. Note that $\mathbf{M}_z \mathbf{C} = \mathbf{C} \mathbf{M}_{\bar{z}}$ if and only if

$$M_z D_i = D_i M_{\bar{z}} \text{ for } i = 1, 2, 4. \quad (2.11)$$

Hence for $i = 1, 2, 4$ we have

$$M_z D_i \tilde{J} = D_i M_{\bar{z}} \tilde{J} = D_i \tilde{J} M_z.$$

Thus the linear operators $D_i \tilde{J}$ commute with $M_{\bar{z}}$, so they have to be of the form $D_i \tilde{J} = M_{\psi_i}$ for $\psi_i \in L^\infty$. Hence $D_i = M_{\psi_i} \tilde{J}$ for $\psi_i \in L^\infty$.

Note that $\mathbf{C}$ has to satisfy conditions (2.1)–(2.3). Condition (2.1) is satisfied automatically since $D_i = M_{\psi_i} \tilde{J}$ are antilinearly self-adjoint for $\psi_i \in L^\infty$. By checking (2.2) we get

$$(M_{\psi_1} \tilde{J})^2 + M_{\psi_2} \tilde{J} \tilde{J} M_{\bar{\psi}_2} = I \text{ and } (M_{\psi_4} \tilde{J})^2 + M_{\psi_2} \tilde{J} \tilde{J} M_{\bar{\psi}_2} = I,$$

which is equivalent to (2.9). Finally by (2.3) we get

$$\tilde{J} M_{\bar{\psi}_2} M_{\psi_1} \tilde{J} + M_{\psi_4} \tilde{J} \tilde{J} M_{\bar{\psi}_2} = 0$$

which is equivalent to (2.10). $\square$

Remark 2.6. Note that if (2.9)–(2.10) are satisfied and $\psi_2 = 0$, then $|\psi_1| = |\psi_4| = 1$. Hence D_1 and D_4 are M_z -conjugations in L^2 . On the other hand, if $\psi_1 = 0$ (which is equivalent to $\psi_4 = 0$), then $|\psi_2| = 1$, which implies that D_2 is an M_z -conjugation.

Example 2.7. The following conjugations satisfy Theorem 2.5:

$$\tilde{\mathbf{J}}_1 = \begin{bmatrix} \tilde{J} & 0 \\ 0 & \tilde{J} \end{bmatrix}, \quad \text{or} \quad \tilde{\mathbf{J}}_2 = \begin{bmatrix} 0 & \tilde{J} \\ \tilde{J} & 0 \end{bmatrix}, \quad \text{or} \quad \mathbf{C} = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{J} & \tilde{J} \\ \tilde{J} & -\tilde{J} \end{bmatrix}.$$

Example 2.8. Let $\psi_1(z) = -\psi_4(z) = \frac{1}{2i}(-\bar{z} + z)$ and $\psi_2(z) = \frac{1}{2}(\bar{z} + z)$. In other words, $\psi_1(e^{it}) = -\psi_4(e^{it}) = \sin t$ and $\psi_2(e^{it}) = \cos t$. Observe that such functions satisfy conditions (2.9)–(2.10), so $\mathbf{C} = \begin{bmatrix} M_{\psi_1} \tilde{J} & M_{\psi_2} \tilde{J} \\ M_{\psi_2} \tilde{J} & M_{\psi_4} \tilde{J} \end{bmatrix}$ is a conjugation satisfying Theorem 2.5.

3. Operator Valued Functions

Let $\mathcal{H}$ be a complex separable Hilbert space. Denote by $L^2(\mathcal{H})$ the space of all (classes of) functions $\mathbf{f}: \mathbb{T} \rightarrow \mathcal{H}$ which are measurable and satisfy

$$\int_{\mathbb{T}} \|\mathbf{f}(z)\|^2 dm(z) < \infty \quad (3.1)$$

(the norm $\|\cdot\|$ under the integral is the norm in $\mathcal{H}$). So $\mathbf{f} \in L^2(\mathcal{H})$ is understood as a class (represented by $\mathbf{f}$) of all measurable functions satisfying (3.1) and equal to $\mathbf{f}$ on $\mathbb{T}$ a.e. with respect to m . Recall that the measurability of $\mathbf{f}$ means that $z \mapsto \|\mathbf{f}(z)\|$ is a measurable function (or that $z \mapsto \langle \mathbf{f}(z), x \rangle$ is measurable for every $x \in \mathcal{H}$, which due to separability of $\mathcal{H}$ is an equivalent definition).

The space $L^2(\mathcal{H})$ is a Hilbert space with the inner product given by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{T}} \langle \mathbf{f}(z), \mathbf{g}(z) \rangle dm(z), \quad \mathbf{f}, \mathbf{g} \in L^2(\mathcal{H}),$$

where the inner product under the integral is the inner product in $\mathcal{H}$ (note that $L^2(\mathcal{H})$ is also separable).

Functions in $L^2(\mathcal{H})$ have an expansion analogous to the Fourier expansion in L^2 . Observe that if $\mathbf{f} \in L^2(\mathcal{H})$, then for each $n \in \mathbb{Z}$ the linear functional $y \mapsto \int_{\mathbb{T}} \langle \mathbf{f}(z), y \rangle \bar{z}^n dm(z)$ is bounded and so there exists $x_n \in \mathcal{H}$ such that

$$\langle x_n, y \rangle = \int_{\mathbb{T}} \langle \mathbf{f}(z), y \rangle \bar{z}^n dm(z) \quad \text{for all } y \in \mathcal{H}. \quad (3.2)$$

The element x_n is called the n -th *Fourier coefficient* of $\mathbf{f}$. It turns out that $\mathbf{f} \in L^2(\mathcal{H})$ can be expressed as $\mathbf{f} = \sum_{n=-\infty}^{\infty} x_n e_n$, where x_n is given by (3.2), $e_n(z) = z^n$ for $z \in \mathbb{T}$ and the series converges in the norm of $L^2(\mathcal{H})$. Moreover, for $\mathbf{f} = \sum_{n=-\infty}^{\infty} x_n e_n \in L^2(\mathcal{H})$ and $\mathbf{g} = \sum_{n=-\infty}^{\infty} y_n e_n \in L^2(\mathcal{H})$ we have

$$\|\mathbf{f}\|^2 = \sum_{n=-\infty}^{\infty} \|x_n\|^2$$

and

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{n=-\infty}^{\infty} \langle x_n, y_n \rangle,$$

and this correspondence between elements of $L^2(\mathcal{H})$ and sequences $\{x_n\}_{n=-\infty}^{\infty} \subset \mathcal{H}$ such that $\sum_{n=-\infty}^{\infty} \|x_n\|^2 < \infty$ is one-to-one (see, e.g., [13, pp. 46–48]).

Denote by $H^2(\mathcal{H})$ the subspace of $L^2(\mathcal{H})$ consisting of those functions from $L^2(\mathcal{H})$ whose Fourier coefficients with negative indices are 0, i.e.,

$$H^2(\mathcal{H}) = \left\{ \mathbf{f} \in L^2(\mathcal{H}) : \mathbf{f} = \sum_{n=0}^{\infty} x_n e_n \right\}.$$

Each $\mathbf{f} \in H^2(\mathcal{H})$ can be also identified with a function

$$\mathbf{f}(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n, \quad \lambda \in \mathbb{D},$$

which is analytic in the unit disk $\mathbb{D}$. Thus $H^2(\mathcal{H})$ can be seen as a subspace of $L^2(\mathcal{H})$ or as a space of functions analytic in $\mathbb{D}$ and with values in $\mathcal{H}$. The boundary values on $\mathbb{T}$ can be then obtained through radial limits (here the radial functions converge to the boundary function in the $L^2(\mathcal{H})$ norm). For more details see also [1, 12].

Now let us recall after [1] some notations. Let $H^\infty(L(\mathcal{H}))$ denote the set of bounded $L(\mathcal{H})$ -valued analytic functions, $\mathbf{F}: \mathbb{D} \rightarrow L(\mathcal{H})$ with the norm $\|\mathbf{F}\|_\infty = \sup\{\|\mathbf{F}(\lambda)\| : |\lambda| < 1\}$. On the other hand, we can consider functions on the unit circle. A function $\mathbf{F}: \mathbb{T} \rightarrow L(\mathcal{H})$ is said to be measurable, if

for every $x \in \mathcal{H}$ the function $z \mapsto \mathbf{F}(z)x$ is measurable. Let us denote by $L^\infty(L(\mathcal{H}))$ the space of (again, classes of) all such measurable functions $\mathbf{F}$ which are essentially bounded, i.e., $\|\mathbf{F}\|_\infty = \text{ess sup}_{z \in \mathbb{T}} \|\mathbf{F}(z)\| < \infty$. Every $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ admits a Fourier expansion given by the formal series

$$\mathbf{F} = \sum_{n=-\infty}^{\infty} F_n e_n, \quad (3.3)$$

with $F_n \in L(\mathcal{H})$ for $n \in \mathbb{Z}$. Each of the coefficients F_n is defined as an integral in the strong sense, i.e.,

$$F_n h = \int_{\mathbb{T}} z^{-n} \mathbf{F}(z) h \, dm(z) \quad \text{for } h \in \mathcal{H}. \quad (3.4)$$

The space $H^\infty(L(\mathcal{H}))$ can be identified with the subspace of $L^\infty(L(\mathcal{H}))$ consisting of those functions $\mathbf{F}$ whose Fourier coefficients F_n vanish if $n < 0$. Namely, every bounded analytic function from $H^\infty(L(\mathcal{H}))$ has boundary values a.e. on $\mathbb{T}$, the boundary function belongs to $L^\infty(L(\mathcal{H}))$ and its Fourier coefficients with negative indices vanish. Conversely, every $\mathbf{F} = \sum_{n=0}^{\infty} F_n e_n \in L^\infty(L(\mathcal{H}))$ can be then extended to a function analytic in $\mathbb{D}$ by the formula

$$\mathbf{F}(\lambda) = \sum_{n=0}^{\infty} F_n \lambda^n, \quad \lambda \in \mathbb{D},$$

and this extension is bounded. Moreover, both $\|\cdot\|_\infty$ coincide. Therefore a function from $H^\infty(L(\mathcal{H}))$ can be seen as an element of $L^\infty(L(\mathcal{H}))$ or as a bounded analytic operator valued function in $\mathbb{D}$ (see [1, p. 232]). Note that for $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ we have that $\mathbf{F} \in H^\infty(L(\mathcal{H}))$ if and only if $M_{\mathbf{F}}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. It is an easy consequence of (3.4).

For each $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ we define a bounded linear operator $M_{\mathbf{F}}$ on $L^2(\mathcal{H})$: for $\mathbf{f} \in L^2(\mathcal{H})$,

$$(M_{\mathbf{F}} \mathbf{f})(z) = \mathbf{F}(z) \mathbf{f}(z) \quad \text{a.e. on } \mathbb{T}.$$

In particular, for $\mathbf{f} \in L^2(\mathcal{H})$, we have, a.e. on $\mathbb{T}$,

$$(\mathbf{M}_z \mathbf{f})(z) = z \mathbf{f}(z) \quad \text{and} \quad (\mathbf{M}_{\bar{z}} \mathbf{f})(z) = \bar{z} \mathbf{f}(z),$$

that is, $\mathbf{M}_z = M_{z \mathbf{I}_{\mathcal{H}}}$ and $\mathbf{M}_{\bar{z}} = M_{\bar{z} \mathbf{I}_{\mathcal{H}}}$. For $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ the adjoint function $\mathbf{F}^*$ is naturally defined as $\mathbf{F}^*(z) = \mathbf{F}(z)^*$ (a.e. on $\mathbb{T}$) and we also define $\mathbf{F}^\#$ by $(\mathbf{F}^\#)(z) = \mathbf{F}(\bar{z})^*$ (a.e. on $\mathbb{T}$). Clearly, $M_{\mathbf{F}^*} = M_{\mathbf{F}}^*$. If $\mathbf{F}: \mathbb{T} \rightarrow L(\mathcal{H})$ is a constant function, $\mathbf{F}(z) = F$ a.e. on $\mathbb{T}$, we will denote by F its action on $L^2(\mathcal{H})$.

Similarly, we define a measurable function $\mathbf{F}: \mathbb{T} \rightarrow LA(\mathcal{H})$ and denote by $L^\infty(LA(\mathcal{H}))$ the space of all measurable essentially bounded functions valued in the space $LA(\mathcal{H})$. For each $\mathbf{C} \in L^\infty(LA(\mathcal{H}))$ we define a bounded antilinear operator $A_{\mathbf{C}}$ on $L^2(\mathcal{H})$: for $\mathbf{f} \in L^2(\mathcal{H})$,

$$(A_{\mathbf{C}} \mathbf{f})(z) = \mathbf{C}(z) \mathbf{f}(z) \quad \text{a.e. on } \mathbb{T}.$$

4. M_z -Commuting and M_z -Conjugations in $L^2(\mathcal{H})$

For an arbitrary conjugation J in $\mathcal{H}$ let us define two conjugations $\tilde{\mathbf{J}}$ and $\mathbf{J}^*$ on $L^2(\mathcal{H})$ given by

$$(\tilde{\mathbf{J}}\mathbf{f})(z) = J(\mathbf{f}(z)) \quad \text{a.e. on } \mathbb{T} \quad (4.1)$$

and

$$(\mathbf{J}^*\mathbf{f})(z) = J(\mathbf{f}(\bar{z})) \quad \text{a.e. on } \mathbb{T}. \quad (4.2)$$

Note that for $\mathbf{f} = \sum_{n=-\infty}^{\infty} x_n e_n \in L^2(\mathcal{H})$,

$$\tilde{\mathbf{J}}\mathbf{f} = \sum_{n=-\infty}^{\infty} J(x_{-n})e_n \quad \text{and} \quad \mathbf{J}^*\mathbf{f} = \sum_{n=-\infty}^{\infty} J(x_n)e_n. \quad (4.3)$$

Indeed, if $\tilde{\mathbf{J}}\mathbf{f} = \sum_{n=-\infty}^{\infty} y_n e_n$ is the Fourier expansion of $\tilde{\mathbf{J}}\mathbf{f}$, then by (3.2), for each $n \in \mathbb{Z}$ and $y \in \mathcal{H}$,

$$\begin{aligned} \langle y_n, y \rangle &= \int_{\mathbb{T}} \langle (\tilde{\mathbf{J}}\mathbf{f})(z), y \rangle \bar{z}^n dm(z) = \int_{\mathbb{T}} \langle J(\mathbf{f}(z)), y \rangle \bar{z}^n dm(z) \\ &= \overline{\int_{\mathbb{T}} \langle \mathbf{f}(z), Jy \rangle z^n dm(z)} = \overline{\langle x_{-n}, Jy \rangle} = \langle J(x_{-n}), y \rangle \end{aligned}$$

and so $y_n = J(x_{-n})$. The Fourier coefficients of $\mathbf{J}^*\mathbf{f}$ can be obtained similarly. Observe that by (4.3),

$$\mathbf{J}^*(H^2(\mathcal{H})) \subset H^2(\mathcal{H}) \quad \text{and} \quad \tilde{\mathbf{J}}(H^2(\mathcal{H})) \subset L^2(\mathcal{H}) \ominus zH^2(\mathcal{H}).$$

Moreover, conjugations $\tilde{\mathbf{J}}$ and $\mathbf{J}^*$ have the following properties:

Proposition 4.1. *For $\mathbf{f} \in L^2(\mathcal{H})$ we have*

1. $(\tilde{\mathbf{J}}\mathbf{J}^*\mathbf{f})(z) = (\mathbf{J}^*\tilde{\mathbf{J}}\mathbf{f})(z) = \mathbf{f}(\bar{z})$ for almost all $z \in \mathbb{T}$,
2. $\tilde{\mathbf{J}}M_z = M_{\bar{z}}\tilde{\mathbf{J}}$,
3. $\mathbf{J}^*M_z = M_z\mathbf{J}^*$.

Proposition 4.2. *Let J be a conjugation in $\mathcal{H}$, and let $\mathbf{F} \in L^\infty(L(\mathcal{H}))$. Then*

1. $M_{\mathbf{F}}$ is $\tilde{\mathbf{J}}$ -symmetric if and only if $\mathbf{F}(z)$ is J -symmetric for almost all $z \in \mathbb{T}$.
2. $M_{\mathbf{F}}$ is $\mathbf{J}^*$ -symmetric if and only if $J\mathbf{F}(z)J = \mathbf{F}^\#(z)$ for almost all $z \in \mathbb{T}$.
3. $\tilde{\mathbf{J}}M_{\mathbf{F}}\tilde{\mathbf{J}} = M_{\mathbf{F}^*}$ if and only if $\mathbf{J}^*M_{\mathbf{F}}\mathbf{J}^* = M_{\mathbf{F}^\#}$.
4. $\tilde{\mathbf{J}}M_{\mathbf{F}}\tilde{\mathbf{J}} = M_{\mathbf{F}^\#}$ if and only if $\mathbf{J}^*M_{\mathbf{F}}\mathbf{J}^* = M_{\mathbf{F}^*}$.
5. if $\mathbf{F}(\bar{z}) = \mathbf{F}(z)$ for almost all $z \in \mathbb{T}$, then $M_{\mathbf{F}}$ is $\tilde{\mathbf{J}}$ -symmetric if and only if it is $\mathbf{J}^*$ -symmetric.

Proof. Note that for $\mathbf{f} \in L^2(\mathcal{H})$ we have a.e. on $\mathbb{T}$

$$\begin{aligned} (\tilde{\mathbf{J}}M_{\mathbf{F}}\tilde{\mathbf{J}}\mathbf{f})(z) &= J((M_{\mathbf{F}}\tilde{\mathbf{J}}\mathbf{f})(z)) \\ &= J(\mathbf{F}(z)(\tilde{\mathbf{J}}\mathbf{f})(z)) = J\mathbf{F}(z)J(\mathbf{f}(z)). \end{aligned} \quad (4.4)$$

Hence (1) is proved. Similarly, (2) follows from the equality

$$\begin{aligned} (\mathbf{J}^* M_{\mathbf{F}} \mathbf{J}^* \mathbf{f})(z) &= J((M_{\mathbf{F}} \mathbf{J}^* \mathbf{f})(\bar{z})) \\ &= J(\mathbf{F}(\bar{z})(\mathbf{J}^* \mathbf{f})(\bar{z})) = J\mathbf{F}(\bar{z})J(\mathbf{f}(z)). \end{aligned} \quad (4.5)$$

Comparing (4.4) with (4.5) we get (3) and (4). Condition (5) follows from (3), since $\mathbf{F}(\bar{z}) = \mathbf{F}(z)$ (a.e. on $\mathbb{T}$) if and only if $\mathbf{F}(z)^* = \mathbf{F}^\#(z)$ (a.e. on $\mathbb{T}$). $\square$

The following theorem gives a characterization of all $\mathbf{M}_z$ -commuting conjugations in $L^2(\mathcal{H})$.

Theorem 4.3. *Let J be a conjugation in $\mathcal{H}$. Then the following conditions are equivalent:*

1. $\mathbf{C}$ is a conjugation in $L^2(\mathcal{H})$ such that $\mathbf{C}\mathbf{M}_z = \mathbf{M}_z\mathbf{C}$,
2. there is $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ such that $\mathbf{U}(z)$ is a unitary operator for almost all $z \in \mathbb{T}$, $M_{\mathbf{U}}$ is $\mathbf{J}^*$ -symmetric and $\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^* = \mathbf{J}^*M_{\mathbf{U}}^*$.

Proof. From the equality $\mathbf{C}\mathbf{M}_z = \mathbf{M}_z\mathbf{C}$ and Proposition 4.1 we have

$$\mathbf{C}\mathbf{J}^*\mathbf{M}_z = \mathbf{C}\mathbf{M}_z\mathbf{J}^* = \mathbf{M}_z\mathbf{C}\mathbf{J}^*. \quad (4.6)$$

Since the linear operator $\mathbf{C}\mathbf{J}^*$ is unitary and commutes with $\mathbf{M}_z$, then (see [13, Theorem 3.17, Corollary 3.19]) it is equal to $M_{\mathbf{U}}$, where $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ and $\mathbf{U}(z)$ is unitary for almost all $z \in \mathbb{T}$. Hence $\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^*$. Since $\mathbf{C}$ is a conjugation, for $\mathbf{f} \in L^2(\mathcal{H})$ a.e. on $\mathbb{T}$ we must have

$$\begin{aligned} \mathbf{f}(z) &= (\mathbf{C}^2\mathbf{f})(z) = (M_{\mathbf{U}}\mathbf{J}^*M_{\mathbf{U}}\mathbf{J}^*\mathbf{f})(z) \\ &= \mathbf{U}(z)J((M_{\mathbf{U}}\mathbf{J}^*\mathbf{f})(\bar{z})) = \mathbf{U}(z)J\mathbf{U}(\bar{z})J\mathbf{f}(z), \end{aligned}$$

which is equivalent to $J\mathbf{U}(\bar{z})J = \mathbf{U}(z)^*$ a.e. on $\mathbb{T}$. Therefore, condition (2) follows from Proposition 4.2 (2).

The implication (2) $\Rightarrow$ (1) is easy. $\square$

Remark 4.4. Let us note that condition (1) in Theorem 4.3 does not depend on the conjugation J in $\mathcal{H}$. Hence if $\mathbf{C}$ satisfies (1), then condition (2) is satisfied for any conjugation J in $\mathcal{H}$. Therefore, for two different conjugations J and J' we obtain two different unitary operator valued functions $\mathbf{U}$ and $\mathbf{U}'$ such that

$$\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^* = M_{\mathbf{U}'}\mathbf{J}'^*.$$

Thus $\mathbf{U}'(z) = \mathbf{U}(z)V_0$ for almost all $z \in \mathbb{T}$, where V_0 is a unitary operator given by $V_0 = JJ'$. It follows that if we have, for a given conjugation $\mathbf{C}$, the operator valued function $\mathbf{U}$ determined by some conjugation J in $\mathcal{H}$, then we can easily obtain the function $\mathbf{U}'$ corresponding to any other conjugation J' in $\mathcal{H}$.

Recall (see [2, Theorem 2.4]) that in the scalar case C commutes with M_z if and only if $CM_\varphi = M_{\varphi^\#}C$ for all $\varphi \in L^\infty$. This is not necessarily true in the general case.

Remark 4.5. Let $\mathbf{C}$ be a conjugation in $L^2(\mathcal{H})$, $\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^*$ for some $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ such that $\mathbf{U}(z)$ is a unitary operator for almost all $z \in \mathbb{T}$ and $M_{\mathbf{U}}$ is $\mathbf{J}^*$ -symmetric. Assume that $\Phi \in L^\infty(L(\mathcal{H}))$. Then

$$\mathbf{C}M_\Phi = M_{\Phi^\#}\mathbf{C}$$

if and only if

$$\mathbf{U}(z)J\Phi(\bar{z}) = \Phi^\#(z)\mathbf{U}(z)J \quad (4.7)$$

for almost all z in $\mathbb{T}$. Indeed, this follows from the fact that a.e. on $\mathbb{T}$,

$$\begin{aligned} (\mathbf{C}M_\Phi\mathbf{f})(z) &= (M_{\mathbf{U}}\mathbf{J}^*M_\Phi\mathbf{f})(z) = \mathbf{U}(z)(\mathbf{J}^*M_\Phi\mathbf{f})(z) \\ &= \mathbf{U}(z)J((M_\Phi\mathbf{f})(\bar{z})) = \mathbf{U}(z)J(\Phi(\bar{z})\mathbf{f}(\bar{z})) \end{aligned}$$

and

$$(M_{\Phi^\#}\mathbf{C}\mathbf{f})(z) = (M_{\Phi^\#}M_{\mathbf{U}}\mathbf{J}^*\mathbf{f})(z) = \Phi^\#(z)\mathbf{U}(z)J(\mathbf{f}(\bar{z})).$$

If $\Phi(z)$ is J -symmetric for almost all $z \in \mathbb{T}$, then $J\Phi(\bar{z}) = \Phi^\#(z)J$ a.e. on $\mathbb{T}$ and condition (4.7) holds provided that for almost all $z \in \mathbb{T}$, $\Phi^\#(z)$ commutes with $\mathbf{U}(z)$.

Example 4.6. If $\mathcal{H} = \mathbb{C}$ and $J(w) = \bar{w}$, $w \in \mathbb{C}$, then $U \in L^\infty$ and, by Proposition 4.2 (2), $\mathbf{J}^*$ -symmetry of U means that $U(z) = U(\bar{z})$ a.e. on $\mathbb{T}$ and we obtain [2, Theorem 2.4].

Example 4.7. Consider $\mathcal{H} = \mathbb{C}^2$ and the conjugation on $\mathbb{C}^2$ defined by $J_1(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Then, by Theorem 2.1, any $\mathbf{M}_z$ -commuting conjugation $\mathbf{C}$ on $L^2(\mathbb{C}^2)$ has a form

$$\mathbf{C} = \begin{bmatrix} M_{\psi_1}\mathbf{J}^* & M_{\psi_2}\mathbf{J}^* \\ M_{\bar{\psi}_2^\#}\mathbf{J}^* & M_{\psi_4}\mathbf{J}^* \end{bmatrix} = \begin{bmatrix} M_{\psi_1} & M_{\psi_2} \\ M_{\bar{\psi}_2^\#} & M_{\psi_4} \end{bmatrix} \begin{bmatrix} \mathbf{J}^* & 0 \\ 0 & \mathbf{J}^* \end{bmatrix} = M_{\mathbf{U}}\mathbf{J}_1^* \quad (4.8)$$

and the functions $\psi_1, \psi_2, \psi_4 \in L^\infty$ satisfy conditions (2.4)–(2.6). Hence in view of Theorem 4.3 the unitary operator valued function is $\mathbf{U} = \begin{bmatrix} \psi_1 & \psi_2 \\ \bar{\psi}_2^\# & \psi_4 \end{bmatrix}$.

By conditions (2.4)–(2.6) we have that $M_{\mathbf{U}}$ is $\mathbf{J}_1^*$ -symmetric and the operator $\mathbf{U}(z)$ is unitary for almost all $z \in \mathbb{T}$.

On the other hand, if the conjugation on $\mathbb{C}^2$ is defined by $J_2(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$, then $\mathbf{J}_2^* = \begin{bmatrix} 0 & \mathbf{J}^* \\ \mathbf{J}^* & 0 \end{bmatrix}$. The conjugation $\mathbf{C}$ on $L^2(\mathbb{C}^2)$ has the form

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} M_{\psi_1} & M_{\psi_2} \\ M_{\bar{\psi}_2^\#} & M_{\psi_4} \end{bmatrix} \begin{bmatrix} \mathbf{J}^* & 0 \\ 0 & \mathbf{J}^* \end{bmatrix} = \begin{bmatrix} M_{\psi_1} & M_{\psi_2} \\ M_{\bar{\psi}_2^\#} & M_{\psi_4} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{J}^* \\ \mathbf{J}^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} M_{\psi_2} & M_{\psi_1} \\ M_{\psi_4} & M_{\bar{\psi}_2^\#} \end{bmatrix} \mathbf{J}_2^* = M_{\mathbf{U}_2}\mathbf{J}_2^*. \end{aligned}$$

By straightforward calculations one can check that $M_{\mathbf{U}_2}$ is $\mathbf{J}_2^*$ -symmetric and $\mathbf{U}_2(z)$ is unitary for almost all $z \in \mathbb{T}$.

The following theorem gives a characterization of all $\mathbf{M}_z$ -conjugations in $L^2(\mathcal{H})$.

Theorem 4.8. *Let $\mathbf{C}$ be an antilinear operator in $L^2(\mathcal{H})$. Then the following are equivalent*

1. $\mathbf{C}$ is a conjugation on $L^2(\mathcal{H})$ such that $\mathbf{M}_z \mathbf{C} = \mathbf{C} \mathbf{M}_{\bar{z}}$,
2. there is $\mathbf{C}_0 \in L^\infty(LA(\mathcal{H}))$ such that $\mathbf{C} = A_{\mathbf{C}_0}$ and $\mathbf{C}_0(z)$ is a conjugation for almost all $z \in \mathbb{T}$,
3. for any conjugation J in $\mathcal{H}$ there is $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ such that $\mathbf{U}(z)$ is a J -symmetric unitary operator for almost all $z \in \mathbb{T}$ and $\mathbf{C} = M_{\mathbf{U}} \tilde{\mathbf{J}} = \tilde{\mathbf{J}} M_{\mathbf{U}^*}$.

Proof. First we will prove that (1) $\Rightarrow$ (3). By Proposition 4.1 (2) we have

$$\mathbf{M}_z \mathbf{C} \tilde{\mathbf{J}} = \mathbf{C} \mathbf{M}_{\bar{z}} \tilde{\mathbf{J}} = \mathbf{C} \tilde{\mathbf{J}} \mathbf{M}_z, \quad (4.9)$$

which implies that the unitary operator $\mathbf{C} \tilde{\mathbf{J}}$ commutes with $\mathbf{M}_z$. Hence, as in the proof of Theorem 4.3, $\mathbf{C} \tilde{\mathbf{J}} = M_{\mathbf{U}}$, for $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ such that $\mathbf{U}(z)$ is unitary for almost all $z \in \mathbb{T}$. Then $\mathbf{C} = M_{\mathbf{U}} \tilde{\mathbf{J}}$. Since $\mathbf{C}$ is a conjugation, for almost all $z \in \mathbb{T}$ we must have

$$\begin{aligned} \mathbf{f}(z) &= (\mathbf{C}^2 \mathbf{f})(z) = (M_{\mathbf{U}} \tilde{\mathbf{J}} M_{\mathbf{U}} \tilde{\mathbf{J}} \mathbf{f})(z) \\ &= \mathbf{U}(z) J((M_{\mathbf{U}} \tilde{\mathbf{J}} \mathbf{f})(z)) = \mathbf{U}(z) J \mathbf{U}(z) J \mathbf{f}(z), \end{aligned}$$

which is equivalent to $J \mathbf{U}(z) J = \mathbf{U}(z)^*$ a.e. on $\mathbb{T}$.

To see that (3) $\Rightarrow$ (2) define $\mathbf{C}_0 \in L^\infty(LA(\mathcal{H}))$ as $\mathbf{C}_0(z) = \mathbf{U}(z) J$ a.e. on $\mathbb{T}$. An easy calculation shows that (2) $\Rightarrow$ (1). $\square$

Remark 4.9. As in Theorem 4.3, condition (1) in Theorem 4.8 does not depend on J . Hence (1) implies that for any conjugations J and J' in $\mathcal{H}$ there exist unitary operator valued functions $\mathbf{U}$ and $\mathbf{U}'$ such that

$$\mathbf{C} = M_{\mathbf{U}} \tilde{\mathbf{J}} = M_{\mathbf{U}'} \tilde{\mathbf{J}}'.$$

Moreover, $\mathbf{U}'(z) = \mathbf{U}(z) J J'$ a.e. on $\mathbb{T}$.

Remark 4.10. Let $\mathbf{C} = A_{\mathbf{C}_0}$ be an $\mathbf{M}_z$ -conjugation in $L^2(\mathcal{H})$. Suppose that $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ is an operator valued function. Then $M_{\mathbf{F}}$ is $\mathbf{C}$ -symmetric if and only if for almost all $z \in \mathbb{T}$, $\mathbf{F}(z)$ is $\mathbf{C}_0(z)$ -symmetric.

Example 4.11. Consider $\mathcal{H} = \mathbb{C}^2$ and the conjugation J_1 in $\mathbb{C}^2$ defined by $J_1(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Then by Theorem 2.5 any $\mathbf{M}_z$ -conjugation $\mathbf{C}_1$ on $L^2(\mathbb{C}^2)$ has a form

$$\mathbf{C}_1 = \begin{bmatrix} M_{\psi_1} \tilde{\mathbf{J}} & M_{\psi_2} \tilde{\mathbf{J}} \\ M_{\psi_2} \tilde{\mathbf{J}} & M_{\psi_4} \tilde{\mathbf{J}} \end{bmatrix} = \begin{bmatrix} M_{\psi_1} & M_{\psi_2} \\ M_{\psi_2} & M_{\psi_4} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{J}} & 0 \\ 0 & \tilde{\mathbf{J}} \end{bmatrix} = M_{\mathbf{U}} \tilde{\mathbf{J}}_1 \quad (4.10)$$

and the functions $\psi_1, \psi_2, \psi_4 \in L^\infty$ satisfy conditions (2.9)–(2.10). Hence in view of Theorem 4.8, the unitary operator valued function is

$$\mathbf{U} = \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_4 \end{bmatrix}.$$

If we consider another conjugation in $\mathbb{C}^2$ given by $J_2(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$, then the conjugation $\mathbf{C}_2$ in $L^2(\mathbb{C}^2)$ has a form

$$\mathbf{C}_2 = \begin{bmatrix} M_{\psi_2} & M_{\psi_1} \\ M_{\psi_4} & M_{\psi_2} \end{bmatrix} \begin{bmatrix} 0 & \tilde{\mathbf{J}} \\ \tilde{\mathbf{J}} & 0 \end{bmatrix} = M_{\mathbf{U}_2} \tilde{\mathbf{J}}_2.$$

5. Conjugations Preserving $H^2(\mathcal{H})$

Let J be any conjugation in $\mathcal{H}$. Then, as noted before, by (4.3) we have $\mathbf{J}^*(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. Now our aim is to characterize all $\mathbf{M}_z$ -commuting conjugations with this property. Observe here that if $\mathbf{C}$ is a conjugation in $L^2(\mathcal{H})$ such that $\mathbf{C}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$, then actually we must have $\mathbf{C}(H^2(\mathcal{H})) = H^2(\mathcal{H})$.

Theorem 5.1. *Let J be a conjugation in $\mathcal{H}$ and let $\mathbf{C}$ be a conjugation in $L^2(\mathcal{H})$ such that $\mathbf{C}\mathbf{M}_z = \mathbf{M}_z\mathbf{C}$. If $\mathbf{C}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$, then there is a unitary J -symmetric operator $U_0 \in L(\mathcal{H})$ such that $\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^* = \mathbf{J}^*M_{\mathbf{U}^*}$, where $\mathbf{U}$ is a constant operator valued function $\mathbf{U}(z) = U_0$ for almost all $z \in \mathbb{T}$.*

Proof. By Theorem 4.3, there exists a unitary valued $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ such that $\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^*$ and $M_{\mathbf{U}}\mathbf{M}_z = \mathbf{M}_z M_{\mathbf{U}}$. Since, by assumption, $M_{\mathbf{U}}\mathbf{J}^*(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$, we have $M_{\mathbf{U}}H^2(\mathcal{H}) \subset H^2(\mathcal{H})$. By the commutativity relation

$$M_{\mathbf{U}}\mathbf{M}_{z^k} = \mathbf{M}_{z^k}M_{\mathbf{U}}$$

we get that

$$M_{\mathbf{U}}(z^k H^2(\mathcal{H})) \subset z^k H^2(\mathcal{H})$$

for every nonnegative integer k . Since $\mathbf{C} = \mathbf{C}^\sharp = \mathbf{J}^*M_{\mathbf{U}^*}$, similarly,

$$M_{\mathbf{U}^*}(z^k H^2(\mathcal{H})) \subset z^k H^2(\mathcal{H}),$$

which implies that all subspaces $z^k H^2(\mathcal{H})$, $k = 0, 1, \dots$, are reducing for $M_{\mathbf{U}}$. Hence also $H^2(\mathcal{H}) \ominus zH^2(\mathcal{H})$ is reducing for $M_{\mathbf{U}}$. Therefore

$$M_{\mathbf{U}}\mathcal{H} \subset \mathcal{H},$$

which means that $\mathbf{U}$ is a constant operator valued function, $\mathbf{U}(z) = U_0$ for almost all $z \in \mathbb{T}$, where $U_0 \in L(\mathcal{H})$ is a unitary operator. Since $M_{\mathbf{U}}$ is $\mathbf{J}^*$ -symmetric, we get that U_0 is J -symmetric. $\square$

Remark 5.2. If $\mathcal{H} = \mathbb{C}$ and $J(w) = \bar{w}$, $w \in \mathbb{C}$, then U_0 is a constant of modulus 1 (see [2, Corollary 3.1]).

Example 5.3. As in Example 4.7 let $\mathcal{H} = \mathbb{C}^2$ and $J_1(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Then, by Theorem 5.1, Example 4.7 and conditions (2.4)–(2.6), an antilinear operator $\mathbf{C} = \begin{bmatrix} D_1 & D_2 \\ D_2^\sharp & D_4 \end{bmatrix}$, $i = 1, 2, 4$, is a conjugation commuting with $\mathbf{M}_z$ and $\mathbf{C}(H^2(\mathbb{C}^2)) \subset H^2(\mathbb{C}^2)$ if and only if $D_i = \lambda_i J^*$ with $\lambda_i \in \mathbb{D}$ such that

$$|\lambda_1|^2 + |\lambda_2|^2 = |\lambda_2|^2 + |\lambda_4|^2 = 1 \quad (5.1)$$

$$\bar{\lambda}_1 \lambda_2 + \bar{\lambda}_2 \lambda_4 = 0. \quad (5.2)$$

Remark 5.4. Note that if $\mathbf{C}$ is an $\mathbf{M}_z$ -commuting conjugation such that $\mathbf{C}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$, then for every conjugation J in $\mathcal{H}$ there is a unitary J -symmetric operator $U_0 \in L(\mathcal{H})$ such that $\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^*$, where $\mathbf{U}$ is a constant operator valued function $\mathbf{U}(z) = U_0$ for almost all $z \in \mathbb{T}$. In view of Remark 4.4 the relation between the values of constant operator valued functions corresponding to conjugations J_1 and J_2 is given by $U_2 = U_1 J_1 J_2$.

Considering $\mathbf{M}_z$ -conjugations preserving $H^2(\mathcal{H})$ note that $\tilde{\mathbf{J}}(H^2(\mathcal{H})) \not\subset H^2(\mathcal{H})$. In fact, $\tilde{\mathbf{J}}(H^2(\mathcal{H})) = L^2(\mathcal{H}) \ominus zH^2(\mathcal{H})$. More generally, we have:

Proposition 5.5. *There are no $\mathbf{M}_z$ -conjugations on $L^2(\mathcal{H})$ for which $H^2(\mathcal{H})$ is invariant.*

Proof. Assume that $\mathbf{C}$ is an $\mathbf{M}_z$ -conjugation and $\mathbf{C}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. Then $\mathbf{C}(H^2(\mathcal{H})) = H^2(\mathcal{H})$. By Theorem 4.8 (3), $\mathbf{C} = M_{\mathbf{U}}\tilde{\mathbf{J}}$, hence

$$M_{\mathbf{U}}(H^2(\mathcal{H})) = \mathbf{C}\tilde{\mathbf{J}}(H^2(\mathcal{H})) = L^2(\mathcal{H}) \ominus zH^2(\mathcal{H}).$$

Let $\mathbf{f} \in H^2(\mathcal{H})$. Then also $z^k \mathbf{f} \in H^2(\mathcal{H})$ for any $k = 1, 2, \dots$. Since $\mathbf{U}$ is an operator valued function, $M_{\mathbf{U}}$ commutes with $\mathbf{M}_z$ and $\mathbf{M}_{\bar{z}}$. Hence $M_{\mathbf{U}}(z^k \mathbf{f}) = z^k M_{\mathbf{U}} \mathbf{f} \perp zH^2(\mathcal{H})$ and $M_{\mathbf{U}} \mathbf{f} \perp \bar{z}^{k-1} H^2(\mathcal{H})$. Since k is arbitrary, then $M_{\mathbf{U}} \mathbf{f} = 0$, so we also have $M_{\mathbf{U}} = 0$, which is a contradiction. $\square$

6. Conjugations and Model Spaces

Assume now that $\dim \mathcal{H} = d < \infty$. A function $\Theta \in H^\infty(L(\mathcal{H}))$ is called *inner*, if its boundary values are unitary operators in $L(\mathcal{H})$ almost everywhere on $\mathbb{T}$ (since $\dim \mathcal{H} < \infty$). Suppose that Θ is a *pure* inner function, i.e., $\|\Theta(0)\| < 1$. Define the corresponding *model space* $K_\Theta = H^2(\mathcal{H}) \ominus \Theta H^2(\mathcal{H})$ and let P_Θ be the orthogonal projection on K_Θ . Note that the subspace K_Θ^∞ of all bounded functions in K_Θ ($K_\Theta^\infty = K_\Theta \cap L^\infty(\mathcal{H})$) is dense in K_Θ .

Lemma 6.1. *Let $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ and let J_1, J_2 be any two conjugations in $\mathcal{H}$. Then $\tilde{\mathbf{F}}$ defined a.e. on $\mathbb{T}$ by $\tilde{\mathbf{F}}(z) = J_1 \mathbf{F}(z) J_2$ belongs to $H^\infty(L(\mathcal{H}))$ if and only if $\mathbf{F}^*$ belongs to $H^\infty(L(\mathcal{H}))$.*

Proof. It is clear that $\tilde{\mathbf{F}} \in L^\infty(L(\mathcal{H}))$ whenever $\mathbf{F} \in L^\infty(L(\mathcal{H}))$. Let $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ be given by expansion (3.3). Then, by (3.4), the n -th Fourier coefficient of $\tilde{\mathbf{F}}$ equals

$$(\tilde{\mathbf{F}})_n h = \int_{\mathbb{T}} z^{-n} J_1 \mathbf{F}(z) J_2 h \, dm(z) = J_1 \int_{\mathbb{T}} z^n \mathbf{F}(z) J_2 h \, dm(z) = J_1 F_{-n} J_2 h$$

for $h \in \mathcal{H}$. On the other hand, also by (3.4), n -th Fourier coefficient of $\mathbf{F}^*$ equals $(\mathbf{F}^*)_n = F_{-n}^*$. Now the rest of the proof is clear. $\square$

Lemma 6.2. (see [1], pp. 118-119) *Let $\Theta \in H^\infty(L(\mathcal{H}))$ be an operator valued inner function and let $\mathbf{F} \in L^\infty(L(\mathcal{H}))$. If $M_{\mathbf{F}}(H^2(\mathcal{H})) \subset \Theta H^2(\mathcal{H})$, then $\mathbf{F} \in H^\infty(L(\mathcal{H}))$ and there exists $\Psi \in H^\infty(L(\mathcal{H}))$ such that $\mathbf{F} = \Theta \Psi$.*

Proof. The proof follows the reasoning presented in [1, pp. 118-119]. The inclusion $M_{\mathbf{F}}(H^2(\mathcal{H})) \subset \Theta H^2(\mathcal{H})$ means that for each $\mathbf{f} \in H^2(\mathcal{H})$ there exists $\mathbf{g} \in H^2(\mathcal{H})$ such that

$$M_{\mathbf{F}} \mathbf{f} = M_{\Theta} \mathbf{g}.$$

Moreover, since M_{Θ} is an isometry, there is only one such $\mathbf{g}$ and

$$\|\mathbf{g}\| = \|M_{\Theta} \mathbf{g}\| = \|M_{\mathbf{F}} \mathbf{f}\| \leq \|\mathbf{F}\|_\infty \|\mathbf{f}\|.$$

We can thus define a bounded linear operator $T: H^2(\mathcal{H}) \rightarrow H^2(\mathcal{H})$ by $\mathbf{f} \mapsto T\mathbf{f} = \mathbf{g}$. Hence, for $\mathbf{f} \in H^2(\mathcal{H})$ we have $M_{\mathbf{F}}\mathbf{f} = M_{\Theta}T\mathbf{f}$ and

$$M_{\Theta}TM_z\mathbf{f} = M_{\mathbf{F}}M_z\mathbf{f} = M_zM_{\mathbf{F}}\mathbf{f} = M_zM_{\Theta}T\mathbf{f} = M_{\Theta}M_zT\mathbf{f},$$

which means that $TM_z\mathbf{f} = M_zT\mathbf{f}$. It follows that $T = M_{\Psi}$ for some $\Psi \in H^{\infty}(L(\mathcal{H}))$ ([1, Chap. 5, Theorem 1.7]) and $M_{\mathbf{F}}\mathbf{f} = M_{\Theta\Psi}\mathbf{f}$. In particular, for every $x \in \mathcal{H}$,

$$\mathbf{F}(z)x = \Theta(z)\Psi(z)x \quad \text{a.e. on } \mathbb{T}$$

and so $\mathbf{F} \in H^{\infty}(L(\mathcal{H}))$. $\square$

We can now give another proof of Lemma 6.1. Namely, if $\tilde{\mathbf{F}} \in H^{\infty}(L(\mathcal{H}))$, then $M_{\tilde{\mathbf{F}}}(zH^2(\mathcal{H})) \subset zH^2(\mathcal{H})$. Since $\mathbf{F}(z) = J_1\tilde{\mathbf{F}}(z)J_2$ a.e. on $\mathbb{T}$, we have that $M_{\mathbf{F}} = \tilde{\mathbf{J}}_1M_{\tilde{\mathbf{F}}}\tilde{\mathbf{J}}_2$ and

$$\begin{aligned} M_{\mathbf{F}}(H^2(\mathcal{H})^{\perp}) &= \tilde{\mathbf{J}}_1M_{\tilde{\mathbf{F}}}\tilde{\mathbf{J}}_2(H^2(\mathcal{H})^{\perp}) = \tilde{\mathbf{J}}_1M_{\tilde{\mathbf{F}}}(zH^2(\mathcal{H})) \\ &\subset \tilde{\mathbf{J}}_1(zH^2(\mathcal{H})) = H^2(\mathcal{H})^{\perp}. \end{aligned}$$

It follows that $M_{\mathbf{F}^*} = M_{\mathbf{F}}^*$ preserves $H^2(\mathcal{H})$ and $\mathbf{F}^* \in H^{\infty}(L(\mathcal{H}))$ by Lemma 6.2. The proof of the other implication is analogous.

Lemma 6.3. *Let $\mathbf{F} \in H^{\infty}(L(\mathcal{H}))$ and let J be a conjugation in $\mathcal{H}$. Then the following are equivalent:*

1. $\mathbf{F}(z)$ is J -symmetric a.e. on $\mathbb{T}$;
2. $\mathbf{F}(\lambda)$ is J -symmetric for all $\lambda \in \mathbb{D}$.

Proof. For the proof it is enough to note that each of the conditions (1) and (2) is equivalent to J -symmetry of all of the coefficients F_n of the Fourier/Taylor expansion of $\mathbf{F}$. $\square$

In what follows, if the operator valued inner function Θ and the conjugation J satisfy one of the conditions (1) or (2) from Lemma 6.3 we will simply say that Θ is J -symmetric.

The model operator $S_{\Theta} \in L(K_{\Theta})$ is given by

$$(S_{\Theta}\mathbf{f}) = P_{\Theta}(M_z\mathbf{f}) \quad \text{for } \mathbf{f} \in K_{\Theta}. \quad (6.1)$$

From [4, Theorem 3.1] it follows that the model operator S_{Θ} is complex symmetric if and only if there is a conjugation J in $\mathcal{H}$ such that Θ is J -symmetric (which implies that $\Theta(z)$ is J -symmetric a.e. on $\mathbb{T}$). In that case $\Theta(z)J$ is a conjugation in $\mathcal{H}$ for almost all $z \in \mathbb{T}$ and as a consequence $M_{\Theta}\tilde{\mathbf{J}}$ is a conjugation in $L^2(\mathcal{H})$ by Theorem 4.8. From now on let us assume that Θ is J -symmetric. Then $\mathbf{C}_{\Theta,J}$ defined by

$$\mathbf{C}_{\Theta,J}\mathbf{f}(z) = \Theta(z)\bar{z}(\tilde{\mathbf{J}}\mathbf{f})(z) = \Theta(z)\bar{z}J(\mathbf{f}(z)) \quad \text{a.e. on } \mathbb{T}$$

(that is, $\mathbf{C}_{\Theta,J} = M_{\Theta}M_{\bar{z}}\tilde{\mathbf{J}}$) is a conjugation on $L^2(\mathcal{H})$ that leaves K_{Θ} invariant. Moreover, the following is true:

Proposition 6.4. *Let $\mathbf{V} \in L^\infty(L(\mathcal{H}))$ be a unitary operator valued function. Then $\mathbf{C} = M_{\mathbf{V}}\mathbf{C}_{\Theta,J}$ is a conjugation in $L^2(\mathcal{H})$ if and only if $M_{\mathbf{V}}$ is $\mathbf{C}_{\Theta,J}$ -symmetric. In other words,*

$$\Theta(z)J\mathbf{V}(z)\Theta(z)J = \mathbf{V}(z)^* \quad (6.2)$$

almost everywhere on $\mathbb{T}$. Moreover, if for almost all $z \in \mathbb{T}$, $\mathbf{V}(z) = V_0$, where V_0 is a unitary operator in $L(\mathcal{H})$, then K_Θ is invariant for $\mathbf{C}$.

Proof. Note that (6.2) is a consequence of $\mathbf{C}$ being an involution. For the proof of the second statement note that by (6.2) and J -symmetry of Θ ,

$$V_0\Theta(z) = \Theta(z)JV_0^*J \quad \text{a.e. on } \mathbb{T}.$$

So for $n \geq 1$ and $x \in \mathcal{H}$,

$$\begin{aligned} M_{\mathbf{V}}\mathbf{C}_{\Theta,J}(xe_{-n}) &= \mathbf{C}_{\Theta,J}M_{\mathbf{V}}^*(xe_{-n}) = \mathbf{C}_{\Theta,J}(V_0^*xe_{-n}) \\ &= M_\Theta \mathbf{M}_{\bar{z}} \tilde{\mathbf{J}}(V_0^*xe_{-n}) = M_\Theta(JV_0^*xe_{n-1}). \end{aligned}$$

Hence $\mathbf{C}(L^2(\mathcal{H}) \ominus H^2(\mathcal{H})) \subset \Theta H^2(\mathcal{H})$. On the other hand, for $n \geq 0$ and $x \in \mathcal{H}$,

$$\begin{aligned} M_{\mathbf{V}}\mathbf{C}_{\Theta,J}M_\Theta(xe_n) &= M_{\mathbf{V}}M_{\Theta^*}\mathbf{C}_{\Theta,J}(xe_n) = M_{\mathbf{V}}M_{\Theta^*}M_\Theta \mathbf{M}_{\bar{z}} \tilde{\mathbf{J}}(xe_n) \\ &= M_{\mathbf{V}}(Jxe_{-n-1}) = V_0Jxe_{-n-1}. \end{aligned}$$

Hence $\mathbf{C}(\Theta H^2(\mathcal{H})) \subset L^2(\mathcal{H}) \ominus H^2(\mathcal{H})$. □

For $\lambda \in \mathbb{D}$ define an operator valued function k_λ^Θ by

$$k_\lambda^\Theta(z) = \frac{1}{1-\bar{\lambda}z}(1-\Theta(z)\Theta(\lambda)^*) \quad \text{a.e. on } \mathbb{T}.$$

For each $x \in \mathcal{H}$ denote the function $z \mapsto k_\lambda^\Theta(z)x$ by $k_\lambda^\Theta x$. Recall from [10] that $k_\lambda^\Theta x \in K_\Theta$ and for each $f \in K_\Theta$,

$$\langle f, k_\lambda^\Theta x \rangle = \langle f(\lambda), x \rangle$$

(the inner product on the left hand side is the $L^2(\mathcal{H})$ inner product while the inner product on the right hand side is the inner product from $\mathcal{H}$). Similarly, denote by $\widetilde{k}_\lambda^\Theta x$ the function $z \mapsto \widetilde{k}_\lambda^\Theta(z)x$, where

$$\widetilde{k}_\lambda^\Theta(z) = \frac{1}{z-\lambda}(\Theta(z) - \Theta(\lambda)) \quad \text{a.e. on } \mathbb{T}.$$

For $\Theta \in H^\infty(L(\mathcal{H}))$ recall that $\Theta^\#(z) = \Theta(\bar{z})^*$, and Θ is inner if and only if $\Theta^\#$ is inner.

Lemma 6.5. *Let J be a conjugation on $\mathcal{H}$ such that Θ is J -symmetric. Then $\mathbf{C}_{\Theta,J}(k_\lambda^\Theta x) = k_\lambda^\Theta Jx$ for $x \in \mathcal{H}$.*

Proof. Note that a.e. on $\mathbb{T}$,

$$\begin{aligned} (\mathbf{C}_{\Theta,J}k_\lambda^\Theta x)(z) &= \Theta(z)\bar{z}J(k_\lambda^\Theta(z)x) = \Theta(z)\bar{z}J\left(\frac{1}{1-\bar{\lambda}z}(1-\Theta(z)\Theta(\lambda)^*)x\right) \\ &= \frac{\bar{z}}{1-\bar{\lambda}\bar{z}}\Theta(z)(1-\Theta(z)^*\Theta(\lambda))Jx = \frac{1}{z-\lambda}(\Theta(z) - \Theta(\lambda))Jx. \quad \square \end{aligned}$$

Proposition 6.4 describes a class of conjugations in $L^2(\mathcal{H})$ which leave model spaces invariant. The following result says that amongst all $\mathbf{M}_z$ -conjugations only conjugations in that class have this property.

Theorem 6.6. *Let $\mathbf{C}$ be an $\mathbf{M}_z$ -conjugation on $L^2(\mathcal{H})$ and let $\Theta \in L^\infty(L(\mathcal{H}))$ be a J -symmetric pure inner function with a conjugation J on $\mathcal{H}$. Suppose that K_Θ is invariant for $\mathbf{C}$. Then $\mathbf{C} = M_{\mathbf{V}}\mathbf{C}_{\Theta,J}$ with $\mathbf{V}$ a unitary valued constant function, $\mathbf{V}(z) = V_0$ a.e. on $\mathbb{T}$, such that $M_{\mathbf{V}}$ is $\mathbf{C}_{\Theta,J}$ -symmetric.*

Proof. By Theorem 4.8 we know that $\mathbf{C} = M_{\mathbf{U}}\tilde{\mathbf{J}} = \tilde{\mathbf{J}}M_{\mathbf{U}^*}$, where $\mathbf{U} \in L^\infty(L(\mathcal{H}))$, and for almost all $z \in \mathbb{T}$, $\mathbf{U}(z)$ is unitary and J -symmetric. Fix $x \in \mathcal{H}$. Note that a.e. on $\mathbb{T}$,

$$\begin{aligned} (\mathbf{C}\mathbf{C}_{\Theta,J}k_\lambda^\Theta x)(z) &= \mathbf{U}(z)J\left(\frac{1}{z-\lambda}(\Theta(z) - \Theta(\lambda))\right)Jx \\ &= \mathbf{U}(z)\frac{1}{\bar{z}-\lambda}(\Theta(z)^* - \Theta(\lambda)^*)x \\ &= \frac{z}{1-\lambda z}\mathbf{U}(z)\Theta(z)^*(1 - \Theta(z)\Theta(\lambda)^*)x \\ &= z\mathbf{U}(z)\Theta(z)^*k_\lambda^\Theta(z)x. \end{aligned}$$

Denote by $\mathbf{W}_1$ the operator valued function given by

$$\mathbf{W}_1(z) = z\mathbf{U}(z)\Theta(z)^*(1 - \Theta(z)\Theta(0)^*) \quad \text{a.e. on } \mathbb{T}.$$

Since $\mathbf{C}\mathbf{C}_{\Theta,J}k_0^\Theta x \in K_\Theta$, then $M_{\mathbf{W}_1}x \in H^2(\mathcal{H})$. By commutativity of $M_{\mathbf{W}_1}$ with $\mathbf{M}_z$, we get that for $n = 0, 1, \dots$,

$$M_{\mathbf{W}_1}(e_n x) \in H^2(\mathcal{H}).$$

Since $x \in \mathcal{H}$ is arbitrary, hence $H^2(\mathcal{H})$ is invariant for $M_{\mathbf{W}_1}$.

Recalling that Θ is a pure analytic function, i.e., $\|\Theta(0)\| < 1$, we have that $z \mapsto (1 - \Theta(z)\Theta(0)^*)^{-1}$ is a bounded analytic function. Hence for any $\mathbf{f} \in H^2(\mathcal{H})$ we have $(1 - \Theta(\cdot)\Theta(0)^*)^{-1}\mathbf{f} \in H^2(\mathcal{H})$ and

$$M_{\mathbf{W}_1}(1 - \Theta(\cdot)\Theta(0)^*)^{-1}\mathbf{f} = zM_{\mathbf{U}}\Theta(\cdot)^*\mathbf{f} \in H^2(\mathcal{H}).$$

Therefore, $\mathbf{V} = \mathbf{M}_z\mathbf{U}\Theta^*$ is analytic. On the other hand, a.e. on $\mathbb{T}$,

$$\begin{aligned} (\mathbf{C}_{\Theta,J}\mathbf{C}k_0^\Theta x)(z) &= (\mathbf{C}_{\Theta,J}\tilde{\mathbf{J}}M_{\mathbf{U}^*}k_0^\Theta x)(z) \\ &= \Theta(z)\bar{z}\mathbf{U}(z)^*(1 - \Theta(z)\Theta(0)^*)x \\ &= \Theta(z)\mathbf{U}(z)^*\bar{z}(1 - \Theta(z)\Theta(0)^*)x. \end{aligned}$$

Similarly to what was done above we define $\mathbf{W}_2(z) = \Theta(z)\mathbf{U}(z)^*\bar{z}(1 - \Theta(z)\Theta(0)^*)$ a.e. on $\mathbb{T}$. Observe that $M_{\mathbf{W}_2}\mathbf{M}_z = \mathbf{M}_z M_{\mathbf{W}_2}$, and consequently as above $M_{\mathbf{W}_2}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. Hence for $\mathbf{f} \in H^2(\mathcal{H})$ we get

$$M_{\mathbf{W}_2}(1 - \Theta(\cdot)\Theta(0)^*)^{-1}\mathbf{f} = M_\Theta M_{\mathbf{U}^*}\mathbf{M}_{\bar{z}}\mathbf{f} \in H^2(\mathcal{H}).$$

Therefore, $\mathbf{V}^* = \Theta\mathbf{U}^*\mathbf{M}_{\bar{z}}$ is also analytic. It follows that $\mathbf{V}$ is a constant unitary operator valued function, $\mathbf{V}(z) = V_0$ a.e. on $\mathbb{T}$. A direct calculation shows that V_0 satisfies (6.2). Hence $\mathbf{C} = M_{\mathbf{V}}\mathbf{C}_{\Theta,J}$. $\square$

Remark 6.7. Let $\Theta \in L^\infty(L(\mathcal{H}))$ be a pure inner function and let $\mathbf{C}$ be an $\mathbf{M}_z$ -conjugation in $L^2(\mathcal{H})$ which leaves K_Θ invariant. Then by Theorem 6.6, for every conjugation J in $\mathcal{H}$ such that Θ is J -symmetric, there exists a unitary operator $V_0 \in L(\mathcal{H})$ such that $\mathbf{C} = M_{\mathbf{V}}\mathbf{C}_{\Theta,J}$, where $\mathbf{V}(z) = V_0$

a.e. on $\mathbb{T}$. Therefore, for two such conjugations J and J' in $\mathcal{H}$ there exist unitary operators V_0, V'_0 and constant operator valued functions $\mathbf{V}(z) = V_0$, $\mathbf{V}'(z) = V'_0$ (a.e. on $\mathbb{T}$) such that

$$\mathbf{C} = M_{\mathbf{V}}\mathbf{C}_{\Theta, J} = M_{\mathbf{V}'}\mathbf{C}_{\Theta, J'}.$$

Moreover, $V'_0 = V_0 J J'$.

Example 6.8. To illustrate Theorem 6.6 consider $\mathcal{H} = \mathbb{C}^2$ and the conjugation $J_1(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Note that $\Theta(z) = \begin{bmatrix} z & 0 \\ 0 & z^2 \end{bmatrix}$ is J_1 -symmetric and defines a pure inner matrix valued function. Then $K_{\Theta} = \{(a_0, b_0 + b_1 z) : a_0, b_0, b_1 \in \mathbb{C}\}$ and the conjugation $\mathbf{C}_{\Theta, J_1}$ is equal to $\begin{bmatrix} \tilde{J} & 0 \\ 0 & z\tilde{J} \end{bmatrix}$.

Assume that $\mathbf{C}$ is an $\mathbf{M}_z$ -conjugation. By Theorem 2.5

$$\mathbf{C} = \begin{bmatrix} M_{\psi_1} \tilde{J} & M_{\psi_2} \tilde{J} \\ M_{\psi_2} \tilde{J} & M_{\psi_4} \tilde{J} \end{bmatrix}$$

with ψ_i satisfying (2.9)–(2.10). Hence

$$\mathbf{C}(a_0, b_0 + b_1 z) = (\psi_1 \bar{a}_0 + \psi_2 (\bar{b}_0 + \bar{b}_1 \bar{z}), \psi_2 \bar{a}_0 + \psi_4 (\bar{b}_0 + \bar{b}_1 \bar{z}))$$

for any $a_0, b_0, b_1 \in \mathbb{C}$. If $\mathbf{C}$ leaves subspace K_{Θ} invariant, then $\psi_1 = \lambda_1$, $\psi_2 = 0$, $\psi_4 = \lambda_4 z$ with $\lambda_i \in \mathbb{T}$. It is clear that $\mathbf{C} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_4 \end{bmatrix} \mathbf{C}_{\Theta, J_1}$.

7. Conjugations Between Model Spaces

In this section we consider conjugations which map one model space into another. Let $\dim \mathcal{H} < \infty$.

Lemma 7.1. *Let $\Theta, \Lambda \in H^\infty(L(\mathcal{H}))$ be two inner functions. Then the following are equivalent:*

1. $\Lambda^* \Theta \in H^\infty(L(\mathcal{H}))$;
2. $\Theta H^2(\mathcal{H}) \subset \Lambda H^2(\mathcal{H})$;
3. $K_{\Lambda} \subset K_{\Theta}$.

Proof. If (1) holds, that is, $\Lambda^* \Theta = \Psi \in H^\infty(L(\mathcal{H}))$, then $\Theta = \Lambda \Psi$ and so (2) is satisfied. On the other hand, if (2) is satisfied, then by Lemma 6.2, $\Theta = \Lambda \Psi$ for some $\Psi \in H^\infty(L(\mathcal{H}))$, $\Psi = \Lambda^* \Theta$ and (1) follows. The equivalence of (2) and (3) is obvious. $\square$

In view of Lemma 7.1 we will say that an operator valued inner function Λ divides an operator valued inner function Θ , and we write $\Lambda \leq \Theta$, if $\Lambda^* \Theta \in H^\infty(L(\mathcal{H}))$. Equivalently, $\Lambda \leq \Theta$, if $\Theta = \Lambda \Psi$ for some $\Psi \in H^\infty(L(\mathcal{H}))$ (clearly $\Psi = \Lambda^* \Theta$ is also an operator valued inner function).

Proposition 7.2. *Let $\Theta, \Lambda \in H^\infty(L(\mathcal{H}))$ be two inner functions. If there exists a conjugation J in $\mathcal{H}$ such that both Θ and Λ are J -symmetric, then the following are equivalent:*

1. $\Lambda^* \Theta \in H^\infty(L(\mathcal{H}))$;

2. $\Theta\Lambda^* \in H^\infty(L(\mathcal{H}))$.

Proof. Assume that $\Theta\Lambda^* \in H^\infty(L(\mathcal{H}))$. Then $\Theta\Lambda^*(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$. Note also that $\Theta\Lambda^*(\Lambda H^2(\mathcal{H})) = \Theta H^2(\mathcal{H})$ and

$$\mathbf{C}_{\Theta,J} = M_{\Theta\Lambda^*}\mathbf{C}_{\Lambda,J}.$$

Since both Λ and Θ are J -symmetric, we have $\mathbf{C}_{\Lambda,J}(K_\Lambda) = K_\Lambda$ and $\mathbf{C}_{\Theta,J}(K_\Theta) = K_\Theta$. Hence

$$\mathbf{C}_{\Theta,J}(K_\Lambda) = M_{\Theta\Lambda^*}\mathbf{C}_{\Lambda,J}(K_\Lambda) = M_{\Theta\Lambda^*}(K_\Lambda) \subset K_\Theta,$$

which implies that

$$K_\Lambda \subset \mathbf{C}_{\Theta,J}(K_\Theta) = K_\Theta.$$

By Lemma 7.1 it follows that $\Lambda^*\Theta \in H^\infty(L(\mathcal{H}))$. The other implication can be proved analogously. $\square$

Theorem 7.3. *Let $\Theta, \Lambda \in H^\infty(L(\mathcal{H}))$ be two pure inner functions and assume that there exist conjugations J_Θ, J_Λ in $\mathcal{H}$ such that Θ is J_Θ -symmetric and Λ is J_Λ -symmetric. Moreover, let $\mathbf{C}$ be an $\mathbf{M}_z$ -conjugation in $L^2(\mathcal{H})$. Then $\mathbf{C}(K_\Lambda) \subset K_\Theta$ if and only if $\mathbf{C} = \mathbf{C}_{\Gamma,J}$ for some inner function $\Gamma \in H^\infty(L(\mathcal{H}))$ and for some conjugation J in $\mathcal{H}$ such that $\Lambda \leq \Gamma \leq \Theta$ and Γ is J -symmetric. In particular, then $\Lambda \leq \Theta$.*

Proof. Assume first that $\mathbf{C} = \mathbf{C}_{\Gamma,J}$, Γ is J -symmetric and $\Lambda \leq \Gamma \leq \Theta$. Then $K_\Lambda \subset K_\Gamma \subset K_\Theta$ and

$$\mathbf{C}(K_\Lambda) = \mathbf{C}_{\Gamma,J}(K_\Lambda) \subset \mathbf{C}_{\Gamma,J}(K_\Gamma) = K_\Gamma \subset K_\Theta.$$

Assume now that $\mathbf{C}$ is an $\mathbf{M}_z$ -conjugation such that $\mathbf{C}(K_\Lambda) \subset K_\Theta$. By Theorem 4.8 there exist $\mathbf{U}_\Lambda, \mathbf{U}_\Theta \in L^\infty(\mathcal{H})$ such that

$$\mathbf{C} = M_{\mathbf{U}_\Lambda} \tilde{\mathbf{J}}_\Lambda = \tilde{\mathbf{J}}_\Theta M_{\mathbf{U}_\Theta},$$

the function $\mathbf{U}_\Lambda$ is unitary valued and J_Λ -symmetric, and the function $\mathbf{U}_\Theta$ is unitary valued and J_Θ -symmetric. Moreover, it follows from Remark 4.9 that

$$\mathbf{U}_\Theta(z) = \mathbf{U}_\Lambda(z)J_\Lambda J_\Theta \quad \text{a.e. on } \mathbb{T}. \quad (7.1)$$

By Lemma 6.5, for $x \in \mathcal{H}$ we have a.e. on $\mathbb{T}$,

$$\begin{aligned} (\mathbf{C}\mathbf{C}_{\Lambda,J_\Lambda}k_0^\Lambda x)(z) &= \mathbf{U}_\Lambda(z)J_\Lambda((\mathbf{C}_{\Lambda,J_\Lambda}k_0^\Lambda x)(z)) \\ &= \mathbf{U}_\Lambda(z)J_\Lambda(\bar{z}(\Lambda(z) - \Lambda(0))J_\Lambda x) \\ &= \mathbf{U}_\Lambda(z)z(\Lambda(z)^* - \Lambda(0)^*)x \\ &= \mathbf{U}_\Lambda(z)z\Lambda(z)^*(1 - \Lambda(z)\Lambda(0)^*)x. \end{aligned}$$

Define

$$\mathbf{V}_1(z) = \mathbf{U}_\Lambda(z)z\Lambda(z)^* \quad \text{and} \quad \mathbf{W}_1(z) = \mathbf{V}_1(z)(1 - \Lambda(z)\Lambda(0)^*)$$

(a.e. on $\mathbb{T}$). Clearly, $\mathbf{V}_1, \mathbf{W}_1 \in L^\infty(L(\mathcal{H}))$. By the calculations above, $M_{\mathbf{W}_1}(xe_0) = \mathbf{C}\mathbf{C}_{\Lambda,J_\Lambda}k_0^\Lambda x \in H^2(\mathcal{H})$. Since $M_{\mathbf{W}_1}$ commutes with $\mathbf{M}_z$, we also get $M_{\mathbf{W}_1}(xe_n) \in H^2(\mathcal{H})$ for $n = 0, 1, \dots$. Hence $M_{\mathbf{W}_1}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$ and $\mathbf{W}_1 \in H^\infty(L(\mathcal{H}))$.

Recalling that Λ is a pure analytic function, we have that $z \mapsto (1 - \Lambda(z)\Lambda(0)^*)^{-1}$ belongs to $H^\infty(L(\mathcal{H}))$. Hence also $\mathbf{V}_1 \in H^\infty(L(\mathcal{H}))$ and $\mathbf{V}_1$ is inner, since its values are unitary operators.

On the other hand, using (7.1) we get (a.e. on $\mathbb{T}$)

$$\begin{aligned} (\mathbf{C}_{\Theta, J_\Theta} \mathbf{C} k_0^\Lambda x)(z) &= (\mathbf{C}_{\Theta, J_\Theta} \tilde{\mathbf{J}}_\Theta M_{\mathbf{U}_\Theta^*} k_0^\Lambda x)(z) \\ &= \Theta(z) \bar{z} \mathbf{U}_\Theta(z)^* (1 - \Lambda(z) \Lambda(0)^*) x \\ &= \Theta(z) \bar{z} J_\Theta J_\Lambda \mathbf{U}_\Lambda(z)^* (1 - \Lambda(z) \Lambda(0)^*) x. \end{aligned}$$

Define

$$\mathbf{V}_2(z) = \Theta(z) \bar{z} J_\Theta J_\Lambda \mathbf{U}_\Lambda(z)^* \quad \text{and} \quad \mathbf{W}_2(z) = \mathbf{V}_2(z) (1 - \Lambda(z) \Lambda(0)^*)$$

(a.e. on $\mathbb{T}$). As above $H^2(\mathcal{H})$ is $M_{\mathbf{W}_2}$ -invariant so $\mathbf{W}_2 \in H^\infty(L(\mathcal{H}))$, and consequently $\mathbf{V}_2 \in H^\infty(L(\mathcal{H}))$ is an inner function. Since

$$\mathbf{V}_2(z) = J_\Theta \Theta(z)^* z \mathbf{U}_\Lambda(z) J_\Lambda \quad \text{a.e. on } \mathbb{T},$$

it follows from Lemma 6.1 that $\mathbf{V}_3(z) = \mathbf{U}_\Lambda(z)^* \bar{z} \Theta(z)$ (a.e. on $\mathbb{T}$) is also an inner function.

Define

$$\Gamma(z) = \mathbf{U}_\Lambda(z) z \quad \text{a.e. on } \mathbb{T}.$$

Then $\mathbf{V}_1 = \Gamma \Lambda^* \in H^\infty(L(\mathcal{H}))$, and so $\Gamma = \mathbf{V}_1 \Lambda$ is an inner function. Observe also that Γ is J_Λ -symmetric. By Proposition 7.2 we also have $\Lambda^* \Gamma \in H^\infty(L(\mathcal{H}))$, and so $\Lambda \leq \Gamma$.

Moreover, $\mathbf{V}_3 = \Gamma^* \Theta \in H^\infty(L(\mathcal{H}))$ and $\Gamma \leq \Theta$. In particular,

$$\Lambda^* \Theta = \Lambda^* \Gamma \Gamma^* \Theta \in H^\infty(L(\mathcal{H})),$$

that is, $\Lambda \leq \Theta$. Finally, a.e. on $\mathbb{T}$,

$$(\mathbf{C}f)(z) = \mathbf{U}_\Lambda(z) J_\Lambda(f(z)) = \Gamma(z) \bar{z} J_\Lambda(f(z)) = (\mathbf{C}_{\Gamma, J_\Lambda} f)(z). \quad \square$$

Remark 7.4. Note that in the proof of Theorem 7.3 we actually did not use the fact that Θ is pure. Moreover, if we assume that Λ and Θ are J -symmetric, it follows from the proof that Γ can be chosen to be J -symmetric as well.

Remark 7.5. Let $\mathbf{C} = \mathbf{C}_{\Gamma, J}$ for an inner function $\Gamma \in H^\infty(L(\mathcal{H}))$, which is J -symmetric and such that $\Lambda \leq \Gamma \leq \Theta$. Then for any conjugation J' in $\mathcal{H}$ the function Γ' defined by

$$\Gamma'(z) = \Gamma(z) J J' \quad \text{a.e. on } \mathbb{T}$$

is J' -symmetric, $\Lambda \leq \Gamma' \leq \Theta$ and $\mathbf{C} = \mathbf{C}_{\Gamma', J'}$.

Remark 7.6. If $\Lambda = \Theta$ is J -symmetric and $\mathbf{C}$ is an $\mathbf{M}_z$ -conjugation in $L^2(\mathcal{H})$ such that $\mathbf{C}(K_\Theta) \subset K_\Theta$, then by Theorem 7.3 and Remark 7.4 there exists an inner J -symmetric function Γ such that $\mathbf{C} = \mathbf{C}_{\Gamma, J}$ and $\Theta \leq \Gamma \leq \Theta$. The last condition implies that $\mathbf{V} = \Gamma \Theta^*$ is a unitary constant and a.e. on $\mathbb{T}$,

$$(\mathbf{C}f)(z) = \Gamma(z) \bar{z} J(f(z)) = \Gamma(z) \Theta(z)^* \Theta(z) \bar{z} J(f(z)) = (M_{\mathbf{V}} \mathbf{C}_{\Theta, J} f)(z).$$

Moreover, $\mathbf{V}$ satisfies (6.2) and $M_{\mathbf{V}}$ is $\mathbf{C}_{\Theta, J}$ -symmetric.

Now recall that $\mathbf{J}^*$ is an $\mathbf{M}_z$ -commuting conjugation. The proposition below shows some basic properties of $\mathbf{J}^*$ as to model spaces.

Proposition 7.7. *Let J be a conjugation in $\mathcal{H}$ and let Θ be a J -symmetric inner function. Then*

1. $\mathbf{J}^* M_\Theta = M_{\Theta^\#} \mathbf{J}^*$;
2. $\mathbf{J}^*(\Theta H^2(\mathcal{H})) = \Theta^\# H^2(\mathcal{H})$;
3. $\mathbf{J}^*(K_\Theta) = K_{\Theta^\#}$,
4. $\mathbf{J}^*(k_0^\Theta x) = k_0^{\Theta^\#} Jx$.

In what follows we describe all $\mathbf{M}_z$ -commuting conjugations mapping one model space into another.

Theorem 7.8. *Let $\Theta, \Lambda \in H^\infty(L(\mathcal{H}))$ be two pure inner functions and assume that there exist conjugations J_Θ, J_Λ in $\mathcal{H}$ such that Θ is J_Θ -symmetric and Λ is J_Λ -symmetric. Assume that $\mathbf{C}$ is an $\mathbf{M}_z$ -commuting conjugation in $L^2(\mathcal{H})$. Then $\mathbf{C}(K_\Lambda) \subset K_\Theta$ if and only if there is a unitary J_Λ -symmetric operator $U_0 \in L(\mathcal{H})$ such that $\mathbf{C} = M_{\mathbf{U}_\Lambda} \mathbf{J}_\Lambda^*$, where $\mathbf{U}_\Lambda$ is a constant operator valued function, $\mathbf{U}_\Lambda(z) = U_0$ for almost all $z \in \mathbb{T}$ and $U_0 \Lambda^\# \leq \Theta$.*

Proof. Since $\mathbf{C}$ is $\mathbf{M}_z$ -commuting, then by Theorem 4.3 there is $\mathbf{U}_\Lambda \in L^\infty(L(\mathcal{H}))$ such that $\mathbf{U}_\Lambda(z)$ is a unitary operator for almost all $z \in \mathbb{T}$, $M_{\mathbf{U}_\Lambda}$ is $\mathbf{J}_\Lambda^*$ -symmetric and $\mathbf{C} = M_{\mathbf{U}_\Lambda} \mathbf{J}_\Lambda^*$. Note that using Lemma 6.5 and Proposition 7.7 (4) a.e. on $\mathbb{T}$ we have

$$\begin{aligned} (\mathbf{C} \mathbf{C}_{\Lambda, J_\Lambda} \widetilde{k_0^\Lambda x})(z) &= \mathbf{U}_\Lambda(z) \mathbf{J}_\Lambda^*(k_0^\Lambda J_\Lambda x)(z) \\ &= \mathbf{U}_\Lambda(z) (1 - \Lambda^\#(z)(\Lambda^\#(0))^*)x. \end{aligned} \quad (7.2)$$

Define $\mathbf{V}(z) = \mathbf{U}_\Lambda(z) (1 - \Lambda^\#(z)(\Lambda^\#(0))^*)$ a.e. on $\mathbb{T}$. It is clear that $\mathbf{V} \in L^\infty(L(\mathcal{H}))$ and by (7.2) we have that $M_{\mathbf{V}}(xe_0) = \mathbf{C} \mathbf{C}_{\Lambda, J_\Lambda} \widetilde{k_0^\Lambda x} \in H^2(\mathcal{H})$ because $\mathbf{C}(K_\Lambda) \subset K_\Theta$. Since $M_{\mathbf{V}}$ commutes with $\mathbf{M}_z$, we have also that $M_{\mathbf{V}}(xe_n) \in H^2(\mathcal{H})$ for $n = 1, 2, \dots$. Hence $M_{\mathbf{V}}(H^2(\mathcal{H})) \subset H^2(\mathcal{H})$ and $\mathbf{V} \in H^\infty(L(\mathcal{H}))$. Since Λ is pure, it follows that $z \mapsto (1 - \Lambda^\#(z)(\Lambda^\#(0))^*)^{-1} \in H^\infty(L(\mathcal{H}))$. By (7.2) we obtain that

$$\mathbf{U}_\Lambda = \mathbf{V} (1 - \Lambda^\#(\Lambda^\#(0))^*)^{-1} \in H^\infty(L(\mathcal{H})),$$

which implies that $\mathbf{C}$ leaves $H^2(\mathcal{H})$ invariant. Applying Theorem 5.1 we get that $\mathbf{U}_\Lambda$ is a constant operator valued function and $\mathbf{U}_\Lambda(z) = U_0$ a.e. on $\mathbb{T}$. Since $M_{\mathbf{U}_\Lambda}$ is $\mathbf{J}_\Lambda^*$ -symmetric thus U_0 is J -symmetric by Proposition 4.2.

Note that by Proposition 7.7 (3), we have

$$\mathbf{C}(K_\Lambda) = U_0 K_{\Lambda^\#}.$$

Since U_0 is unitary, then $U_0 K_{\Lambda^\#} = K_{U_0 \Lambda^\#}$. Hence by Lemma 7.1,

$$\mathbf{C}(K_\Lambda) \subset K_\Theta$$

if and only if $U_0 \Lambda^\# \leq \Theta$. □

Remark 7.9. Consider a pure inner function $\Theta \in H^\infty(L(\mathcal{H}))$ and a conjugation J_Θ in $\mathcal{H}$ such that Θ is J_Θ -symmetric. As a consequence of Theorem 7.8 we have that if $\mathbf{C}$ is an $\mathbf{M}_z$ -commuting conjugation in $L^2(\mathcal{H})$, then $\mathbf{C}(K_\Theta) \subset K_{\Theta^\#}$ if and only if there is a unitary operator $U_0 \in L(\mathcal{H})$ such that $\mathbf{C} = M_{\mathbf{U}_\Theta} \mathbf{J}_\Theta^*$, where $\mathbf{U}_\Theta$ is a constant J_Θ -symmetric operator valued

function, $\mathbf{U}_\Theta(z) = U_0$ for almost all $z \in \mathbb{T}$ and $U_0\Theta^\# \leq \Theta^\#$. Note that if U_0 commutes with Θ , the last condition is always satisfied, since then $(U_0\Theta^\#)^*\Theta^\# = U_0^* \in H^\infty(L(\mathcal{H}))$.

8. Conjugations and Shift Invariant Subspaces

Let $\dim \mathcal{H} < \infty$ and let $\Theta, \Lambda \in H^\infty(L(\mathcal{H}))$ be two inner functions. For any fixed conjugation J in $\mathcal{H}$ define

$$\mathbf{C}_J^{\Lambda, \Theta} = M_\Theta \mathbf{J}^* M_{\Lambda^*}.$$

Clearly, $\mathbf{C}_J^{\Lambda, \Theta}$ is an antilinear isometry. Moreover, it is easy to see that $\mathbf{C}_J^{\Lambda, \Theta}$ maps $\Lambda H^2(\mathcal{H})$ onto $\Theta H^2(\mathcal{H})$.

Proposition 8.1. *The antilinear operator $\mathbf{C}_J^{\Lambda, \Theta}$ is an involution (and hence a conjugation in $L^2(\mathcal{H})$) if and only if*

$$\Theta(z)J\Lambda^\#(z) = \Lambda(z)J\Theta^\#(z) \quad \text{a.e. on } \mathbb{T}. \quad (8.1)$$

Proof. We have

$$(\mathbf{C}_J^{\Lambda, \Theta})^2 = M_\Theta \mathbf{J}^* M_{\Lambda^*} M_\Theta \mathbf{J}^* M_{\Lambda^*} = I_{L^2(\mathcal{H})}$$

if and only if

$$M_\Theta \mathbf{J}^* M_{\Lambda^*} = M_{\Lambda^*} \mathbf{J}^* M_{\Theta^*}. \quad (8.2)$$

Since for $\mathbf{f} \in L^2(\mathcal{H})$ a.e. on $\mathbb{T}$,

$$(M_\Theta \mathbf{J}^* M_{\Lambda^*} \mathbf{f})(z) = \Theta(z)J(\Lambda(\bar{z})^* \mathbf{f}(\bar{z})) = (M_{\Theta J \Lambda^\#} \tilde{\mathbf{J}} \mathbf{J}^* \mathbf{f})(z)$$

and

$$(M_{\Lambda^*} \mathbf{J}^* M_{\Theta^*} \mathbf{f})(z) = \Lambda(z)J(\Theta(\bar{z})^* \mathbf{f}(\bar{z})) = (M_{\Lambda J \Theta^\#} \tilde{\mathbf{J}} \mathbf{J}^* \mathbf{f})(z),$$

we see that (8.2) is equivalent to (8.1). $\square$

For $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ and a conjugation J in $\mathcal{H}$ define

$$\mathbf{F}_J(z) = J\mathbf{F}(\bar{z})J, \quad \text{a.e. on } \mathbb{T}.$$

Clearly, $\mathbf{F}_J \in L^\infty(L(\mathcal{H}))$. Moreover, $\mathbf{F}_J \in H^\infty(L(\mathcal{H}))$ if and only if $\mathbf{F} \in H^\infty(L(\mathcal{H}))$ (see the proof of Lemma 6.1), and $\mathbf{F}_J$ is an inner function if and only if $\mathbf{F}$ is. It is easy to verify the following.

Lemma 8.2. *Let $\mathbf{F}, \mathbf{G} \in L^\infty(L(\mathcal{H}))$ and let J be a conjugation in $\mathcal{H}$. Then*

1. $(\mathbf{F}_J)_J = \mathbf{F}$;
2. $(\mathbf{F}\mathbf{G})_J = \mathbf{F}_J \mathbf{G}_J$;
3. $(\mathbf{F}_J)^*(z) = J\mathbf{F}^\#(z)J = (\mathbf{F}^*)_J(z)$ a.e. on $\mathbb{T}$;
4. $(\mathbf{F}_J)^\#(z) = J\mathbf{F}^*(z)J = (\mathbf{F}^\#)_J(z)$ a.e. on $\mathbb{T}$.

Moreover, from the proof of Proposition 4.2 we get:

Lemma 8.3. *Let $\mathbf{F} \in L^\infty(L(\mathcal{H}))$ and let J be a conjugation in $\mathcal{H}$. Then*

$$\mathbf{J}^* M_{\mathbf{F}} \mathbf{J}^* = M_{\mathbf{F}_J}.$$

Remark 8.4. Note that condition (8.1) can be expressed as

$$\Theta\Lambda_J^* = \Lambda\Theta_J^*. \quad (8.3)$$

Indeed, (8.1) is equivalent to

$$\Theta(z)J\Lambda^\#(z)J = \Lambda(z)J\Theta^\#(z)J \quad \text{a.e. on } \mathbb{T},$$

and by Lemma 8.2 (3), for almost all $z \in \mathbb{T}$, $J\Lambda^\#(z)J = \Lambda_J^*(z)$ and $J\Theta^\#(z)J = \Theta_J^*(z)$.

Remark 8.5. If $\mathcal{H} = \mathbb{C}$ and $J(w) = \bar{w}$, $w \in \mathbb{C}$, then for $\varphi \in L^\infty(\mathbb{T})$ we have

$$\varphi_J = \varphi^\#.$$

Moreover, for scalar inner functions θ and α condition (8.1) (or (8.3)) takes form

$$\theta(z)\alpha(\bar{z}) = \alpha(z)\theta(\bar{z}) \quad \text{a.e. on } \mathbb{T},$$

which in this case is equivalent to $\theta\theta^\# = \alpha\alpha^\#$ (see [2, Theorem 5.2]).

Theorem 8.6. Let $\Theta, \Lambda \in H^\infty(L(\mathcal{H}))$ be two inner functions and let J be a conjugation in $\mathcal{H}$. There exists an $\mathbf{M}_z$ -commuting conjugation $\mathbf{C}$ in $L^2(\mathcal{H})$ such that $\mathbf{C}(\Lambda H^2(\mathcal{H})) \subset \Theta H^2(\mathcal{H})$ if and only if there is an inner function $\Psi \in H^\infty(L(\mathcal{H}))$ such that

$$(\Psi\Lambda^*\Theta)_J = (\Psi\Lambda^*\Theta)^*. \quad (8.4)$$

In that case, $\mathbf{C} = \mathbf{C}_J^{\Lambda, \Gamma}$ for some inner function $\Gamma \in H^\infty(L(\mathcal{H}))$ such that $\Theta \leq \Gamma$ and

$$\Gamma\Lambda_J^* = \Lambda\Gamma_J^*. \quad (8.5)$$

Assume at first that there exists an inner function $\Psi \in H^\infty(L(\mathcal{H}))$ such that (8.4) holds. Then

$$\Theta^*\Lambda\Psi^* = (\Psi\Lambda^*\Theta)^* = (\Psi\Lambda^*\Theta)_J = \Psi_J\Lambda_J^*\Theta_J,$$

and so

$$\Lambda\Psi^*\Theta_J^* = \Theta\Psi_J\Lambda_J^*. \quad (8.6)$$

Put $\Gamma = \Theta\Psi_J$. Then $\Gamma \in H^\infty(L(\mathcal{H}))$ is an inner function, $\Theta \leq \Gamma$ and by (8.6),

$$\Lambda\Gamma_J^* = \Lambda(\Theta_J\Psi)^* = \Lambda\Psi^*\Theta_J^* = \Theta\Psi_J\Lambda_J^* = \Gamma\Lambda_J^*.$$

Thus (8.5) holds and, by Proposition 8.1 and Remark 8.4, $\mathbf{C} = \mathbf{C}_J^{\Lambda, \Gamma}$ is a conjugation in $L^2(\mathcal{H})$. Moreover, it is an $\mathbf{M}_z$ -commuting conjugation such that

$$\mathbf{C}(\Lambda H^2(\mathcal{H})) = \Gamma H^2(\mathcal{H}) \subset \Theta H^2(\mathcal{H}).$$

Assume now that $\mathbf{C}$ is an $\mathbf{M}_z$ -commuting conjugation in $L^2(\mathcal{H})$ such that $\mathbf{C}(\Lambda H^2(\mathcal{H})) \subset \Theta H^2(\mathcal{H})$. By Theorem 4.3 and Proposition 4.2 (2) there exists a unitary valued $\mathbf{U} \in L^\infty(L(\mathcal{H}))$ such that

$$\mathbf{C} = M_{\mathbf{U}}\mathbf{J}^* = \mathbf{J}^*M_{\mathbf{U}^*}$$

and $J\mathbf{U}(z)J = \mathbf{U}^\#(z)$ a.e. on $\mathbb{T}$. Therefore,

$$\mathbf{C}(\Lambda H^2(\mathcal{H})) = \mathbf{J}^* M_{\mathbf{U}^*} M_\Lambda (H^2(\mathcal{H})) \subset M_\Theta (H^2(\mathcal{H})),$$

and by Lemma 8.3,

$$M_{\mathbf{U}^* \Lambda} (H^2(\mathcal{H})) \subset \mathbf{J}^* M_\Theta (H^2(\mathcal{H})) = M_{\Theta_J} \mathbf{J}^* (H^2(\mathcal{H})) = \Theta_J H^2(\mathcal{H}).$$

By Lemma 6.2, $\mathbf{U}^* \Lambda \in H^\infty(L(\mathcal{H}))$ and there exists an inner function $\Psi \in H^\infty(L(\mathcal{H}))$ such that $\mathbf{U}^* \Lambda = \Theta_J \Psi$. Note that by the fact that $J\mathbf{U}(z)J = \mathbf{U}^\#(z)$ a.e. on $\mathbb{T}$ and Lemma 8.2 (3), we get $\mathbf{U}_J^* = \mathbf{U}$. It follows that

$$\Theta \Psi_J \Lambda_J^* = (\Theta_J \Psi \Lambda^*)_J = (\mathbf{U}^* \Lambda \Lambda^*)_J = \mathbf{U} = \Lambda \Psi^* \Theta_J^*,$$

which is an equivalent form of (8.4). Moreover, the above means that the function $\Gamma = \Theta \Psi_J = \mathbf{U}_J^* \Lambda_J$ satisfies (8.5) (since $\Gamma_J^* = \Psi^* \Theta_J^*$). Clearly, $\Gamma \in H^\infty(L(\mathcal{H}))$ is an inner function and $\Theta \leq \Gamma$. Moreover,

$$\mathbf{C} = M_{\mathbf{U}} \mathbf{J}^* = M_{\mathbf{U}_J^*} \mathbf{J}^* M_\Lambda M_{\Lambda^*} = M_{\mathbf{U}_J^* \Lambda_J} \mathbf{J}^* M_{\Lambda^*} = M_\Gamma \mathbf{J}^* M_{\Lambda^*} = \mathbf{C}_J^{\Lambda, \Gamma}. \quad \square$$

Remark 8.7. Let $\Theta \in H^\infty(L(\mathcal{H}))$ be an inner function. Assume that $\mathbf{C}$ is an $\mathbf{M}_z$ -commuting conjugation in $L^2(\mathcal{H})$ such that $\mathbf{C}(\Theta H^2(\mathcal{H})) \subset \Theta H^2(\mathcal{H})$. By Theorem 8.6 we obtain that $\mathbf{C} = \mathbf{C}_J^{\Theta, \Gamma}$ for some inner function $\Gamma \in H^\infty(L(\mathcal{H}))$ such that $\Theta \leq \Gamma$ and

$$\Gamma \Theta_J^* = \Theta \Gamma_J^*. \quad (8.7)$$

Therefore there exists $\Psi \in H^\infty(L(\mathcal{H}))$ such that $\Gamma = \Theta \Psi$ and by (8.7),

$$\Theta \Psi \Theta_J^* = \Theta \Psi_J^* \Theta_J^*.$$

It follows that $\Psi = \Psi_J^*$. Since $\Psi_J, \Psi_J^* \in H^\infty(L(\mathcal{H}))$, so Ψ must be a unitary constant. Assume that $\Psi(z) = U_0 \in L(\mathcal{H})$ a.e. on $\mathbb{T}$, then $\Gamma(z) = \Theta(z)U_0$ a.e. on $\mathbb{T}$ and

$$\mathbf{C} = M_\Gamma \mathbf{J}^* M_{\Theta^*} = M_\Theta M_{U_0} \mathbf{J}^* M_{\Theta^*} = M_{\Theta U_0 \Theta^*} M_\Theta \mathbf{J}^* M_{\Theta^*}.$$

Note that by (8.7) we now have

$$\Theta(z)U_0 J \Theta(\bar{z})^* J = \Theta(z) J U_0^* \Theta(\bar{z})^* J \quad \text{a.e. on } \mathbb{T},$$

which implies that $U_0 J = J U_0^*$, i.e., U_0 is J -symmetric. Recalling the scalar case considered in [2, Corollary 5.4] one can expect that $\Theta U_0 \Theta^*$ is a unitary constant. This is not necessarily true (see Example 8.8).

Example 8.8. Let $\mathcal{H} = \mathbb{C}^2$ and consider the conjugation $J(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$ in $\mathbb{C}^2$. If we take $\Theta = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$ and $U_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is easy to see that both Θ and U_0 are J -symmetric, but $\Theta U_0 \Theta^* = \begin{bmatrix} 0 & \bar{z} \\ z & 0 \end{bmatrix}$, so it is not constant.

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