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Correction

Correction to: Unbounded Hankel Operators and Moment Problems

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We use freely the notation of [5]. It was observed in [1] that some a priori condition on moments q_n was omitted in Theorem 1.2 of [5]. Our goal is to give a corrected version of this theorem.

Let us consider the quadratic form

$$q[g,g] = \sum_{n,m\geq 0} q_{n+m} g_m \bar{g}_n \tag{1}$$

defined on a set $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$ of sequences $g = (g_0, g_1, \dots)$ with only a finite number of nonzero components. We suppose that

$$q_n = \int_{-\infty}^{\infty} \mu^n dM(\mu), \quad \forall n = 0, 1, \dots, \tag{2}$$

with a non-negative measure $dM(\mu)$ on $\mathbb{R}$ satisfying the condition

$$\int_{-\infty}^{\infty} |\mu|^n dM(\mu) < \infty, \quad \forall n = 0, 1, \dots. \tag{3}$$

For any interval $\Delta \subset \mathbb{R}$, we consider a class $C^\infty(\Delta; \{\varkappa_n\}) \subset C^\infty(\Delta)$ of functions satisfying the condition

$$|f^{(n)}(x)| \leq C(f)^n n! \varkappa_n^n, \quad n \geq 1, \tag{4}$$

for some sequence $\varkappa_n > 0$. This class is called quasi-analytic if, for all $f \in C^\infty(\Delta; \{\varkappa_n\})$, the conditions $f^{(n)}(x_0) = 0$ for some $x_0 \in \Delta$ and all $n \in \mathbb{Z}_+$ imply that $f(x) = 0$ for all $x \in \Delta$. It was shown by Carleman (see, e.g., his book [2] or the paper [3]) that the class $C^\infty(\Delta; \{\varkappa_n\})$ is quasi-analytic if and only if the condition

$$\sum_{n\geq 2} \gamma_n^{-1} = \infty \quad \text{where} \quad \gamma_n = \inf_{m\geq n} m\varkappa_m, \tag{5}$$

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is satisfied. Of course, this condition holds if

$$\sum_{n\geq 2}(n\varkappa_n)^{-1}=\infty. \tag{6}$$

Obviously, analytic functions belong to the class $C^\infty(\Delta;\{\varkappa_n\})$ provided $\varkappa_n \geq 1$. If $\varkappa_n = const$, then this class consists of analytic functions. In the cases $\varkappa_n = \varkappa_0 \ln n$, $\varkappa_n = \varkappa_0 \ln n \ln(\ln n)$, etc., estimates (4) are known as the Denjoy conditions.

Let us now give a corrected version of Theorem 1.2 of [5].

Theorem 1. *Let the moments q_n be defined by (2). Then the following conditions are equivalent:*

(i) *The form $q[g,g]$ defined on $\mathcal{D}$ by (1) is closable in $\ell^2(\mathbb{Z}_+)$ and*

$$q_{2n}\leq (n!)^2\varkappa_n^{2n},\quad n\geq 1, \tag{7}$$

for some sequence $\varkappa_n \geq 1$ obeying condition (5).

(ii) *The matrix elements $q_n \to 0$ as $n\to\infty$.*

(iii) *The measure $dM(\mu)$ defined by equations (2) satisfies the condition*

$$M(\mathbb{R}\backslash(-1,1))=0 \tag{8}$$

(*to put it differently,* $\operatorname{supp} M\subset[-1,1]$ *and* $M(\{-1\})=M(\{1\})=0$).

Remark 2. (i) In the previous version of this paper [5], condition (7) was omitted. It was pointed out in [1] that, without some kind of an a priori assumption, the closability of $q[g,g]$ does not imply (ii) or (iii).

(ii) A priori conditions (5), (7) permit very rapid growth of the moments q_n as $n\to\infty$, for example, as $(n\ln n)^n$. However for closable forms $q[g,g]$, we prove that $q_n\to 0$ as $n\to\infty$.

(iii) Let the Carleman condition

$$\sum_{n\geq 1}q_{2n}^{-1/(2n)}=\infty \tag{9}$$

be satisfied. In accordance with (7) set

$$\varkappa_n=(n!)^{-1/n}q_{2n}^{1/(2n)}.$$

It follows from the Stirling formula that

$$\lim_{n\to\infty}\frac{q_{2n}^{1/(2n)}}{n\varkappa_n}=e^{-1},$$

and hence condition (9) implies (6).

As far as the proof of Theorem 1 is concerned, we note that only the implication

$$(i)\Longrightarrow(ii)\text{ or }(iii) \tag{10}$$

is sufficiently non-trivial. The proof of this statement is practically the same as that of Theorem 1.2 in [5] if condition (7) is properly taken into account. Below we repeat this proof with necessary modifications.

Let $L^2(M) = L^2(\mathbb{R}; dM)$ be the space of functions $u(\mu)$ with the norm $\|u\|_{L^2(M)}$. Observe that under assumption (3) for an arbitrary $u \in L^2(M)$, all the integrals

$$\int_{-\infty}^{\infty} u(\mu)\mu^n dM(\mu) =: u_n, \quad n \in \mathbb{Z}_+,$$

are absolutely convergent. We denote by $\mathcal{D}_* \subset L^2(M)$ the set of all $u \in L^2(M)$ such that the sequence $\{u_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{Z}_+)$. We use the following result which combines Lemmas 2.1 and 2.2 of [5].

Lemma 3. *The form $q[g, g]$ defined on $\mathcal{D}$ is closable in the space $\ell^2(\mathbb{Z}_+)$ if and only if the set $\mathcal{D}_*$ is dense in $L^2(M)$.*

Thus, for the proof of (10), we only have to check that if the set $\mathcal{D}_*$ is dense in $L^2(M)$ and condition (7) is satisfied, then relation (8) holds.

For an arbitrary $u \in L^2(M)$, we put

$$f(x) = \int_{-\infty}^{\infty} e^{i\mu x} u(\mu) dM(\mu), \quad x \in \mathbb{R}. \tag{11}$$

Then, for all $n \in \mathbb{Z}_+$, we have

$$f^{(n)}(x) = i^n \int_{-\infty}^{\infty} e^{i\mu x} \mu^n u(\mu) dM(\mu) \tag{12}$$

and hence, by the Schwarz inequality,

$$|f^{(n)}(x)| \leq \|u\|_{L^2(M)} \sqrt{q_{2n}}. \tag{13}$$

It now follows from condition (7) that the function $f \in C^\infty(\mathbb{R}; \{\varkappa_n\})$.

Assume now that $u \in \mathcal{D}_*$. Then according to formula (12) for $x = 0$ the sequence $f^{(n)}(0)$ is bounded and hence the function

$$\widetilde{f}(z) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{14}$$

is entire and satisfies the estimate

$$|\widetilde{f}(z)| \leq C_0 \sum_{n=0}^{\infty} \frac{1}{n!} |z|^n = C_0 e^{|z|}, \quad z \in \mathbb{C}, \quad C_0 = \max_{n \geq 0} |f^{(n)}(0)|. \tag{15}$$

Since $\widetilde{f}^{(n)}(0) = f^{(n)}(0)$ for all $n \in \mathbb{Z}_+$ and both functions $\widetilde{f}(x)$ and $f(x)$ belong to the class $C^\infty(\Delta; \{\varkappa_n\})$ for any bounded interval $\Delta \subset \mathbb{R}$, they coincide on Δ and hence for all $x \in \mathbb{R}$. Using the Phragmén–Lindelöf principle, it is easy to deduce from estimates (13) for $\widetilde{f}(x)$ and (15) that

$$|\widetilde{f}(z)| \leq C e^{|\operatorname{Im} z|}, \quad z \in \mathbb{C}, \tag{16}$$

for some $C > 0$.

According to the Paley–Wiener theorem (see, e.g., Theorem IX.12 of [4]), it follows from estimate (16) that the Fourier transform of $\widetilde{f}(x)$ (considered as a distribution in the Schwartz class $\mathcal{S}'(\mathbb{R})$) is supported by the interval $[-1,1]$. Therefore formula (11) for $f(x) = \widetilde{f}(x)$ implies that for every $u \in \mathcal{D}_*$, the distribution $u(\mu)dM(\mu)$ is also supported by $[-1,1]$, that is

$$\int_{-\infty}^{\infty} \varphi(\mu)u(\mu)dM(\mu) = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}\setminus[-1,1]). \tag{17}$$

If $\mathcal{D}_*$ is dense in $L^2(M)$, we can approximate 1 by functions $u \in \mathcal{D}_*$ in this space. Hence equality (17) is true with $u(\mu) = 1$. It follows that

$$\operatorname{supp} M \subset [-1,1]$$

because $\varphi \in C_0^\infty(\mathbb{R}\setminus[-1,1])$ is arbitrary. Since, as shown in [5] $M(\{-1\}) = M(\{1\}) = 0$, this concludes the proof of relation (8).

Remark 4. The condition (7) in Theorem 1 can be replaced by an estimate

$$\int_{-\infty}^{\infty} e^{2\epsilon|\mu|} dM(\mu) < \infty$$

for some $\epsilon > 0$. In this case the function $f(z)$ given by (11) is analytic and bounded in the strip $|\operatorname{Im} z| < \epsilon$. Therefore the functions $\widetilde{f}(z)$ (defined by (14)) and $f(z)$ coincide as analytic functions so that the theory of quasi-analytic functions is not required.

Finally, we note that, in Propositions 4.1 and 4.3, the phrase "(or, equivalently, the form (1) is closable)" should be replaced by "(or, equivalently, the form (1) is closable and condition (7) is satisfied)".

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