Spectral Determinants and an Ambarzumian Type Theorem on Graphs

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Abstract. We consider an inverse problem for Schrödinger operators on connected equilateral graphs with standard matching conditions. We calculate the spectral determinant and prove that the asymptotic distribution of a subset of its zeros can be described by the roots of a polynomial. We verify that one of the roots is equal to the mean value of the potential and apply it to prove an Ambarzumian type result, i.e., if a specific part of the spectrum is the same as in the case of zero potential, then the potential has to be zero.

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1. Introduction

Quantum graphs arise naturally as simplified models in mathematics, physics, chemistry, and engineering [3]. Ambarzumian's theorem in inverse spectral theory refers to a setting when a differential operator can be reconstructed from at most one spectrum due to the presence of a constant eigenfunction. The original theorem from 1929 states for $q \in C[0, \pi]$ that if the eigenvalues of

$$\left. \begin{aligned} -y'' + q(x)y &= \lambda y \\ y'(0) &= y'(1) = 0 \end{aligned} \right\} \quad (1.1)$$

are $\lambda_n = n^2\pi^2$ ($n \geq 0$), then $q = 0$ [2]. Neumann boundary conditions are crucial here for otherwise the statement is not true. Eigenvalues other than zero are used only through eigenvalue asymptotics to get $\int_0^1 q = 0$; hence a subsequence $\lambda_r = r^2\pi^2 + o(1)$ of them is sufficient to reach the same conclusion even if $q \in L^1(0, \pi)$. On finite intervals inverse eigenvalue problems have a

vast literature. In general we mention Horváth [18] and the fundamental paper of Borg [7] as well as the works referenced by and referencing these. For Ambarzumian's theorem a recent stability result is found in Horváth [19]. Boman et al. [6] proved that an interval with zero potential can be identified by its spectrum even if we only know a priori that the underlying space is a graph with appropriate boundary conditions.

Let us turn to the list of extensions when the underlying space is a fixed graph. On a tree with edges of equal length knowing the smallest eigenvalue 0 exactly and a specific part of the spectrum approximately is enough for recover to the zero potential (Carlson and Pivovarchik [8]). On a tree with different edge lengths this is still true (see Lemma 4.4 of Law and Yanagida [25]), however, the required set of eigenvalues is given by an existence proof.

An ultimate generalization of Ambarzumian type scenarios with a fixed underlying space is found by Davies [12]. In an abstract framework which contains arbitrary graphs with arbitrary edge lengths as particular special cases he was able to recover to the zero potential by determining the mean value of the potential from heat trace asymptotics. In the graph setting there are also trace formulas, for example (8.14) of Bolte et al. [5] that could be used to reach the same conclusion. In the context of inverse eigenvalue problems this is not entirely satisfactory however, as spectral informations are the eigenvalues themselves and it is not clear how to apply trace formulas when only a small part of the spectrum is known. We look for instead a minimal set of eigenvalues which determines the mean value of the potential and give an answer to that question if all edge lengths are equal.

For a summary on differential operators on graphs, see Pokornyi and Borovskikh [30] and Kuchment [23].

In this paper we consider a connected graph $G(V, E)$ with edges of equal length. The graph can contain loops and multiple edges. We parametrize each edge with $x \in (0, 1)$. This gives an orientation on G . We consider a Schrödinger operator with potential $q_j(x) \in L^1(0, 1)$ on the edge e_j and with Neumann (or Kirchhoff) boundary conditions (sometimes called standard matching conditions), i.e., solutions are required to be continuous at the vertices and, in the local coordinate pointing outward, the sum of derivatives is zero. More formally, consider the eigenvalue problem

$$-y'' + q_j(x)y = \lambda y \quad (1.2)$$

on e_j for all j with the conditions

$$y_j(\kappa_j) = y_k(\kappa_k) \quad (1.3)$$

if e_j and e_k are incident edges attached to a vertex v where $\kappa = 0$ for outgoing edges, $\kappa = 1$ for incoming edges (and can be both 0 or 1 for loops); and in every vertex v

$$\sum_{e_j \text{ leaves } v} y'_j(0) = \sum_{e_j \text{ enters } v} y'_j(1) \quad (1.4)$$

(loops are counted on both sides).

The *spectral determinant* or alternatively *functional determinant* or *characteristic function* of the problem (1.2)–(1.4) is a meromorphic function whose zeros coincide with its spectrum. Spectral determinants have been subject to continuous attention in the theoretical physics literature for the last 20 years [1, 9, 11, 14, 15, 17, 20, 28, 31]. As a tool for proving our Ambarzumian type results, we give a formula for the spectral determinant of Schrödinger operators (see (2.9)) and for the asymptotic distribution of some of its zeros. It is already known that for finite connected graphs, the main term of the eigenvalue asymptotics can be obtained from Weyl's law and the next terms depend on complicated combinations of $\int_0^1 q_j$, $j = 1, \dots, |E|$ (Möller and Pivovarchik [27], p. 213). We express some of these combinations as the roots of a polynomial (see (2.11)) and prove that for *any* connected, equilateral graph there is a root equal to the mean value of the potential. For the convenience of the reader, we emphasize this result in the context of spectral determinants:

Theorem 1.1' *Consider a connected graph $G(V, E)$ with edges of equal length. The spectral determinant of Schrödinger operators on G with standard matching conditions has a sequence of roots which asymptotically differ by the mean value of the potential from the corresponding sequence of roots of the spectral determinant of the free Schrödinger operator. Precisely, the problem (1.2)–(1.4) has a sequence of eigenvalues $\lambda_k = (2k\pi)^2 + \frac{1}{|E|} \sum_j \int_0^1 q_j + o(1)$ ($k \in \mathbb{Z}^+$). Moreover, if G is a bipartite graph, the problem (1.2)–(1.4) has a sequence of eigenvalues $\lambda_k = (k\pi)^2 + \frac{1}{|E|} \sum_j \int_0^1 q_j + o(1)$.*

Remark. It is important to note at this point that the presence of the potential does not simply shift the spectrum of the Laplacian. This will be clear from (2.11). A related result is Theorem 2.1 of de Verdière [13]. One might also expect that a variant of Theorem 1.1 holds in the case of possibly different edge lengths though it seems hard to describe the asymptotic distribution of the eigenvalues in question.

For our Ambarzumian type result, we need an improved version of this theorem:

Theorem 1.1. *Consider the eigenvalue problem (1.2)–(1.4). There are exactly $|E| - |V| + 2$ eigenvalues (counting multiplicities) such that $\lambda = (2k\pi)^2 + O(1)$ as $k \rightarrow \infty$ ($k \in \mathbb{Z}^+$). Among these eigenvalues at least one has the asymptotics $\lambda = (2k\pi)^2 + \frac{1}{|E|} \sum_j \int_0^1 q_j + o(1)$. Moreover, if G is a bipartite graph, the same is true for k instead of $2k$, i.e., there are exactly $|E| - |V| + 2$ eigenvalues (counting multiplicities) such that $\lambda = (k\pi)^2 + O(1)$ and at least one has the asymptotics $\lambda = (k\pi)^2 + \frac{1}{|E|} \sum_j \int_0^1 q_j + o(1)$.*

Remark. Although G is a directed graph, if we reverse an edge e_j and change the potential to $q_j(1 - x)$ on it, we get an eigenvalue problem with the same eigenvalues and eigenfunctions (which are reversed with respect to the new direction on e_j).

Remark. For $q = 0$ and $\lambda = 2k^2\pi^2$ ($k \in \mathbb{Z}^+$) one eigenfunction is $\cos 2k\pi x$ and in each cycle there is a Dirichlet eigenfunction $\pm \sin 2k\pi x$ on its edges

FIGURE 1. A graph with equal effective resistances ($\frac{2}{3}$)

(depending on the direction) and 0 everywhere else (Kurasov [24], Theorem 2(4)). In the bipartite case we can assume that V is a disjoint union of V_1 and V_2 , and that every edge points from V_1 to V_2 . Then for k odd, we can take $\cos k\pi x$ on all edges; besides, there are Dirichlet eigenfunctions, $\pm \sin k\pi x$ alternately on the edges of a cycle and 0 elsewhere. Hence in both cases there are $|E| - |V| + 2$ independent eigenfunctions. We shall prove that the multiplicity is not greater.

Theorem 1.2. *Consider the eigenvalue problem (1.2)–(1.4). If $\lambda = 0$ is the smallest eigenvalue and for infinitely many $k \in \mathbb{Z}^+$ there are $|E| - |V| + 2$ eigenvalues (counting multiplicities) such that $\lambda = (2k\pi)^2 + o(1)$, then $q = 0$ a.e. on G . Moreover, if G is a bipartite graph, the same is true for k instead of $2k$, i.e., if $\lambda = 0$ is the smallest eigenvalue and for infinitely many $k \in \mathbb{Z}^+$ there are $|E| - |V| + 2$ eigenvalues (counting multiplicities) such that $\lambda = (k\pi)^2 + o(1)$, then $q = 0$ a.e. on G .*

Remark. For a tree $|E| - |V| + 2 = 1$, and a tree is bipartite, hence Theorem 1.2 is a generalization of Theorem 1.2 in Carlson and Pivovarchik [8].

If the graph represents an electrical circuit in which each edge has a unit resistance, the *effective resistance* of an edge can be computed (or in mathematics, defined) by terms of the graph Laplacian. A result of Kirchhoff [22] is that the effective resistance of an edge e can be expressed as the number of spanning trees containing e divided by the number of all spanning trees.

Theorem 1.3. *Consider the eigenvalue problem (1.2)–(1.4). If every non-loop edge of G has the same effective resistance $r < 1$, the multiplicities required by Theorem 1.2 can be weakened to $|E| - |V| + 1$.*

Interesting examples are the complete graph (if $|V| > 2$) or a graph with one point and a loop, which corresponds to the case of periodic boundary conditions treated in Cheng et al. [10]. Figure 1 shows a non edge-transitive example (Göbel [16]).

2. The Proof

Denote by $c_j(x, \lambda)$ the solution of (1.2) which satisfies the conditions $c_j(0, \lambda) - 1 = c'_j(0, \lambda) = 0$ and by $s_j(x, \lambda)$ the solution of (1.2) which satisfies the conditions $s_j(0, \lambda) = s'_j(0, \lambda) - 1 = 0$. Each $y_j(x, \lambda)$ may be written as a linear combination

$$y_j(x, \lambda) = A_j(\lambda)c_j(x, \lambda) + \tilde{B}_j(\lambda)s_j(x, \lambda). \quad (2.1)$$

Then $y_j(0, \lambda) = A_j(\lambda)$ is the same on each outgoing edge; hence we choose to index the functions $A(\lambda)$ by vertices, and then

$$y_j(x, \lambda) = A_v(\lambda)c_j(x, \lambda) + \tilde{B}_j(\lambda)s_j(x, \lambda), \quad (2.2)$$

if e_j starts from v . If the eigefunctions are normalized, i.e., $\sum_j \|y_j(x, \lambda)\|_2^2 = 1$, then $A_v(\lambda) = O(1)$, $\tilde{B}_j(\lambda) = O(\sqrt{\lambda})$ [8]. For later calculations it is more convenient to work with the $O(1)$ -variables $A_v(\lambda)$ and $B_j(\lambda) = \frac{\tilde{B}_j(\lambda)}{\sqrt{\lambda}}$.

The coefficients A_v and B_j form a $(|V| + |E|)$ -dimensional vector, which satisfies $|V|$ Kirchhoff conditions at the vertices and $|E|$ continuity conditions at the incoming ends of edges, namely, for all $v \in V(G)$,

$$\sum_{e_j: \dots \rightarrow v} \frac{1}{\sqrt{\lambda}} A_{v_j}(\lambda) c'_j(1, \lambda) + B_j(\lambda) s'_j(1, \lambda) - \sum_{e_j: v \rightarrow \dots} B_j(\lambda) = 0, \quad (2.3)$$

where in the first sum v_j denotes the starting point of e_j ; and for all $e_j \in E(G)$,

$$A_u(\lambda) c_j(1, \lambda) + \sqrt{\lambda} B_j(\lambda) s_j(1, \lambda) - A_v(\lambda) = 0, \quad (2.4)$$

if e_j points from u to v .

The matrix of this homogeneous linear system of equations has the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

- A is like an adjacency matrix; $a_{vu} = \frac{1}{\sqrt{\lambda}} \sum c'_j(1, \lambda)$, the sum is taken on edges pointing from u to v ;
- B and C are like incidence matrices;

$$b_{vj} = \begin{cases} s'_j(1, \lambda) & \text{if } e_j \text{ ends in } v \\ -1 & \text{if } e_j \text{ starts from } v \\ s'_j(1, \lambda) - 1 & \text{if } e_j \text{ is a loop in } v \\ 0 & \text{otherwise} \end{cases}$$

$$c_{jv} = \begin{cases} -1 & \text{if } e_j \text{ ends in } v \\ c_j(1, \lambda) & \text{if } e_j \text{ starts from } v \\ -1 + c_j(1, \lambda) & \text{if } e_j \text{ is a loop in } v \\ 0 & \text{otherwise} \end{cases}$$
- D is a diagonal matrix, $d_{jj} = \sqrt{\lambda} s_j(1, \lambda)$.

The determinant of the matrix M is the so-called spectral determinant of the problem (1.2)–(1.4).

Example 1. Consider a single vertex with a loop. Then

$$M = M_1 = \left[\begin{array}{c|c} \frac{1}{\sqrt{\lambda}} c'(1, \lambda) & s'(1, \lambda) - 1 \\ \hline -1 + c(1, \lambda) & \sqrt{\lambda} s(1, \lambda) \end{array} \right].$$

Example 2. Consider a star graph with root r and vertices u, v and w . Let e_1, e_2 and e_3 point from u, v and w to r , respectively. We choose to index rows and columns by $r, u, v, w, e_1, e_2, e_3$, in that order. Then the matrix $M = M_2$ is:

$$\left[\begin{array}{cccc|ccc} 0 & \frac{1}{\sqrt{\lambda}} c'_1(1, \lambda) & \frac{1}{\sqrt{\lambda}} c'_2(1, \lambda) & \frac{1}{\sqrt{\lambda}} c'_3(1, \lambda) & s'_1(1, \lambda) & s'_2(1, \lambda) & s'_3(1, \lambda) \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline -1 & c_1(1, \lambda) & 0 & 0 & \sqrt{\lambda} s_1(1, \lambda) & 0 & 0 \\ -1 & 0 & c_2(1, \lambda) & 0 & 0 & \sqrt{\lambda} s_2(1, \lambda) & 0 \\ -1 & 0 & 0 & c_3(1, \lambda) & 0 & 0 & \sqrt{\lambda} s_3(1, \lambda) \end{array} \right],$$

with determinant $\frac{1}{\sqrt{\lambda}} \sum_{j=1}^3 c'_j(1, \lambda) \prod_{p \neq j} c_p(1, \lambda)$ (corresponding to formula (5) of Pivovarchik [29]).

The elements of M have the following asymptotics for $\lambda = k^2 \pi^2 + d + o(1)$ (see Eq. (2.3) [8] or Lemma 3.1 [25]):

$$\frac{1}{\sqrt{\lambda}} c'_j(1, \lambda) = (-1)^k \frac{1}{2\sqrt{\lambda}} \left(\int_0^1 q_j - d \right) + o\left(\frac{1}{\sqrt{\lambda}}\right), \quad (2.5)$$

$$s'_j(1, \lambda) = (-1)^k + o\left(\frac{1}{\sqrt{\lambda}}\right), \quad (2.6)$$

$$c_j(1, \lambda) = (-1)^k + o\left(\frac{1}{\sqrt{\lambda}}\right), \quad (2.7)$$

$$\sqrt{\lambda} s_j(1, \lambda) = (-1)^k \frac{1}{2\sqrt{\lambda}} (d - \int_0^1 q_j) + o\left(\frac{1}{\sqrt{\lambda}}\right). \quad (2.8)$$

Remark. Using these asymptotics, we get $\det M_1 = c'(1, \lambda) s(1, \lambda) + o(\frac{1}{\lambda})$ for k even (using the Wronskian would yield only $o(\frac{1}{\sqrt{\lambda}})$), and $\det M_2 = \frac{1}{\sqrt{\lambda}} \sum_{j=1}^3 c'_j(1, \lambda) + o(\frac{1}{\sqrt{\lambda}})$, for all k . These are special cases of (2.9) below.

Lemma 2.1. *If $\lambda = (2k)^2 \pi^2 + O(1)$, or $\lambda = k^2 \pi^2 + O(1)$ and G is bipartite, then every $|V| \times |V|$ submatrix of C (and of B) has determinant at most $o(\frac{1}{\sqrt{\lambda}})$.*

Proof. Leaving out the $o(\frac{1}{\sqrt{\lambda}})$ terms from the submatrix we make only $o(\frac{1}{\sqrt{\lambda}})$ error in its determinant. What we get is an incidence matrix of a graph with $|V|$ vertices and $|V|$ edges. This must contain a cycle, hence the corresponding rows are dependent. The proof for B is similar. $\square$

Lemma 2.2. *The determinant of M is $O(\lambda^{-\frac{1}{2}(|E|-|V|+2)})$.*

Proof. Look at the terms in the Leibniz formula for the determinant.

Taking $(|V| - 1)$ factors from B (and consequently from C) we have to take $(|E| - |V| + 2)$ factors of magnitude $O(\frac{1}{\sqrt{\lambda}})$ from A and D ; otherwise, we get smaller terms, using the previous lemma. $\square$

The next statement is a variant of the Matrix Tree Theorem (Kirchhoff [22]; see also Lovász [26], p. 252, Bollobás [4], Theorem II.12, Tutte [32], Theorem VI.29).

Theorem 2.3. *If $\lambda = k^2\pi^2 + O(1)$ and k is even or G is bipartite, then*

$$\det M = (-1)^{k|V|} \sum_{\tau} \left(\frac{1}{\sqrt{\lambda}} \sum_{e_j \in G} c'_j(1, \lambda) \right) \prod_{e_j \notin \tau} \sqrt{\lambda} s_j(1, \lambda) + o(\lambda^{-\frac{|E|-|V|+2}{2}}). \quad (2.9)$$

where the sum is taken for all spanning trees τ of G .

Proof. The main terms in the Leibniz formula for the determinant are those which contain exactly $(|E| - |V| + 1)$ elements from D . The product of a fixed set of $(|E| - |V| + 1)$ elements in D is weighted by the determinant of the respective minor, with all other elements of D substituted by zero. The remaining rows in C and columns in B look like an ordered (or unordered) incidence matrix of the graph τ spanned by the remaining $(|V| - 1)$ edges for k even (or odd, respectively). If τ contains a cycle, then the determinant of the minor is $o(\frac{1}{\sqrt{\lambda}})$. Otherwise τ is a spanning tree of G ; then the determinant is $\frac{1}{\sqrt{\lambda}} \sum_{e_j \in G} c'_j(1, \lambda)$, the sum of the elements in A (plus $o(\frac{1}{\sqrt{\lambda}})$), as it follows from the next two lemmas. $\square$

Lemma 2.4. *Let τ be a spanning tree of G and R be the ordered vertex-edge incidence matrix for τ . Consider the matrix $M_1 = \begin{bmatrix} X & R \\ -R^T & Y \end{bmatrix}$, where Y is a $(|V| - 1) \times (|V| - 1)$ zero matrix and X has only one nonzero element s . Then the determinant of M_1 is s , independently of the position of the nonzero element in X .*

Proof. In the Leibniz formula every nonzero term (in fact there is only one) contains exactly one element from X , hence it is enough to prove this for $s = 1$. Let the indices of the nonzero element in X be uv . First we prove that the determinant of M_1 is independent of v . Indeed, there is a path in the tree between two arbitrary vertices, hence we can add to (or subtract from) row u the rows corresponding to the vertices of that path. Similarly, the determinant does not depend on u . Reversing an edge in G does not change the determinant; using this and by adding rows (and corresponding columns) we can assume that τ is a path. Then taking $u = |V|$, $v = 1$ and expanding the determinant from left to the right we get $((-1)^{|V|})^{|V|-1} = 1$.

Lemma 2.5. *Let G be a bipartite graph, τ a spanning tree of G and R the unordered incidence matrix for τ . Consider the matrix $M_1 = \begin{bmatrix} X & -R \\ -R^T & Y \end{bmatrix}$, where Y is a $(|V| - 1) \times (|V| - 1)$ zero matrix and X has only one nonzero element, $x_{uv} = s$, such that $(uv) \in E(G)$. Then the determinant of M_1 is $(-1)^{|V|} s$, independently of u and v .*

Proof. There is only one nonzero term in the Leibniz formula for the determinant, which contains only ± 1 's besides s , hence $\det M_1 = \pm s$. If V is a

disjoint union of V_1 and V_2 such that all edges connect V_1 to V_2 , then let us multiply by (-1) the rows in R corresponding to V_1 and the columns in R^T corresponding to V_2 , respectively. This multiplies the determinant by $(-1)^{|V|}$ leaving the nonzero element of X unchanged for $(uv) \in E(G)$. Hence the statement follows from the previous lemma. $\square$

Substituting the asymptotics (2.5)–(2.8) we get

Corollary 2.6. *If $\lambda = k^2\pi^2 + d + o(1)$ and k is even or G is bipartite, then*

$$\det M = (-1)^{k|E|} \left(\frac{1}{2\sqrt{\lambda}} \right)^{|E|-|V|+2} p(d) + o(\lambda^{-\frac{1}{2}(|E|-|V|+2)}), \quad (2.10)$$

where

$$p(d) = \sum_{\tau} \left(\sum_{e_j \in G} \int_0^1 q_j - |E|d \right) \prod_{e_j \notin \tau} (d - \int_0^1 q_j), \quad (2.11)$$

the outer sum is taken for all spanning trees τ of G .

Proof of Theorem 1.1. A point $\lambda \in \mathbb{C}$ is an eigenvalue of the problem (1.2)–(1.4) if and only if $\det M(\lambda) = 0$. Let the distinct roots of $p(d)$ be $d_1, \dots, d_l$. By the previous corollary for $\lambda = k^2\pi^2 + O(1)$ (if k is even or G is bipartite) the distinct roots of $\det M(\lambda)$ are exactly of the form $\lambda = k^2\pi^2 + d_j + o(1)$ ($1 \leq j \leq l$). As it is seen from (2.11), one of them is $\lambda = k^2\pi^2 + \frac{1}{|E|} \sum_{e_j \in G} \int_0^1 q_j + o(1)$.

It remains to prove that the total multiplicities of the eigenvalues $\lambda = k^2\pi^2 + O(1) \iff \sqrt{\lambda} = k\pi + o(1)$ are exactly $|E| - |V| + 2$. A direct calculation shows that this is true for $q = 0$. Indeed, then $\det M$ is a polynomial of $\cos \sqrt{\lambda}$ and $\sin \sqrt{\lambda}$, hence its zeros are 2π -periodic (in the bipartite case π -periodic) in $\sqrt{\lambda}$. Periodicity with $\sqrt{\lambda} = k\pi + o(1)$ implies $\sqrt{\lambda} = k\pi$ with finitely many exceptions. For $\lambda = k^2\pi^2$ A and D are zero matrices, thus the rank of M is $2(|V| - 1)$, and its nullspace is exactly $(|E| - |V| + 2)$ -dimensional.

Then consider the eigenvalue problem $-y'' + t q_j(x)y = \lambda y$ with the boundary conditions (1.3)–(1.4) for $t \in [0, 1]$. The corresponding operator $T(t)$ forms a holomorphic family and its eigenvalues $\lambda_n(t)$ and normalized eigenfunctions $g_n(t)$ can be represented by holomorphic functions of t (applying Theorems VII-1.8 and II-6.1 from Kato's book [21]).

$$\lambda'_n(t) = \langle g_n(t), T'(t)g_n(t) \rangle = \int_G g_n^2(t) q \quad (2.12)$$

is bounded by $\|g_n\|_{\infty}^2 \|q\|_1 = O(\|q\|_1)$ (see (2.2) and the paragraph below it). Hence $|\lambda_n(1) - \lambda_n(0)| = O(1)$ and the total multiplicity of eigenvalues $\lambda = k^2\pi^2 + O(1)$ is the same for all $q \in L^1$. $\square$

Proof of Theorem 1.2.. According to Theorem 1.1,

$$\int_G q = \sum_{e_j \in G} \int_0^1 q_j = 0. \quad (2.13)$$

Let us denote the operator of the eigenvalue problem (1.2)–(1.4) by L .

$$\frac{\langle \varphi, L\varphi \rangle}{\langle \varphi, \varphi \rangle} \geq \lambda_0 = 0 \quad (2.14)$$

and equality holds if and only if φ is an eigenfunction of L . It follows that the constant 1 must be an eigenfunction corresponding to the eigenvalue 0. Substituting this to (1.2) gives $q(x) = 0$. $\square$

Proof of Theorem 1.3.. The equality of the effective resistances $r < 1$ implies that the number of spanning trees not containing a fixed edge is the same nonzero integer for every non-loop edge.

Suppose that $\sum_{e_j \in G} \int_0^1 q_j \neq 0$. Then by (2.11)

$$\prod_{e_j \text{ is a loop}} (d - \int_0^1 q_j) \sum_{\tau} \prod_{\substack{e_j \text{ is not a loop} \\ e_j \notin \tau}} (d - \int_0^1 q_j) = \sum_{\tau} d^{|E|-|V|+1}, \quad (2.15)$$

as loops are not in spanning trees. Hence if e_j is a loop then $\int_0^1 q_j = 0$, and the sum of the roots of the second factor must also be zero. This is $(1-r) \sum_{e_j \text{ is not a loop}} \int_0^1 q_j$ multiplied by the number of spanning trees. Thus $\sum_{e_j \in G} \int_0^1 q_j = 0$ and we can proceed as in the proof of Theorem 1.2. $\square$

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