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Integral Equations and Operator Theory

Characterizations of Centrality in C^*-algebras via Local Monotonicity and Local Additivity of Functions

Gergő Nagy

Abstract. In this paper, we give two characterizations of central elements in a C^*-algebra $\mathcal{A}$ in terms of local properties of maps on $\mathcal{A}$ given by the function calculus. We prove that for a strictly convex increasing function f defined on an open interval which is unbounded from above, an element $a \in \mathcal{A}$ is central if and only if f is locally monotone at a. That result significantly improves similar theorems by Ogasawara, Pedersen, Wu, Molnár and Virosztek. An analogous statement on local additivity is also presented.

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1. Introduction and Statement of the Main Results

Extension of functions has a considerable literature and it gives ways to plug elements in them which originally did not belong to their domains. One such way is provided by the function calculi which were developed to provide the possibility of plugging elements of abstract structures in scalar functions. Among them, the continuous function calculus can be used to plug normal elements of a C^*-algebra in complex-valued continuous functions defined on the spectra of such elements. It is a natural question whether certain properties of such functions are preserved, meaning that they still possess them on subsets of those operators. One of the most basic property of scalar functions is monotonicity, which is investigated on the space of self-adjoint elements of a C^*-algebra. It turned out that for a large class of monotone functions, quite surprisingly, the preservation of this property can occur only when the underlying algebra is commutative. This shows that the monotonicity of certain maps on C^*-algebras defined by the continuous function calculus gives us important information about the algebraic structure of the algebra considered.

In what follows, we mention some results on the connection between the commutativity of a C^*-algebra and the property of a monotone function that its extension to a set of self-adjoint elements of that algebra is also monotone. Ogasawara [3] proved that a C^*-algebra is commutative exactly when squaring is monotone on the set of its positive elements. Next, Pedersen [4] extended this result to power functions with exponent greater than 1. Continuing this line of investigations, Wu [8] showed that the exponential function can also be used to characterize commutative C^*-algebras in the above way, and Ji and Tomiyama [1] determined a large class of functions having this property.

In these results, the monotonicity property of the functions in question yields consequences on the underlying C^*-algebra. It also happens that monotonicity on sets of self-adjoint elements has implications on those functions. An important example of such cases is the one where the considered function f is defined on an interval D, nonnegative-valued and operator monotone, meaning that $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators A, B on a Hilbert space with spectra in D. Such maps are widely investigated and have significant applications in the theory of operator means. They have very strong regularity properties, for example each of them has an analytic continuation to the upper half plane. This shows that in the case of such functions f, the monotonicity property of the map $A \mapsto f(A)$ on the set of self-adjoint elements of the C^*-algebra of bounded operators on a Hilbert space has considerable consequences on f.

Recently, "localized" versions of the results mentioned in the second paragraph were also published. First, Molnár [2] has proved that a self-adjoint element a of a C^*-algebra is central (,i.e., it commutes with all of its members) exactly when the exponential function is locally monotone at a. Then Virosztek [6] has succeeded in finding a quite large class of functions (including the power functions with exponent greater than 1 and the exponential function) which have the property that central elements can be characterized via their local monotonicity at such points. The main aim of this paper is to considerably extend that result in order to cover all strictly convex increasing functions.

Investigations of the above type were carried out also concerning convexity. The study of relations between the commutativity of a C^*-algebra (or locally, the centrality of an element) and the global (or local) convexity of some functions is of interest, too. Here, we mention a corresponding result [5, Thm. 4.] stating that a C^*-algebra $\mathcal{A}$ is commutative exactly when one has a convex function $f\colon]0, \infty[\to \mathbb{R}$ such that the map $A \mapsto f(A)$ is not convex on the cone of complex positive definite 2×2 matrices, but it is so on the set of such elements of $\mathcal{A}$. Recently, Virosztek [7] has shown that a quite large class of functions possesses the property that each member of that class is locally convex exactly at central elements.

Now we turn to our results. Before we present them, we shall introduce some definitions and notation that will be used in the paper. Let $\mathcal{A}$ be a C^*-algebra, denote by $\mathcal{A}_{sa}$ the space of all self-adjoint elements of $\mathcal{A}$ and let $\sigma(a)$ stand for the spectrum of an element $a \in \mathcal{A}$. We say that a is positive if it is self-adjoint and $\sigma(a) \subset [0, \infty)$. The usual order $\leq$ on $\mathcal{A}_{sa}$ is defined as follows: $a \leq b$ if $b-a$ is positive $(a, b \in \mathcal{A}_{sa})$. Let f be a real-valued continuous function defined on some interval $D \subset \mathbb{R}$ which is unbounded from above and $a \in \mathcal{A}_{sa}$ be an element for which $\sigma(a) \subset D$. Then we say that f is locally monotone at a, in other words a is a "point of operator monotonicity" of f exactly when for any $b \in \mathcal{A}_{sa}$ satisfying $a \leq b$, one has $f(a) \leq f(b)$. The identity operator on a complex Hilbert space $\mathcal{H}$ is denoted by I. Now our first result follows in which we establish that among the self-adjoint elements of the C^*-algebra $\mathcal{C}(\mathcal{H})$ of all compact operators on $\mathcal{H}$, central elements (which are easily seen to be the scalar operators in $\mathcal{C}(\mathcal{H})_{sa}$) can be characterized in terms of a weakened version of this property.

Proposition 1. *Let D be an open interval which is unbounded from above and $f\colon D \to \mathbb{R}$ be a strictly convex increasing function. Then for any operator $A \in \mathcal{C}(\mathcal{H})_{sa}$ with $\sigma(A) \subset D$ one has that*

$$\begin{aligned} A \in \mathbb{R}I \iff \forall\ X \in \mathcal{C}(\mathcal{H})_{sa},\ \text{rank}\ (X - A) = 1,\ A \leq X : \\ f(A) \leq f(X). \end{aligned}$$

Consequently, if $\dim \mathcal{H} = \infty$*, then each statement in this relation is equivalent to the equality $A = 0$.*

As for this result and the next two, we remark that, as it is well-known, real convex functions are continuous on the interior of their domain. For the case of boundary points of the domains, we mention that, as it can be seen very easily from Sect. 2, the openness of D can be omitted from those statements provided that continuity of f is assumed on $D \cap \partial D$. The general result on local monotonicity in C^*-algebras reads as follows.

Theorem 1. *Let D be an open interval which is unbounded from above and $f\colon D \to \mathbb{R}$ be a strictly convex increasing function. Then for any element $a \in \mathcal{A}_{sa}$ satisfying $\sigma(a) \subset D$, we have that*

$$a \text{ is central} \iff f \text{ is locally monotone at } a.$$

Clearly, this theorem is a considerable extension and unification of the similar results mentioned in the second and fourth paragraphs. In fact, the most general among them is that of Virosztek [6] which states the same conclusion as the one in Theorem 1 with the additional hypotheses that f is continuously differentiable and its derivative is logarithmically concave. Theorem 1 has the following global version which establishes a characterization of commutativity of C^*-algebras in terms of the order $\leq$.

Corollary 1. *Let D be an open interval which is unbounded from above and $f\colon D \to \mathbb{R}$ be a strictly convex increasing function. Then $\mathcal{A}$ is commutative exactly when the map $a \mapsto f(a)$ $(a \in \mathcal{A}_{sa},\ \sigma(a) \subset D)$ is monotone.*

In the particularly important special case where $\mathcal{A}$ is the operator algebra $\mathcal{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$, this corollary tells us that there are no functions f which satisfy its conditions and the second assertion in the latter equivalence.

In the following results, analogously to local monotonicity, we consider local additivity of functions at points in C^*-algebras. The corresponding definition reads as follows. We say that the continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is locally additive at the element $a \in \mathcal{A}_{sa}$ provided that $f(a+x) = f(a) + f(x)$ for all elements $x \in \mathcal{A}_{sa}$. In what follows, we state that among the operators in $\mathcal{C}(\mathcal{H})_{sa}$, the central elements are distinguished in terms of a weakened form of this property.

Proposition 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonaffine function which is continuous at 0 in the case $\dim \mathcal{H} = \infty$ and $A \in \mathcal{C}(\mathcal{H})_{sa}$ be an operator such that $f(\alpha+x) = f(\alpha) + f(x)$ for all numbers $\alpha \in \sigma(A),\ x \in \mathbb{R}$. Then we have that*

$$A \in \mathbb{R}I \iff \forall\ X \in \mathcal{C}(\mathcal{H})_{sa},\ \text{rank}\ X = 1:\ f(A+X) = f(A) + f(X)$$

Therefore in the case $\dim \mathcal{H} = \infty$, both assertions in the latter relation are equivalent to the equality $A = 0$.

We remark that compact operators can only have 0 as the accumulation point of their spectra, and the element $g(Y)$ is well-defined for any $Y \in \mathcal{C}(\mathcal{H})_{sa}$ and for any complex-valued function g which is continuous on $\sigma(Y)$. Our statement concerning local additivity on C^*-algebras is

Theorem 2. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a nonaffine continuous function and $a \in \mathcal{A}_{sa}$ is an element such that for each numbers $\alpha \in \sigma(a),\ x \in \mathbb{R}$, we have $f(\alpha + x) = f(\alpha) + f(x)$. Then*

$$a \text{ is central} \iff f \text{ is locally additive at } a.$$

Observe that a real-valued continuous affine function is locally additive at a self-adjoint element of a C^*-algebra exactly when it is linear, so it is clear that the condition on nonaffinity in the last two results is indispensable. As for a corresponding global version of Theorem 2, we remark that, as it is well-known, continuous additive real functions are scalar multiples of the identity. This implies that for a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ fixed arbitrarily, the map $a \mapsto f(a)$ $(a \in \mathcal{A}_{sa})$ is additive exactly when $f(x) = cx$ $(x \in \mathbb{R})$ with some number $c \in \mathbb{R}$. We immediately conclude that Theorem 2 does not possess a corresponding global version.

2. Proofs

This section is devoted to the verification of our results formulated in the previous one. We use the following notation in it. For any vectors u, v in a Hilbert space $\mathcal{H}$, the symbol $u \otimes v$ stands for the map $z \mapsto \langle z, v\rangle u$ $(z \in \mathcal{H})$, moreover Tr denotes the trace of matrices. In the next four statements (Lemmas 1–3 and Corollary 2), we shall use the following notation and assumptions.

(*) Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} = 2$ and pick an operator $A \in \mathcal{B}(\mathcal{H})_{sa}$ with $\sigma(A) = \{a_1, a_2\}$. Let $u, v \in \mathcal{H}$ be orthogonal normalized eigenvectors of A, and define $B = (u+v) \otimes (u+v)$. For any number $x \in \mathbb{R} \setminus \{0\}$, set

$$\lambda_1(x) = \frac{a_1 + a_2 + 2x - \sqrt{(a_1 - a_2)^2 + 4x^2}}{2},$$

$$\begin{aligned}
\lambda_2(x) &= \frac{a_1 + a_2 + 2x + \sqrt{(a_1 - a_2)^2 + 4x^2}}{2}, \\
\alpha(x) &= (f(\lambda_2(x)) - f(\lambda_1(x))) \left(\frac{a_1 - a_2}{2(\lambda_2(x) - \lambda_1(x))} + \frac{1}{2} \right) + f(\lambda_1(x)), \\
\beta(x) &= x \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, \\
\gamma(x) &= (f(\lambda_2(x)) - f(\lambda_1(x))) \left(\frac{a_2 - a_1}{2(\lambda_2(x) - \lambda_1(x))} + \frac{1}{2} \right) + f(\lambda_1(x)). \quad (1)
\end{aligned}$$

In this section, we will use the identity

$$\lambda_2(x) - \lambda_1(x) = \sqrt{(a_1 - a_2)^2 + 4x^2}. \quad (2)$$

In our first lemma, we present a useful formula for functions of the sum of A and a scalar multiple of B.

Lemma 1. *Let D be an interval which is unbounded from above, $f\colon D \to \mathbb{R}$ be a function and $x \in \mathbb{R} \setminus \{0\}$ be a number such that $x > 0$, if $D \neq \mathbb{R}$. Then with the notation and conditions in* (*)*, in the case $\sigma(A) \subset D$,*

$$f(A + xB) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \beta(x) & \gamma(x) \end{pmatrix}.$$

Proof. One can check that the matrix of $A + xB$ with respect to the basis $\{u, v\}$ is

$$\begin{pmatrix} x + a_1 & x \\ x & x + a_2 \end{pmatrix}$$

and that its eigenvalues are $\lambda_1(x), \lambda_2(x)$. The function

$$e\colon t \mapsto \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}(t - \lambda_1(x)) + f(\lambda_1(x))$$

equals f on $\sigma(L(x))$, thus $f(L(x)) = e(L(x))$. Now using (2), the assertion in Lemma 1 follows very easily. □

The statement below has a crucial role in verifying the main results on local monotonicity.

Lemma 2. *Let D be an open interval which is unbounded from above and $f\colon D \to \mathbb{R}$ be a strictly convex increasing function. Then with the notation and conditions in* (*)*, we have that the hypotheses $\sigma(A) \subset D$, $A \notin \mathbb{R}I$ imply the existence of elements $t_0 > 0$, $w \in \mathcal{H}$ for which $\|w\| = 1$, the quantity*

$$\Delta(a_1, a_2) = \langle f(A)w, w\rangle - \langle f(A + t_0 B)w, w\rangle \quad (3)$$

is positive and t_0, $\Delta(a_1, a_2)$ depend only on $\sigma(A)$.

Proof. In this argument, we mainly identify operators on $\mathcal{H}$ and elements of $\mathcal{H}$ with their matrices and coordinate row vectors, respectively with respect to the basis $\{u, v\}$. We are going to show that there is a number $t_0 > 0$ such that $\det(f(A + t_0 B) - f(A)) < 0$. In order to do so, select an arbitrary scalar $x > 0$. Then using Lemma 1 and the notation in (1), we compute

$$\begin{aligned}\det(f(A+xB)-f(A)) &= (\alpha(x)-f(a_1))(\gamma(x)-f(a_2))-\beta(x)^2\\ &= \left[\left(1+\frac{a_1-a_2}{\lambda_2(x)-\lambda_1(x)}\right)\frac{f(\lambda_2(x))}{2}+(a_2-a_1)\frac{f(\lambda_1(x))}{2(\lambda_2(x)-\lambda_1(x))}\right.\\ &\quad \left.+\left(\frac{f(\lambda_1(x))}{2}-f(a_1)\right)\right]\left[\left(1+\frac{a_2-a_1}{\lambda_2(x)-\lambda_1(x)}\right)\frac{f(\lambda_2(x))}{2}\right.\\ &\quad \left.+(a_1-a_2)\frac{f(\lambda_1(x))}{2(\lambda_2(x)-\lambda_1(x))}+\left(\frac{f(\lambda_1(x))}{2}-f(a_2)\right)\right]\\ &\quad -\frac{x^2}{(\lambda_2(x)-\lambda_1(x))^2}(f(\lambda_2(x))^2-2f(\lambda_1(x))f(\lambda_2(x))\\ &\quad +f(\lambda_1(x))^2).\end{aligned}$$

By performing the multiplications in this difference and grouping the obtained terms with respect to the exponent $i \in \{0,1,2\}$ in their factors of the form $f(\lambda_2(x))^i$, we see that the ones with $i=2$ cancel. Upon merging the ones with $i=1$ and $i=0$, we get two terms which are denoted by $S_1(x)$ and $S_0(x)$, respectively. Defining

$$\begin{aligned}s_1(x) = &\frac{(a_1-a_2)^2 f(\lambda_1(x))}{2(\lambda_2(x)-\lambda_1(x))^2}\\ &+\frac{1}{2}\left[\left(1+\frac{a_1-a_2}{\lambda_2(x)-\lambda_1(x)}\right)\left(\frac{1}{2}f(\lambda_1(x))-f(a_2)\right)\right.\\ &\left.+\left(1+\frac{a_2-a_1}{\lambda_2(x)-\lambda_1(x)}\right)\left(\frac{1}{2}f(\lambda_1(x))-f(a_1)\right)\right]\\ &+2f(\lambda_1(x))\frac{x^2}{(\lambda_2(x)-\lambda_1(x))^2},\end{aligned}$$

it is straightforward to see that

$$S_1(x) = s_1(x)f(\lambda_2(x)), \tag{4}$$

$$\begin{aligned}S_0(x) = &\left((a_2-a_1)\frac{f(\lambda_1(x))}{2(\lambda_2(x)-\lambda_1(x))}+\frac{1}{2}f(\lambda_1(x))-f(a_1)\right)\\ &\cdot\left((a_1-a_2)\frac{f(\lambda_1(x))}{2(\lambda_2(x)-\lambda_1(x))}+\frac{1}{2}f(\lambda_1(x))-f(a_2)\right)\\ &-\frac{x^2}{(\lambda_2(x)-\lambda_1(x))^2}f(\lambda_1(x))^2, \det(f(A+xB)-f(A))=S_1(x)+S_0(x).\end{aligned} \tag{5}$$

Now we are going to examine the limit $\lim_{x\to\infty}(S_1(x)+S_0(x))$. To this end, using (2) and the continuity of f, we compute

$$\lim_{x\to\infty}(\lambda_2(x)-\lambda_1(x)) = \infty,\ \lim_{x\to\infty}\frac{x^2}{(\lambda_2(x)-\lambda_1(x))^2}=\frac{1}{4},$$
$$\lim_{x\to\infty} f(\lambda_1(x)) = f\left(\frac{a_1+a_2}{2}\right).$$

Then we see that

$$\lim_{x\to\infty} S_0(x) = \left(\frac{1}{2}f\left(\frac{a_1+a_2}{2}\right) - f(a_1)\right)\left(\frac{1}{2}f\left(\frac{a_1+a_2}{2}\right) - f(a_2)\right) - \frac{1}{4}f\left(\frac{a_1+a_2}{2}\right)^2, \quad \lim_{x\to\infty} s_1(x) = f\left(\frac{a_1+a_2}{2}\right) - \frac{f(a_1)+f(a_2)}{2}. \tag{6}$$

Now we distinguish the two possible cases concerning the limit of the increasing function f at ∞. In the first one, suppose that it is ∞. Then clearly, $\lim_{x\to\infty} f(\lambda_2(x)) = \infty$. Since A is not a scalar matrix, $a_1 \neq a_2$, and therefore by the strict convexity of f and (6), $\lim_{x\to\infty} s_1(x) < 0$. Then it follows that $\lim_{x\to\infty} S_1(x) = -\infty$ which, together with (6), gives us that $\lim_{x\to\infty}(S_1(x) + S_0(x)) = -\infty < 0$.

Suppose now that $\lim_{x\to\infty} f(x) < \infty$. In this case, by considering $f - \lim\limits_{x\to\infty} f(x)$ instead of f, we may and do assume that

$$\lim_{x\to\infty} f(x) = 0. \tag{7}$$

Observe that regarding the conditions and conclusions in Lemma 2, there is no loss of generality in this assumption. Then by (4) and (6), we deduce that

$$\lim_{x\to\infty}(S_1(x) + S_0(x)) = \frac{1}{4}\left(2f(a_1)f(a_2) - f\left(\frac{a_1+a_2}{2}\right)(f(a_1) + f(a_2))\right)$$

The assumption (7) and the condition that f is increasing imply that $f \leq 0$. Since f is strictly convex, then it follows that

$$\lim_{x\to\infty}(S_1(x) + S_0(x)) \leq \frac{1}{8}\left(4f(a_1)f(a_2) - (f(a_1)+f(a_2))^2\right) = -\frac{1}{8}(f(a_1) - f(a_2))^2.$$

Using the plain fact that strictly convex increasing functions are injective and the relation $a_1 \neq a_2$, which is due to the condition $A \notin \mathbb{R}I$, we deduce from the last displayed chain of equalities that $\lim_{x\to\infty}(S_1(x) + S_0(x)) < 0$ in this case, too. This inequality and (5) yield that $\lim_{x\to\infty} \det(f(A+xB) - f(A)) = -\infty$, hence we conclude that there is a number $t_0 > 0$ for which

$$\det(f(A + t_0 B) - f(A)) < 0, \tag{8}$$

implying that $f(A + t_0B) - f(A)$ is not positive semidefinite in both cases.

To sum up, we have shown that, identifying A with the matrix $\mathrm{diag}(a_1, a_2)$, for the latter matrix, one has a scalar $t_0 \in \mathbb{R} \setminus \{0\}$ and a vector $w \in \mathbb{C}^2$ such that the quantity in (3) is positive. Moreover, clearly, that quantity depends only on w, $\mathrm{diag}(a_1, a_2)$ and t_0. But the last three objects depend only on $\sigma(A)$, thus so does $\Delta(a_1, a_2)$. Now the statement of Lemma 2 follows easily. □

We have seen in this proof that if the conditions of that result hold, then (8) is satisfied by all large enough numbers. Therefore, it is straightforward to see that the following statement is valid, which we think interesting on its own right.

Corollary 2. *Let D be an open interval which is unbounded from above and $f: D \to \mathbb{R}$ be a strictly convex increasing function. Then with the notation and conditions in (*), if $\sigma(A) \subset D$ and for each number $t > 0$ there is a* *scalar $t_0 > t$ satisfying that $f(A+t_0B)$ and $f(A)$ are comparable with respect to the order $\leq$, then $A \in \mathbb{R}I$.*

The next assertion is the key tool for the proofs of our results concerning local additivity.

Lemma 3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonaffine function. Then with the notation and conditions in (*), one has that the relation $A \notin \mathbb{R}I$ implies the existence of elements $t_0 \in \mathbb{R} \setminus \{0\},\ w \in \mathcal{H}$ for which $\|w\| = 1$, the quantity*

$$\delta(a_1, a_2) = \langle (f(A) + f(t_0B))w, w\rangle - \langle f(A+t_0B)w, w\rangle \tag{9}$$

is not 0 and $t_0, \delta(a_1, a_2)$ depend only on $\sigma(A)$.

Proof. In the sequel, we identify operators on $\mathcal{H}$ and elements of $\mathcal{H}$ with their matrices and coordinate row vectors, respectively with respect to the basis $\{u, v\}$. Assume that for all admissible choices of t_0, w, the quantity in (9) is 0. We shall show that this hypothesis leads to a contradiction. In order to do so, choose an arbitrary scalar $x \in \mathbb{R} \setminus \{0\}$ and observe that the latter assumption implies

$$f(A + xB) = f(A) + f(xB).$$

Using the notation in (1) and the formula in Lemma 1, it follows easily that the left-hand and right-hand side of the latter equation is

$$L(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \beta(x) & \gamma(x) \end{pmatrix}$$

and

$$R(x) = \begin{pmatrix} (f(2x) + f(0))/2 + f(a_1) & (f(2x) - f(0))/2 \\ (f(2x) - f(0))/2 & (f(2x) + f(0))/2 + f(a_2) \end{pmatrix},$$

respectively, furthermore one clearly has

$$\operatorname{Tr} R(x) = f(2x) + f(0) + f(a_1) + f(a_2). \tag{10}$$

From the equality of the first rows of $L(x)$ and $R(x)$, we deduce

$$\begin{aligned} &(f(\lambda_2(x)) - f(\lambda_1(x)))\left(\frac{a_1 - a_2}{2(\lambda_2(x) - \lambda_1(x))} + \frac{1}{2}\right) + f(\lambda_1(x)) \\ &\quad = \frac{f(2x) + f(0)}{2} + f(a_1) \end{aligned} \tag{11}$$

and

$$x\frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)} = (f(2x) - f(0))/2. \tag{12}$$

Using (12) and (10), we compute the left hand-side of (11) in order to get

$$\begin{aligned} &(f(\lambda_2(x)) - f(\lambda_1(x)))\left(\frac{a_1 - a_2}{2(\lambda_2(x) - \lambda_1(x))} + \frac{1}{2}\right) + f(\lambda_1(x)) \\ &= (a_1 - a_2)\frac{f(2x) - f(0)}{4x} + \frac{f(\lambda_2(x)) + f(\lambda_1(x))}{2} \end{aligned}$$

$$= (a_1 - a_2)\frac{f(2x) - f(0)}{4x} + \frac{\operatorname{Tr} L(x)}{2} = (a_1 - a_2)\frac{f(2x) - f(0)}{4x} + \frac{\operatorname{Tr} R(x)}{2} = (a_1 - a_2)\frac{f(2x) - f(0)}{4x} + \frac{f(2x) + f(0) + f(a_1) + f(a_2)}{2}.$$

That side is equal to $(f(2x) + f(0))/2 + f(a_1)$ by (11). Furthermore, since $A \notin \mathbb{R}I$, one has $a_1 \neq a_2$, so we get very easily that $f(2x) - f(0) = \alpha x$ for a real number α. Since $x \in \mathbb{R} \setminus \{0\}$ was an arbitrary scalar, it follows that f is affine, a contradiction. Now we can conclude that our initial hypothesis was false, therefore there must exist a number $t_0 \in \mathbb{R} \setminus \{0\}$ and a vector $w \in \mathbb{C}^2$ such that the quantity in (9) is not 0. Finally, the proof can be completed in the same way as in the last paragraph of the verification of Lemma 2. □

Now we are in a position to prove our result concerning local monotonicity on $\mathcal{C}(\mathcal{H})$.

Proof of Proposition 1. To prove the nontrivial implication $\Longleftarrow$, assume that $A \notin \mathbb{R}I$. Then since $A \in \mathcal{C}(\mathcal{H})_{sa}$, it is straightforward to see that A has at least two different eigenvalues with a corresponding eigenvector for both. In addition, the 2-dimensional subspace $\mathcal{L} \subset \mathcal{H}$ spanned by those vectors and its orthogonal complement $\mathcal{L}^\perp$ are invariant under A. Now Lemma 2 applies to $A|_\mathcal{L}$ and we get very easily that there is a self-adjoint operator X_0 on $\mathcal{L}$ for which $A|_\mathcal{L} \leq X_0$ and $X_0 - A|_\mathcal{L}$ has rank 1, but $f(A|_\mathcal{L}) \leq f(X_0)$ does not hold. Then the element $\tilde{X}_0 \in \mathcal{C}(\mathcal{H})_{sa}$ with $\tilde{X}_0|_\mathcal{L} = X_0$, $\tilde{X}_0|_{\mathcal{L}^\perp} = A|_{\mathcal{L}^\perp}$ satisfies rank $(\tilde{X}_0 - A) = 1$, $A \leq \tilde{X}_0$, but does not fulfill the relation $f(A) \leq f(\tilde{X}_0)$. Now the proof can be completed in a trivial way. □

The statement of Proposition 2 can be shown using the same argument as above. Now we proceed with the verification of our second main result.

Proof of Theorem 1. In the sequel, we use the argument in the proof of [6, Theorem1] and we shall skip some steps of our argument which can be done in the same way as there. This applies to the verification of the implication $\Longrightarrow$ in Theorem 1. Hence, we turn to the proof of the other implication there, which will be carried out by contraposition. Accordingly, assume that a is not central. Then the mentioned argument in [6] applies and just as there, we infer that one has a complex Hilbert space $\mathcal{H}$ and an irreducible representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\sigma(\pi(a))$ has (at least) two elements, say x and y. Furthermore, for any integer $n \in \mathbb{N}$, there are orthogonal unit vectors $u_n, v_n \in \mathcal{H}$ that satisfy

$$\lim_{n\to\infty} (\pi(a)u_n - xu_n) = 0, \ \lim_{n\to\infty} (\pi(a)v_n - yv_n) = 0.$$

Now let $n \in \mathbb{N}$ be a number, $\mathcal{K}_n$ be the subspace of $\mathcal{H}$ generated by $\{u_n, v_n\}$ and E_n be the projection on $\mathcal{H}$ onto the orthogonal complement of $\mathcal{K}_n$. Define

$$\psi_n(a) = xu_n \otimes u_n + yv_n \otimes v_n + E_n \pi(a) E_n.$$

Observe that for any given number $n \in \mathbb{N}$, the set $\sigma(\psi_n(a)|_{\mathcal{K}_n})$ is the same. Therefore, Lemma 2 applies and, using the notation $B_n = (u_n + v_n) \otimes$

$(u_n + v_n)$, we infer the existence of elements $t_0 > 0$, $w_n \in \mathcal{K}_n$ for which $\|w_n\| = \sqrt{2}$ and

$$\langle f(\psi_n(A))w_n, w_n\rangle - \langle f(\psi_n(A) + t_0 B_n)w_n, w_n\rangle > 0,$$

moreover this inner product and t_0 do not depend on n. The rest of the proof is the same as the corresponding part of the argument in [6]. □

The statement of Theorem 2 can be verified using the previous proof with some trivial modifications and applying Lemma 3 instead of Lemma 2 in its corresponding part.

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Gergő Nagy (✉)
Institute of Mathematics
University of Debrecen
P.O. Box 400 Debrecen 4002
Hungary
e-mail: nagyg@science.unideb.hu

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