On the Model and Invariant Subspaces for Pairs of Commuting Isometries

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Abstract. The paper is devoted to a model and joint invariant subspaces under a pair of commuting isometries. A certain class of pairs of commuting isometries is defined. We give a model for such pairs and show that an arbitrary pair of commuting isometries has a minimal extension to a pair in the defined class. Subsequently we investigate a model for a general commuting pair of isometries via joint invariant subspaces of this extension. As an application operators of multiplication by independent variables on the Hardy space over the torus are extended to a pair in the defined class and joint invariant subspaces of the extension are described.

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1. Introduction

The Wold decomposition [18], precisely the abstract version of Halmos [6], decomposes an isometry between a unitary operator and a unilateral shift. Unitary operators are described by the spectral theorem while the result of Beurling [2] and the following extensions by Lax [8] and Halmos [6] give the model for a unilateral shift and its invariant subspaces. A single isometry is fully characterized by these results.

Neither the Wold-Halmos decomposition nor the Beurling's result generalize to pairs of isometries fully. The most natural extension of Wold's result to a pair of isometries is a quaternary decomposition among: a pair of unitary operators, a pair of unilateral shifts and two mixed pairs. A general pair of commuting isometries may not admit such a decomposition. The example of pairs satisfying it are, by the result of Słociński [17], pairs of doubly commuting isometries (i.e., operators commute and each of them commute

with the adjoint of the other operator). Hereof, the above quaternary decomposition is called the Wold-Słociński decomposition. There are non doubly commuting pairs that admit the Wold-Słociński decomposition. However, an arbitrary pair of commuting isometries decomposes with an additional fifth summand. A precise description of the additional summand remains a subject of research.

Another unknown part of a model for pairs of commuting isometries concerns pairs of unilateral shifts. A two dimensional Beurling type result provides a model for a pair of doubly commuting unilateral shifts. Moreover, according to examples of Rudin [11], the Beurling type result for pairs of doubly commuting isometries does not describe all joint invariant subspaces, but only such where the operators continue to doubly commute—see [9]. There are, however, other invariant subspaces, some of them are described by inner functions used in an extended way (cf. [4, 5, 7, 16]). Even less is known about commuting, non doubly commuting unilateral shifts. It is noteworthy that the results involving double commutativity mentioned above extend to n -tuples (cf. [14, 15]). For a general n -tuple of commuting isometries probably the best known model is the Berger-Coburn-Lebow result in [1] based on the Beurling-Lax-Halmos model for the product of the n -tuple.

Let $\mathcal{B}(H)$ denote the algebra of bounded, linear operators on a separable Hilbert space H . A subspace $H_0 \subset H$ is a closed linear manifold, $\overline{\text{span}}\{\dots\}$ denotes the closure of a linear manifold spanned by vectors or subspaces listed in brackets and P_{H_0} denotes an orthogonal projection onto H_0 (i.e., $P_{H_0}^2 = P_{H_0} = P_{H_0}^*$). Whenever a minimal extension is mentioned, the minimality refers to a space where the extension operate, as a minimal element of a chain with the inclusion order. Denote by $\text{Lat}(V_1)$, $\text{Lat}(V_2)$, $\text{Lat}(V_1, V_2)$ lattices of (joint) invariant subspaces under isometries $V_1, V_2 \in \mathcal{B}(H)$. Let us recall Wold-Halmos decomposition theorem [6, 18].

Theorem 1.1. (Wold-Halmos) *Let $V \in \mathcal{B}(H)$ be an isometry. There is a unique decomposition of H between orthogonal, reducing under V subspaces H_u, H_s , such that $V|_{H_u}$ is a unitary operator and $V|_{H_s}$ is a unilateral shift. Moreover,*

$$H_u = \bigcap_{n \geq 0} V^n H, \quad H_s = \bigoplus_{n \geq 0} V^n (\ker V^*).$$

Let $\mathbb{D} \subset \mathbb{C}$ stands for the unit disk with a boundary $\mathbb{T}$ and $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$ denote the space of scalar valued, square integrable functions and Hardy space, respectively. The space of square integrable functions valued in a separable Hilbert space $\mathcal{H}$ with the inner product induced by the norm $\|f\| = (\int_{\mathbb{T}} \|f(z)\|_{\mathcal{H}}^2 dz)^{1/2}$ is unitarily equivalent to $L^2(\mathbb{T}) \otimes \mathcal{H}$ and is denoted with this symbol. Similarly $H^2(\mathbb{T}) \otimes \mathcal{H}$ denotes the space of analytic, square integrable, $\mathcal{H}$ valued functions. Set $P_n \in \mathcal{B}(H^2(\mathbb{T}))$ for a projection onto the subspace $\mathbb{C}z^n$, and then $P_n \otimes I$ is a projection onto $z^n \otimes \mathcal{H}$ for $n \geq 0$ where z^n stands for the monomial $\{z \mapsto z^n\}$ in this context. Further, let $L^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ and $H^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ consist of $\mathcal{B}(\mathcal{H})$ valued functions and analytic functions, respectively, measurable in the sense that $\{z \mapsto \theta(z)x\}$ is a measurable function for any $x \in \mathcal{H}$ and bounded in the essential supremum

norm. It is a normed algebra with an involution where the operations are defined pointwise. Any function $\theta \in L^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ corresponds to the operator $\hat{\theta} \in \mathcal{B}(L^2(\mathbb{T}) \otimes \mathcal{H})$ defined as $(\hat{\theta}f)\{z \mapsto \theta(z)f(z)\}$. The operation " $\hat{\cdot}$ " is an algebra isomorphism onto $\hat{L} := \{\hat{\theta} : \theta \in L^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))\}$ and $\hat{H} := \{\hat{\theta} : \theta \in H^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))\}$, respectively. Operators of multiplication by independent variable $M_z \otimes I \in \mathcal{B}(L^2(\mathbb{T}) \otimes \mathcal{H})$ and $T_z \otimes I \in \mathcal{B}(H^2(\mathbb{T}) \otimes \mathcal{H})$ are models for a bilateral shift and a unilateral shift, respectively, of multiplicities equal to $\dim \mathcal{H}$. The spaces $\hat{L}, \hat{H}$ are commutants of $M_z \otimes I, T_z \otimes I$, respectively. A function $\theta \in L^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))$ is a unimodular function, if $\theta(z)$ is a partial isometry with the same initial space $\mathcal{H}_0$ for almost every $z \in \mathbb{T}$ (equivalently $\hat{\theta}$ is a partial isometry with the initial space $L^2(\mathbb{T}) \otimes \mathcal{H}_0$). If, additionally, $\theta \in H^\infty(\mathbb{T}, \mathcal{B}(\mathcal{H}))$, then it is called an inner function. Inner functions describe all subspaces invariant under a unilateral shift by the following result [6, Theorem 3].

Theorem 1.2. (Beurling-Lax-Halmos) *Subspaces invariant under the operator $T_z \otimes I \in \mathcal{B}(H^2(\mathbb{T}) \otimes \mathcal{H})$ are precisely those of the form $\hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H})$ where $\hat{\theta}$ is an inner function.*

Most of the facts recalled above are proved by P. R. Halmos [6], see also [12, 13].

Note that $\text{Lat}(V_1, V_2) = \text{Lat}(V_1) \cap \text{Lat}(V_2)$. However, the Beurling-Lax-Halmos result represents $\text{Lat}(V_i)$ by $\mathcal{B}(\ker V_i^*)$ valued inner functions. It makes $\text{Lat}(V_1)$ and $\text{Lat}(V_2)$ difficult to compare since the appropriate inner functions are disjointly valued for $\ker V_1^* \neq \ker V_2^*$. The main idea of the paper is to extend an arbitrary pair of commuting isometries (V_1, V_2) on a Hilbert space H to a pair $(\tilde{V}_1, \tilde{V}_2)$ on a Hilbert space $\tilde{H}$ of a certain form, which allows to investigate both $\text{Lat}(\tilde{V}_1)$ and $\text{Lat}(\tilde{V}_2)$ by inner function valued in the same algebra. Then the initial pair is described by the following relations: $H \in \text{Lat}(\tilde{V}_1, \tilde{V}_2)$ and $\text{Lat}(V_1, V_2) = \{H \cap \mathcal{M} : \mathcal{M} \in \text{Lat}(\tilde{V}_1, \tilde{V}_2)\} \subset \text{Lat}(\tilde{V}_1, \tilde{V}_2)$. Precisely, an arbitrary pair of commuting isometries (V_1, V_2) is extended to a pair $(U^k V^m, U^l V^n)$ where U is a unitary operator commuting with an isometry V and k, l, m, n are appropriate nonnegative integers. Then $\text{Lat}(U^k V^m, U^l V^n)$ is investigated via $\text{Lat}(V)$.

In Sect. 2 $\text{Lat}(V^m, V^n)$ is described for a unilateral shift V and m, n positive integers. In Sect. 3 an arbitrary pair of commuting isometries (V_1, V_2) is proved to have a minimal extension to a commuting pair of isometries $(\tilde{V}_1, \tilde{V}_2)$ satisfying $\tilde{V}_1^m = \tilde{U} \tilde{V}_2^n$, where $\tilde{U}$ is a unitary operator commuting with $\tilde{V}_1, \tilde{V}_2$ and m, n are positive integers. In Sect. 4 a pair of commuting isometries (V_1, V_2) satisfying $V_1^m = U V_2^n$, for m, n relatively prime, positive integers, is extended to a pair $(\tilde{U}^k \tilde{V}^m, \tilde{U}^l \tilde{V}^n)$ where $\tilde{V}$ is an isometry, $\tilde{U}$ is a unitary operator commuting with $\tilde{V}$ and $km - ln = 1$. Moreover, the extension is minimal and if V_1, V_2 are unilateral shifts, then $\tilde{V}$ is a unilateral shift. Applications are presented in the last section.

1.1. Preliminary Results

Let us remark several technical results which are used throughout the paper.

By Fuglede-Putnam theorem and [6, Theorem 1] we get:

Remark 1.3. Unitary operators commuting with a unilateral shift $T_z \otimes I \in \mathcal{B}(H^2(\mathbb{T}) \otimes \mathcal{H})$ are precisely those of the form $I \otimes \mathcal{U}$ where $\mathcal{U} \in \mathcal{B}(\mathcal{H})$ is a unitary operator.

We take advantage of the following facts:

Remark 1.4. Let (V_1, V_2) be a pair of commuting isometries and $i, j, p \in \mathbb{Z}_+$.

1. The operator $V_2^{*j} V_1^i$ commutes with V_2 if and only if $V_2^{*j} V_1^i$ is an isometry. Then $V_2^{*j} V_1^i$ also commutes with V_1 and $(V_2^{*j} V_1^i)^p = V_2^{*pj} V_1^{pi}$.
2. If $i \leq j$ then $(I - V_2^i V_2^{*i}) V_1 (I - V_2^j V_2^{*j}) = (I - V_2^i V_2^{*i}) V_1$ and consequently $((I - V_2 V_2^*) V_1)^j = (I - V_2 V_2^*) V_1^j$.

Indeed, if $V_2^{*j} V_1^i$ commutes with V_2 , then

$$(V_2^{*j} V_1^i)^* V_2^{*j} V_1^i = V_1^{*i} V_2^j V_2^{*j} V_1^i = V_1^i V_2^{*j} V_1^i V_2^j = V_1^{*i} V_2^{*j} V_2^j V_1^i = I,$$

so $V_2^{*j} V_1^i$ is an isometry.

For the reverse implication note that if $V_2^{*j} V_1^i$ is an isometry then

$$\|V_2^{*j-1} V_1^i f\| \geq \|V_2 V_2^* V_2^{*j-1} V_1^i f\| = \|V_2^{*j} V_1^i f\| = \|f\| \geq \|V_2^{*j-1} V_1^i f\|.$$

Hence $\|V_2 V_2^* V_2^{*j-1} V_1^i f\| = \|V_2^{*j-1} V_1^i f\|$. Since for any projection P , if $\|Px\| = \|x\|$, then $Px = x$ and $V_2 V_2^*$ is a projection we get $V_2 V_2^* V_1^i f = V_2^{*j-1} V_1^i f = V_2^{*j} V_1^i V_2 f$, so $V_2^{*j} V_1^i$ commutes with V_2 .

If $V_2^{*j} V_1^i$ commutes with V_2 , then

$$V_1 V_2^{*j} V_1^i = V_2^{*j} V_2^j V_1 V_2^{*j} V_1^i = V_2^{*j} V_1 V_2^{*j} V_1^i V_2^j = V_2^{*j} V_1 V_2^{*j} V_2^j V_1^i = V_2^{*j} V_1^i V_1,$$

so $V_2^{*j} V_1^i$ commutes with V_1 .

The condition $(V_2^{*j} V_1^i)^p = V_2^{*pj} V_1^{pi}$ is a consequence of the commutativity between $V_2^{*j} V_1^i$ and V_1 . We argue by induction, the base step $p = 1$ is obvious. The inductive step: if $(V_2^{*j} V_1^i)^p = V_2^{*pj} V_1^{pi}$, then

$$\begin{aligned} (V_2^{*j} V_1^i)^{p+1} &= V_2^{*j} V_1^i (V_2^{*j} V_1^i)^p = V_2^{*j} (V_2^{*j} V_1^i)^p V_1^i \\ &= V_2^{*j} V_2^{*pj} V_1^{pi} V_1^i = V_2^{*(p+1)j} V_1^{(p+1)i}. \end{aligned}$$

For the second claim note that

$$\begin{aligned} (I - V_2^i V_2^{*i}) V_1 f &= (I - V_2^i V_2^{*i}) V_1 ((I - V_2^j V_2^{*j}) f + V_2^j V_2^{*j} f) \\ &= (I - V_2^i V_2^{*i}) V_1 (I - V_2^j V_2^{*j}) f + (I - V_2^i V_2^{*i}) V_2^j V_1 V_2^{*j} f \\ &= (I - V_2^i V_2^{*i}) V_1 (I - V_2^j V_2^{*j}) f. \end{aligned}$$

The last remark concerns relatively prime numbers.

Remark 1.5. If m, n are relatively prime, positive integers, then:

- for any $i = 1, \dots, n-1$ there are unique integers k, l such that $0 < k < n$, $0 \leq l < m$ and $i = km - ln$,
- the set $A = \mathbb{Z}_+ \setminus \{km + ln : k, l \in \mathbb{Z}_+ \cup \{0\}\}$ is bounded and $mn - m - n$ is the maximal element of A .

Since m, n are relatively prime, the remainders of division of $m, 2m, \dots, nm$ by n are pairwise different. Hence they form a set of cardinality n , so the whole set $\{0, \dots, n-1\}$. Excluding nm and the corresponding reminder 0 we get that for any $i \in \{1, \dots, n-1\}$ there is a unique $0 < k < n$ such that $km = ln + i$ for some nonnegative integer l . Hence l is also unique and $0 \leq l < m$.

For the second claim, denote an integer greater than $mn - m - n$ as $mn - m - n + pn + i$ where p is a nonnegative integer and $i \in \{0, \dots, n-1\}$. By the first claim $i = km - ln$ for the appropriate k, l and $mn - m - n + pn + i = mn - m - n + pn + km - ln = (m - 1 + p - l)n + (k - 1)m$. Note that $0 < k$ and $l < m$ yield $m + p - 1 - l, k - 1 \in \mathbb{Z}_+ \cup \{0\}$. Thus the maximal element of A is at most $mn - m - n$. On the other hand, if $mn - m - n \notin A$, then $mn - m - n = km + ln$, and so $mn = (k+1)m + (l+1)n$. Since m, n are relatively prime, the last equality implies $k+1 = pn, (l+1) = qm$ for positive integers p, q . This leads to a contradiction: $mn = (k+1)m + (l+1)n = (p+q)mn > mn$ as $p+q \geq 2$.

2. Subspaces Invariant Under a Pair of Powers of a Unilateral Shift

In the section we describe $\text{Lat}(V^m, V^n)$ where V is a unilateral shift. The main tool is Beurling-Lax-Halmos characterization of $\text{Lat}(V)$. Since invariant subspaces of any unilateral shift are described by this result, we may assume m, n to be relatively prime, for if not, we replace V by $V^{GCD(m,n)}$.

We start with an obvious observation that $\text{Lat}(V) \subset \text{Lat}(V^m, V^n)$ but there is no equality in general.

Example 2.1. Let $V \in L(H)$ be a unilateral shift and $\mathcal{M} \in \text{Lat}(V)$. By [6] $\mathcal{M} \ominus V\mathcal{M}$ is a wandering subspace for $V|_{\mathcal{M}}$ and $\mathcal{M} = \bigoplus_{n \geq 0} V^n(\mathcal{M} \ominus V\mathcal{M})$. Consider $\mathcal{M}_0 = \mathcal{M} \ominus V(\mathcal{M} \ominus V\mathcal{M}) = \bigoplus_{\substack{n \geq 0 \\ n \neq 1}} V^n(\mathcal{M} \ominus V\mathcal{M})$. Then $\mathcal{M}_0 \notin \text{Lat}(V)$ but $\mathcal{M}_0 \in \text{Lat}(V^m, V^n)$ provided neither of m, n equals 1.

Obviously, if any of m, n equals 1, then $\text{Lat}(V^m, V^n) = \text{Lat}(V)$.

Corollary 2.2. *Let V be a unilateral shift. Then $\text{Lat}(V) \subset \text{Lat}(V^m, V^n)$ and they are equal if and only if $m = 1$ or $n = 1$.*

Let us investigate a correspondence between $\text{Lat}(V)$ and $\text{Lat}(V^m, V^n)$ more thoroughly.

Lemma 2.3. *Let $V \in L(H)$ be a unilateral shift and m, n be relatively prime, positive integers. For any $\mathcal{M} \in \text{Lat}(V^m, V^n)$ there is a subspace $\mathcal{N} \in \text{Lat}(V)$ such that $V^{(m-1)(n-1)}\mathcal{N} \subset \mathcal{M} \subset \mathcal{N}$.*

Proof. Observe that the minimal, invariant under V subspace containing $\mathcal{M}$ is $\mathcal{N} = \overline{\text{span}}\{V^j \mathcal{M} : j \geq 0\}$. By Remark 1.5 for any $j \geq (m-1)(n-1)$ there are $k_j, l_j \in \mathbb{Z}_+$ such that $j = k_j m + l_j n$. Hence

$$\begin{aligned}
V^{(m-1)(n-1)}\mathcal{N} &= \overline{\text{span}}\{V^j\mathcal{M} : j \geq (m-1)(n-1)\} \\
&= \overline{\text{span}}\{V^{k_j m + l_j n}\mathcal{M} : j \geq (m-1)(n-1)\} \\
&= \overline{\text{span}}\{(V^m)^{k_j}(V^n)^{l_j}\mathcal{M} : j \geq (m-1)(n-1)\} \subset \mathcal{M}. \quad \square
\end{aligned}$$

Note that different subspaces in $\text{Lat}(V^m, V^n)$ may correspond to the same subspace in $\text{Lat}(V)$ where the correspondence is in the sense of Lemma 2.3. Hence a description of $\text{Lat}(V^m, V^n)$ requires an additional invariant.

Theorem 2.4. *Let V be a unilateral shift and m, n be relatively prime, positive integers. Subspaces jointly invariant under (V^m, V^n) are of the form*

$$V^{(m-1)(n-1)}\mathcal{N} \oplus \mathcal{N}_0$$

where $\mathcal{N} \in \text{Lat}(V)$ and $\mathcal{N}_0$ is a subspace of $\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}$ invariant under $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^\alpha$ for $\alpha = m, n$.

Proof. Note that $V^{(m-1)(n-1)}\mathcal{N}$ is closed, so it is a subspace.

Let $\mathcal{M} = V^{(m-1)(n-1)}\mathcal{N} \oplus \mathcal{N}_0$ where $\mathcal{N}, \mathcal{N}_0$ are as in the statement. Then $\mathcal{M} \subset \mathcal{N}$ and $V^\alpha x = P_{\mathcal{N}}V^\alpha x = P_{V^{(m-1)(n-1)}\mathcal{N}}V^\alpha x + P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^\alpha x \in \mathcal{M}$ for $\alpha = m, n$ and $x \in \mathcal{M}$. Hence $\mathcal{M} \in \text{Lat}(V^m, V^n)$.

Let $\mathcal{M} \in \text{Lat}(V^m, V^n)$ be an arbitrary subspace and $\mathcal{N} \in \text{Lat}(V)$ be a subspace corresponding to $\mathcal{M}$ as in Lemma 2.3. The inclusions $V^{(m-1)(n-1)}\mathcal{N} \subset \mathcal{M} \subset \mathcal{N}$ yield $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}x = x - P_{V^{(m-1)(n-1)}\mathcal{N}}x \in \mathcal{M}$ for any $x \in \mathcal{M}$. Hence $\mathcal{M}$ is invariant under $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}$ (so reducing) and therefore $P_{\mathcal{M}}$ commutes with $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}$. Hence $\mathcal{N}_0 := \text{ran}(P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}P_{\mathcal{M}}) \subset \mathcal{M}$ is closed as the product $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}P_{\mathcal{M}}$ is a projection. Note that $\mathcal{M} = \mathcal{N}_0 \oplus V^{(m-1)(n-1)}\mathcal{N}$ and $V^\alpha \mathcal{N}_0 \subset V^\alpha \mathcal{M} \subset \mathcal{M}$ for $\alpha = m, n$. Consequently $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^\alpha \mathcal{N}_0 \subset P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}\mathcal{M} = \mathcal{N}_0$. $\square$

Theorem 2.4 describes $\text{Lat}(V^m, V^n)$ by $\text{Lat}(V)$ characterized by Theorem 1.2 and $\text{Lat}(P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^m, P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^n)$ which need to be characterized. The latter lattice may be easier to investigate than $\text{Lat}(V^m, V^n)$ since the operators act on a smaller space. In particular they act on a finite dimensional space for V of finite multiplicity. The other example where a description of $\text{Lat}(P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^m, P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^n)$ is easier is the case $m = 2, n = 3$.

Corollary 2.5. *For a unilateral shift V we have*

$$\text{Lat}(V^2, V^3) = \{V^2\mathcal{N} \oplus \mathcal{N}_0 : \mathcal{N} \in \text{Lat}(V), \mathcal{N}_0 \subset \mathcal{N} \ominus V^2\mathcal{N}\}.$$

Proof. Since $P_{\mathcal{N} \ominus V^2\mathcal{N}}V^\alpha|_{\mathcal{N}} = 0$ for $\alpha = 2, 3$, any subspace $\mathcal{N}_0 \subset \mathcal{N} \ominus V^2\mathcal{N}$ satisfies the assumptions of Theorem 2.4. $\square$

Corollary 2.5 and Theorem 1.2 provide the full description of $\text{Lat}(V^2, V^3)$. Corollary 2.5 may be extended to (m, n) satisfying $(m-1)(n-1) \leq \min\{m, n\}$. However, the only such pairs are $(1, n), (m, 1), (2, 3), (3, 2)$. Since $\text{Lat}(V, V^n), \text{Lat}(V^m, V)$ are described by Corollary 2.2, Corollary 2.5 may not be extended to other interesting cases.

For m, n arbitrary, we start with subspaces in $\text{Lat}(V^m, V^n)$ corresponding, in the sense of Lemma 2.3, to subspaces reducing under V . The next result describes $\text{Lat}(P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^m, P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}}V^n)$ in such a case. The following Theorem 2.7 extends the result to the general case.

Remark 2.6. Let $V = T_z \otimes I \in \mathcal{B}(H^2(\mathbb{T}) \otimes \mathcal{H})$, m, n be relatively prime, positive integers and $\mathcal{H}_0 \subset \mathcal{H}$. Then $\mathcal{N} = H^2(\mathbb{T}) \otimes \mathcal{H}_0 \in \text{Lat}(V)$. In order to describe all subspaces in $\text{Lat}(V^m, V^n)$ corresponding, in the sense of Lemma 2.3, to $\mathcal{N}$ we need to describe all subspaces $\mathcal{N}_0$ satisfying conditions of Theorem 2.4. It is convenient to denote $\mathcal{N} = \sum_{i=0}^{\infty} z^i \otimes \mathcal{H}_0$. Then $\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N} = \sum_{i=0}^{mn-m-n} z^i \otimes \mathcal{H}_0$ and

$$P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}} V^\alpha \left(\sum_{i=0}^{mn-m-n} z^i h_i \right) = \sum_{i=0}^{mn-m-n-\alpha} z^{i+\alpha} h_i$$

for $\alpha < (m-1)(n-1)$. Let $f(z) = \sum_{i=0}^{mn-m-n} z^i h_i$ be an arbitrary vector in $\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}$. The minimal subspace of $\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}$ invariant under $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}} V^\alpha$, for $\alpha = m, n$, generated by f is

$$\mathcal{N}_f = \overline{\text{span}} \left\{ \sum_{i=0}^{mn-m-n-\beta} z^{i+\beta} h_i : \beta = km + ln < (m-1)(n-1) \right\}.$$

Note that $\{\beta : \beta = km + ln < (m-1)(n-1)\}$ is of finite cardinality and so $\dim \mathcal{N}_f < \infty$. On the other hand, since $\mathcal{H}$, and so $H^2(\mathbb{T}) \otimes \mathcal{H}$ are separable, subspaces of $\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}$ invariant under $P_{\mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}} V^\alpha$, for $\alpha = m, n$, are precisely those of the form

$$\overline{\text{span}}\{\mathcal{N}_{f_i} : \{f_i\}_{i \geq 0} \subset \mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N}\}.$$

Note that if $\dim \mathcal{H} < \infty$, then $\dim \mathcal{N} \ominus V^{(m-1)(n-1)}\mathcal{N} < \infty$.

The following result, Remark 2.6 and Theorem 1.2 describe $\text{Lat}(V^m, V^n)$ in the general case.

Theorem 2.7. *Let $V = T_z \otimes I \in \mathcal{B}(H^2(\mathbb{T}) \otimes \mathcal{H})$ and m, n be relatively prime, positive integers. The subspaces jointly invariant under (V^m, V^n) are precisely those of the form*

$$\hat{\theta} \left(H_0 \oplus \left(z^{(m-1)(n-1)} H^2(\mathbb{T}) \otimes \mathcal{H}_0 \right) \right)$$

where $\hat{\theta}$ is an inner function with initial space $H^2(\mathbb{T}) \otimes \mathcal{H}_0$ and H_0 is a subspace of $\sum_{i=0}^{mn-m-n} z^i \otimes \mathcal{H}_0$ invariant under $P_{\sum_{i=0}^{mn-m-n} z^i \otimes \mathcal{H}_0} V^\alpha$ for $\alpha = m, n$.

Proof. By Theorem 1.2 an arbitrary subspace invariant under V is of the form $\mathcal{N} = \hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H})$ where $\hat{\theta}$ is an inner function. Recall that $(\hat{\theta}f)\{z \mapsto \theta(z)f(z)\}$ for $f \in H^2(\mathbb{T}) \otimes \mathcal{H}$ where $\theta(z)$ are partial isometries on $\mathcal{H}$ with the same initial space $\mathcal{H}_0$ for almost every $z \in \mathbb{T}$. Hence $\{z \mapsto \theta(z)f(z)\} = \{z \mapsto \theta(z)P_{\mathcal{H}_0}f(z)\}$ where $P_{\mathcal{H}_0} \in \mathcal{B}(\mathcal{H})$. In other words, $\hat{\theta}f = \hat{\theta}(I \otimes P_{\mathcal{H}_0})f$ for any $f \in H^2(\mathbb{T}) \otimes \mathcal{H}$. Since $I \otimes P_{\mathcal{H}_0} = P_{H^2(\mathbb{T}) \otimes \mathcal{H}_0}$, we get $\mathcal{N} = \hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H}) = \hat{\theta}(I \otimes P_{\mathcal{H}_0})(H^2(\mathbb{T}) \otimes \mathcal{H}) = \hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H}_0)$.

Let us denote for convenience $M = (m-1)(n-1)$. Let $f \in \mathcal{N} \ominus V^M \mathcal{N}$ and $g \in \mathcal{N}$. Then $f = (\hat{\theta}f_0)\{z \mapsto \theta(z)f_0(z)\}$ and $g = (\hat{\theta}g_0)\{z \mapsto \theta(z)g_0(z)\}$ for some $f_0, g_0 \in H^2(\mathbb{T}) \otimes \mathcal{H}_0$. Since $\hat{\theta}$ commutes with V , we get $V^M g : \{z \mapsto$

$\theta(z)z^M g_0(z)\}$. Since $f_0(z), g_0(z) \in \mathcal{H}_0$ and $\theta(z)|_{\mathcal{H}_0}$ is an isometry for almost any $z \in \mathbb{T}$, we get

$$\begin{aligned} 0 &= \langle f, V^M g \rangle = \int_{\mathbb{T}} \langle \theta(z) f_0(z), \theta(z) z^M g_0(z) \rangle_{\mathcal{H}} dm \\ &= \int_{\mathbb{T}} \langle f_0(z), z^M g_0(z) \rangle_{\mathcal{H}} dm = \langle f_0, V^M g_0 \rangle. \end{aligned}$$

Hence $f_0 \in (H^2(\mathbb{T}) \otimes \mathcal{H}_0) \ominus V^M(H^2(\mathbb{T}) \otimes \mathcal{H}_0) = \sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0$ and so

$$\mathcal{N} \ominus V^M \mathcal{N} = \hat{\theta} \left(\sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0 \right).$$

Since $\hat{\theta}$ is an isometry on $H^2(\mathbb{T}) \otimes \mathcal{H}_0$, the subspace $\mathcal{N} \ominus V^M \mathcal{N}$ is orthogonal to $\hat{\theta}(\sum_{i=M}^{\infty} z^i \otimes \mathcal{H}_0)$ and

$$P_{\mathcal{N} \ominus V^M \mathcal{N}} \hat{\theta} = P_{\mathcal{N} \ominus V^M \mathcal{N}} \hat{\theta} P_{\sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0} = \hat{\theta} P_{\sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0}. \quad (2.1)$$

Note that for any $\mathcal{N}_0 \subset \mathcal{N} \ominus V^M \mathcal{N}$ there is $H_0 \subset \sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0$ such that $\mathcal{N}_0 = \hat{\theta} H_0$. Since $\hat{\theta}$ commutes with V and from (2.1) we get $P_{\mathcal{N} \ominus V^M \mathcal{N}} V^\alpha \hat{\theta} H_0 = P_{\mathcal{N} \ominus V^M \mathcal{N}} \hat{\theta} V^\alpha H_0 = \hat{\theta} P_{\sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0} V^\alpha H_0$. Therefore $\mathcal{N}_0$ is invariant under $P_{\mathcal{N} \ominus V^M \mathcal{N}} V^\alpha$ if and only if H_0 is invariant under $P_{\sum_{i=0}^{M-1} z^i \otimes \mathcal{H}_0} V^\alpha$. Theorem 2.4 finishes the proof. $\square$

3. Pairs Satisfying $V_1^m = UV_2^n$

In the section we show that for any pair of commuting isometries there is a minimal extension to a pair of commuting isometries satisfying the equality

$$V_1^m = UV_2^n \quad (\dagger)$$

where m, n are positive integers and U is a unitary operator commuting with V_1, V_2 . One can check that one way that isometric pairs (V_1, V_2) satisfying $(\dagger)$ arise is as $(V_1, V_2) = (U^k V^n, U^l V^m)$ where V is an isometry and U is a unitary operator commuting with V . In the next section we extend pairs satisfying $(\dagger)$ to pairs of the form as above, i.e. $(U^k V^m, U^l V^n)$ where U is a unitary operator commuting with an isometry V .

The extension to a pair satisfying $(\dagger)$ does exist since there exists an extension to a pair of unitary operators, which clearly satisfy $(\dagger)$. However, the minimality of the extension is important. Precisely, we are mostly interested in the case when the isometric factor V in the formula of the final extension $(U^k V^m, U^l V^n)$ is a unilateral shift. An example of a pair of unilateral shifts satisfying $(\dagger)$ are generalized powers defined in [3]. In fact pairs satisfying $(\dagger)$ were motivated by generalized powers. Both examples are compatible pairs of isometries (i.e. such that $P_{\text{ran}(V_1^m)} P_{\text{ran}(V_2^n)} = P_{\text{ran}(V_2^n)} P_{\text{ran}(V_1^m)}$ for any $m, n > 0$). The model of compatible pairs is known, described in [3, 10]. However, not all pairs satisfying $(\dagger)$ are compatible, as in the following example.

Example 3.1. Let $H = H^2(\mathbb{T}) \ominus \mathbb{C}(1 - z)$ which is invariant under T_{z^i} for $i \geq 2$. Hence we may define $V_1 = T_{z^2}|_H, V_2 = T_{z^3}|_H$. Then $V_1^3 = V_2^2$. One can check that $\ker V_1^* = \overline{\text{span}}\{\frac{\sqrt{2}}{2}(1+z), \frac{\sqrt{2}}{2}(z^2 - z^3)\}$ and $\ker V_2^* = \overline{\text{span}}\{\frac{\sqrt{2}}{2}(1+z), z^2, \frac{\sqrt{2}}{2}(z^3 - z^4)\}$. Hence $\text{ran}(V_1) = \overline{\text{span}}\{\frac{\sqrt{2}}{2}(z^2 + z^3), z^n : n \geq 4\}$ and $\text{ran}(V_2) = \overline{\text{span}}\{\frac{\sqrt{2}}{2}(z^3 + z^4), z^n : n \geq 5\}$. Thus $P_{\text{ran}(V_1)}P_{\text{ran}(V_2)}z^4 = P_{\text{ran}(V_1)}\frac{1}{2}(z^3 + z^4) = \frac{\sqrt{2}}{4}(z^2 + z^3) + \frac{1}{2}z^4$ while $P_{\text{ran}(V_2)}P_{\text{ran}(V_1)}z^4 = P_{\text{ran}(V_2)}z^4 = \frac{1}{2}(z^3 + z^4)$.

Before we construct the extension, let us investigate properties of pairs satisfying $(\dagger)$.

Remark 3.2. Let (V_1, V_2) be a pair of commuting isometries. The following conditions are equivalent.

- (V_1, V_2) satisfy $(\dagger)$,
- $V_2^{*n}V_1^m$ is unitary,
- $\text{ran}(V_1^m) = \text{ran}(V_2^n)$.

Moreover, a pair satisfying the conditions above admit the Wold-Słociński decomposition. Precisely, it decomposes between a pair of unitary operators and a pair of unilateral shifts.

Note that $U = V_2^{*n}V_1^m$ from $(\dagger)$ and so $V_2^{*n}V_1^m$ is unitary. Conversely, if $V_2^{*n}V_1^m$ is unitary, then, by Remark 1.4, it commutes with V_1, V_2 . Hence the pair satisfies $(\dagger)$ with $U := V_2^{*n}V_1^m$. We have shown that the pair satisfies $(\dagger)$ if and only if $V_2^{*n}V_1^m$ is unitary. On the other hand, $V_2^{*n}V_1^m$ is an isometry if and only if $\text{ran}(V_1^m) \subset \text{ran}(V_2^n)$. Thus $V_2^{*n}V_1^m$ is unitary if and only if $\text{ran}(V_1^m) = \text{ran}(V_2^n)$.

Recall that by Theorem 1.1 a maximal subspace reducing an isometry V_i to a unitary operator is $H_{ui} = \bigcap_{i \geq 0} \text{ran}(V_i^n)$. Since the sequence $\{\text{ran}(V_i^n)\}_{i \geq 0}$ is decreasing the equality $\text{ran}(V_1^m) = \text{ran}(V_2^n)$ yields $H_{u1} = H_{u2}$. Hence $H_{u1} = H_{u2}$ reduces (V_1, V_2) to a pair of unitary operators and $H \ominus H_{u1} = H \ominus H_{u2}$ reduces (V_1, V_2) to a pair of unilateral shifts.

By Remark 3.2 for any extension of (V_1, V_2) to a pair $(\widehat{V}_1, \widehat{V}_2)$ satisfying $(\dagger)$ the operator $\widehat{V}_2^{*n}\widehat{V}_1^m$ is unitary. However, note that $\widehat{V}_2^{*n}\widehat{V}_1^m$ is not a unitary dilation of $V_2^{*n}V_1^m$. Indeed, by Remark 1.4

$$P_H(\widehat{V}_2^{*n}\widehat{V}_1^m)^j|_H = P_H(\widehat{V}_2^{*n})^j(\widehat{V}_1^m)^j|_H = (V_2^{*n})^j(V_1^m)^j.$$

Since $(V_2^{*n})^j(V_1^m)^j \neq (V_2^{*n}V_1^m)^j$ for $j \geq 2$ in general $\widehat{V}_2^{*n}\widehat{V}_1^m$ is not a dilation of $V_2^{*n}V_1^m$. However, if $V_2^{*n}V_1^m$ is an isometry then by Remark 1.4 $(V_2^{*n})^j(V_1^m)^j = (V_2^{*n}V_1^m)^j$ for $j \in \mathbb{Z}$ and $\widehat{V}_2^{*n}\widehat{V}_1^m$ is a unitary dilation of $V_2^{*n}V_1^m$.

Let us investigate the case when $V_2^{*n}V_1^m$ is an isometry more thoroughly.

Remark 3.3. If a commuting pair of isometries (V_1, V_2) on a Hilbert space H is extended to a commuting pair of isometries $(\widehat{V}_1, \widehat{V}_2)$ on a Hilbert space $\widehat{H}$ satisfying $\widehat{V}_1^m = \widehat{U}\widehat{V}_2^n$ where $\widehat{U} \in \mathcal{B}(\widehat{H})$ is a unitary operator commuting with $\widehat{V}_1, \widehat{V}_2$ then the following conditions are equivalent:

1. $V_2^{*n}V_1^m$ is an isometry.
2. $\widehat{U}H \subset H$,
3. $V_2^{*n}V_1^mh = \widehat{U}h$ for any $h \in H$,
4. $\widehat{U}|_H$ may be considered as an operator on H and $\widehat{U}|_H$ commutes with V_1, V_2

(1) $\Rightarrow$ (2) Note that $P_H\widehat{U}h = P_H\widehat{V}_2^{*n}\widehat{V}_1^mh = V_2^{*n}V_1^mh$ for any $h \in H$. Since $\widehat{U}$ and $V_2^{*n}V_1^m$ are isometries, we may reduce the projection and get $\widehat{U}h = V_2^{*n}V_1^mh \in H$.

(2) $\Rightarrow$ (3) Note that $\widehat{V}_2^{*n}\widehat{V}_1^m = \widehat{U}$. If $\widehat{U}H \subset H$ then $V_2^{*n}V_1^mh = P_H\widehat{V}_2^{*n}\widehat{V}_1^mh = P_H\widehat{U}h = \widehat{U}h$ for any $h \in H$.

(3) $\Rightarrow$ (1) is trivial.

(2) $\Rightarrow$ (4) By the assumption $\widehat{U}|_H$ indeed may be considered as an operator on H . Hence and since $\widehat{V}_1h = V_1h$ for any $h \in H$ we get $\widehat{V}_1\widehat{U}h = V_1\widehat{U}h$ for any $h \in H$. Eventually, $\widehat{U}V_1h = \widehat{U}\widehat{V}_1h = \widehat{V}_1\widehat{U}h = V_1\widehat{U}h$. The proof for V_2 is similar.

(4) $\Rightarrow$ (2) is trivial.

Let us show an example where H is not invariant under $\widehat{U}$.

Example 3.4. Let $H = H^2(\mathbb{T}) \otimes \mathbb{C}^2$ and $V_1 = T_z \otimes I, U = I \otimes \mathcal{U}$ where $\mathcal{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Set $\mathcal{M} = H \ominus \mathbb{C}f$ where $f = 1 \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $U \left(1 \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = f$, so $\mathcal{M}$ is not invariant under U . However, $\text{ran}(V_1U) \subset \text{ran}(V_1) = \text{ran}(T_z \otimes I) = \sum_{i=1}^{\infty} z^i \otimes \mathbb{C}^2 \subset \mathcal{M}$. Thus $\mathcal{M}$ is invariant under V_1 and $V_2 := V_1U = T_z \otimes \mathcal{U}$ but it is not invariant under U .

Assume that a pair (V_1, V_2) and an extension $(\widehat{V}_1, \widehat{V}_2)$ satisfying $(\dagger)$ do not meet conditions in Remark 3.3. Since $\widehat{V}_i$ commutes with $\widehat{U}$ and $\widehat{V}_iH = V_iH \subset H$ for $i = 1, 2$ the space $\tilde{H} := \overline{\text{span}}\{\widehat{U}^nH : n \geq 0\}$ is invariant under $\widehat{V}_1, \widehat{V}_2$. Hence, we may define $\tilde{V}_1 := \widehat{V}_1|_{\tilde{H}}, \tilde{V}_2 := \widehat{V}_2|_{\tilde{H}} \in \mathcal{B}(\tilde{H})$. Note that $(\widehat{V}_1, \widehat{V}_2)$ is an extension of $(\tilde{V}_1, \tilde{V}_2)$ and $\tilde{H}$ is invariant under $\widehat{U}$. Hence, the pair $(\tilde{V}_1, \tilde{V}_2)$ and the extension $(\widehat{V}_1, \widehat{V}_2)$ satisfy conditions of Remark 3.3. We construct the extension of an arbitrary pair (V_1, V_2) to a pair $(\widehat{V}_1, \widehat{V}_2)$ satisfying $(\dagger)$ in two steps:

1. An arbitrary pair of commuting isometries (V_1, V_2) is extended to a commuting pair $(\tilde{V}_1, \tilde{V}_2)$ such that $\tilde{V}_2^{*n}\tilde{V}_1^m$ is an isometry. Then $\tilde{V}_2^{*n}\tilde{V}_1^m$ commutes with $\tilde{V}_1, \tilde{V}_2$ by Remark 1.4.
2. Since $\tilde{V}_2^{*n}\tilde{V}_1^m$ is an isometry, its unitary dilation $\widehat{U}$ is in fact an extension. Moreover, $\widehat{U}$ commutes with $(\widehat{V}_1, \widehat{V}_2)$ —an isometric extensions of $(\tilde{V}_1, \tilde{V}_2)$.

Proposition 3.5. *For any pair of commuting isometries (V_1, V_2) on a Hilbert space H and positive integers m, n , there is an extension to a commuting pair of isometries $(\widehat{V}_1, \widehat{V}_2)$ on a Hilbert space $\widehat{H}$ satisfying $\widehat{V}_1^m = \widehat{U}\widehat{V}_2^n$ where $\widehat{U} \in \mathcal{B}(\widehat{H})$ is a unitary operator commuting with $\widehat{V}_1, \widehat{V}_2$. Moreover, the extension may be chosen to be minimal.*

Proof. The construction is done in the two steps described before the proposition.

1. Let $\check{V}_2 \in \mathcal{B}(\check{H})$ be the minimal unitary extension of V_2 , and let $\check{V}_1$ be an isometric extension of V_1 on $\check{H}$ commuting with $\check{V}_2$. Define $\check{H} = \overline{\text{span}}\{\check{U}^i H, i \geq 0\}$ and $\tilde{U} = \check{U}|_{\check{H}}$ where $\check{U} = \check{V}_2^{*n} \check{V}_1^m$. Note that $\check{U}$ and consequently also $\tilde{U}$ are isometries. Thus, by Remark 1.4, $\check{U}$ commutes with $\check{V}_1, \check{V}_2$. Hence and since $H \in \text{Lat}(\check{V}_1, \check{V}_2)$ we get $\check{H} \in \text{Lat}(\check{V}_1, \check{V}_2)$. Subsequently $\tilde{V}_1 := \check{V}_1|_{\check{H}}, \tilde{V}_2 := \check{V}_2|_{\check{H}}$ are commuting isometries on $\check{H}$ commuting also with $\tilde{U} \in \mathcal{B}(\check{H})$. Moreover, $\tilde{V}_1^m = \tilde{U} \tilde{V}_2^n$ and hence $\tilde{V}_2^{*n} \tilde{V}_1^m = \tilde{U}$ is an isometry.
2. Let $\hat{U} \in \mathcal{B}(\hat{H})$ be the minimal unitary extension of $\tilde{U}$ and $(\hat{V}_1, \hat{V}_2)$ be a commuting isometric extension of $(\tilde{V}_1, \tilde{V}_2)$ on $\hat{H}$, commuting also with $\hat{U}$ —see [13, Proposition 6.3, chap.1] for details of such a construction. Note that $\hat{V}_1^m \hat{U}^i|_{\hat{H}} = \hat{U}^i \hat{V}_1^m|_{\hat{H}} = \hat{U}^i \tilde{V}_1^m = \hat{U}^i \tilde{U} \tilde{V}_2^n = \hat{U}^{i+1} \tilde{V}_2^n|_{\hat{H}} = \hat{U} \tilde{V}_2^n \hat{U}^i|_{\hat{H}}$, and so $\hat{V}_1^m|_{\hat{U}^i \hat{H}} = \hat{U} \tilde{V}_2^n|_{\hat{U}^i \hat{H}}$ for any $i \in \mathbb{Z}$. On the other hand, by the minimality $\hat{H} = \overline{\text{span}}\{\hat{U}^i \hat{H} : i \leq 0\}$. Hence $\hat{V}_1^m = \hat{U} \hat{V}_2^n$.

For the minimality, let $\hat{\mathcal{H}} \subset \hat{H}$ be such that $(\hat{V}_1 = \hat{V}_1|_{\hat{\mathcal{H}}}, \hat{V}_2 = \hat{V}_2|_{\hat{\mathcal{H}}})$ on $\hat{\mathcal{H}}$ is an extensions of (V_1, V_2) satisfying the statement. Note that $\hat{U} := \hat{V}_2^{*n} \hat{V}_1^m = P_{\hat{\mathcal{H}}} \hat{V}_2^{*n} \hat{V}_1^m = P_{\hat{\mathcal{H}}} \hat{U}|_{\hat{\mathcal{H}}}$. However, by the assumption $\hat{U}$ as well as $\hat{U}$ are unitary operators and so we may reduce the projection. Hence $\hat{U} = \hat{U}|_{\hat{\mathcal{H}}}$ and in particular $\hat{U}|_H = \hat{U}|_H = \check{U}|_H$. Thus $\hat{H} = \overline{\text{span}}\{\hat{U}^i H, i \geq 0\} \subset \hat{\mathcal{H}}$ and $\tilde{U} = \hat{U}|_{\hat{H}}$. Consequently $\hat{U} = \hat{U}|_{\hat{\mathcal{H}}}$ is a unitary extension of $\tilde{U}$. Since $\hat{\mathcal{H}} \subset \hat{H}$ by the minimality of $\hat{U}$ we get $\hat{\mathcal{H}} = \hat{H}$. $\square$

Let us show an example of a pair of unilateral shifts such that the extension constructed in Proposition 3.5, so a minimal extension, is a pair of unitary operators.

Example 3.6. Let (L_w, L_z) denote operators of multiplication by independent variables w, z on $L^2(\mathbb{T}^2)$ and $\mathcal{M} := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } (i,j) \in \mathbb{Z}^2 \setminus Z\}$ where $Z := \{(i,j) \in \mathbb{Z}^2 : j \geq 0 \text{ and } i \geq -(0 + \dots + j)\}$ as in the picture.

Then $\mathcal{M} \in \text{Lat}(L_w, L_z)$ and $V_1 = L_w|_{\mathcal{M}}, V_2 = L_z|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$ are unilateral shifts. Moreover, $V_2^{*n} V_1^m = P_{\mathcal{M}} L_z^n L_w^m|_{\mathcal{M}}$ for any positive integers m, n . Note that a minimal isometric extension of $P_{\mathcal{M}} L_z^n L_w^m|_{\mathcal{M}}$ is the restriction $L_z^n L_w^m|_{\hat{\mathcal{M}}}$ where $\hat{\mathcal{M}} = \overline{\text{span}}\{(L_z^n L_w^m)^s \mathcal{M} : s \geq 0\}$. By Remark 1.4, $(L_z^n L_w^m)^s =$

$L_{\bar{z}}^{sn} L_w^{sm}$ for $s \geq 1$. Hence $(L_{\bar{z}}^n L_w^m|_{\tilde{\mathcal{M}}})^s = (L_{\bar{z}}^n L_w^m)^s|_{\tilde{\mathcal{M}}} = L_{\bar{z}}^{sn} L_w^{sm}|_{\tilde{\mathcal{M}}}$. Let us show that $\tilde{\mathcal{M}} = L^2(\mathbb{T}^2)$. Consider $w^k z^l$ for an arbitrary $(k, l) \in \mathbb{Z}^2$. Note that there are $(i, j) \in \mathbb{Z}$ and a positive integer s such that $(k, l) = (i + sm, j - sn)$. Hence $w^i z^j \in \mathcal{M}$ and $(L_{\bar{z}}^n L_w^m)^s w^i z^j = L_{\bar{z}}^{sn} L_w^{sm} w^i z^j = w^{i+sm} z^{j-sn} = w^k z^l$. Thus indeed $\tilde{\mathcal{M}} = L^2(\mathbb{T}^2)$, and so a minimal extension of (V_1, V_2) to a pair of commuting isometries satisfying $(\dagger)$ is a pair of unitary operators (L_w, L_z) .

Operators of multiplication by independent variables on the Hardy space over the torus have an extension satisfying $(\dagger)$ which is a pair of unilateral shifts.

Example 3.7. Let (T_w, T_z) denote operators of multiplication by independent variables w, z on the Hardy space over the torus $H^2(\mathbb{T}^2)$. The minimal extension of (T_w, T_z) to a commuting pair of isometries $(\hat{T}_w, \hat{T}_z)$ satisfying $\hat{T}_w = \hat{U}\hat{T}_z$ is $\hat{T}_w = L_w|_{\Delta^2(\mathbb{T}^2)}, \hat{T}_z = L_z|_{\Delta^2(\mathbb{T}^2)}$ where

$$\Delta^2(\mathbb{T}^2) := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } j < -i\}$$

and $\hat{U} = L_{\bar{z}w}|_{\Delta^2(\mathbb{T}^2)}$.

To get the extension of (T_w, T_z) to a pair satisfying $(\dagger)$ for arbitrary m, n redefine $\Delta^2(\mathbb{T}^2) := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } mj < -ni\}$ in Example 3.7.

4. Extension to a Generalization of Powers of an Isometry

In this section we construct a minimal extension of an arbitrary pair of commuting isometries (V_1, V_2) to a pair of the form

$$(U^k V^n, U^l V^m) \quad (\ddagger)$$

where U is a unitary operator commuting with an isometry V and integers, positive m, n, k and nonnegative l , satisfy $km - ln = 1$. We also describe a model for such pairs. Note that for a pair of positive integers (m, n) there are integers (k, l) such that $km - ln = 1$ if and only if m, n are relatively prime. Moreover, by Remark 1.5 such (k, l) satisfying $0 < k < n, 0 \leq l < m$ are unique, hence determined by (m, n) .

Consider a pair of commuting isometries (V_1, V_2) satisfying $(\dagger)$ for the unitary operator U commuting with V_1, V_2 and relatively prime, positive integers m, n . By Fuglede-Putnam theorem U^* commutes with V_1, V_2 . Hence the condition $(\dagger)$ which by $km - ln = 1$ may be stated as $V_1^m = U^{km-ln} V_2^n$ is equivalent to

$$(U^{*k} V_1)^m = (U^{*l} V_2)^n. \quad (\dagger')$$

Note that conditions $(\dagger)$ and $(\dagger')$ are equivalent but $(\dagger')$ requires m, n to be relatively prime. The pair $(U^k V^n, U^l V^m)$ satisfies $(\dagger')$ for any isometry V and a unitary operator U provided they commute. Indeed, as assumed for pairs satisfying $(\dagger)$, U commutes with $V_1 = U^k V^n, V_2 = U^l V^m$ and hence it commutes with V^m, V^n . However, by Fuglede-Putnam theorem it doubly commutes with V^m, V^n and so with $V = V^{*ln} V^{km}$. Obviously $(U^k V^n, U^l V^m)$ are unilateral shifts if and only if V is a unilateral shift.

Before we construct the extension, let us investigate properties of pairs of the form $(\ddagger)$ and give their model.

Remark 4.1. Let (V_1, V_2) be a pair of commuting isometries and m, n be relatively prime, positive integers with the corresponding integers $0 < k < n, 0 \leq l < m$ such that $km - ln = 1$. The pair (V_1, V_2) is of the form $(\ddagger)$ if and only if $V_2^{*n}V_1^m$ is a unitary operator and $V_1^{*l}V_2^k$ is an isometry. Then $U = V_2^{*n}V_1^m$ and $V = V_1^{*l}V_2^k$.

Indeed, let $V_1 = U^kV^n, V_2 = U^lV^m$ for some unitary operator U commuting with an isometry V . Then $V^n = U^{*k}V_1, V^m = U^{*l}V_2$ and U doubly commutes with V_1, V_2 . Hence

$$\begin{aligned} V &= V^{km-ln} = (V^n)^{*l}(V^m)^k = (U^{*k}V_1)^{*l}(U^{*l}V_2)^k \\ &= V_1^{*l}U^{kl}U^{*lk}V_2^k = V_1^{*l}V_2^k \end{aligned}$$

and

$$\begin{aligned} V_2^{*n}V_1^m &= V^{*mn}U^{*ln}U^{km}V^{nm} = V^{*mn}U^{km-ln}V^{nm} \\ &= V^{*mn}V^{nm}U^{km-ln} = U. \end{aligned}$$

Conversely, assume that $V_2^{*n}V_1^m$ is a unitary operator and $V_1^{*l}V_2^k$ is an isometry. Any commuting pair of isometries satisfies $V_1^m = UV_2^n$ for $U := V_2^{*n}V_1^m$. Since U is unitary, it doubly commutes with V_1, V_2 by Remark 1.4 and Fuglede-Putnam theorem. Hence U commutes with $V := V_1^{*l}V_2^k$. By Remark 1.4 V commutes with V_1, V_2 and $V^m = V_1^{*ml}V_2^{mk}$. Further, $mk = nl + 1$ yields

$$V_1^{*ml}V_2^{mk} = V_1^{*ml}V_2^{nl+1} = V_1^{*ml}(V_2^n)^lV_2 = V_1^{*ml}(U^*V_1^m)^lV_2 = U^{*l}V_2.$$

Hence $V_2 = U^lV^m$. The proof of $V_1 = U^kV^n$ is similar.

We next describe the model for a pair of the form $(\ddagger)$. By Remark 3.2 any pair of commuting isometries satisfying $(\ddagger)$ (in particular any pair of the form $(\ddagger)$) decomposes between a pair of unitary operators and a pair of unilateral shifts. By Remark 4.1 any pair of commuting unitary operators may be represented in the form $(\ddagger)$ for any relatively prime, positive integers m, n . On the other hand, if a pair of commuting unilateral shifts (V_1, V_2) equals to (U^kV^n, U^lV^m) for a unitary operator U commuting with an isometry V then V is a unilateral shift. Hence we may assume $V = T_z \otimes I \in \mathcal{B}(H^2(\mathbb{T}) \otimes \mathcal{H})$ and by Remark 1.3 $U = I \otimes \mathcal{U}$ where $\mathcal{U} \in \mathcal{B}(\mathcal{H})$ is a unitary operator. Eventually, $V_1 = T_z^n \otimes \mathcal{U}^k, V_2 = T_z^m \otimes \mathcal{U}^l$. Summing up we get the model:

Remark 4.2. Pairs of the form $(\ddagger)$ are precisely:

$$V_1 = U_1 \oplus (T_z^n \otimes \mathcal{U}^k), V_2 = U_2 \oplus (T_z^m \otimes \mathcal{U}^l) \in \mathcal{B}(H_u \oplus (H^2(\mathbb{T}) \otimes \mathcal{H}))$$

where $U_1, U_2 \in \mathcal{B}(H_u), \mathcal{U} \in \mathcal{B}(\mathcal{H})$ are unitary operators, m, n are relatively prime, positive integers and $0 < k < n, 0 \leq l < m$ are such that $km - ln = 1$. Precisely, $V_1 = U^kV^n, V_2 = U^lV^m$ where $U = U_2^{*n}U_1^m \oplus (I \otimes \mathcal{U}), V = U_1^{*l}U_2^k \oplus (T_z \otimes I)$. Thus invariants of pairs of the form $(\ddagger)$ for a fixed pair of relatively prime, positive integers m, n are Hilbert spaces $H_u, \mathcal{H}$ and unitary operators $U_1, U_2 \in \mathcal{B}(H_u), \mathcal{U} \in \mathcal{B}(\mathcal{H})$ where U_1, U_2 commute.

Let us remark that the model above is not the special case of Berger-Coburn-Lebow model in [1]. To compare the models we assume that a pair of commuting isometries (V_1, V_2) is of the form $(\dagger)$ and is completely non unitary (which is assumed in Berger-Coburn-Lebow model). Then it is a pair of unilateral shifts and both models are defined on $H^2(\mathbb{T}) \otimes \mathcal{H}$. However, $T_z \otimes I = V_1 V_2$ in Berger-Coburn-Lebow model which is not true in the model above.

The main aim of the section is to construct a minimal extension of an arbitrary pair of commuting isometries to a pair of the form $(\dagger)$. According to the previous section we may reduce the problem to the case where the initial pair satisfy $(\dagger)$. An example of a pair satisfying $(\dagger)$ which is not of the form $(\dagger)$ is $(V_1|_{\mathcal{M}_0}, V_2|_{\mathcal{M}_0})$ in Example 2.1. Indeed, it satisfies $(\dagger)$ for $U = I$ but is not a pair of powers of the same isometry since V defined in the example acts on a superspace of $\mathcal{M}_0$.

By Remark 4.1 we need to construct an extension of an arbitrary pair of commuting isometries to a pair $(\tilde{V}_1, \tilde{V}_2)$ such that $\tilde{V}_2^{*\beta} \tilde{V}_1^\alpha$ are isometric/unitary for the appropriate pairs of powers (α, β) . In the further part, in the proof of the main theorem of the section we show that for a commuting pair of isometries (V_1, V_2) satisfying $(\dagger)$ operators $V_2^{*b} V_1^a$ are isometries for sufficiently large (a, b) . On the other hand, (α, β) required in Remark 4.1 may not be large enough. Therefore we need the following proposition. Note that if $V_2^{*\beta} V_1^\alpha$ is an isometry, then, by Remark 1.4, $(V_2^{*\beta} V_1^\alpha)^2 = V_2^{*2\beta} V_1^{2\alpha}$ and so $V_2^{*2\beta} V_1^{2\alpha}$ is an isometry, but the converse implication may not hold. For example, if $m = 3, n = 2$ in Example 2.1, then $(V_2|_{\mathcal{M}_0})^{*2} (V_1|_{\mathcal{M}_0})^2 = V^2|_{\mathcal{M}_0}$, so it is an isometry, but $(V_2|_{\mathcal{M}_0})^* (V_1|_{\mathcal{M}_0})$ vanishes on $\mathcal{M} \ominus V\mathcal{M}$.

Proposition 4.3. *Let (V_1, V_2) be a commuting pair of isometries on a Hilbert space H such that $V_2^{*2\beta} V_1^{2\alpha}$ is an isometry. There is a minimal extension $(\tilde{V}_1, \tilde{V}_2)$ on a Hilbert space $\tilde{H}$ such that $\tilde{V}_2^{*\beta} \tilde{V}_1^\alpha$ is an isometry.*

Moreover, the extension preserves the following properties:

- if V_i is a unilateral shift, then $\tilde{V}_i$ is a unilateral shift for $i = 1, 2$,
- if $V_2^{*\beta} V_1^\alpha$ is a co-isometry, then $\tilde{V}_2^{*\beta} \tilde{V}_1^\alpha$ is a unitary operator,
- if $V_2^{*n} V_1^m$ is unitary, then $\tilde{V}_2^{*n} \tilde{V}_1^m$ is unitary for $m, n \in \mathbb{Z}_+$.

Proof. Let

$$\tilde{H} := \{(h, f) : h \in H, f \in \mathcal{D}\}$$

be a cartesian product Hilbert space with $\langle (h_1, f_1), (h_2, f_2) \rangle_{\tilde{H}} = \langle h_1, h_2 \rangle + \langle f_1, f_2 \rangle$, and so $\|(h, f)\|_{\tilde{H}}^2 = \|h\|^2 + \|f\|^2$ where

$$\mathcal{D} = \overline{(I - V_2^\beta V_2^{*\beta}) V_1^\alpha H}$$

and $\langle \cdot, \cdot \rangle, \|\cdot\|$ denote the scalar product, the norm on H , respectively. Note that $H \subset \tilde{H}$ in the sense of the equivalence $h \simeq (h, 0)$. Define

$$\tilde{W}(h, f) = (Wh + V_2^{*\beta} Wf, (I - V_2^\beta V_2^{*\beta}) Wf) \quad (4.1)$$

for any isometry W commuting with V_1, V_2 . In particular (4.1) defines $(\tilde{V}_1, \tilde{V}_2)$ which we show to satisfy the statement.

For an arbitrary $f = \lim_{n \rightarrow \infty} (I - V_2^\beta V_2^{*\beta}) V_1^\alpha g_n \in \mathcal{D}$ by the second claim in Remark 1.4 we get

$$\begin{aligned} (I - V_2^\beta V_2^{*\beta}) W f &= \lim_{n \rightarrow \infty} (I - V_2^\beta V_2^{*\beta}) W (I - V_2^\beta V_2^{*\beta}) V_1^\alpha g_n \\ &= \lim_{n \rightarrow \infty} (I - V_2^\beta V_2^{*\beta}) V_1^\alpha W g_n \in \mathcal{D}. \end{aligned}$$

Hence $\widetilde{W}$ is a well defined operator.

Note that if $f \in \mathcal{D} \subset \ker V_2^{*\beta}$ then $V_2^{*\beta} W f \in \ker W^*$ and so

$$\begin{aligned} \|\widetilde{W}(h, f)\|_{\widetilde{H}}^2 &= \|Wh\|^2 + \|V_2^{*\beta} W f\|^2 + \|(I - V_2^\beta V_2^{*\beta}) W f\|^2 \\ &= \|Wh\|^2 + \|V_2 V_2^{*\beta} W f\|^2 + \|(I - V_2^\beta V_2^{*\beta}) W f\|^2 \\ &= \|h\|^2 + \|W f\|^2 = \|(h, f)\|_{\widetilde{H}}^2. \end{aligned}$$

Consequently $\widetilde{W}$ is an isometry.

One can check that for any pair of isometries W, V commuting with V_1, V_2 and between each other

$$\widetilde{W}\widetilde{V}(h, f) = (WVh + V_2^{*\beta} WVf, (I - V_2^\beta V_2^{*\beta}) WVf) = \widetilde{W}\widetilde{V}(h, f) \quad (4.2)$$

where the first coordinate in (4.2) follows by a direct calculation, while the second is a consequence of the second claim in Remark 1.4. Hence $\widetilde{W}, \widetilde{V}$ inherit commutativity. In particular $\widetilde{V}_1$ commutes with $\widetilde{V}_2$.

Summing up, $(\widetilde{V}_1, \widetilde{V}_2)$ is a commuting pair of isometries extending (V_1, V_2) .

Next we show that $\text{ran}(\widetilde{V}_1^\alpha)$ is orthogonal to $\ker \widetilde{V}_2^{*\beta}$ which implies that $\widetilde{V}_2^{*\beta} \widetilde{V}_1^\alpha$ is an isometry.

From (4.2) we get $\widetilde{V}_i \widetilde{V}_i^j = \widetilde{V}_i^{j+1}$ and hence inductively $\widetilde{V}_i^j = \widetilde{V}_i^j$ for $j \geq 0$ and $i = 1, 2$. Precisely,

$$\widetilde{V}_i^j(h, f) = (V_i^j h + V_2^{*\beta} V_i^j f, (I - V_2^\beta V_2^{*\beta}) V_i^j f). \quad (4.3)$$

Assume $j \geq \alpha$. Since $V_2^{*2\beta} V_1^{2\alpha}$ is an isometry $\text{ran}(V_1^{j+\alpha}) \subset \text{ran}(V_1^{2\alpha}) \subset \text{ran}(V_2^{2\beta}) \subset \text{ran}(V_2^\beta)$. Hence and by the second claim in Remark 1.4

$$(I - V_2^\beta V_2^{*\beta}) V_1^j (I - V_2^\beta V_2^{*\beta}) V_1^\alpha g = (I - V_2^\beta V_2^{*\beta}) V_1^{j+\alpha} g = 0$$

for any $g \in H$. Since $(I - V_2^\beta V_2^{*\beta}) V_1^\alpha g$ represents an arbitrary vector in a dense subset of $\mathcal{D}$ by continuity $(I - V_2^\beta V_2^{*\beta}) V_1^j|_{\mathcal{D}} = 0$. Hence and from (4.3) we get

$$\begin{aligned} \widetilde{V}_2^j(h, f) &= (V_2^j h + V_2^{j-\beta} f, 0) \text{ for } j \geq \beta, \\ \widetilde{V}_1^j(h, f) &= (V_1^j h + V_2^{*\beta} V_1^j f, 0) \text{ for } j \geq \alpha, \end{aligned} \quad (4.4)$$

and subsequently

$$\text{ran}(\widetilde{V}_1^\alpha) = \{(h, 0) : h \in \text{ran}(V_1^\alpha) \oplus V_2^{*\beta} V_1^\alpha \mathcal{D}\}, \quad (4.5)$$

$$\text{ran}(\widetilde{V}_2^\beta) = \{(h, 0) : h \in \text{ran}(V_2^\beta) \oplus \mathcal{D}\}, \quad (4.6)$$

$$\ker \widetilde{V}_2^{*\beta} = \{(h, f) : h \in \ker V_2^{*\beta} \ominus \mathcal{D}, f \in \mathcal{D}\}.$$

Note that $V_2^{*\beta}V_1^\alpha\mathcal{D}$ is closed since $(I - V_2^\beta V_2^{*\beta})V_1^j|_{\mathcal{D}} = 0$ yields that $V_2^{*\beta}V_1^\alpha|_{\mathcal{D}}$ is an isometry.

We investigate $\mathcal{D}$ to get a more convenient description of $\ker \tilde{V}_2^{*\beta}$. Since $(I - V_2^\beta V_2^{*\beta})V_1^\alpha V_2^\beta = 0$ we get $\mathcal{D} = \overline{(I - V_2^\beta V_2^{*\beta})V_1^\alpha \ker V_2^{*\beta}}$. Moreover, for $h \in \ker V_2^{*\beta} \ominus V_1^{*\alpha} \ker V_2^{*\beta}$ and $f \in \ker V_2^{*\beta}$ we get $0 = \langle h, V_1^{*\alpha} f \rangle = \langle V_1^\alpha h, f \rangle$. Thus $V_1^\alpha(\ker V_2^{*\beta} \ominus V_1^{*\alpha} \ker V_2^{*\beta}) \subset \text{ran } V_2^\beta$ and we get further simplification

$$\mathcal{D} = \overline{(I - V_2^\beta V_2^{*\beta})V_1^\alpha V_1^{*\alpha} \ker V_2^{*\beta}} = \overline{P_{\ker V_2^{*\beta}} P_{\text{ran } V_1^\alpha} \ker V_2^{*\beta}}.$$

On the other hand,

$$\langle v, (I - V_2^\beta V_2^{*\beta})V_1^\alpha V_1^{*\alpha} w \rangle = \langle v, V_1^\alpha V_1^{*\alpha} w \rangle = \langle V_1^{*\alpha} v, V_1^{*\alpha} w \rangle \quad (\diamond)$$

for $v, w \in \ker V_2^{*\beta}$. For $w = v \in \ker V_2^{*\beta} \ominus \mathcal{D}$ the left hand side of $(\diamond)$ equals 0 and the right hand side is equal to $\|V_1^{*\alpha} v\|^2$. Hence $\|V_1^{*\alpha} v\|^2 = 0$ and so $\ker V_2^{*\beta} \ominus \mathcal{D} \subset \ker V_1^{*\alpha} \cap \ker V_2^{*\beta}$. On the other hand, for $v \in \ker V_1^{*\alpha} \cap \ker V_2^{*\beta}$ and an arbitrary $w \in \ker V_2^{*\beta}$ the right hand side of $(\diamond)$ equals 0 which implies that $\ker V_1^{*\alpha} \cap \ker V_2^{*\beta}$ is orthogonal to a dense subset of $\mathcal{D}$, so the converse inclusion $\ker V_1^{*\alpha} \cap \ker V_2^{*\beta} \subset \ker V_2^{*\beta} \ominus \mathcal{D}$. Thus $\ker V_2^{*\beta} \ominus \mathcal{D} = \ker V_1^{*\alpha} \cap \ker V_2^{*\beta}$ and

$$\ker \tilde{V}_2^{*\beta} = \{(h, f) : h \in \ker V_1^{*\alpha} \cap \ker V_2^{*\beta}, f \in \mathcal{D}\}.$$

Recall that our aim is to show that $\text{ran}(\tilde{V}_1^\alpha)$ is orthogonal to $\ker \tilde{V}_2^{*\beta}$ which by the above and (4.5) is equivalent to $V_2^{*\beta}V_1^\alpha\mathcal{D} \subset H \ominus \ker V_1^{*\alpha} \cap \ker V_2^{*\beta} = \overline{\text{span}\{\text{ran}(V_1^\alpha), \text{ran}(V_2^\beta)\}}$. By the definition of $\mathcal{D}$ it is enough to prove that $V_2^{*\beta}V_1^\alpha(I - V_2^\beta V_2^{*\beta})V_1^\alpha g = V_2^{*\beta}V_1^{2\alpha}g - V_1^\alpha V_2^{*\beta}V_1^\alpha g \in \overline{\text{span}\{\text{ran}(V_1^\alpha), \text{ran}(V_2^\beta)\}}$ for any $g \in H$. Since $V_2^{*2\beta}V_1^{2\alpha}$ is an isometry, by Remark 1.4 it commutes with V_2 and consequently $V_2^{*\beta}V_1^{2\alpha}g = V_2^{*2\beta}V_1^{2\alpha}V_2^\beta g = V_2^\beta V_2^{*2\beta}V_1^{2\alpha}g \in \text{ran}(V_2^\beta)$. The second summand $V_1^\alpha V_2^{*\beta}V_1^\alpha g$ is trivially in $\text{ran}(V_1^\alpha)$.

We showed that $\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha$ is an isometry. Let us check the minimality. Assume that $(\tilde{V}_1|_{\tilde{H}}, \tilde{V}_2|_{\tilde{H}})$ satisfy the statement where $H \subset \tilde{H} \subset \tilde{\tilde{H}}$ and $\tilde{\tilde{H}} \in \text{Lat}(\tilde{V}_1, \tilde{V}_2)$. Note that $(\tilde{V}_2|_{\tilde{H}})^{*\beta}(\tilde{V}_1|_{\tilde{H}})^\alpha = P_{\tilde{H}}\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha|_{\tilde{H}}$. Since $(\tilde{V}_2|_{\tilde{H}})^{*\beta}(\tilde{V}_1|_{\tilde{H}})^\alpha$ and $\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha$ are isometries we may reduce the projection and get $(\tilde{V}_2|_{\tilde{H}})^{*\beta}(\tilde{V}_1|_{\tilde{H}})^\alpha = \tilde{V}_2^{*\beta}\tilde{V}_1^\alpha|_{\tilde{H}}$. Hence $\tilde{H}$ is invariant under $\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha$. Let us check that the minimal $\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha$ invariant subspace containing H is the whole $\tilde{H}$ and so $\tilde{H} = \tilde{\tilde{H}}$. By the definition of $\tilde{H}$ and $\mathcal{D}$ it is enough to show that $(0, (I - V_2^\beta V_2^{*\beta})V_1^\alpha h)$ belongs to the minimal $\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha$ invariant subspace containing H for any $h \in H$. By (4.4)

$$\tilde{V}_2^\beta(V_2^{*\beta}V_1^\alpha h, (I - V_2^\beta V_2^{*\beta})V_1^\alpha h) = (V_1^\alpha h, 0) = \tilde{V}_1^\alpha(h, 0).$$

Hence

$$\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha(h, 0) = (V_2^{*\beta}V_1^\alpha h, (I - V_2^\beta V_2^{*\beta})V_1^\alpha h)$$

and so

$$(0, (I - V_2^\beta V_2^{*\beta})V_1^\alpha h) = \tilde{V}_2^{*\beta}\tilde{V}_1^\alpha(h, 0) - (V_2^{*\beta}V_1^\alpha h, 0) \in \overline{\text{span}\{\tilde{V}_2^{*\beta}\tilde{V}_1^\alpha H, H\}}.$$

Let us show the properties of the extensions.

Recall that by Wold-Halmos theorem an isometry V is a unilateral shift if and only if $\bigcap_{\gamma>0} V^\gamma H = \{0\}$. By (4.5) $\text{ran}(\tilde{V}_1^\alpha) \subset H$ and since the sequence $\{\text{ran}(\tilde{V}_1^\alpha)\}_{\alpha \geq 0}$ is decreasing

$$\bigcap_{\gamma>0} \tilde{V}_1^\gamma \tilde{H} = \bigcap_{\gamma>\alpha} \tilde{V}_1^\gamma \tilde{H} \subset \bigcap_{\gamma>0} \tilde{V}_1^\gamma H = \bigcap_{\gamma>0} V_1^\gamma H = \{0\}.$$

The proof for V_2 follows similarly by (4.6).

The second property is equivalent to the implication: if $V_1^{*\alpha} V_2^\beta$ is an isometry, then $\tilde{V}_1^{*\alpha} \tilde{V}_2^\beta$ is an isometry. However, $V_1^{*\alpha} V_2^\beta$ is an isometry if and only if $\text{ran}(V_2^\beta) \subset \text{ran}(V_1^\alpha)$ and similarly for the extensions. Thus we need to show that if $\text{ran}(V_2^\beta) \subset \text{ran}(V_1^\alpha)$, then $\text{ran}(\tilde{V}_2^\beta) \subset \text{ran}(\tilde{V}_1^\alpha)$. Recall that $\mathcal{D} \subset H \ominus (\ker V_1^{*\alpha} \cap \ker V_2^{*\beta}) = \overline{\text{span}}\{\text{ran}(V_1^\alpha), \text{ran}(V_2^\beta)\}$. Thus if $\text{ran}(V_2^\beta) \subset \text{ran}(V_1^\alpha)$ then $\mathcal{D} \subset \text{ran}(V_1^\alpha)$ and from (4.5), (4.6) $\text{ran}(\tilde{V}_2^\beta) = \{(h, 0) : h \in \text{ran}(V_2^\beta) \oplus \mathcal{D}\} \subset \{(h, 0) : h \in \text{ran}(V_1^\alpha)\} \subset \text{ran}(\tilde{V}_1^\alpha)$.

The third property is equivalent to the implication: if $\text{ran}(V_2^n) = \text{ran}(V_1^m)$, then $\text{ran}(\tilde{V}_2^n) = \text{ran}(\tilde{V}_1^m)$. Denote $U = V_2^{*n} V_1^m$ which, as a unitary operator, by Remark 1.4 and Fuglede-Putnam theorem doubly commutes with V_1, V_2 . Then $V_1^m = V_2^n U$ and $U\mathcal{D} = \mathcal{D}$. By (4.1) we get

$$\begin{aligned} \tilde{V}_1^m(h, f) &= (V_1^m h + V_2^{*\beta} V_1^m f, (I - V_2^\beta V_2^{*\beta}) V_1^m f) \\ &= (V_2^n U h + V_2^{*\beta} V_2^n U f, (I - V_2^\beta V_2^{*\beta}) V_2^n U f) = \tilde{V}_2^n(Uh, Uf). \end{aligned}$$

Eventually, $UH = H$ and $U\mathcal{D} = \mathcal{D}$ implies that $\text{ran}(\tilde{V}_2^n) = \text{ran}(\tilde{V}_1^m)$. $\square$

A result similar to Proposition 4.3 but for a unitary extension follows by the properties of the isometric extension proved in Proposition 4.3.

Corollary 4.4. *Let (V_1, V_2) be a pair of commuting isometries such that $V_2^{*2\beta} V_1^{2\alpha}$ is a unitary operator. There is a minimal extension to a pair of commuting isometries $(\tilde{V}_1, \tilde{V}_2)$ such that $\tilde{V}_2^{*\beta} \tilde{V}_1^\alpha$ is a unitary operator.*

Proof. By Proposition 4.3 there is an extension $(\tilde{V}_1, \tilde{V}_2)$ such that $\tilde{V}_1^{*2\alpha} \tilde{V}_2^{2\beta}$ is unitary (it follows by the third property for $m = 2\alpha, n = 2\beta$) and $\tilde{V}_1^{*\alpha} \tilde{V}_2^\beta$ is an isometry. Thus $\tilde{V}_2^{*2\beta} \tilde{V}_1^{2\alpha}$ is unitary and $\tilde{V}_2^{*\beta} \tilde{V}_1^\alpha$ is a co-isometry. By Proposition 4.3 (the second property) $(\tilde{V}_1, \tilde{V}_2)$ can be extended to $(\hat{V}_1, \hat{V}_2)$ such that $\hat{V}_2^{*\beta} \hat{V}_1^\alpha$ is a unitary operator. Obviously the extension is minimal as both steps provide a minimal extension by Proposition 4.3. $\square$

Now we are ready to show that any pair satisfying (†) may be extended to a pair of the form (‡).

Theorem 4.5. *Any pair of commuting isometries (V_1, V_2) on a Hilbert space H satisfying*

$$V_1^m = UV_2^n$$

for a unitary operator U commuting with V_1, V_2 and relatively prime, positive integers m, n may be extended to a pair

$$(\tilde{U}^k \tilde{V}^n, \tilde{U}^l \tilde{V}^m)$$

where:

- $\tilde{U}$ is a unitary operator commuting with an isometry $\tilde{V}$,
- $H \in \text{Lat}(\tilde{V}^m, \tilde{V}^n)$ and
- (k, l) are unique integers such that $0 < k < n, 0 \leq l < m$ and $km - ln = 1$.

Moreover, the extension may be chosen to be minimal, and for a minimal extension if V_1, V_2 are unilateral shifts, then $\tilde{V}$ is a unilateral shift.

Proof. Assume that (V_1, V_2) extends to a commuting pair of isometries $(\tilde{V}_1, \tilde{V}_2)$ such that $\tilde{V}_1^{*l} \tilde{V}_2^k$ is an isometry and $\tilde{V}_2^{*n} \tilde{V}_1^m$ is unitary. By Remark 4.1 $\tilde{V}_1 = \tilde{U}^k \tilde{V}^m, \tilde{V}_2 = \tilde{U}^l \tilde{V}^n$ where $\tilde{U} = \tilde{V}_2^{*n} \tilde{V}_1^m$ and $\tilde{V} = \tilde{V}_1^{*l} \tilde{V}_2^k$. We need to show that $H \in \text{Lat}(\tilde{V}^m, \tilde{V}^n)$. Since $\tilde{V}_2$ is an extension of V_2 then $U = V_2^{*n} V_1^m = P_H \tilde{V}_2^{*n} \tilde{V}_1^m|_H = P_H \tilde{U}|_H$ and similarly $U^* = P_H \tilde{U}^*|_H$. Since both U and $\tilde{U}^*$ are unitary, we may reduce the projection. Thus $\tilde{U}|_H = U$ and $\tilde{U}^*|_H = U^*$ and so H reduces $\tilde{U}$. Hence and since $H \in \text{Lat}(\tilde{V}_1, \tilde{V}_2)$ we get $H \in \text{Lat}(\tilde{V}^m, \tilde{V}^n)$.

It remains to prove that (V_1, V_2) may be extended to a commuting pair of isometries $(\tilde{V}_1, \tilde{V}_2)$ such that $\tilde{V}_1^{*l} \tilde{V}_2^k$ is an isometry and $\tilde{V}_2^{*n} \tilde{V}_1^m$ is unitary. The integers k, l exist and are unique by Remark 1.5. For any positive integer q let

$$qk = pn + r$$

where p, r denote the quotient and the remainder of qk/n , respectively. On the other hand, $km = ln + 1$. Hence $pmn + rm = (pn + r)m = qkm = q(ln + 1) = qln + q$. Dividing $pmn + rm = qln + q$ by n we get

$$ql = pm - \rho$$

where $\rho := \frac{q}{n} - \frac{r}{n}m$. Thus $\rho = pm - ql \in \mathbb{Z}$ and since r , as a reminder of division by n , is less than n , we get $\rho > 0$ for $q \geq mn$. The assumption $V_1^m = UV_2^n$ implies $U^* = V_1^{*m} V_2^n$, and by Remark 1.4 $U^{*p} = V_1^{*pm} V_2^{pn}$. Hence $V_1^{*ql} V_2^{qk} = V_1^{*pm} V_1^\rho V_2^{pn} V_2^r = U^{*p} V_1^\rho V_2^r$, and so $V_1^{*ql} V_2^{qk}$ is an isometry for any $q \geq mn$. Consequently, there is $i \geq 0$ minimal, such that $V_1^{*2^i l} V_2^{2^i k}$ is an isometry. Using Proposition 4.3 i -times we obtain a minimal extension to a pair $\tilde{V}_1, \tilde{V}_2$ such that $\tilde{V}_1^{*l} \tilde{V}_2^k$ is an isometry. Note that by the third property in Proposition 4.3 the operator $\tilde{V}_2^{*n} \tilde{V}_1^m$ is unitary since $V_2^{*n} V_1^m = U$ is unitary.

If V_1, V_2 are unilateral shifts, then by the first property in Proposition 4.3 extensions are unilateral shifts and so $\tilde{V}$ is a unilateral shift as well. $\square$

Since the proof of Theorem 4.5 is based on Proposition 4.3, the extension preserves all the properties described in Proposition 4.3. As a conclusion of Proposition 3.5 and Theorem 4.5 we get the extension of an arbitrary pair of commuting isometries.

Theorem 4.6. Any pair of commuting isometries (V_1, V_2) , for any relatively prime, positive integers m, n may be extended to a pair $(\hat{U}^k \hat{V}^n, \hat{U}^l \hat{V}^m)$ where $\hat{U}$ is a unitary operator commuting with an isometry $\hat{V}$ and (k, l) are unique integers such that $0 < k < n, 0 \leq l < m$ and $km - ln = 1$. Moreover, the extension may be chosen minimal.

5. Applications

By Theorem 4.6, any pair of commuting isometries may be extended to a pair of the form (‡). Hence an arbitrary pair is of the form $(U^l V^n|_H, U^l V^m|_H)$ where $H \in \text{Lat}(U^k V^n, U^l V^m)$. Since the model for pairs of the form $(U^l V^n, U^l V^m)$ is given by Remark 4.2, a description of $\text{Lat}(U^k V^n, U^l V^m)$ provides a model for an arbitrary pair of commuting isometries. Unfortunately $\text{Lat}(U^k V^n, U^l V^m)$ is characterized only for some cases like in Theorem 2.7. We show applications of this case and investigate $\text{Lat}(U^k V^n, U^l V^m)$ more generally.

Let a pair of commuting isometries (V_1, V_2) satisfy $V_1^m = UV_2^n$ where U is a unitary operator commuting with V_1, V_2 . By Remark 3.2 such a pair decomposes between a pair of unitary operators and a pair of unilateral shifts. In fact, any pair of unitary operators (U_1, U_2) satisfies the equality $U_1^m = UU_2^n$ where $U = U_2^{*n}U_1^m$. We focus on pairs of unilateral shifts satisfying $V_1^m = UV_2^n$. If U has an eigenvalue λ , then the corresponding eigenspace H_λ is reducing under V_1, V_2 by commutativity of U with V_1, V_2 . Moreover, $V_1^m|_{H_\lambda} = \lambda V_2^n|_{H_\lambda}$. Invariant subspaces of such pairs are described by the following corollary of Theorem 4.5.

Corollary 5.1. *Let a pair of commuting unilateral shifts (V_1, V_2) on a Hilbert space H satisfy $V_1^m = \lambda V_2^n$ for relatively prime, positive integers m, n and a complex number λ . Then there is a unilateral shift $\tilde{V} \in \mathcal{B}(\tilde{H})$ such that $H \subset \tilde{H}$ and $\text{Lat}(V_1, V_2) = \{H \cap \mathcal{N} : \mathcal{N} \in \text{Lat}(\tilde{V}^m, \tilde{V}^n)\}$ where $\text{Lat}(\tilde{V}^n, \tilde{V}^m)$ is described by Theorem 2.7.*

In the general case a similar but weaker result follows from Theorem 4.6. The result is weaker since there is no description similar to Theorem 2.7. We assume a general pair of commuting isometries. However, even if we assume a pair of unilateral shifts, an extension may be a general pair of isometries, recall Example 3.6.

Corollary 5.2. *For a commuting pair of isometries (V_1, V_2) on a Hilbert space H and positive, relatively prime integers m, n there are an isometry $\hat{V}$ commuting with a unitary operator $\hat{U}$ on a Hilbert space $\hat{H}$ such that*

$$V_1 = \hat{U}^k \hat{V}^n|_H, \quad V_2 = \hat{U}^l \hat{V}^m|_H \quad \text{and} \quad H \in \text{Lat}(\hat{U}^k \hat{V}^m, \hat{U}^l \hat{V}^n)$$

where the integers $0 < k < m, 0 \leq l < n$ are such that $km - ln = 1$. Moreover, $\text{Lat}(V_1, V_2) = \{H \cap \mathcal{N} : \mathcal{N} \in \text{Lat}(\hat{U}^k \hat{V}^n, \hat{U}^l \hat{V}^m)\}$.

Let us focus on $\text{Lat}(U^k V^n, U^l V^m)$. Since Theorem 4.6 provides the extension for any relatively prime, positive integers m, n , we may assume $m = n = 1$ and so $k = 1, l = 0$. In other words, we make an approach to a description of $\text{Lat}(UV, V)$ where U is a unitary operator commuting with a unilateral shift V . The advantage is that UV, V are unitarily equivalent. The following remark describes the lattice of subspaces invariant under a pair of unitarily equivalent isometries.

Remark 5.3. Let $V, W \in \mathcal{B}(H)$ where V is an isometry and W is a unitary operator (not commuting with V). Then

$$\text{Lat}(WVW^*, V) = \text{Lat}(V) \cap W\text{Lat}(V)$$

where $W\text{Lat}(V) = \{W\mathcal{M} : \mathcal{M} \in \text{Lat}(V)\}$. Indeed, it is an easy observation that $\mathcal{M}$ is invariant under WVW^* if and only if $W^*\mathcal{M}$ is invariant under V . Since W is unitary, $\{\mathcal{M} : W^*\mathcal{M} \in \text{Lat}(V)\} = W\text{Lat}(V)$.

Remark 5.3 is shown in the context of the pair (UV, V) . However, isometries $U^k V^n, U^l V^m$ are unilateral shifts of multiplicities $m \dim \ker V^*$ and $n \dim \ker V^*$ respectively, so are unitarily equivalent if $\dim \ker V^* = \infty$. Hence Remark 5.3 has wider applications. We continue investigating $\text{Lat}(UV, V)$. Assuming V to be a unilateral shift, the model for the pair (UV, V) given by Remark 4.2 is $(T_z \otimes \mathcal{U}, T_z \otimes I)$ on $H^2(\mathbb{T}) \otimes \mathcal{H}$. Recall that $P_i \in \mathcal{B}(H^2(\mathbb{T}))$ denotes the projection onto the space $\mathbb{C}z^i$.

Theorem 5.4. Let $V = T_z \otimes I, U = I \otimes \mathcal{U}$ and $W = \sum_{i \geq 0} P_i \otimes \mathcal{U}^i$ operate on $H^2(\mathbb{T}) \otimes \mathcal{H}$. The subspaces jointly invariant under (UV, V) are precisely those of the form $\hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H})$ where $\hat{\theta}$ is an inner function satisfying

$$\hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H}) = W\hat{\eta}(H^2(\mathbb{T}) \otimes \mathcal{H})$$

for some inner function $\hat{\eta}$.

Proof. Note that W is a unitary operator not commuting with V . Let $\hat{\eta}$ be an inner function. Then $W\hat{\eta}(z)$ is a partial isometry for almost every z with the same initial space as the initial space of $\hat{\eta}(z)$, so the same initial space for almost every z . However, $W\hat{\eta}$ may not commute with V and so it may not be an inner function. On the other hand, if $W\hat{\eta}(H^2(\mathbb{T}) \otimes \mathcal{H}) = \hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H})$ for $\hat{\theta}$ an inner function then $\hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H}) \in \text{Lat}(V) \cap W\text{Lat}(V)$. Moreover, there are no other subspaces in $\text{Lat}(V) \cap W\text{Lat}(V)$. Hence the theorem follows by Remark 5.3 provided we show $UV = WVW^*$. Check that

$$WVW^* = \left(\sum_{j \geq 0} P_j \otimes \mathcal{U}^j \right) (T_z \otimes I) \left(\sum_{i \geq 0} P_i \otimes \mathcal{U}^{*i} \right) = \sum_{i \geq 0} \sum_{j \geq 0} P_j T_z P_i \otimes \mathcal{U}^j \mathcal{U}^{*i}.$$

On the other hand, $T_z P_i H^2(\mathbb{T}) = P_{i+1} H^2(\mathbb{T})$ yields $P_j T_z P_i = 0$ for $j \neq i+1$ and $P_{i+1} T_z P_i = T_z P_i$. Thus

$$\begin{aligned} \sum_{i \geq 0} \sum_{j \geq 0} P_j T_z P_i \otimes \mathcal{U}^j \mathcal{U}^{*i} &= \sum_{i \geq 0} T_z P_i \otimes \mathcal{U} \\ &= (T_z \otimes \mathcal{U}) \left(\sum_{i \geq 0} P_i \otimes I \right) = T_z \otimes \mathcal{U} = UV. \quad \square \end{aligned}$$

As a corollary of Theorem 5.4 and Theorem 4.6, we get the following model.

Theorem 5.5. Any pair of commuting isometries have a minimal extension to a pair (UV, V) where U is a unitary operator commuting with an isometry V . Those of them for which V is a unilateral shift are of the form

$$((T_z \otimes I)|_{\mathcal{M}}, (T_z \otimes \mathcal{U})|_{\mathcal{M}})$$

where

- $\mathcal{U} \in \mathcal{B}(\mathcal{H})$ is a unitary operator,
- $\mathcal{M} = \hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H})$ where $\hat{\theta}$ is an inner function satisfying

$$\hat{\theta}(H^2(\mathbb{T}) \otimes \mathcal{H}) = W\hat{\eta}(H^2(\mathbb{T}) \otimes \mathcal{H})$$

for some inner function $\hat{\eta}$ and $W = \sum_{i \geq 0} P_i \otimes \mathcal{U}^i$.

According to Example 3.6 the theorem above does not give a universal model even for pairs of unilateral shifts. However, it describes interesting examples, like operators of multiplication by independent variables on the Hardy space over the torus and their restrictions to invariant subspaces. We finish by a detailed investigation of this example. According to Example 3.7 the subspace $\mathcal{H}$ described in the model in Theorem 5.5 equals $L^2(\mathbb{T})$. Hence we get operators on $H^2(\mathbb{T}) \otimes L^2(\mathbb{T})$ which, via a unitary equivalence, may be considered as a subspace of $L^2(\mathbb{T}^2)$.

Theorem 5.6. *Let (L_w, L_z) be operators of multiplication by independent variables on $L^2(\mathbb{T}^2)$. Set $\Delta^2(\mathbb{T}^2) := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } j < -i\}$ and $W \in \mathcal{B}(\Delta^2(\mathbb{T}^2))$ defined by $Ww^iz^j = w^{2i+j}z^{-i}$. Then*

$$\text{Lat}(L_w|_{\Delta^2(\mathbb{T}^2)}, L_z|_{\Delta^2(\mathbb{T}^2)}) = \text{Lat}(L_w|_{\Delta^2(\mathbb{T}^2)}) \cap W\text{Lat}(L_w|_{\Delta^2(\mathbb{T}^2)}).$$

Proof. Note that $L_w = L_{w\bar{z}}L_z$ and the subspace $\Delta^2(\mathbb{T}^2)$ is reducing under $L_{w\bar{z}}$ and invariant under L_w, L_z . The model for $L_z|_{\Delta^2(\mathbb{T}^2)}$ yields $\Delta^2(\mathbb{T}^2) \simeq H^2(\mathbb{T}) \otimes \mathcal{H}$ and $L_z|_{\Delta^2(\mathbb{T}^2)} \simeq T_v \otimes I$ where

$$\mathcal{H} = \ker(L_z|_{\Delta^2(\mathbb{T}^2)})^* = \ker(L_w|_{\Delta^2(\mathbb{T}^2)})^* = \overline{\text{span}}\{w^iz^{-i} : i \in \mathbb{Z}\}$$

and v is the variable in $\mathbb{T}$ while (w, z) remains assigned to $\mathbb{T}^2$. Moreover, $L_z^t(\ker(L_z|_{\Delta^2(\mathbb{T}^2)})^*) \simeq (T_v \otimes I)^t(1 \otimes \mathcal{H}) = v^t \otimes \mathcal{H}$ and so

$$\Delta^2(\mathbb{T}^2) \ni w^iz^j = w^iz^{-i+(i+j)} = L_z^{i+j}w^iz^{-i} \simeq v^{i+j} \otimes w^iz^{-i} \in H^2(\mathbb{T}) \otimes \mathcal{H}.$$

One can check that $\mathcal{H}$ reduces $L_{w\bar{z}}$ and $L_{w\bar{z}}|_{\Delta^2(\mathbb{T}^2)} \simeq I \otimes L_{w\bar{z}}|_{\mathcal{H}}$ and $L_w|_{\Delta^2(\mathbb{T}^2)} \simeq T_v \otimes L_{w\bar{z}}|_{\mathcal{H}}$. Then by Theorem 5.4 $L_w|_{\Delta^2(\mathbb{T}^2)} = WL_z|_{\Delta^2(\mathbb{T}^2)}W^*$ where $W \simeq \sum_{n \geq 0} P_n \otimes (L_{w\bar{z}}|_{\mathcal{H}})^n$. Hence

$$\begin{aligned} Ww^iz^j &\simeq \sum_{n \geq 0} P_n \otimes (L_{w\bar{z}}|_{\mathcal{H}})^n (v^{i+j} \otimes w^iz^{-i}) = \sum_{n \geq 0} (P_n v^{i+j} \otimes w^{i+n}z^{-i-n}) \\ &= v^{i+j} \otimes w^{2i+j}z^{-2i-j} \simeq w^{2i+j}z^{-i}. \end{aligned}$$

Remark 5.3 finishes the proof. $\square$

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References

- [1] Berger, C.A., Coburn, L.A., Lebow, A.: Representation and index theory for C^* -algebras generated by commuting isometries. *J. Funct. Anal.* **27**, 51–99 (1978)
- [2] Beurling, : On two problems concerning linear transformations in Hilbert spaces. *Acta. Math.* **81**, 239–255 (1949)
- [3] Burdak, Z., Kosiek, M., Słociński, M.: Compatible pairs of commuting isometries. *Linear Algebra Appl.* **479**, 216–259 (2015)
- [4] Burdak, Z., Kosiek, M., Pagacz, P., Słociński, M.: Invariant subspaces of $\mathcal{H}^2(\mathbb{T}^2)$ and $L^2(\mathbb{T}^2)$ preserving compatibility. *J. Math. Anal. Appl.* **455**, 1706–1719 (2017)
- [5] Gaspar, D., Suciú, N.: On invariant subspaces on bitorus. *J. Oper. Theory* **30**, 227–241 (1993)
- [6] Halmos, P.R.: Shifts on Hilbert spaces. *J. Reine Angew. Math.* **208**, 102–112 (1961)
- [7] Izuchi, K., Nakazi, T., Seto, M.: Backward shift invariant subspaces in the bidisc. II. *J. Oper. Theory* **51**, 361–376 (2004)
- [8] Lax, P.D.: Translation invariant spaces. *Acta Math.* **101**, 163–178 (1959)
- [9] Mandrekar, V.: The validity of beurling theorems in poldiscs. *Proc. Am. Math. Soc.* **103**, 145–148 (1988)
- [10] Horák, Karel, Müller, Vladimír: Functional model for commuting isometries. *Czechoslovak Math. J.* **39**(2), 370–379 (1989)
- [11] Rudin, W.: *Function Theory in Polydiscs*. Benjamin, New York (1969)
- [12] Radjavi, H., Rosenthal, P.: *Invariant Subspaces*. Springer, Berlin (1973)
- [13] Sz.-Nagy, B., Foiaş, C.: *Harmonic Analysis of operators on Hilbert space*, North-holland publishing company, Amsterdam, London, (1970)
- [14] Sarkar, J.: Wold decomposition for doubly commuting isometries. *Linear Algebra Appl.* **445**, 289–301 (2014)
- [15] Sarkar, J., Sasane, A., Wick, B.: Doubly commuting submodules of the Hardy module over polydiscs. *Studia Math.* **2791**, 179–192 (2013)
- [16] Seto, M., Yang, R.W.: Inner sequence based invariant subspaces in $H^2(D^2)$. *Proc. Am. Math. Soc.* **135**, 2519–2526 (2007)
- [17] Słociński, M.: On the Wold type decomposition of a pair of commuting isometries. *Ann. Polon. Math.* **37**, 255–262 (1980)
- [18] Wold, H.: *A Study in the Analysis of Stationary Time Series*. Almkvist and Wiksell, Stockholm (1954)

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