Sharp Gaussian Estimates for Heat Kernels of Schrödinger Operators

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Abstract. We characterize functions $V \leq 0$ for which the heat kernel of the Schrödinger operator $\Delta + V$ is comparable with the Gauss–Weierstrass kernel uniformly in space and time. In dimension 4 and higher the condition turns out to be more restrictive than the condition of the boundedness of the Newtonian potential of V . This resolves the question of V. Liskevich and Y. Semenov posed in 1998. We also give specialized sufficient conditions for the comparability, showing that local L^p integrability of V for $p > 1$ is not necessary for the comparability.

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1. Introduction and Main Results

Let $d = 1, 2, \dots$. We consider the Gauss–Weierstrass kernel,

$$g(t, x, y) = (4\pi t)^{-d/2} e^{-|y-x|^2/(4t)}, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

It is well known that g is the fundamental solution of the equation $\partial_t u = \Delta u$, and time-homogeneous probability transition density – the heat kernel of Δ . For Borel measurable function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ we call $G : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty]$ the heat kernel of $\Delta + V$ or the Schrödinger perturbation of g by V , if the following Duhamel or perturbation formula holds for $t > 0$, $x, y \in \mathbb{R}^d$,

$$G(t, x, y) = g(t, x, y) + \int_0^t \int_{\mathbb{R}^d} G(s, x, z) V(z) g(t-s, z, y) dz ds.$$

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Under appropriate assumptions on V , explicit definition of G may be given by means of the Feynmann-Kac formula [5, Section 6], the Trotter formula [29, p. 467], the perturbation series [5], or by means of quadratic forms on L^2 spaces [10, Section 4]. In particular the assumption $V \in L^p(\mathbb{R}^d)$ with $p > d/2$ was used by Aronson [2], Zhang [29, Remark 1.1(b)] and by Dziubański and Zienkiewicz [11]. Aizenman and Simon [1, 23] proposed functions $V(z)$ from the Kato class, which contains $L^p(\mathbb{R}^d)$ for every $p > d/2$ [1, Chapter 4], see also Chung and Zhao [9, Chapter 3, Example 2]. An enlarged Kato class was used by Voigt [25] in the study of Schrödinger semigroups on L^1 [25, Proposition 5.1]. For perturbations by time-dependent functions $V(u, z)$, Zhang [26, 28] introduced the so-called parabolic Kato condition. The condition was then generalized and employed by Schnaubelt and Voigt [21], Liskevich and Semenov [18], Milman and Semenov [20], Liskevich et al. [19], and Gulisashvili and van Casteren [14].

Given the function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ we ask if there are positive numbers, i.e., *constants* $0 < c_1 \leq c_2 < \infty$ such that the following two-sided bound holds,

$$c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad t > 0, \quad x, y \in \mathbb{R}^d. \quad (1)$$

One can also ponder a weaker property – if for a given $T \in (0, \infty)$,

$$c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad 0 < t \leq T, \quad x, y \in \mathbb{R}^d. \quad (2)$$

We call (1) and (2) *sharp Gaussian estimates* or bounds, respectively *global* (or uniform) and *local* in time. We observe that the inequalities in (1) and (2) are stronger than the plain *Gaussian estimates*:

$$c_1 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t\varepsilon_1}} \leq G(t, x, y) \leq c_2 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t\varepsilon_2}}, \quad x \in \mathbb{R}^d,$$

where $0 < \varepsilon_1, c_1 \leq 1 \leq \varepsilon_2, c_2 < \infty$, which can also be global or local in time.

Berenstein proved the plain Gaussian estimates for $V \in L^p$ with $p > d/2$ (see [17]). Simon [23, Theorem B.7.1] resolved them for V in the Kato class, Zhang [28] and Milman and Semenov [20] applied the parabolic Kato class for this purpose. For further discussion we refer the reader to [18–20, 29] and [6, Lemma 4]. We also refer to Bogdan and Szczypkowski [7, Section 1, 4] for a survey of the plain Gaussian bounds for Schrödinger heat kernels along with a streamlined approach, new results and explicit constants based on the so-called 4G inequality.

The plain Gaussian estimates are ubiquitous in analysis but (1) and (2) provide precious qualitative information, if they hold for V . It is intrinsically difficult to characterize (1) and (2) for those V that take on positive values, while the case of $V \leq 0$ is more manageable. Arsen'ev proved (2) for $V \in L^p + L^\infty$ with $p > d/2, d \geq 3$. Van Casteren [24] proved (2) for V in the intersection of the Kato class and $L^{d/2} + L^\infty$ for $d \geq 3$ (see [20]). Arsen'ev also obtained (1) for $V \in L^p$ with $p > d/2$ under additional smoothness assumptions (see [17]). Liskevich and Semenov stated sufficient conditions for (1) and (2) in [17, Theorem 1, Corollary 1, Theorem 2]. Zhang [29, Theorem 1.1] and Milman

and Semenov [20, Theorem 1C, Remark (2)] gave sufficient integro-supremal conditions for (1) and (2) for general V and characterized (1) and (2) for $V \leq 0$. It will be convenient to state the conditions by means of

$$S(V, t, x, y) = \int_0^t \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t-s, z, y)}{g(t, x, y)} |V(z)| dz ds, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

The motivation for using this quantity comes from Zhang [29, Lemma 3.1 and Lemma 3.2] and from Bogdan et al. [5, (1)]. We often write $S(V)$ if we do not need to specify t, x, y . As explained in Sect. 4, $S(V)$ is the potential of $|V|$ for the so-called Gaussian bridges. We also note that [5, Section 6] uses $S(V)$ for general transition densities. The next two results indicate why $S(V)$ is important. Their proofs are given in Sect. 2.

Lemma 1.1. *Let $V \leq 0$. Then (1) is equivalent to*

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty. \quad (3)$$

Also, for each $T \in (0, \infty)$, (2) is equivalent to

$$\sup_{0 < t \leq T, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty. \quad (4)$$

We say that V satisfying (3) or (4) has bounded potential for bridges (is bridge-potential bounded) globally or locally in time, respectively.

Lemma 1.2. *If for some $h > 0$ and $0 \leq \eta < 1$ we have*

$$\sup_{0 < t \leq h, x, y \in \mathbb{R}^d} S(V^+, t, x, y) \leq \eta,$$

and if $S(V^-)$ is bounded on bounded subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, then

$$e^{-S(V^-, t, x, y)} \leq \frac{G(t, x, y)}{g(t, x, y)} \leq \left(\frac{1}{1 - \eta} \right)^{1+t/h}, \quad t > 0, \quad x, y \in \mathbb{R}^d. \quad (5)$$

We also notice the following consequence of the Duhamel formula.

Remark 1.3. If $V \geq 0$, then (1) implies (3) and (2) implies (4).

For clarity we note that $S(V)$ is unbounded for all nontrivial V in dimensions $d = 1$ and 2 , hence (1) is impossible for nontrivial $V \geq 0$ and nontrivial $V \leq 0$ in these dimensions. This is explained at the end of Sect. 2 below.

1.1. Characterization of Sharp Gaussian Estimates

The conditions in Lemmas 1.1 and 1.2 may be cumbersome to verify. For this reason we propose a simpler integro-supremal test for (1). For $d \geq 3$ and $x, y \in \mathbb{R}^d$ we define

$$K(V, x, y) = \int_{\mathbb{R}^d} |V(z)| K(z - x, y) dz,$$

where

$$K(x, y) = \frac{e^{-(|x||y| - x \cdot y)/2}}{|x|^{d-2}} (1 + |x||y|)^{d/2-3/2}, \quad (6)$$

and $x \cdot y$ is the usual scalar product. We denote, as usual,

$$\begin{aligned} \|S(V)\|_\infty &= \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y), \\ \|K(V)\|_\infty &= \sup_{x, y \in \mathbb{R}^d} K(V, x, y). \end{aligned}$$

These two integro-supremal quantities turn out to be comparable, as follows.

Theorem 1.4. *There are constants M_1, M_2 depending only on d , such that*

$$M_1 \|K(V)\|_\infty \leq \|S(V)\|_\infty \leq M_2 \|K(V)\|_\infty. \quad (7)$$

The proof of Theorem 1.4 is given in Sect. 3. By (7) and Lemma 1.1 we get the following characterization of the sharp global Gaussian estimates.

Corollary 1.5. *If $V \leq 0$, then (1) holds if and only if $K(V)$ is bounded.*

Similarly, for general (signed) V we get (1) provided $\|K(V^-)\|_\infty < \infty$ and $\|K(V^+)\|_\infty < 1/M_2$. This follows from Lemma 1.2 and Theorem 1.4.

We next elaborate on more specific applications of $K(V)$ to sharp global Gaussian estimates. In particular we resolve a long-standing open problem posed by Liskevich and Semenov. For $d \geq 3$ we let $C_d = \Gamma(d/2 - 1)/(4\pi^{d/2})$. The Newtonian kernel is $\int_0^\infty g(s, x, z) ds = C_d |z - x|^{2-d}$, $x, z \in \mathbb{R}^d$, and the Newtonian potential of (a nonnegative) function f at $x \in \mathbb{R}^d$ is denoted

$$-\Delta^{-1}f(x) := \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z) f(z) dz ds = \int_{\mathbb{R}^d} \frac{C_d}{|z - x|^{d-2}} f(z) dz.$$

For $d = 3$ the formula (6) considerably simplifies and we easily obtain

$$\|K(V)\|_\infty = C_d^{-1} \|\Delta^{-1}|V|\|_\infty \quad \text{for } d = 3. \quad (8)$$

Thus if $d = 3$ and $V \leq 0$, then by Corollary 1.5 the sharp global Gaussian bounds (1) are equivalent to the condition of potential-boundedness, namely $\|\Delta^{-1}V\|_\infty < \infty$. This is classical [20, Remark (3) on p. 4] but remarkable, because the Newtonian kernel is isotropic, that is rotation-invariant, while $K(V)$ and $S(V)$ have a certain anisotropy-sensitivity.

Putting aside the exceptional, the main focus of the present paper is on $d \geq 4$. As usual, we let

$$\|V\|_{d/2} = \left(\int_{\mathbb{R}^d} |V(z)|^{d/2} dz \right)^{2/d}.$$

By [17, Theorem 2] and [20, Remark (1) and (4) on p. 4] we have (1) for $d \geq 4$ if $\|\Delta^{-1}V^-\|_\infty + \|V^-\|_{d/2} < \infty$, $\|\Delta^{-1}V^+\|_\infty < 1$ and $\|V^+\|_{d/2}$ is small enough. A long-standing open problem for (1) with $V \leq 0$ posed in 1998 by Liskevich and Semenov [17, p. 602] reads as follows: “The validity of the two-sided estimates for the case $d > 3$ without the additional assumption $V \in L^{d/2}$ is an open question.” In view of Theorem 1.4 and Lemma 1.1 the question is whether for $d \geq 4$ the finiteness of $\|\Delta^{-1}V\|_\infty$ implies the finiteness of $\|K(V)\|_\infty$. Our next estimate is a variant of [17, Corollary 1] and closely relates $\|K(V)\|_\infty$ to $\|\Delta^{-1}V\|_\infty$ and $\|V\|_{d/2}$ for $d \geq 4$:

$$C_d^{-1} \|\Delta^{-1}|V|\|_\infty \leq \|K(V)\|_\infty \leq 2^{(d-3)/2} \left(C_d^{-1} \|\Delta^{-1}|V|\|_\infty + \kappa_d \|V\|_{d/2} \right). \quad (9)$$

In Sect. 3 we prove (9) and the following result, which points out a gap between $\|K(V)\|_\infty$ and $\|\Delta^{-1}|V|\|_\infty$ in (9).

Proposition 1.6. *Let $d \geq 4$. For $z = (z_1, z_2, \dots, z_d) \in \mathbb{R}^d$ we write $z = (z_1, \mathbf{z}_2)$, where $\mathbf{z}_2 = (z_2, \dots, z_d) \in \mathbb{R}^{d-1}$. We define*

$$A = \{(z_1, \mathbf{z}_2) \in \mathbb{R}^d : z_1 > 4, |\mathbf{z}_2| \leq \sqrt{z_1}\}, \quad \text{and} \\ V(z_1, \mathbf{z}_2) = -\frac{1}{z_1} \mathbf{1}_A(z_1, \mathbf{z}_2).$$

Then $\|\Delta^{-1}V\|_\infty < \infty$ but $\|K(V)\|_\infty = \infty$. There is even a function $V \leq 0$ with compact support such that $\|\Delta^{-1}V\|_\infty < \infty$ but $\|K(V)\|_\infty = \infty$.

From Lemma 1.1 we conclude that for $d \geq 4$ neither finiteness nor smallness of $\|\Delta^{-1}V\|_\infty$ are sufficient for (1). Therefore the answer to the question of Liskevich and Semenov is negative.

Here are a few more comments that relate our result to existing literature and serve as preparation for the proofs. Due to the work of Zhang [29], the following quantity is a proxy for $S(V)$,

$$N(V, t, x, y) := \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ + \int_{t/2}^t \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4(t-\tau))}}{(t-\tau)^{d/2}} |V(z)| dz d\tau = N(V, t, y, x), \quad (10)$$

where $t > 0$, $x, y \in \mathbb{R}^d$. Indeed, by [29, Lemma 3.1, Lemma 3.2, and line 11 on p. 469] there are constants m_1, m_2 depending only on d such that

$$S(V, t, x, y) \geq m_1 N(V, t/2, x, y), \quad t > 0, \quad x, y \in \mathbb{R}^d, \quad (\text{L})$$

$$S(V, t, x, y) \leq m_2 N(V, t, x, y), \quad t > 0, \quad x, y \in \mathbb{R}^d. \quad (\text{U})$$

We also let $\|N(V)\|_\infty = \sup_{t>0, x, y \in \mathbb{R}^d} N(V, t, x, y)$. By (L) and (U) we get

$$m_1 \|N(V)\|_\infty \leq \|S(V)\|_\infty \leq m_2 \|N(V)\|_\infty.$$

In [20, Theorem 1C] and [17, (8)] another quantity was used to study (1) and (2),

$$e_*(V, \lambda) = \sup_{\alpha \in \mathbb{R}^d} \|V(\lambda - \Delta + 2\alpha \cdot \nabla)^{-1}\|_{1 \rightarrow 1} \\ = \sup_{\alpha \in \mathbb{R}^d} \|(\lambda - \Delta + 2\alpha \cdot \nabla)^{-1}|V|\|_\infty,$$

where $\lambda \geq 0$. It may be given in terms of the Gauss–Weierstrass kernel, e.g.,

$$e_*(V, 0) = (4\pi)^{-d/2} \sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - x, y) |V(z)| dz,$$

where

$$J(x, y) = \int_0^\infty \tau^{-d/2} e^{-\frac{|x-\tau y|^2}{4\tau}} d\tau, \quad x, y \in \mathbb{R}^d.$$

There is a certain anisotropy-sensitivity of $e_*(V, 0)$ due to $2\alpha \cdot \nabla$ above, which is similar to that of $K(V)$. In fact, in Lemma 3.2 below we prove that there are constants n_1, n_2 depending only on $d \geq 3$ such that

$$n_1 e_*(V, 0) \leq \|S(V)\|_\infty \leq n_2 e_*(V, 0). \quad (11)$$

In view of Theorem 1.4, the quantities $\|S(V)\|_\infty$, $\|K(V)\|_\infty$, $\|N(V)\|_\infty$, $e_*(V, 0)$ are all comparable, which makes them equivalent for studying (1) with $V \leq 0$.

We add a few comments on the exceptional case $d = 3$. By [20, (3) in Remark on p. 4] and (8) we have $e_*(V, 0) = \|\Delta^{-1}|V|\|_\infty = C_d \|K(V)\|_\infty$. Also, for $d = 3$ by [20, Remark (1) and (3) on p. 4], the condition

$$\|\Delta^{-1}V^-\|_\infty < \infty, \quad \|\Delta^{-1}V^+\|_\infty < 1, \quad (12)$$

suffices for (1). Furthermore, if $V \leq 0$, then the condition $\|\Delta^{-1}V\|_\infty < \infty$ characterizes the plain global Gaussian bounds, see [22] and [30, p. 556 and Corollary A]. Therefore by (8), for $d = 3$ the plain global Gaussian bounds hold for $V \leq 0$ if and only if the sharp global Gaussian bounds hold. In contrast, for $d \geq 4$ by Proposition 1.6 the plain global Gaussian bounds may occur in the absence of the sharp global Gaussian bounds (1).

We recall that $\sup_{0 < t \leq T, x, y \in \mathbb{R}^d} S(V, t, x, y)$ with $T < \infty$ is useful for the local in time sharp Gaussian estimates (2), see Lemmas 1.1 and 1.2. In a similar fashion $\sup_{0 < t \leq T, x, y \in \mathbb{R}^d} N(V, t, x, y)$ is used in [29, Theorem 1.1], while in [20, Theorem 1C] the authors make use of $e_*(V, \lambda)$ for $\lambda > 0$. In this connection see also Corollary 2.4 below.

1.2. Sufficient Conditions for Sharp Gaussian Estimates

In this section we propose sufficient conditions for (1) and (2) for functions V which have a form of the tensor product. Such conditions are the second main topic of the paper—they culminate in Theorem 1.8 below. We also show that L^p integrability for $p > 1$ is not necessary for (1) or (2). Let $p, p_1, p_2 \in [1, \infty]$.

Definition 1.7. We write $f \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$ if there are $f_1 \in L^{p_1}(\mathbb{R}^{d_1})$ and $f_2 \in L^{p_2}(\mathbb{R}^{d_2})$, such that

$$f(x_1, x_2) = f_1(x_1)f_2(x_2), \quad x_1 \in \mathbb{R}^{d_1}, \quad x_2 \in \mathbb{R}^{d_2}.$$

We note that $L^p(\mathbb{R}^{d_1}) \times L^p(\mathbb{R}^{d_2}) \subset L^p(\mathbb{R}^{d_1+d_2})$, in fact $\|f\|_p = \|f_1\|_p \|f_2\|_p$ if f is the tensor product $f(x_1, x_2) = f_1(x_1)f_2(x_2)$.

Theorem 1.8. Let $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, $V: \mathbb{R}^d \rightarrow \mathbb{R}$, $p_1, p_2 \in [1, \infty]$ and

$$\frac{d_1}{2p_1} + \frac{d_2}{2p_2} = 1.$$

(a) If $r \in (p_1, \infty]$ and $V \in L^r(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$, then

$$\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \leq c t^{1-d_1/(2r)-d_2/(2p_2)},$$

where $c = C(d_1, r)C(d_2, p_2) \frac{[\Gamma(1-d_1/(2r)-d_2/(2p_2))]^2}{\Gamma(2-d_1/r-d_2/p_2)} \|V_1\|_r \|V_2\|_{p_2}$.

(b) If $1 \leq q < p_1 < r \leq \infty$ and $V \in [L^q(\mathbb{R}^{d_1}) \cap L^r(\mathbb{R}^{d_1})] \times L^{p_2}(\mathbb{R}^{d_2})$, then (3) holds.

The proof of Theorem 1.8 is given in Sect. 4, where we use in a crucial way the tensorization of the Gauss–Weierstrass kernel and its bridges. Lemmas 1.1 and 1.2 provide the following conclusion.

Corollary 1.9. *Under the assumptions of Theorem 1.8(a), G satisfies the sharp local Gaussian bounds (2). If $V \leq 0$ and the assumptions of Theorem 1.8(b) hold, then G has the sharp global Gaussian bounds (1).*

Clearly, if $|U| \leq |V|$, then $S(U) \leq S(V)$. This may be used to extend the conclusions of Theorem 1.8 and Corollary 1.9 beyond tensor products.

Proposition 1.10. *For every $d \geq 3$ there is a function $V \leq 0$ such that (1) holds but $V \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L^p_{loc}(\mathbb{R}^d)$.*

In particular (1) does not necessitate $\|V\|_{d/2} < \infty$, i.e., the finiteness of $\|K(V)\|_\infty$ does not imply that of $\|V\|_{d/2}$; see also (9) in this connection. We note in passing that local L^1 integrability is necessary for (2) if V does not change sign, cf. Lemmas 1.1 and 2.1, and Remark 1.3. The function V in Proposition 1.10 is constructed in Sect. 5 from highly anisotropic tensor products of power functions.

The structure of the remainder of the paper is as follows. In Sect. 2 we provide definitions and preliminaries, in particular we prove Lemmas 1.1 and 1.2. In Sect. 3 we prove Theorem 1.4 and Proposition 1.6. In Sect. 4 we prove Theorem 1.8. In Sect. 5 we prove Proposition 1.10 and give examples which illustrate and comment on our results.

2. Preliminaries

We let $\mathbb{N} = \{1, 2, \dots\}$, $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$. Recall that $d \in \mathbb{N}$ and V is an arbitrary Borel measurable function: $\mathbb{R}^d \rightarrow \mathbb{R}$.

We begin with the following observations on integrability and potential-boundedness (14) of functions V which are bridges potential-bounded.

Lemma 2.1. *If $S(V, t, x, y) < \infty$ for some $t > 0$, $x, y \in \mathbb{R}^d$, then $V \in L^1_{loc}(\mathbb{R}^d)$. If (4) holds, then*

$$\sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s, x, z) |V(z)| dz ds < \infty. \quad (13)$$

If (3) even holds, then

$$\sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z) |V(z)| dz ds < \infty. \quad (14)$$

Proof. The first statement follows, because $g(t, x, y)$ is locally bounded from below on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ (see [13, Lemma 3.7] for a quantitative general result). Since $\int_{\mathbb{R}^d} S(V, t, x, y) g(t, x, y) dy = \int_0^t \int_{\mathbb{R}^d} g(s, x, z) |V(z)| dz ds$, we see that (4) implies (13) and (3) implies (14). $\square$

We shall use the following functions:

$$f(t) = \sup_{x,y \in \mathbb{R}^d} S(V, t, x, y), \quad t \in (0, \infty),$$

$$F(t) = \sup_{0 < s < t} f(s) = \sup_{\substack{x,y \in \mathbb{R}^d \\ 0 < s < t}} S(V, s, x, y), \quad t \in (0, \infty].$$

We fix V and $x, y \in \mathbb{R}^d$. For $0 < \varepsilon < t$, we consider

$$S(V, t - \varepsilon, x, y) = \int_0^t \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t - \varepsilon - s, z, y)}{g(t - \varepsilon, x, y)} |V(z)| \mathbf{1}_{[0, t - \varepsilon]}(u) dz ds.$$

By Fatou's lemma we get

$$S(V, t, x, y) \leq \liminf_{\varepsilon \rightarrow 0} S(V, t - \varepsilon, x, y),$$

meaning that $(0, \infty) \ni t \mapsto S(V, t, x, y)$ is lower semicontinuous on the left. It follows that f is lower semi-continuous on the left, too. In consequence, $f(t) \leq F(t)$ and $F(t) = \sup_{0 < s \leq t} f(s)$ for $0 < t < \infty$.

We next claim that f is sub-additive, that is,

$$f(t_1 + t_2) \leq f(t_1) + f(t_2), \quad t_1, t_2 > 0. \quad (15)$$

This follows from the Chapman–Kolmogorov equations for g . Indeed, we have $S(V, t_1 + t_2, x, y) = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_0^{t_1} \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t_1 + t_2 - s, z, y)}{g(t_1 + t_2, x, y)} |V(z)| dz ds \\ &= \int_0^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t_1 - s, z, w)g(t_2, w, y)g(t_1, x, w)}{g(t_1 + t_2, x, y)g(t_1, x, w)} |V(z)| dw dz ds \\ &\leq \int_{\mathbb{R}^d} \frac{g(t_2, w, y)g(t_1, x, w)}{g(t_1 + t_2, x, y)} S(V, t_1, x, w) dw \leq f(t_1), \end{aligned}$$

and I_2 equals

$$\begin{aligned} &\int_{t_1}^{t_1+t_2} \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t_1 + t_2 - s, z, y)}{g(t_1 + t_2, x, y)} |V(z)| dz ds \\ &= \int_{t_1}^{t_1+t_2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(t_1, x, w)g(s - t_1, w, z)g(t_2 - (s - t_1), z, y)g(t_2, w, y)}{g(t_1 + t_2, x, y)g(t_2, w, y)} |V(z)| dw dz ds \\ &\leq \int_{\mathbb{R}^d} \frac{g(t_1, x, w)g(t_2, w, y)}{g(t_1 + t_2, x, y)} S(V, t_2, w, y) dw \leq f(t_2). \end{aligned}$$

This yields (15).

Lemma 2.2. *For all $t, h > 0$ we have $f(t) \leq F(h) + t f(h)/h$.*

Proof. Let $k \in \mathbb{N}$ be such that $(k - 1)h < t \leq kh$, and let $\theta = t - (k - 1)h$. Then $t = \theta + (k - 1)h$, and by (15) we get

$$f(t) \leq f(\theta) + t f(h)/h \leq F(h) + t f(h)/h,$$

since $0 < \theta \leq h$. □

Corollary 2.3. *$F(t) \leq F(h) + t F(h)/h$ and $F(2t) \leq 2F(t)$ for $t, h > 0$.*

We may now prove Lemmas 1.1 and 1.2.

Proof of Lemma 1.2. If $V \leq 0$ then $0 \leq G \leq g$ is constructed in [29, p. 470], and the Duhamel formula follows from the finiteness of $S(V^-)$ and the discussion after [29, (3.3)]. Then the left-hand side of (5) follows from [29, pp. 467-468], or we can use [5, (41)], which results therein from Jensen's inequality and the second displayed formula on page 252 of [5]. For general, i.e., for signed V , the kernel G is constructed by applying the above procedure to g and $-V^-$, and then perturbing the resulting kernel by V^+ . The latter is done by means of the perturbation series, cf. [5, Lemma 2]; then the Duhamel formula is obtained without further conditions. We now prove the right hand side of (5), and without loss of generality we may assume that $V \geq 0$. For $0 < s < t$, $x, y \in \mathbb{R}^d$, we let $p_0(s, x, t, y) = g(t - s, x, y)$ and $p_n(s, x, t, y) = \int_s^t \int_{\mathbb{R}^d} p_{n-1}(s, x, u, z) V(z) p_0(u, z, t, y) dz du$, $n \in \mathbb{N}$. Let $Q : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ satisfy $Q(u, r) + Q(r, v) \leq Q(u, v)$. By [15, Theorem 1] (see also [6, Theorem 3]) if there is $0 < \eta < 1$ such that

$$p_1(s, x, t, y) \leq [\eta + Q(s, t)] p_0(s, x, t, y), \quad (16)$$

then

$$\tilde{p}(s, x, t, y) := \sum_{n=0}^{\infty} p_n(s, x, t, y) \leq \left(\frac{1}{1-\eta} \right)^{1+\frac{Q(s,t)}{\eta}} p_0(s, x, t, y). \quad (17)$$

Corollary 2.3 and the assumptions of the lemma imply that (16) is satisfied with $\eta = F(h) < 1$ and $Q(s, t) = (t-s)F(h)/h$. Since $G(t, x, y) = \tilde{p}(0, x, t, y)$, the proof of (5) is complete (see also [5, (17)]). $\square$

Proof of Lemma 1.1. If (2) holds then Duhamel formula and nonnegativity of G yield (4). Similarly, (1) implies (3). The reverse implications follow from (5). $\square$

As a consequence of Corollary 2.3 we also obtain the following result.

Corollary 2.4. *Let $V \leq 0$ and $T > 0$. Then (2) holds if and only if*

$$C e^{-ct} g(t, x, y) \leq G(t, x, y), \quad t > 0, x, y \in \mathbb{R}^d, \quad (18)$$

for some constants C and c . In fact we can take

$$\ln C = - \sup_{\substack{x, y \in \mathbb{R}^d \\ 0 < t \leq T}} S(V, t, x, y) \quad \text{and} \quad c = \frac{1}{T} \sup_{x, y \in \mathbb{R}^d} S(V, T, x, y).$$

Proof. (18) implies (2) for every fixed $T > 0$. Conversely, if (2) holds for fixed $T > 0$, then by Lemmas 1.2 and 2.2 we have

$$\frac{G(t, x, y)}{g(t, x, y)} \geq e^{-S(V, t, x, y)} \geq e^{-f(t)} \geq e^{-F(T)} e^{-tf(T)/T}. \quad \square$$

We note in passing that the above proof shows that (2) is determined by the behavior of $\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y)$ for small $t > 0$. We also see that (14) and thus (3) fail in dimensions $d = 1$ and $d = 2$, because then $\int_0^\infty g(s, x, z) ds \equiv \infty$, unless $V = 0$ a.e. From Lemma 1.1 and Remark 1.3 it follows that (1) fails for nontrivial $V \leq 0$ and for nontrivial $V \geq 0$ if $d = 1$ or 2.

3. Characterization of the Sharp Global Gaussian Estimates

In this section we prove our main result, i.e., Theorem 1.4. We start by using $N(V, t)$, (U) and (L), to estimate $S(V, t)$.

Lemma 3.1. *Let $t > 0$. We have*

$$\int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \leq N(V, t, x, y), \quad x, y \in \mathbb{R}^d,$$

and

$$\sup_{x, y} N(V, t, x, y) \leq 2 \sup_{x, y} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau.$$

Proof. The first inequality follows by the definition of $N(V, t, x, y)$. For the proof of the second one we note that

$$\begin{aligned} & \int_{t/2}^t \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4(t-\tau))}}{(t-\tau)^{d/2}} |V(z)| dz d\tau \\ &= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-x+(\tau/t)(x-y)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau. \end{aligned} \quad \square$$

Lemma 3.2. *We have*

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 (4\pi)^{d/2} e_*(V, 0),$$

and

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq 2 m_2 (4\pi)^{d/2} e_*(V, 0).$$

Proof. By (L) and Lemma 3.1,

$$\begin{aligned} & \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \geq m_1 \sup_{t>0, x, y \in \mathbb{R}^d} N(V, t/2, x, y) \\ & \geq m_1 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(2\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ & = m_1 \sup_{t>0, y, w \in \mathbb{R}^d} \int_0^{t/4} \int_{\mathbb{R}^d} \frac{e^{-|z-y+\tau w|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ & = m_1 \sup_{y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z-y, w) |V(z)| dz = m_1 (4\pi)^{d/2} e_*(V, 0). \end{aligned}$$

By (U) and Lemma 3.1,

$$\begin{aligned} & \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq m_2 \sup_{t>0, x, y \in \mathbb{R}^d} N(V, t, x, y) \\ & \leq 2 m_2 \sup_{t>0, x, y \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+(\tau/t)(y-x)|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \\ & \leq 2 m_2 \sup_{t>0, y, w \in \mathbb{R}^d} \int_0^{t/2} \int_{\mathbb{R}^d} \frac{e^{-|z-y+\tau w|^2/(4\tau)}}{\tau^{d/2}} |V(z)| dz d\tau \end{aligned}$$

$$= 2m_2 \sup_{y, w \in \mathbb{R}^d} \int_{\mathbb{R}^d} J(z - y, w) |V(z)| dz = 2m_2 (4\pi)^{d/2} e_*(V, 0). \quad \square$$

Proof of Theorem 1.4. For $x, w \in \mathbb{R}^d$ and $\tau > 0$ we have

$$\frac{|x - \tau w|^2}{4\tau} = \frac{|x|^2}{4\tau} - \frac{x \cdot w}{2} + \frac{\tau |w|^2}{4}.$$

We change the variables $\tau = 4u/|w|^2$ and use [12, 8.432, formula 6.] to get

$$J(x, w) = \int_0^\infty \tau^{-d/2} e^{-\frac{|x - \tau w|^2}{4\tau}} d\tau = 2e^{\frac{x \cdot w}{2}} \left(\frac{|x|}{|w|} \right)^{-d/2+1} K_{d/2-1} \left(\frac{|x||w|}{2} \right).$$

Here, as usual, K_ν is the modified Bessel function of the second kind. We claim that for each $\nu \geq 1/2$,

$$K_\nu(z) \approx z^{-\nu} e^{-z} (1+z)^{\nu-1/2}, \quad z > 0.$$

Here $\approx$ means that the ratio of both sides is bounded above and below by constants independent of z . The comparison is obtained by putting $x = 1$ in [12, 8.432, formulas 9. and 8.] and considering cases $z \leq 1$ and $z > 1$, correspondingly. Therefore,

$$\begin{aligned} J(x, w) &\approx e^{-(|x||w| - x \cdot w)/2} \left(\frac{|x|}{|w|} \right)^{-d/2+1} \left(\frac{|x||w|}{2} \right)^{-d/2+1} \left(1 + \frac{|x||w|}{2} \right)^{d/2-3/2} \\ &\approx K(x, w), \end{aligned}$$

and so $K(V) \approx e_*(V, 0)$. The result follows by Lemma 3.2. $\square$

Proof of (9). The left hand side inequality follows from the identity $K(V, x, 0) = C_d^{-1}(-\Delta^{-1})|V|(x)$. If $y = 0$, then the upper bound trivially holds. For $y \neq 0$ we consider two domains of integration. We have

$$\begin{aligned} \int_{|z-x||y| \leq 1} K(z-x, y) |V(z)| dz &\leq 2^{(d-3)/2} \int_{|z-x||y| \leq 1} \frac{1}{|z-x|^{d-2}} |V(z)| dz \\ &\leq \frac{2^{(d-3)/2}}{C_d} \|\Delta^{-1} V\|_\infty. \end{aligned}$$

Furthermore, by a change of variables and the Hölder inequality,

$$\begin{aligned} \int_{|z-x||y| \geq 1} K(z-x, y) |V(z)| dz &\leq 2^{(d-3)/2} \int_{|z-x||y| \geq 1} \frac{e^{-\frac{1}{2}(|z-x||y| - (z-x) \cdot y)}}{(|z-x||y|)^{(d-1)/2}} |y|^{d-2} |V(z)| dz \\ &\leq 2^{(d-3)/2} \kappa_d \|V\|_{d/2}, \end{aligned}$$

where

$$\kappa_d = \left(\int_{|w| > 1} \left(e^{-\frac{1}{2}(|w| - w_1)} |w|^{-(d-1)/2} \right)^{d/(d-2)} dw \right)^{(d-2)/d} < \infty.$$

We skip the proof of the finiteness of κ_d ; it can be found in the first version of the paper on [arXiv:1706.06172v1](https://arxiv.org/abs/1706.06172v1). $\square$

Proof of Proposition 1.6. We first prove that $\|K(V)\|_\infty = \infty$. Let $y = (1, \mathbf{0}) \in \mathbb{R}^d$. For $z = (z_1, \mathbf{z}_2) \in A$ we have

$$0 \leq |z||y| - z \cdot y = |z| - z_1 = \frac{|\mathbf{z}_2|^2}{\sqrt{z_1^2 + |\mathbf{z}_2|^2} + z_1} \leq \frac{z_1}{\sqrt{z_1^2 + |\mathbf{z}_2|^2} + z_1} \leq 1$$

and thus also $z_1 \leq |z| \leq 2z_1$. Then,

$$\begin{aligned} \|K(V)\|_\infty &\geq \int_{\mathbb{R}^d} e^{-\frac{1}{2}(|z||y| - z \cdot y)} |V(z)| \frac{(1 + |z||y|)^{\frac{d}{2} - \frac{3}{2}}}{|z|^{d-2}} dz \geq c \int_A \frac{1}{z_1} \frac{z_1^{\frac{d}{2} - \frac{3}{2}}}{z_1^{d-2}} dz \\ &= c \int_4^\infty \int_{|\mathbf{z}_2| < \sqrt{z_1}} z_1^{-1+2-d+\frac{d}{2}-\frac{3}{2}} d\mathbf{z}_2 dz_1 \\ &= c \int_4^\infty z_1^{-1+2-d+\frac{d}{2}-\frac{3}{2}+\frac{1}{2}(d-1)} dz_1 = \infty. \end{aligned}$$

We now prove that $\|\Delta^{-1}|V|\|_\infty < \infty$. By the symmetric rearrangement inequality (see [16, Chapter 3]) we have

$$\begin{aligned} &\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|z-x|^{d-2}} |V(z)| dz \\ &= \sup_{x_1 \in \mathbb{R}} \int_4^\infty \int_{\mathbb{R}^{d-1}} \frac{\mathbf{1}_{|\mathbf{z}_2| < \sqrt{z_1}}}{[(z_1 - x_1)^2 + |\mathbf{z}_2|^2]^{(d-2)/2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1. \end{aligned}$$

In other words, it suffices to consider $x = (x_1, \mathbf{0})$. We only need to show that the following three integrals are uniformly bounded for $x_1 \geq 4$. The first integral is

$$\begin{aligned} I_1 &= \int_{x_1+\sqrt{x_1}}^\infty \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2} + |\mathbf{z}_2|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &\leq \int_{x_1+\sqrt{x_1}}^\infty \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &= c \int_{x_1+\sqrt{x_1}}^\infty \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} z_1^{\frac{1}{2}(d-1)} dz_1 = c \int_{\sqrt{x_1}}^\infty \frac{1}{z_1^{d-2}} (z_1 + x_1)^{\frac{d}{2} - \frac{3}{2}} dz_1 \\ &\leq c' \int_{\sqrt{x_1}}^{x_1} \frac{1}{z_1^{d-2}} x_1^{\frac{d}{2} - \frac{3}{2}} dz_1 + c' \int_{x_1}^\infty \frac{1}{z_1^{d-2}} z_1^{\frac{d}{2} - \frac{3}{2}} dz_1 \leq c'' < \infty. \end{aligned}$$

The second integral we consider is

$$\begin{aligned} I_2 &= \int_2^{x_1-\sqrt{x_1}} \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2} + |\mathbf{z}_2|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &\leq \int_2^{x_1-\sqrt{x_1}} \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{|z_1 - x_1|^{d-2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\ &= c \int_2^{x_1-\sqrt{x_1}} \frac{1}{(x_1 - z_1)^{d-2}} z_1^{\frac{d}{2} - \frac{3}{2}} dz_1 \\ &= c \int_{\sqrt{x_1}}^{x_1-2} \frac{1}{w^{d-2}} (x_1 - w)^{\frac{d}{2} - \frac{3}{2}} dw \leq c \int_{\sqrt{x_1}}^{x_1-2} \frac{1}{w^{d-2}} x_1^{\frac{d}{2} - \frac{3}{2}} dw \leq c' < \infty. \end{aligned}$$

The remaining integral is

$$\begin{aligned}
 I_3 &= \int_{x_1 - \sqrt{x_1}}^{x_1 + \sqrt{x_1}} \int_{|\mathbf{z}_2| < \sqrt{z_1}} \frac{1}{[(z_1 - x_1)^2 + |\mathbf{z}_2|^2]^{(d-2)/2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\
 &\leq 2 \int_{x_1 - \sqrt{x_1}}^{x_1 + \sqrt{x_1}} \int_{|\mathbf{z}_2| < 2\sqrt{x_1}} \frac{1}{[(z_1 - x_1)^2 + |\mathbf{z}_2|^2]^{(d-2)/2}} \frac{1}{z_1} d\mathbf{z}_2 dz_1 \\
 &\leq 2 \int_{B(0, 3\sqrt{x_1})} \frac{1}{|z|^{d-2}} \frac{1}{x_1} dz \leq c < \infty.
 \end{aligned}$$

To prove the second statement of Proposition 1.6, for $s > 0$ we let $d_s f(x) = s f(\sqrt{s}x)$. Note that the dilatation does not change the norms:

$$\|\Delta^{-1}(d_s f)\|_\infty = \|\Delta^{-1}f\|_\infty, \quad \|K(d_s f)\|_\infty = \|K(f)\|_\infty.$$

Furthermore, $\text{supp}(d_s f) \subseteq B(0, r/\sqrt{s})$ if $\text{supp}(f) \subseteq B(0, r)$, $r > 0$. Since $\|\Delta^{-1}V\|_\infty = C < \infty$ and $\|K(V)\|_\infty = \infty$, therefore $\|\Delta^{-1}(V\mathbf{1}_{B_r})\|_\infty \leq C$ for every $r > 0$ and $\|K(V\mathbf{1}_{B_r})\|_\infty \rightarrow \infty$ as $r \rightarrow \infty$. For $n \in \mathbb{N}$ we define

$$V_n = d_{r_n^2}(V\mathbf{1}_{B_{r_n}}),$$

where r_n is chosen such that $\|K(V\mathbf{1}_{B_{r_n}})\|_\infty \geq 4^n$. Also, $\text{supp}(V_n) \subseteq B(0, 1)$. We define $\tilde{V} = \sum_{n=1}^\infty V_n/2^n$. Then,

$$\|K(\tilde{V})\|_\infty \geq \|K(V_n)\|_\infty/2^n \geq 2^n \rightarrow \infty,$$

as $n \rightarrow \infty$, and

$$\|\Delta^{-1}\tilde{V}\|_\infty \leq \sum_{n=1}^\infty \|\Delta^{-1}V_n\|_\infty/2^n \leq C. \quad \square$$

Similarly, (1) fails for $-\varepsilon\tilde{V} \geq 0$ with any $\varepsilon > 0$, cf. Remark 1.3.

4. Sufficient Conditions for the Sharp Gaussian Estimates

Recall from [8, (2.5)] that for $p \in [1, \infty]$,

$$\|P_t f\|_\infty \leq C(d, p) t^{-d/(2p)} \|f\|_p, \quad t > 0,$$

where $P_t f(x) = \int_{\mathbb{R}^d} g(t, x, z) f(z) dz$, $f \in L^p(\mathbb{R}^d)$ and

$$C(d, p) = \begin{cases} (4\pi)^{-d/2}, & \text{if } p = 1, \\ (4\pi)^{-d/(2p)} (1 - p^{-1})^{(1-p^{-1})d/2}, & \text{if } p \in (1, \infty]. \end{cases}$$

We will give an analogue for the *bridges* $T_s^{t,y}$. Here $t > 0$, $y \in \mathbb{R}^d$, and

$$T_s^{t,y} f(x) = \int_{\mathbb{R}^d} \frac{g(s, x, z) g(t-s, z, y)}{g(t, x, y)} f(z) dz, \quad 0 < s < t, \quad x \in \mathbb{R}^d.$$

Clearly,

$$T_s^{t,y} f(x) = T_{t-s}^{t,x} f(y), \quad 0 < s < t, \quad x, y \in \mathbb{R}^d. \quad (19)$$

By the Chapman–Kolmogorov equations (the semigroup property) for the kernel g , we have $T_s^{t,y}1 = 1$. We also note that $S(V)$ is related to the potential (0-resolvent) operator of T as follows,

$$S(V, t, x, y) = \int_0^t T_s^{t,y} |V|(x) ds.$$

Lemma 4.1. *For $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$ we have*

$$\|T_s^{t,y} f\|_\infty \leq C(d, p) \left[\frac{(t-s)s}{t} \right]^{-d/(2p)} \|f\|_p, \quad 0 < s < t, y \in \mathbb{R}^d.$$

Proof. We note that

$$\frac{g(s, x, z) g(t-s, z, y)}{(4\pi)^{-d/2} g(t, x, y)} = \left[\frac{(t-s)s}{t} \right]^{-\frac{d}{2}} \exp \left[-\frac{|z-x|^2}{4s} - \frac{|y-z|^2}{4(t-s)} + \frac{|y-x|^2}{4t} \right].$$

As in [27, (3.4)], we have

$$\frac{|z-x|^2}{4s} + \frac{|y-z|^2}{4(t-s)} \geq \frac{|y-x|^2}{4t}. \quad (20)$$

Indeed, (20) follows from the triangle and Cauchy-Schwarz inequalities:

$$|y-x| \leq \sqrt{s} \frac{|z-x|}{\sqrt{s}} + \sqrt{t-s} \frac{|y-z|}{\sqrt{t-s}} \leq \sqrt{t} \left(\frac{|z-x|^2}{s} + \frac{|y-z|^2}{t-s} \right)^{1/2}.$$

For $p = 1$, the assertion of the lemma results from (20). For $p \in (1, \infty)$, we let $p' = p/(p-1)$, apply Hölder's inequality, the identity $g(s, x, z)^{p'} = g(s/p', x, z)(4\pi s)^{(1-p')d/2}(p')^{-d/2}$, and the semigroup property, to get

$$\begin{aligned} |T_s^{t,y} f(x)| &\leq g(t, x, y)^{-1} \left[\int_{\mathbb{R}^d} g(s, x, z)^{p'} g(t-s, z, y)^{p'} dz \right]^{1/p'} \|f\|_p \\ &= g(t, x, y)^{-1} \left[(4\pi)^{(1-p')d} (p')^{-d} [s(t-s)]^{(1-p')d/2} \right. \\ &\quad \times \left. \int_{\mathbb{R}^d} g(s/p', x, z) g((t-s)/p', z, y) dz \right]^{1/p'} \|f\|_p \\ &= C(d, p) \left[\frac{s(t-s)}{t} \right]^{-d/(2p)} \|f\|_p. \end{aligned}$$

For $p = \infty$, the assertion follows from the identity $T_s^{t,y}1 = 1$. $\square$

Zhang [29, Proposition 2.1] showed that (1) and (2) hold for V in specific L^p spaces (see also [29, Theorem 1.1 and Remark 1.1]). In Proposition 4.2 and Corollary 4.3 below we prove his result by a different method.

Proposition 4.2. *Let $V: \mathbb{R}^d \rightarrow \mathbb{R}$ and $p, q \in [1, \infty]$.*

(a) *If $V \in L^p(\mathbb{R}^d)$, $p > d/2$ and $c = C(d, p) \frac{[\Gamma(1-d/(2p))]}{\Gamma(2-d/p)} \|V\|_p$, then*

$$\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \leq c t^{1-d/(2p)}, \quad t > 0.$$

(b) *If $V \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $q < d/2 < p$, then (3) holds.*

Proof. Part (a) follows from Lemma 4.1, so we proceed to (b). For $t > 2$,

$$\int_0^t T_s^{t,y} |V|(x) ds = \int_0^{t/2} T_s^{t,y} |V|(x) ds + \int_0^{t/2} T_s^{t,x} |V|(y) ds. \quad (21)$$

By Lemma 4.1, the first term of the sum can be estimated as follows:

$$\begin{aligned} \int_0^{t/2} T_s^{t,y} |V|(x) ds &\leq c \|V\|_p \int_0^1 \left[\frac{(t-s)s}{t} \right]^{-d/(2p)} ds + c \|V\|_q \int_1^{t/2} \left[\frac{(t-s)s}{t} \right]^{-d/(2q)} ds \\ &\leq c' \|V\|_p \int_0^1 s^{-d/(2p)} ds + c' \|V\|_q \int_1^\infty s^{-d/(2q)} ds. \end{aligned} \quad (22)$$

By (19), the second term has the same bound. For $t \in (0, 2]$ we use (a). $\square$

Lemmas 1.1 and 1.2 provide the following conclusion:

Corollary 4.3. *Under the assumptions of Proposition 4.2(a), G satisfies the sharp local Gaussian bounds (2). If $V \leq 0$ and the assumptions of Proposition 4.2(b) hold, then G has the sharp global Gaussian bounds (1).*

Recall from Sect. 1 that (1) holds for $d = 3$ if $\|\Delta^{-1}V^-\|_\infty < \infty$ and $\|\Delta^{-1}V^+\|_\infty < 1$, and it holds for $d \geq 4$ if $\|\Delta^{-1}V^-\|_\infty + \|V^-\|_{d/2} < \infty$, $\|\Delta^{-1}V^+\|_\infty < 1$ and $\|V^+\|_{d/2}$ is small enough. This yields another proof of the second statement of Corollary 4.3, because of the following observation:

Lemma 4.4. *The assumptions of Proposition 4.2(b) necessitate that $d \geq 3$, $\|\Delta^{-1}|V|\|_\infty < \infty$ and $\|V\|_{d/2} < \infty$.*

Proof. Plainly, the assumptions of Proposition 4.2(b) imply $d > 2$ and $V \in L^{d/2}(\mathbb{R}^d)$. We now verify that $\|\Delta^{-1}|V|\|_\infty < \infty$. By Hölder's inequality,

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \int_{B(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} dz &\leq \| |z|^{2-d} \mathbf{1}_{B(0,1)}(z) \|_{p'} \|V\|_p < \infty, \\ \sup_{x \in \mathbb{R}^d} \int_{B^c(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} dz &\leq \| |z|^{2-d} \mathbf{1}_{B^c(0,1)}(z) \|_{q'} \|V\|_q < \infty, \end{aligned}$$

where p', q' are the exponents conjugate to p, q , respectively. $\square$

In what follows, we develop sufficient conditions for (1) and (2). Let $d_1, d_2 \in \mathbb{N}$ and $d = d_1 + d_2$.

Remark 4.5. The Gauss–Weierstrass kernel $g(t, x)$ in $\mathbb{R}^d$ can be represented as a tensor product:

$$g(t, x) = (4\pi t)^{-d_1/2} e^{-|x_1|^2/(4t)} (4\pi t)^{-d_2/2} e^{-|x_2|^2/(4t)},$$

where $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$ and $x = (x_1, x_2)$. The kernels of the bridges factorize accordingly:

$$\begin{aligned} &\frac{g(s, x, z) g(t-s, z, y)}{g(t, x, y)} \\ &= \frac{(4\pi s)^{-d_1/2} e^{-|z_1-x_1|^2/(4s)} (4\pi(t-s))^{-d_1/2} e^{-|y_1-z_1|^2/(4(t-s))}}{(4\pi t)^{-d_1/2} e^{-|y_1-x_1|^2/(4t)}} \\ &\quad \times \frac{(4\pi s)^{-d_2/2} e^{-|z_2-x_2|^2/(4s)} (4\pi(t-s))^{-d_2/2} e^{-|y_2-z_2|^2/(4(t-s))}}{(4\pi t)^{-d_2/2} e^{-|y_2-x_2|^2/(4t)}}. \end{aligned}$$

Corollary 4.6. *Let $V_1: \mathbb{R}^{d_1} \rightarrow \mathbb{R}$, $V_2: \mathbb{R}^{d_2} \rightarrow \mathbb{R}$, and $V(x) = V_1(x_1)V_2(x_2)$, where $x = (x_1, x_2) \in \mathbb{R}^d$, $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$. Assume that $V_1 \in L^\infty(\mathbb{R}^{d_1})$ and $\sup_{t>0, x_2, y_2 \in \mathbb{R}^{d_2}} S(V_2, t, x_2, y_2) < \infty$. Then (3) holds.*

Proof. In estimating $S(V, t, x, y)$ we first use the factorization of the bridges and the boundedness of V_1 , and then the Chapman–Kolmogorov equations and the boundedness of $S(V_2)$. $\square$

Lemma 4.7. *For $f(x_1, x_2) = f_1(x_1)f_2(x_2) \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$, $0 < s < t$ and $y \in \mathbb{R}^d$, we have*

$$\|T_s^{t,y} f\|_\infty \leq C(d_1, p_1) C(d_2, p_2) \left[\frac{(t-s)s}{t} \right]^{-d_1/(2p_1) - d_2/(2p_2)} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Proof. We proceed as in the proof of Lemma 4.1, using Remark 4.5. $\square$

Proof of Theorem 1.8. We follow the proof of Proposition 4.2, replacing Lemma 4.1 by Lemma 4.7. $\square$

We note in passing that Theorem 1.8 is an extension of Proposition 4.2.

5. Examples

Let $\mathbf{1}_A$ denote the indicator function of A . In what follows, G in (1) is the Schrödinger perturbation of g by V .

Example 5.1. Let $d \geq 3$ and $1 < p < \infty$. For $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{d-1}$ we let $V(x_1, x_2) = -|x_1|^{-1/p} \mathbf{1}_{|x_1|<1} \mathbf{1}_{|x_2|<1}$. Then (1) holds but $V \notin L_{loc}^p(\mathbb{R}^d)$.

Indeed, $V(x_1, x_2) = V_1(x_1)V_2(x_2)$, where

$$\begin{aligned} V_1(x_1) &= -|x_1|^{-1/p} \mathbf{1}_{|x_1|<1}, & x_1 \in \mathbb{R}, \\ V_2(x_2) &= \mathbf{1}_{|x_2|<1}, & x_2 \in \mathbb{R}^{d-1}. \end{aligned}$$

Let

$$1 \leq q < p_1 < r < p,$$

and

$$p_2 = \frac{d-1}{2} \frac{p_1}{p_1 - 1/2}.$$

Since $d \geq 3$, $p_2 > 1$. In the notation of Theorem 1.8 we have $d_1 = 1$, $d_2 = d - 1$, and indeed $d_1/(2p_1) + d_2/(2p_2) = 1$. Since $V_1 \in L^r(\mathbb{R}) \cap L^q(\mathbb{R})$ and $V_2 \in L^{p_2}(\mathbb{R}^{d-1})$, the assumptions of Theorem 1.8(b) are satisfied, and (1) follows by Corollary 1.9. Clearly, $V \notin L_{loc}^p(\mathbb{R}^d)$.

Example 5.2. For $d \geq 3$, $n = 2, 3, \dots$, let $V_n(x) = |x_1|^{-1+1/n} \mathbf{1}_{|x_1|<1} \mathbf{1}_{|x_2|<1}$, where $x = (x_1, x_2)$, $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}^{d-1}$. Let $a_n = \sup_{t>0, x, y \in \mathbb{R}^d} S(V_n, t, x, y)$,

$$V(x) = - \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{V_n(x)}{a_n}, \quad x \in \mathbb{R}^d.$$

Then (1) holds but $V \notin \bigcup_{p>1} L_{loc}^p(\mathbb{R}^d)$.

Indeed, $0 < a_n < \infty$ by Example 5.1, and so

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

This yields the global sharp Gaussian bounds. Fix $p > 1$. Since the function $x_1 \rightarrow |x_1|^{-1+1/n}$ is not in $L^p(-1, 1)$ for large n , we get that $V \notin L^p(B(0, 1))$.

Example 5.3. Let $d \geq 3$ and $V(x_1, x_2) = \frac{-1}{(|x_2|+1)^3}$ for $x_1 \in \mathbb{R}^{d-3}$, $x_2 \in \mathbb{R}^3$. Then (1) holds but $V \notin L^1(\mathbb{R}^d)$.

Indeed, $V \notin L^1(\mathbb{R}^d)$. We let $V_2(x_2) = \frac{-1}{(|x_2|+1)^3}$, $x_2 \in \mathbb{R}^3$. By the symmetric rearrangement inequality [16, Chapter 3], in dimension $d = 3$ we have

$$0 \leq \Delta^{-1}V_2 \leq C_3 \int_{\mathbb{R}^3} \frac{1}{|z|(|z|+1)^3} dz < \infty.$$

By (7) and (8),

$$\sup_{t>0, x_2, y_2 \in \mathbb{R}^3} S(V_2, t, x_2, y_2) < \infty.$$

By Corollary 4.6 and Lemma 1.1 we see that (1) holds for V .

Proof of Proposition 1.10. Add the functions from Examples 5.2 and 5.3. $\square$

We can have nonnegative examples, too. Namely, let $V \leq 0$ be as in Proposition 1.10. Then $M = \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty$. We let $\tilde{V} = |V|/(M+1)$. Then $\tilde{V} \geq 0$, $\tilde{V} \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L^p_{loc}(\mathbb{R}^d)$ and

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(\tilde{V}, t, x, y) = M/(M+1) < 1.$$

Therefore (5) holds for $\tilde{V}$ with $h = \infty$ and $\eta = M/(M+1)$, which yields (1).

Let $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, $V_1: \mathbb{R}^{d_1} \rightarrow \mathbb{R}$, $V_2: \mathbb{R}^{d_2} \rightarrow \mathbb{R}$, and $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$, where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$. Let $G_1(t, x_1, y_1)$, $G_2(t, x_2, y_2)$ be the Schrödinger perturbations of the Gauss–Weierstrass kernels on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ by V_1 and V_2 , respectively. Then $G(t, (x_1, x_2), (y_1, y_2)) := G_1(t, x_1, y_1)G_2(t, x_2, y_2)$ is the Schrödinger perturbation of the Gauss–Weierstrass kernel on $\mathbb{R}^d$ by V . Clearly, if the sharp Gaussian estimates hold for G_1 and G_2 , then they hold for G . Our next example shows that the situation is quite different for tensor products.

Example 5.4. Let $V(x_1, x_2) = -V_1(x_1)V_2(x_2)$, where $x_1, x_2 \in \mathbb{R}^3$,

$$V_1(x) = V_2(x) = -\frac{1-\varepsilon}{2} |x|^{-1-\varepsilon} \mathbf{1}_{|x|<1},$$

and $\varepsilon \in [0, 1)$. Then the heat kernels in $\mathbb{R}^3$ of $\Delta + V_1$ and $\Delta + V_2$ satisfy (1) and (2), but that of $\Delta + V$ in $\mathbb{R}^6$ satisfies neither (1) nor (2).

Indeed, by the symmetric rearrangement inequality [16, Chapter 3],

$$0 \leq -\Delta^{-1}V_1(x) \leq -\Delta^{-1}V_1(0) = \frac{1-\varepsilon}{8\pi} \int_{\{z \in \mathbb{R}^3: |z|<1\}} \frac{1}{|z|} |z|^{-1-\varepsilon} dz = 1/2,$$

for all $x \in \mathbb{R}^3$. Thus, $\|\Delta^{-1}V_1\|_\infty = \|\Delta^{-1}V_2\|_\infty < \infty$. Using the comment following (8), we get (1) for the heat kernels in $\mathbb{R}^3$ of $\Delta + V_1$ and $\Delta + V_2$. However, the heat kernel in $\mathbb{R}^6$ of $\Delta + V$ fails even (2). Indeed, if we let $T \leq 1$, $a \in \mathbb{R}^6$, $|a| = 1$, and $c = \int_0^1 p(s, 0, a) ds$, then by [9, Lemma 3.5],

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^6} g(s, 0, x) |V(x)| dx ds &\geq \int_{\{x \in \mathbb{R}^6: |x|^2 \leq T\}} \int_0^T g(s, 0, x) ds |V(x)| dx \\ &\geq c \int_{\{x \in \mathbb{R}^6: |x|^2 \leq T\}} \frac{1}{|x|^4} |V(x)| dx \\ &\geq c \int_{\{x_1 \in \mathbb{R}^3: |x_1|^2 < T/2\}} |V_1(x_1)| \int_{\{x_2 \in \mathbb{R}^3: |x_2|^2 < T/2\}} \frac{|V_2(x_2)|}{(|x_1|^2 + |x_2|^2)^2} dx_2 dx_1 \\ &\geq \frac{c(1-\varepsilon)}{2} \int_{\{x_1 \in \mathbb{R}^3: |x_1|^2 < T/2\}} |V_1(x_1)| \int_{\{x_2 \in \mathbb{R}^3: |x_2|^2 < T/2\}} \frac{|x_2|^{-1}}{(|x_1|^2 + |x_2|^2)^2} dx_2 dx_1 \\ &= \frac{c(1-\varepsilon)}{2} \int_{\{x_1 \in \mathbb{R}^3: |x_1|^2 < T/2\}} |V_1(x_1)| \frac{\pi T}{|x_1|^2(T/2 + |x_1|^2)} dx_1 \\ &= \pi^2 c T (1-\varepsilon)^2 \int_0^{\sqrt{T/2}} \frac{r^{-1-\varepsilon}}{T/2 + r^2} dr = \infty. \end{aligned}$$

Therefore by Lemma 2.1, (4) fails, and so does (2), according to Lemma 1.1. Thus, the sharp Gaussian estimates may hold for the Schrödinger perturbations of the Gauss–Weierstrass kernels by V_1 and V_2 but fail for the Schrödinger perturbation of the Gauss–Weierstrass kernel by $V(x_1, x_2) = -V_1(x_1)V_2(x_2)$.

In passing we note that the functions $-V_1$, $-V_2$ and $-V$ give a similar counterexample with nonnegative factors, because $1/2 < 1$, cf. (12). Let us also remark that the sharp global Gaussian estimates may hold for $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ but fail for V_1 or V_2 . Indeed, it suffices to consider $V_1(x_1) = -\mathbf{1}_{|x_1| < 1}$ on $\mathbb{R}^3$ and $V_2 \equiv 1$ on $\mathbb{R}$, and to apply Theorem 1.8. We see that it is indeed the combined effect of the factors V_1 and V_2 that matters—as captured in Sect. 4.

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