



Schrödinger Operators on Graphs and Geometry II. Spectral Estimates for L_1 -potentials and an Ambartsumian Theorem

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Abstract. In this paper we study Schrödinger operators with absolutely integrable potentials on metric graphs. Uniform bounds—i.e. depending only on the graph and the potential—on the difference between the n^{th} eigenvalues of the Laplace and Schrödinger operators are obtained. This in turn allows us to prove an extension of the classical Ambartsumian Theorem which was originally proven for Schrödinger operators with Neumann conditions on an interval. We also extend a previous result relating the spectrum of a Schrödinger operator to the Euler characteristic of the underlying metric graph.

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1. Introduction

Quantum graphs—i.e. Schrödinger operators acting on metric graphs—have become an increasingly important branch of mathematical physics in the last 20 years or so. They serve as models of branched thin networks, e.g. nanotubes, and complex molecules. Apart from applications they also serve as a rich source of objects suitable for mathematical inquiry. In particular Schrödinger operators on graphs exhibit spectral properties both of partial differential and ordinary differential operators.

The aim of this paper is twofold. First we work towards proving a spectral estimate, i.e. a comparison between the spectra of the Laplacian and Schrödinger operators acting on the same metric graph. The main motivation for obtaining such estimates is that the spectrum of the Laplacian is

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much easier to calculate. We will prove that just as in the case of a single interval the difference between the Laplace and Schrödinger eigenvalues is uniformly bounded, provided the potential is just absolutely integrable. The classical proof of this relied heavily on the explicit formula for the resolvent kernel of the Laplacian [7]. The corresponding kernel for metric graphs is not in general given explicitly and we will instead work with general perturbation theory. The second goal is to extend results—relating the Laplace spectrum to geometric properties of the graph—to Schrödinger operators with L_1 -potentials. In particular we obtain an inverse spectral theorem that generalizes the celebrated theorem of Ambartsumian (see Sect. 6), and show that the Euler characteristic of the underlying graph is reflected in the spectrum of the Schrödinger operator in the case of standard vertex conditions (see (2.1) below).

1.1. Spectral Estimates and Inverse Spectral Theory

Inverse spectral theory for the Schrödinger equation in \mathbb{R}^n has classically had as a goal to determine the potential given a spectrum. To solve the inverse problem for a Schrödinger operator on a metric graph completely one has in general to determine not only the potential but the underlying metric graph and vertex conditions. It appears that this complete inverse problem is rather difficult, especially since the set of spectral data is not obvious. Therefore it seems attractive to start investigations assuming that the metric graph and the vertex conditions are fixed. The simplest graph to be considered is the star graph formed by a finite number of compact edges, and the corresponding inverse problem resembles very much the inverse problem on a single interval, where the potential is determined by two spectra [14, 30, 33–35]. The case of general trees has also been studied and we have a rather good understanding of the problem [2, 8, 9, 31, 37]. The case of graphs with cycles is much more involved—major difficulties are related to the reconstruction of the potential on the cycles. To assure the uniqueness of the potential one may either use the dependence of the spectral data on the magnetic fluxes [21, 23], or add extra spectral data like the Dirichlet spectrum [38–41].

The problem to reconstruct the metric graph has not been addressed in full generality. Topological characteristics of the graph may be reconstructed [19, 20]. Assuming that the edge lengths are rationally independent one may even reconstruct the graph using the trace formula [15, 18], but under the condition that the potential is zero. Explicit examples of isospectral graphs have been constructed [4, 5, 15]. Employing the boundary control method one may reconstruct metric trees [2] without assuming zero potential. The problem of reconstructing vertex conditions, as well as the influence of vertex conditions on the solvability of the inverse problem, is even less understood [3]. For example a metric tree is not always reconstructable, e.g. if the vertex conditions are not standard [22]. On the other hand more general vertex conditions may help to solve the inverse problem as it is done in [23, 24].

One result in the inverse spectral theory—the Ambartsumian’s celebrated theorem from 1929 stating that the spectrum of a (Neumann) Schrödinger operator on an interval coincides with the spectrum of the (Neumann)

Laplacian if and only if the potential is zero—is of great importance (see Theorem 6.1). This theorem is rather special, since in order to reconstruct a non-zero potential, knowledge of two spectra is required. Several authors generalized this theorem for the case of metric graphs. The results can be divided into two categories. Some authors proved that if the spectrum of the standard Schrödinger operator on a fixed metric graph Γ coincides with the spectrum of the standard Laplacian on Γ , then the potential is zero. This result was first proved for trees [10, 27, 29, 36] and [30], and then for arbitrary graphs by Davies [12]. It was also noted that the Laplacian on a metric graph Γ is isospectral to the Laplacian on an interval if and only if Γ is formed by a single interval. This can be seen as a geometric version of Ambartsumian’s theorem and it is based on the fact that a single interval maximizes the spectral gap for the Laplacian among graphs of fixed total length [13, 25, 28].

The inverse spectral result that we obtain—Theorem 6.5—is that the spectrum of a finite interval is unique among connected finite compact quantum graphs with standard conditions. More precisely, if the spectrum of a Schrödinger operator on a connected metric graph coincides with the spectrum of the Laplacian with Neumann conditions on an interval, then the graph is the interval, and the potential of the Schrödinger operator is zero. Here we assume that standard conditions are imposed on the graph. In the case of the graph being a compact interval, standard conditions coincide with Neumann conditions at the end-points. The result can therefore be seen as an extension of Ambartsumian’s theorem.

1.2. Outline of the Paper

Section 2 contains some fundamentals on quantum graphs. In particular we give some elementary spectral estimates and prove that normalized eigenfunctions of the Laplacian L_0^{st} are uniformly bounded in the L_∞ norm. Section 3 contains the definition of the operator L_q^{st} that we associate with the formal expression $L_0^{\text{st}} + q$, for $q \in L_1(\Gamma)$. The operator is defined via quadratic form methods. Section 4 deals with spectral estimates, and we prove that there exists a uniform bound on the difference of eigenvalues of Laplace and Schrödinger operators on metric graphs. This is done by using the Max–Min and Min–Max principles, along with a Sobolev estimate for functions $\psi \in W_2^1(\Gamma)$. Section 5 gives a result on the zeros of trigonometric polynomials, namely that if the zeros asymptotically tend to the integers, then all the zeros are in fact exactly the integers. By combining this fact with the spectral estimate of Sect. 4 we are able to prove the inverse spectral theorem given in Sect. 6. Section 7 extends a previous result that the Euler characteristic of a graph is reflected in the spectrum of a Schrödinger operator with L_∞ potential to the case of L_1 potentials.

2. Preliminaries—Basics on Quantum Graphs

A quantum graph is a metric graph equipped with a Schrödinger operator, or more formally, a triple (Γ, L, \mathbf{vc}) with Γ a metric graph, L a differential operator, and \mathbf{vc} a set of vertex conditions imposed to connect the edges and

ensure the self-adjointness of L . In this paper we limit ourselves to compact finite graphs (see below). We give a brief overview of these - for a thorough treatment of the theory of quantum graphs, see for example [6] and [26].

Metric Graphs: A compact finite metric graph is a finite collection of compact intervals of \mathbb{R} glued together at the endpoints. More precisely, let, $\{E_n\}_{n=1}^N$ be a finite set of compact intervals, each E_n considered as a subset of a separate copy of \mathbb{R} ,

$$E_n = [x_{2n-1}, x_{2n}], \quad 1 \leq n \leq N.$$

Let $\mathbf{V} = \{x_j\} = \{x_{2n-1}, x_{2n}\}_{n=1}^N$ denote the set of endpoints of the intervals. Fix a partition of \mathbf{V} into equivalence classes

$$\mathbf{V} = V_1 \cup \dots \cup V_M.$$

Identifying the endpoints yields a graph with vertices given by the equivalence classes V_i .

We let $\ell_n = x_{2n} - x_{2n-1}$ denote the length of the edge E_n and define the total length \mathcal{L} of a graph as the sum of its edge lengths:

$$\mathcal{L} = \sum_{n=1}^N \ell_n.$$

Differential Operators on Graphs: L_q acts as a differential operator on each edge separately, in our case $-d^2/dx^2 + q$ for some real potential function q —subscripts will denote the potential of the operator, and L_0 means that the potential is identically 0, so that L_0 is the Laplacian. The action is in the Hilbert spaces $L_2(E_i)$ of square integrable functions on the edges E_i . The measure on each edge is given naturally by the identification of the edge with an interval. L_q then acts on

$$L_2(\Gamma) = \bigoplus_{i=1}^N L_2(E_i).$$

Note that if $q \in L_1(\Gamma)$ then the formal expression $-d^2/dx^2 + q$ should be understood as a sum of quadratic forms (see Sect. 3).

Vertex Conditions: We shall solely study quantum graphs with standard conditions—also known as Kirchoff, Neumann, natural or free conditions. Operators with standard conditions imposed will be written as L_q^{st} . Let us first discuss the case of a bounded potential $q \in L_\infty(\Gamma)$, where the domain of the Schrödinger operator can be given explicitly. For any $\psi \in L_2(\Gamma)$, also $q\psi \in L_2(\Gamma)$, so $L_q\psi = -(d/dx)^2\psi + q\psi \in L_2(\Gamma)$ if and only if $\psi \in W_2^2(\Gamma \setminus V)$. Let $V_j = \{x_{j_1}, \dots, x_{j_k}\}$ and let $\psi(x_{j_i})$ denote the limit $\psi(x_{j_i}) = \lim_{x \rightarrow x_{j_i}} \psi(x)$, where the limit is taken over x inside the interval with x_{j_i} as an end-point.

Standard conditions are then given by imposing the following relations at each vertex V_j

$$\begin{cases} \psi(x_{j_1}) = \psi(x_{j_2}) = \dots = \psi(x_{j_k}) \\ \sum_{x_j \in V_m} \partial_n \psi_{E_n(j)}(x_j) = 0, \end{cases} \tag{2.1}$$

where ∂_n denotes the normal derivative, i.e.

$$\partial_n \psi(x_j) = \begin{cases} \psi'(x_j) & x_j \text{ left end-point} \\ -\psi'(x_j) & x_j \text{ right end-point.} \end{cases}$$

In other words, ψ is required to be continuous in each vertex V_j and the sum of normal derivatives should vanish there. This yields a self-adjoint operator L_q^{st} on Γ .

We will study L_q^{st} for $q \in L_1(\Gamma)$. This means that in general $q\psi \notin L_2(\Gamma)$, so the formal expression $L_q = -d^2/dx^2 + q$ lacks an immediate meaning as a sum of operators from $L_2(\Gamma)$ to $L_2(\Gamma)$. We will however establish that the perturbation by q is infinitesimally form-bounded (see Proposition 3.2) with respect to the quadratic form of L_0 . More precisely, the quadratic form $Q_0(\psi, \psi) = (L_0^{st}\psi, \psi) = (\psi', \psi')$ is defined on the domain of functions from $W_2^1(\Gamma \setminus V)$, which are in addition continuous at all vertices. Then the expression $L_0^{st} + q = L_q^{st}$ can be assigned a meaning via the KLMN theorem (see e.g. [32]), as a self-adjoint operator with the same form domain as L_0^{st} . Note that the domain of the quadratic form only includes the continuity condition from (2.1).

The Spectrum: We denote the spectrum of L by $\sigma(L)$. For finite compact graphs the spectrum of L_0 is discrete, for any self-adjoint boundary conditions, and furthermore satisfy Weyl asymptotics. This also holds for L_q , for $q \in L_1(\Gamma)$ (see Sect. 4). The n^{th} eigenvalue, counting multiplicities, of $L_q^{st}(\Gamma)$ is denoted by $\lambda_n(L_q^{st}) = \lambda_n(L_q^{st}(\Gamma))$. When we write $\sigma(L_{q_1}^{st}(\Gamma_1)) = \sigma(L_{q_2}^{st}(\Gamma_2))$ we mean not only equality as sets, but also that the multiplicities of all eigenvalues are equal.

For a given finite compact graph of total length \mathcal{L} with N edges and M vertices, the following lower and upper estimates can be proven (see ‘‘Appendix’’ for the proof)

$$\left(\frac{\pi}{\mathcal{L}}\right)^2 (n - M)^2 \leq \lambda_n(L_0^{st}) \leq \left(\frac{\pi}{\mathcal{L}}\right)^2 (n + N - 1)^2. \tag{2.2}$$

In particular this estimate implies that the multiplicity of the eigenvalues is uniformly bounded by $M + N$, since (2.2) implies

$$\lambda_{n-N+1} \leq \frac{\pi^2}{\mathcal{L}^2} n^2 \leq \lambda_{n+M}.$$

The above estimate is enough for our purposes, but it may be improved further taking into account the structure of the graph [17].

Another implication of (2.2) is the well-known Weyl asymptotics:

$$\lim_{n \rightarrow \infty} \frac{\lambda_n(L_0^{st})}{(\pi n / \mathcal{L})^2} = 1. \tag{2.3}$$

We shall also need the following uniform bound on the amplitude of eigenfunctions of the Laplacian on finite metric graphs:

$$\|\psi\|_{L_\infty} \leq c(\Gamma) \|\psi\|_{L_2}, \tag{2.4}$$

where the constant $c(\Gamma)$ is determined by the graph Γ but is independent of n .

One can find explicit proofs of all the above mentioned formulas (2.2), (2.3), and (2.4) in the ‘‘Appendix’’.

3. Definition of the Operator

The following well-known Sobolev estimate (a special case of Gagliardo-Nirenberg estimate) is valid for any function $\psi \in W_2^1(0, \ell)$ on an interval of finite length $\ell < \infty$ (see proof in the ‘‘Appendix’’)

$$\|\psi\|_{L_\infty}^2 \leq \epsilon \|\psi'\|_{L_2}^2 + \frac{2}{\epsilon} \|\psi\|_{L_2}^2 \tag{3.1}$$

for sufficiently small $\epsilon > 0$. The constants, though sufficient for our needs, may be improved if one is willing to sacrifice some elegance in the proof.

In particular any function $\psi \in W_2^1(\Gamma \setminus V)$ will satisfy the estimate (3.1) on each edge E_n of Γ , with $\ell = \ell_n$. As the set of edges is finite there is an edge that has minimal length so we obtain a global estimate of $|\psi(x)|^2$ on Γ :

Corollary 3.1. *Let Γ be a finite compact graph and $\psi \in W_2^1(\Gamma)$. Then*

$$\|\psi\|_{L_\infty(\Gamma)}^2 \leq \epsilon \|\psi'\|_{L_2(\Gamma)}^2 + \frac{2}{\epsilon} \|\psi\|_{L_2(\Gamma)}^2 \tag{3.2}$$

for sufficiently small $\epsilon > 0$ and $x \in \Gamma$.

Proposition 3.2. *Let $q \in L_1(\Gamma)$, and let Q_q be the quadratic form given by*

$$Q_q(\psi, \psi) = \int_\Gamma q(x) |\psi(x)|^2 dx,$$

and $Q_{L_0^{st}}$ the quadratic form associated with L_0^{st} . Then for sufficiently small $\epsilon > 0$ there exists $b(\epsilon)$ such that

$$|Q_q(\psi, \psi)| \leq \epsilon Q_{L_0^{st}}(\psi, \psi) + b(\epsilon)(\psi, \psi).$$

In other words Q_q is infinitesimally bounded by $Q_{L_0^{st}}$.

Proof. Corollary 3.1 shows that for sufficiently small $\epsilon > 0$

$$|\psi(x)|^2 \leq \epsilon \|\psi'\|_{L_2(\Gamma)}^2 + \frac{2}{\epsilon} \|\psi\|_{L_2(\Gamma)}^2.$$

Multiplying by q and integrating we obtain

$$\begin{aligned} \left| \int_\Gamma q(x) |\psi(x)|^2 dx \right| &\leq \int_\Gamma |q(x)| |\psi(x)|^2 dx \\ &\leq \epsilon \|q\|_{L_1(\Gamma)} \|\psi'\|_{L_2(\Gamma)}^2 + \frac{2}{\epsilon} \|q\|_{L_1(\Gamma)} \|\psi\|_{L_2(\Gamma)}^2 \\ &= \epsilon \|q\|_{L_1(\Gamma)} Q_{L_0^{st}}(\psi, \psi) + \frac{2}{\epsilon} \|q\|_{L_1(\Gamma)} \|\psi\|_{L_2(\Gamma)}^2. \end{aligned}$$

Replacing ϵ with $\epsilon/\|q\|_{L_1(\Gamma)}$, we may chose $b(\epsilon) = \frac{2}{\epsilon} \|q\|_{L_1(\Gamma)}^2$. □

The KLMN theorem [32] now lets us conclude that there is a unique bounded from below self-adjoint operator associated with the form $Q_{L_0^{\text{st}}} + Q_q$:

Definition 3.1. For $q \in L_1(\Gamma)$ we denote by $L_q^{\text{st}}(\Gamma)$ the operator associated with the form $Q_{L_0^{\text{st}}} + Q_q$.

We note that the form domains of L_q^{st} and L_0^{st} coincide, and

$$(L_q^{\text{st}}\psi, \phi) = (L_0^{\text{st}}\psi, \phi) + \int_{\Gamma} q(x)\psi(x)\overline{\phi(x)}dx,$$

for all $\psi, \phi \in \text{Dom}(Q_{L_q}) = \text{Dom}(Q_{L_0^{\text{st}}})$.

4. Spectral Estimates

We recall the following standard variational theorems, see e.g. [32] for proofs. The lowest eigenvalue will be denoted by λ_0 .

Proposition 4.1. (*Min-Max*). Let A be a self-adjoint, bounded from below, operator with discrete spectrum, then the n^{th} eigenvalue of A is given by

$$\lambda_{n-1}(A) = \min_{\mathcal{V}_n} \max_{\substack{u \in \mathcal{V}_n \\ \|u\|_{L_2}=1}} Q_A(u, u),$$

where \mathcal{V}_n ranges over all n -dimensional subspaces of $\text{Dom}(Q_A)$, the domain of the quadratic form Q_A associated with A .

Proposition 4.2. (*Max-Min*). Let A be a self-adjoint bounded from below, operator with discrete spectrum, then the n^{th} eigenvalue of A is given by

$$\lambda_{n-1}(A) = \max_{\mathcal{V}_{n-1}} \min_{\substack{u \perp \mathcal{V}_{n-1} \\ \|u\|_{L_2}=1}} Q_A(u, u),$$

where \mathcal{V}_{n-1} ranges over all $(n - 1)$ -dimensional subspaces of $\text{Dom}(Q_A)$, the domain of the quadratic form Q_A associated with A .

In order to apply Propositions 4.1 and 4.2 as we do in the following it is of course required that the spectrum of $L_q^{\text{st}}(\Gamma)$ is discrete. For Schrödinger operators with L_1 potentials on finite intervals this is well known, so it is true for finite metric graphs as well since these are just finite-rank perturbations—in the resolvent sense—of the Dirichlet Schrödinger operators (defined by Dirichlet conditions at all vertices), which is nothing else than an orthogonal sum of Dirichlet Schrödinger operators on a collection of finite intervals.

We now proceed to prove the spectral estimate for finite compact graphs, i.e. we show that the difference between the Laplace and the Schrödinger eigenvalues is uniformly bounded:

Theorem 4.3. Let Γ be a finite compact metric graph, and let $q \in L_1(\Gamma)$. Then the difference between the eigenvalues $\lambda_n(L_0^{\text{st}})$ and $\lambda_n(L_q^{\text{st}})$ is bounded by a constant, i.e.

$$|\lambda_n(L_0^{\text{st}}) - \lambda_n(L_q^{\text{st}})| \leq C, \tag{4.1}$$

where $C = C(\Gamma, \|q\|_{L_1(\Gamma)})$ is independent of n .

Similar questions have been discussed in [11] for the case of equilateral metric graphs.

We are going to prove the Theorem using Propositions 4.1 and 4.2. To illustrate the strategy let us first try to derive an upper estimate for $\lambda_n(L_q^{\text{st}})$ using a more naive approach. The quadratic form is

$$Q_{L_q^{\text{st}}}(u, u) = \int_{\Gamma} |u'(x)|^2 dx + \int_{\Gamma} q(x)|u(x)|^2 dx.$$

It can be estimated from above by

$$Q_{L_q^{\text{st}}}(u, u) \leq Q_{L_{q_+}^{\text{st}}}(u, u) = \int_{\Gamma} |u'(x)|^2 dx + \int_{\Gamma} q_+(x)|u(x)|^2 dx, \tag{4.2}$$

where q_+ is the positive part of the potential q :

$$q(x) = q_+(x) - q_-(x), \quad q_{\pm}(x) \geq 0. \tag{4.3}$$

This step cannot be improved much, since the new estimate coincides with the original one in the case where q is nonnegative.

The idea how to proceed is to choose a particular n -dimensional subspace \mathcal{V}_n^0 . Then the Rayleigh quotient gives not an exact value for $\lambda_{n-1}(L_q^{\text{st}})$, but an upper estimate when Proposition 4.1 is used

$$\lambda_{n-1}(L_q^{\text{st}}) = \min_{\mathcal{V}_n} \max_{u \in \mathcal{V}_n} \frac{Q_{L_q^{\text{st}}}(u, u)}{\|u\|_{L_2}^2} \leq \max_{u \in \mathcal{V}_n^0} \frac{Q_{L_q^{\text{st}}}(u, u)}{\|u\|_{L_2}^2}.$$

The only reasonable candidate for \mathcal{V}_n^0 we have at hand is the linear span of the Laplacian eigenfunctions corresponding to the n lowest eigenvalues

$$\mathcal{V}_n^0 = \mathcal{L} \left\{ \psi_0^{L_0^{\text{st}}}, \psi_1^{L_0^{\text{st}}}, \dots, \psi_{n-1}^{L_0^{\text{st}}} \right\}. \tag{4.4}$$

If $q \equiv 0$ then this estimate gives the exact value for λ_{n-1} . Therefore it is natural to split the quadratic form as follows:

$$\lambda_{n-1}(L_q^{\text{st}}) \leq \max_{u \in \mathcal{V}_n^0} \frac{Q_{L_q^{\text{st}}}(u, u)}{\|u\|_{L_2}^2} \leq \max_{u \in \mathcal{V}_n^0} \frac{\int_{\Gamma} |u'(x)|^2 dx}{\|u\|_{L_2}^2} + \max_{u \in \mathcal{V}_n^0} \frac{\int_{\Gamma} q_+(x)|u(x)|^2 dx}{\|u\|_{L_2}^2}.$$

Then the first quotient is equal to $\lambda_{n-1}(L_0^{\text{st}})$ and the maximum is attained on

$$u = \psi_{n-1}^{L_0^{\text{st}}}.$$

If nothing about q is known, then to estimate the second quotient one may use

$$\int_{\Gamma} q_+(x)|u(x)|^2 dx \leq \|q_+\|_{L_1(\Gamma)} \left(\max_{x \in \Gamma} |u(x)| \right)^2. \tag{4.5}$$

We need to estimate $|u(x)|^2$, provided $u = \sum_{j=1}^{n-1} \alpha_j \psi_j^{L_0^{\text{st}}}$. Since $\max |\psi_j^{L_0^{\text{st}}}(x)| \leq c$ (formula (2.4)) we obtain with the Schwarz inequality

$$\begin{aligned} \max_{x \in \Gamma} |u(x)| &\leq \sum_0^{n-1} |\alpha_j| \max |\psi_j^{L_0^{\text{st}}}(x)| \leq c \sum_0^{n-1} |\alpha_j| \\ &\leq c\sqrt{n} \left(\sum_0^{n-1} |\alpha_j|^2 \right)^{1/2} = c\sqrt{n} \|u\|_{L_2}. \end{aligned}$$

Hence

$$\frac{\int_{\Gamma} q_+(x) |u(x)|^2 dx}{\|u\|_{L_2}^2} \leq \|q_+\|_{L_1} c^2 n,$$

and

$$Q_{L_q^{\text{st}}}(u, u) \leq (\lambda_{n-1}(L_0^{\text{st}}) + c^2 n \|q_+\|_{L_1}) \|u\|_{L_2}^2, \tag{4.6}$$

which implies

$$\lambda_{n-1}(L_q^{\text{st}}) - \lambda_{n-1}(L_0^{\text{st}}) \leq \|q_+\|_{L_1} c^2 n, \tag{4.7}$$

i.e. we do not get an estimate uniform in n —the estimate grows linearly with n . The reason is the splitting of the quadratic form of L_q^{st} into two parts. To obtain the upper bounds we used two intrinsically different vectors: the first term is maximized if $u = \psi_{n-1}^{L_0^{\text{st}}}$, while to estimate the second term we used $u = \psi_0^{L_0^{\text{st}}} + \psi_1^{L_0^{\text{st}}} + \dots + \psi_{n-1}^{L_0^{\text{st}}}$. This is the reason that the estimate (4.7) is not optimal.

Proof of Theorem 4.3. We divide the proof into two parts deriving upper and lower estimates separately.

Upper Estimate

As before we use the estimate

$$\lambda_{n-1}(L_q^{\text{st}}) \leq \max_{u \in \mathcal{V}_n^0} \frac{\int_{\Gamma} |u'(x)|^2 dx + \int_{\Gamma} q_+(x) |u(x)|^2 dx}{\|u\|_{L_2}^2}, \tag{4.8}$$

where \mathcal{V}_n^0 is defined by (4.4). Every function $u = \sum_{j=0}^{n-1} \alpha_j \psi_j^{L_0^{\text{st}}}$ from \mathcal{V}_n^0 can be written as a sum $u = u_1 + u_2$, where

$$\begin{aligned} u_1 &:= \alpha_0 \psi_0^{L_0^{\text{st}}} + \alpha_1 \psi_1^{L_0^{\text{st}}} + \dots + \alpha_{n-p-1} \psi_{n-p-1}^{L_0^{\text{st}}}, \\ u_2 &:= \alpha_{n-p} \psi_{n-p}^{L_0^{\text{st}}} + \alpha_{n-p+1} \psi_{n-p+1}^{L_0^{\text{st}}} + \dots + \alpha_{n-1} \psi_{n-1}^{L_0^{\text{st}}}. \end{aligned}$$

Here p is a natural number to be fixed later (independent of n , but depending on Γ and q). Therefore as n increases the first function u_1 will contain an increasing number of terms, while the second function will always be given by a sum of p terms.

From the inequality $\int |u_1 + u_2|^2 dx \leq 2 \int |u_1|^2 dx + 2 \int |u_2|^2 dx$ and the fact that q_+ is nonnegative we have

$$\int_{\Gamma} q_+(x) |u_1(x) + u_2(x)|^2 dx \leq 2 \int_{\Gamma} q_+(x) |u_1(x)|^2 dx + 2 \int_{\Gamma} q_+(x) |u_2(x)|^2 dx. \tag{4.9}$$

That u_1, u_2 are orthogonal is clear, and from this the orthogonality of u'_1 and u'_2 also follows:

$$(u'_1, u'_2) = -(u''_1, u_2) = - \sum_{i=0}^{n-p-1} \lambda_i(L_0^{\text{st}}) (\alpha_i \psi_i^{L_0^{\text{st}}} u_1, u_2) = 0.$$

Taking this into account we arrive at

$$\begin{aligned} Q_{L_{2q_+}^{\text{st}}}(u, u) &\leq \underbrace{\int_{\Gamma} |u'_1(x)|^2 dx + 2 \int_{\Gamma} q_+(x) |u_1(x)|^2 dx}_{=: Q_{L_{2q_+}^{\text{st}}}(u_1, u_1)} \\ &\quad + \underbrace{\int_{\Gamma} |u'_2(x)|^2 dx + 2 \int_{\Gamma} q_+(x) |u_2(x)|^2 dx}_{=: Q_{L_{2q_+}^{\text{st}}}(u_2, u_2)}. \end{aligned} \tag{4.10}$$

To estimate the first form we use (4.5) and the Sobolev estimate (3.2) for $\max |u(x)|$. We get

$$\begin{aligned} Q_{L_{2q_+}^{\text{st}}}(u_1, u_1) &= \int_{\Gamma} |u'_1(x)|^2 dx + 2 \int_{\Gamma} q_+(x) |u_1(x)|^2 dx \\ &\leq \|u'_1\|_{L_2}^2 + 2 \|q_+\|_{L_1} \max_{x \in \Gamma} |u_1(x)|^2 \\ &\leq \|u'_1\|_{L_2}^2 + 2 \|q_+\|_{L_1} (\epsilon \|u'_1\|_{L_2}^2 + \frac{2}{\epsilon} \|u_1\|_{L_2}^2) \\ &= (1 + 2\epsilon \|q_+\|_{L_1}) \|u'_1\|_{L_2}^2 + \frac{4}{\epsilon} \|q_+\|_{L_1} \|u_1\|_{L_2}^2. \end{aligned}$$

Using

$$\begin{aligned} \|u'_1\|_{L_2}^2 = (u'_1, u'_1) &= -(u''_1, u_1) = \sum_{j=0}^{n-p-1} \lambda_j(L_0^{\text{st}}) (\alpha_j \psi_j, \alpha_j \psi_j) \\ &= \sum_{j=0}^{n-p-1} \lambda_j(L_0^{\text{st}}) |\alpha_j|^2 \leq \lambda_{n-p-1}(L_0^{\text{st}}) \sum_{j=0}^{n-p-1} |\alpha_j|^2 = \lambda_{n-p-1}(L_0^{\text{st}}) \|u_1\|_{L_2}^2, \end{aligned} \tag{4.11}$$

we get

$$\begin{aligned} Q_{L_{2q_+}^{\text{st}}}(u_1, u_1) &\leq (1 + 2\epsilon \|q_+\|_{L_1}) \|u'_1\|_{L_2}^2 + \frac{4}{\epsilon} \|q_+\|_{L_1} \|u_1\|_{L_2}^2 \\ &\leq ((1 + 2\epsilon \|q_+\|_{L_1}) \lambda_{n-p-1}(L_0^{\text{st}}) + \frac{4}{\epsilon} \|q_+\|_{L_1}) \|u_1\|_{L_2}^2. \end{aligned}$$

The key point is that ϵ and p can be chosen in such a way that

$$(1 + 2\epsilon \|q_+\|_{L_1}) \lambda_{n-p-1}(L_0^{\text{st}}) + \frac{4}{\epsilon} \|q_+\|_{L_1} < \lambda_{n-1}(L_0^{\text{st}})$$

holds (see (4.14) below).

On the other hand, our naive approach (4.6) can be applied to the second form with the only difference being that the number of eigenfunctions involved is p , not n

$$Q_{L_{2q_+}^{\text{st}}}(u_2, u_2) \leq (\lambda_{n-1}(L_0^{\text{st}}) + 2c^2 p \|q_+\|_{L_1}) \|u_2\|_{L_2}^2. \tag{4.12}$$

Putting together the obtained estimates in (4.10) and using that $\|u_2\|_{L_2}^2 = \|u\|_{L_2}^2 - \|u_1\|_{L_2}^2$ we get

$$\begin{aligned} Q_{L_q^{\text{st}}}(u, u) &\leq ((1 + 2\epsilon\|q_+\|_{L_1})\lambda_{n-p-1}(L_0^{\text{st}}) + \frac{4}{\epsilon}\|q_+\|_{L_1}) \|u_1\|_{L_2}^2 \\ &\quad + (\lambda_{n-1}(L_0^{\text{st}}) + 2c^2p\|q_+\|_{L_1}) \|u_2\|_{L_2}^2 \\ &\leq \lambda_{n-1}(L_0^{\text{st}})\|u\|_{L_2}^2 + 2c^2p\|q_+\|_{L_1}p\|u\|_{L_2}^2 \\ &\quad - (\lambda_{n-1}(L_0^{\text{st}}) - (1 + 2\epsilon\|q_+\|_{L_1})\lambda_{n-p-1}(L_0^{\text{st}}) - \frac{4}{\epsilon}\|q_+\|_{L_1}) \|u_1\|_{L_2}^2. \end{aligned}$$

We would get the desired estimate

$$\lambda_{n-1}(L_q^{\text{st}}) \leq \max_{u \in \mathcal{V}_n^0} \frac{Q_{L_q^{\text{st}}}(u, u)}{\|u\|_{L_2}^2} \leq \lambda_{n-1}(L_0^{\text{st}}) + C \tag{4.13}$$

with $C = 2c^2p\|q_+\|_{L_1}$ if we manage to prove that

$$\lambda_{n-1}(L_0^{\text{st}}) - (1 + 2\epsilon\|q_+\|_{L_1})\lambda_{n-p-1}(L_0^{\text{st}}) - \frac{4}{\epsilon}\|q_+\|_{L_1} > 0 \tag{4.14}$$

for a certain ϵ that may depend on n and p . We use the estimate for Laplacian eigenvalues given in (2.2):

$$\left(\frac{\pi}{\mathcal{L}}\right)^2 (n - M)^2 \leq \lambda_n(L_0^{\text{st}}) \leq \left(\frac{\pi}{\mathcal{L}}\right)^2 (n + N - 1)^2. \tag{4.15}$$

Substituting $\lambda_{n-1}(L_0^{\text{st}})$ with the lower bound and $\lambda_{n-p-1}(L_0^{\text{st}})$ with the upper and setting $\epsilon = 1/n$, we get the following inequality for the left-hand side of (4.14)

$$\begin{aligned} &\lambda_{n-1}(L_0^{\text{st}}) - (1 + 2/n\|q_+\|_{L_1})\lambda_{n-p-1}(L_0^{\text{st}}) - 4n\|q_+\|_{L_1} \\ &\geq \left(\frac{\pi}{\mathcal{L}}\right)^2 (n - 1 - M)^2 - (1 + 2/n\|q_+\|_{L_1}) \\ &\quad \times \left(\frac{\pi}{\mathcal{L}}\right)^2 (n - 1 - p + N - 1)^2 - 4n\|q_+\|_{L_1} \\ &= 2n \left(\frac{\pi}{\mathcal{L}}\right)^2 \left(p - M - N + 1 - \left(1 + 2\left(\frac{\mathcal{L}}{\pi}\right)^2\right)\|q_+\|_{L_1}\right) + O(1). \end{aligned}$$

We see that for any integer $p > M + N - 1 + (1 + 2(\mathcal{L}/\pi)^2)\|q_+\|_{L_1}$ the expression is positive for sufficiently large n and the difference between the eigenvalues possesses the uniform upper estimate:

$$\lambda_n(L_q^{\text{st}}) - \lambda_n(L_0^{\text{st}}) \leq C. \tag{4.16}$$

If one is interested in the difference between the eigenvalues for large n only, then the constant C can be taken equal to

$$C = 2c^2\|q\|_{L_1}(M + N - 1 + (1 + 2(\mathcal{L}/\pi)^2)\|q\|_{L_1}),$$

but this value of C may be too small in order to ensure that (4.16) holds for all n , since proving (4.14) we assumed that n is sufficiently large. The latter assumption does not affect the final result, since for a finite number of eigenvalues (4.16) is always satisfied, but the value of the constant C may be affected.

Lower Estimate

To obtain a lower estimate we are going to use the *Max-Min* principle (Proposition 4.2). The first step is to notice that

$$Q_{L_q^{st}}(u, u) \geq \int_{\Gamma} |u'(x)|^2 dx - \int_{\Gamma} q_-(x) |u(x)|^2 dx. \tag{4.17}$$

Using the subspace \mathcal{V}_{n-1}^0 defined in (4.4) we get

$$\lambda_{n-1}(L_q^{st}) \geq \min_{u \perp \mathcal{V}_{n-1}^0} \frac{Q_{L_q^{st}}(u, u)}{\|u\|_{L_2}^2}.$$

Since u is orthogonal to \mathcal{V}_{n-1}^0 it possesses the representation

$$u = \sum_{j=n-1}^{\infty} \alpha_j \psi_j^{L_0^{st}}.$$

As before let us split the function $u = u_1 + u_2$

$$\begin{aligned} u_1 &:= \alpha_{n-1} \psi_{n-1}^{L_0^{st}} + \alpha_n \psi_n^{L_0^{st}} + \dots + \alpha_{n+p-2} \psi_{n+p-2}^{L_0^{st}}, \\ u_2 &:= \alpha_{n+p-1} \psi_{n+p-1}^{L_0^{st}} + \alpha_{n+p} \psi_{n+p}^{L_0^{st}} + \dots \end{aligned}$$

Note two important differences:

- the function u_1 is given by the sum of p terms, where the number p independent of n will be chosen later, so the functions u_1 and u_2 exchange roles compared with the proof of the upper estimate;
- the function u_2 is given by an infinite series, not by an increasing number of terms as the function u_1 in the proof of upper estimate.

Using the fact that q_- is nonnegative we may split the quadratic form (compare (4.10))

$$\begin{aligned} Q_{L_{q_-}^{st}}(u, u) &\geq \underbrace{\int_{\Gamma} |u'_1(x)|^2 dx - 2 \int_{\Gamma} q_-(x) |u_1(x)|^2 dx}_{=: Q_{L_{-2q_-}^{st}}(u_1, u_1)} \\ &\quad + \underbrace{\int_{\Gamma} |u'_2(x)|^2 dx - 2 \int_{\Gamma} q_-(x) |u_2(x)|^2 dx}_{=: Q_{L_{-2q_-}^{st}}(u_2, u_2)}. \end{aligned} \tag{4.18}$$

Now the function u_1 is given by a finite number of terms and we may similarly to (4.12) estimate

$$Q_{L_{-2q_-}^{st}}(u_1, u_1) \geq (\lambda_{n-1}(L_0^{st}) - 2c^2 p \|q_-\|_{L_1(\Gamma)}) \|u_1\|_{L_2}^2. \tag{4.19}$$

To estimate the second form we use (4.5) and the Sobolev estimate (3.2) for $\max |u(x)|^2$. We get

$$\begin{aligned} Q_{L_{-2q_-}^{st}}(u_2, u_2) &\geq \|u'_2\|_{L_2}^2 - 2 \|q_-\|_{L_1} \max |u_2(x)|^2 \\ &\geq \|u'_2\|_{L_2}^2 - 2 \|q_-\|_{L_1} \left(\epsilon \|u'_2\|_{L_2}^2 + \frac{2}{\epsilon} \|u_2\|_{L_2}^2 \right) \\ &= (1 - 2\epsilon \|q_-\|_{L_1}) \|u'_2\|_{L_2}^2 - \frac{4 \|q_-\|_{L_1}}{\epsilon} \|u_2\|_{L_2}^2. \end{aligned}$$

Taking into account

$$\begin{aligned} \|u'_2\|_{L_2}^2 &= (u'_2, u'_2) = -(u''_2, u_2) = \sum_{j=n+p-1}^{\infty} \lambda_j(L_0^{\text{st}}) (\alpha_j \psi_j, \alpha_j \psi_j) \\ &= \sum_{j=n+p-1}^{\infty} \lambda_j(L_0^{\text{st}}) |\alpha_j|^2 \geq \lambda_{n+p-1}(L_0^{\text{st}}) \sum_{j=n+p-1}^{\infty} |\alpha_j|^2 \\ &= \lambda_{n+p-1}(L_0^{\text{st}}) \|u_2\|_{L_2}^2, \end{aligned} \tag{4.20}$$

we arrive at

$$\begin{aligned} &Q_{L_{-2q_-}^{\text{st}}}(u_2, u_2) \\ &\geq \left((1 - 2\epsilon \|q_-\|_{L_1(\Gamma)}) \lambda_{n+p-1}(L_0^{\text{st}}) - \frac{4\|q_-\|_{L_1(\Gamma)}}{\epsilon} \right) \|u_2\|_{L_2}^2. \end{aligned} \tag{4.21}$$

Summing the estimates (4.19) and (4.21) and taking into account that

$$\|u_2\|_{L_2}^2 = \|u\|_{L_2}^2 - \|u_1\|_{L_2}^2 \text{ we get}$$

$$\begin{aligned} Q_{L_q^{\text{st}}}(u, u) &\geq (\lambda_{n-1}(L_0^{\text{st}}) - 2c^2 p \|q_-\|_{L_1}) \|u_1\|_{L_2}^2 \\ &\quad + \left((1 - 2\epsilon \|q_-\|_{L_1}) \lambda_{n+p-1}(L_0^{\text{st}}) - \frac{4\|q_-\|_{L_1}}{\epsilon} \right) \|u_2\|_{L_2}^2 \\ &\geq \lambda_{n-1}(L_0^{\text{st}}) \|u\|_{L_2}^2 - 2c^2 p \|q_-\|_{L_1} \|u\|_{L_2}^2 \\ &\quad + \left((1 - 2\epsilon \|q_-\|_{L_1}) \lambda_{n+p-1}(L_0^{\text{st}}) - \frac{4\|q_-\|_{L_1}}{\epsilon} - \lambda_{n-1}(L_0^{\text{st}}) \right) \|u_2\|_{L_2}^2. \end{aligned}$$

As before, to prove the desired uniform estimate it is sufficient to show that for large enough n the following expression can be made positive by choosing an appropriate ϵ :

$$(1 - 2\epsilon \|q_-\|_{L_1}) \lambda_{n+p-1}(L_0^{\text{st}}) - \frac{4\|q_-\|_{L_1}}{\epsilon} - \lambda_{n-1}(L_0^{\text{st}}) > 0. \tag{4.22}$$

Again we use (4.15): we substitute $\lambda_{n+p-1}(L_0^{\text{st}})$ with the lower bound and $\lambda_{n-1}(L_0^{\text{st}})$ with the upper. As before we choose $\epsilon = 1/n$, so the left-hand side of (4.22) becomes

$$\begin{aligned} &(1 - 2\epsilon \|q_-\|_{L_1}) \lambda_{n+p-1}(L_0^{\text{st}}) - \frac{4}{\epsilon} \|q_-\|_{L_1} - \lambda_{n-1}(L_0^{\text{st}}) \\ &\geq (1 - 2\|q_-\|_{L_1}/n) \left(\frac{\pi}{\mathcal{L}} \right)^2 (n + p - 1 - M)^2 \\ &\quad - 4n \|q_-\|_{L_1} - \left(\frac{\pi}{\mathcal{L}} \right)^2 (n - 1 + N - 1)^2 \\ &= 2n \left(\frac{\pi}{\mathcal{L}} \right)^2 \left(p - M - N + 1 - \left(1 + 2 \left(\frac{\mathcal{L}}{\pi} \right)^2 \right) \|q_-\|_{L_1} \right) + O(1). \end{aligned}$$

If $p > M + N - 1 + (1 + 2(\mathcal{L}/\pi)^2) \|q_-\|_{L_1}$, then for sufficiently large n the expression is positive, hence the following lower estimate holds

$$\lambda_n(L_q^{\text{st}}) - \lambda_n(L_0^{\text{st}}) \geq C, \tag{4.23}$$

where the exact value of C is determined by the difference between the first few eigenvalues as described above. □

Theorem 4.3 allows us to conclude that the spectra of Schrödinger operators satisfy Weyl asymptotics as well:

Corollary 4.4. *Let $q \in L_1(\Gamma)$, then $\lambda_n(L_q^{\text{st}}(\Gamma))$ satisfies Weyl asymptotics.*

Proof. This is an immediate consequence of (2.3) and Theorem 4.3. □

More importantly we may now show that the effect of an L_1 -perturbation on the eigenvalues will tend to zero in n in the scale of square roots. This step is critical in the proof of Theorem 6.5.

Corollary 4.5. *Let $\lambda_n(L_q^{\text{st}}) = k_{n,q}^2$ and $\lambda_n(L_0^{\text{st}}) = k_{n,0}^2$. If $|\lambda_n(L_q^{\text{st}}) - \lambda_n(L_0^{\text{st}})| \leq C \in \mathbb{R}$ then $|k_{n,q} - k_{n,0}| \leq \frac{C_0}{n}$, for some constant $C_0 \in \mathbb{R}$. In particular,*

$$|k_{n,q} - k_{n,0}| \rightarrow 0, \quad n \rightarrow \infty. \tag{4.24}$$

Proof. Since the eigenvalues satisfy Weyl asymptotics that depend only on the length of Γ we have that $k_{n,q}, k_{n,0} \geq nD$ for some constant D and sufficiently large n . We have

$$\begin{aligned} |k_{n,q} - k_{n,0}| &= \left| \frac{(k_{n,q} + k_{n,0})(k_{n,q} - k_{n,0})}{k_{n,q} + k_{n,0}} \right| = \left| \frac{k_{n,q}^2 - k_{n,0}^2}{k_{n,q} + k_{n,0}} \right| \\ &= \left| \frac{\lambda_n(L_q^{\text{st}}) - \lambda_n(L_0^{\text{st}})}{k_{n,q} + k_{n,0}} \right| \leq \frac{C}{|k_{n,q} + k_{n,0}|} \leq \frac{C_0}{n}, \end{aligned}$$

for some constant $C_0 \in \mathbb{R}$. □

5. On the Zeros of Trigonometric Polynomials

In this section we recall the secular equation for the spectrum $\sigma(L_0^{\text{st}}(\Gamma))$: it is given as the squares of the zeros of a trigonometric polynomial. We then prove that if the zeros k_m of such a finite trigonometric polynomial with constant coefficients are close to a certain equispaced sequence, i.e. satisfy $|k_m - m\pi/\mathcal{L}| \rightarrow 0$ then in fact $k_m = m\pi/\mathcal{L}$ for all m (Theorem 5.2). From this we then prove the geometric Ambartsumian Theorem 6.5.

Theorem 5.1. *Let Γ be a finite compact metric graph. The eigenvalues $\lambda_n(L_0^{\text{st}})$ are given by the squares of the zeros of a certain trigonometric polynomial*

$$p(k) = \sum_{n=1}^N a_n e^{i\omega_n k}$$

with k -independent coefficients $a_n \in \mathbb{C}$ and $\omega_n \in \mathbb{R}$: $\lambda_n(L_0^{\text{st}}) = k_{n,0}^2$ if and only if $p(k_{n,0}) = 0$.

Proof. We sketch the proof: for each non-zero eigenvalue the corresponding eigenfunction is edge-wise just a sum of sine and cosine functions: $\psi(x) = a_j \cos(kx) + b_j \sin(kx)$, $x \in E_j$. The solutions have to satisfy the vertex conditions. Continuity can at each V be written as

$$a_i \cos kx + b_i \sin kx = a_j \cos ky + b_j \sin ky,$$

if $x \in E_i, y \in E_j$, and $x, y \in V$. This yields $2N - M$ equations, where N is the number of edges and M is the number of vertices. The conditions on the normal derivatives can at each vertex V be written as

$$\sum_{x_j \in V} (-1)^j (-a_{[(j+1)/2]} \sin kx_j + b_{[(j+1)/2]} \cos kx_j) = 0,$$

where k has been factored out. This yields an additional M equations, so that we in total have $2N$ equations, which may be written in the form:

$$T(k)\vec{c} = 0,$$

with \vec{c} a vector of the coefficients a_i, b_i and $T(k)$ a matrix with trigonometric entries depending on k . A real number $\lambda = k^2$ is an eigenvalue of L_0^{st} if and only if k is a root of the trigonometric polynomial $\det T(k) = 0$. We refer to [6, 18] for details. See also eg. [15] and [26]. □

Note that it is crucial that the vertex conditions were standard: in the case of more general vertex conditions the secular equation is given by a quasipolynomial instead of the trigonometric polynomial.

Theorem 5.2. *Let f be the trigonometric polynomial*

$$p(k) = \sum_{j=1}^J a_j e^{i\omega_j k} \tag{5.1}$$

with all $\omega_j \in \mathbb{R}, a_j \in \mathbb{C}$. If the zeros k_m of f satisfy

$$\lim_{m \rightarrow \infty} (k_m - m) = 0$$

then $k_m = m$ for all m .

First we need a Lemma:

Lemma 5.3. *Given $\omega_1, \dots, \omega_J \in \mathbb{R}$ there exists a subsequence $\{m_n\}$ of the natural numbers such that, for each ω_j ,*

$$e^{i\omega_j m_n} \rightarrow 1.$$

Proof. For $\vec{\omega} := (\omega_1, \dots, \omega_J) \in \mathbb{R}^J$ let $[\vec{\omega}] := ([\omega_1], \dots, [\omega_J])$ denote the image of ω under the standard projection to the J -torus: $\mathbb{R}^J \rightarrow (\mathbb{R}/2\pi\mathbb{Z})^J$. The statement of the Lemma is equivalent to the existence of an increasing sequence of integers m_n such that $[m_n \vec{\omega}] \rightarrow \vec{0} := (0, \dots, 0) \in (\mathbb{R}/2\pi\mathbb{Z})^J$. Consider the set of points $m\vec{\omega}$, with $m \in \mathbb{N}$ and its projection $[m\vec{\omega}] = ([m\omega_1], \dots, [m\omega_J])$. Since the J -torus is compact this set has a limit point \vec{z} and an increasing subsequence $(m_i) \subset \mathbb{N}$ such that $[m_i \vec{\omega}] \rightarrow \vec{z}$. This is a Cauchy sequence so for any $\epsilon > 0$ there exists $\mathbf{I}(\epsilon)$ such that for any $i_1, i_2 \geq \mathbf{I}(\epsilon)$

$$d([n_{i_1} \vec{\omega}], [n_{i_2} \vec{\omega}]) < \epsilon,$$

where $d(\cdot, \cdot)$ denotes the metric on the J -torus. Taking a sequence $\epsilon_i \rightarrow 0$ we may chose $i_1(\epsilon_i) = \mathbf{I}(\epsilon_i)$ and in each step $i_2(\epsilon_i) > \mathbf{I}(\epsilon_i)$ so large that the difference

$$m_i := n_{i_2(\epsilon_i)} - n_{i_1(\epsilon_i)},$$

is an increasing sequence. It follows that

$$[m_i \vec{\omega}] = [(n_{i_2(\epsilon_i)} - n_{i_1(\epsilon_i)}) \vec{\omega}] \rightarrow \vec{0}. \quad \square$$

Remark 5.1. Note that in the special case where $2\pi, \omega_1, \dots, \omega_J$ are rationally independent the classical theorem of Kronecker (see e.g. [16]) can be used.

Proof of Theorem 5.2. Consider the trigonometric polynomial $p(k)$ in (5.1). Denote $k_m - m =: \gamma_m$ so that γ_m tends to zero as $m \rightarrow \infty$. We have $k_m = m + \gamma_m$ so for each ω_j :

$$e^{i\omega_j k_m} = e^{i\omega_j m} e^{i\omega_j \gamma_m}.$$

Choose a subsequence $\{m_n\}$ as described in Lemma 5.3 and pass, for an arbitrary $r \in \mathbb{N}$, to the $(r + m_n)$:th zero of p . We have

$$\begin{aligned} 0 = p(k_{r+m_n}) &= \sum_{j=1}^J a_j e^{i\omega_j(r+m_n)} e^{i\omega_j \gamma(r+m_n)} \\ &= \sum_{j=1}^J a_j e^{i\omega_j r} e^{i\omega_j m_n} e^{i\omega_j \gamma(r+m_n)} \rightarrow \sum_{j=1}^J a_j e^{i\omega_j r} = p(r), \end{aligned}$$

as $n \rightarrow \infty$. The limit follows from the choice of m_n and the fact that $\gamma_{(r+m_n)}$ tends to 0.

The above calculation shows that $p(r) = 0$. But $p(r) = 0$ and $k_r - r \rightarrow 0$ together imply that $k_r = r$, since even a single extra zero would make the asymptotic behaviour impossible. \square

Remark 5.2. It is not important in the above theorem that k_n tends to the integers. A scaling argument allows one to extend it to the case where k_n are close to integer-multiples of an arbitrary real number.

6. An Ambartsumian Theorem

With the result of the previous two sections we are now in a position to prove an inverse spectral theorem that may be seen as a generalization of Ambartsumian’s classical theorem. For the proof we recall the classical result as well as its geometric version for Laplacians.

The following theorem has been a source of inspiration for researchers in inverse problems for almost a century. In the original article [1] it was assumed that the potential is continuous, but the result holds even if $q \in L_1$. We adjusted the formulation to our notations.

Proposition 6.1. (Ambartsumian’s theorem [1]) *Let q be a real-valued absolutely integrable function on an interval I . Then the spectrum of the standard Schrödinger operator $L_q^{\text{st}}(I)$ coincides with the spectrum of the standard Laplacian $L_0^{\text{st}}(I)$ if and only if the potential q is identically equal to zero.*

Standard conditions on a single compact interval is of course just the classical Neumann conditions at both end-points. It appears that the theorem is still valid if instead of the interval I we have arbitrary connected finite compact metric graph Γ . This result was proven step-by-step by several authors

[10, 27, 29, 36] and [30], but the most general version was given by E.B. Davies [12] (Davies proved it for $q \in L_\infty(\Gamma)$ but noted that this condition surely can be weakened).

Proposition 6.2. (following [12]) *Let $L_q^{\text{st}}(\Gamma)$ and $L_0^{\text{st}}(\Gamma)$ be the standard Schrödinger and Laplace operators on a connected finite compact metric graph Γ . Assume that the potential q is absolutely integrable. Then the eigenvalues of the two operators coincide if and only if the potential q is equal to zero almost everywhere.*

The second theorem is a geometric version of Ambartsumian theorem for standard Laplacians. We start by recalling the result that among graphs with fixed total length the spectral gap is minimized by the single interval [13, 28] and [25]:

Proposition 6.3. (Theorem 3 from [25]) *Let $L_0^{\text{st}}(\Gamma)$ be the standard Laplace operator on a connected finite compact metric graph Γ of total length $\mathcal{L}(\Gamma)$. Assume that the first (nonzero) eigenvalue of $L_0^{\text{st}}(\Gamma)$ coincides with the first (nonzero) eigenvalue of the Laplacian on the interval I of length $\mathcal{L}(\Gamma)$*

$$\lambda_1(L_0^{\text{st}}(\Gamma)) = \lambda_1(L_0^{\text{st}}(I));$$

then the graph Γ coincides with the interval I .

The multiplicity of the eigenvalue zero for the standard Laplacian is equal to the number of connected components in the metric graph and the asymptotics of the spectrum determines the total length of the graph. Hence the above proposition implies:

Theorem 6.4. *Let Γ be a finite compact metric graph. The spectrum of the standard Laplacian on Γ coincides with the spectrum of the standard Laplacian on the interval I*

$$\lambda_j(L_0^{\text{st}}(\Gamma)) = \lambda_j(L_0^{\text{st}}(I)), \quad j = 0, 1, 2, \dots, \quad (6.1)$$

if and only if the graph Γ coincides with the interval I .

The assumptions of the theorem can be weakened, since to ensure that Γ and I have the same total length it is enough to check the asymptotics.

Our goal is to prove that if the spectrum of a Schrödinger operator on a metric graph coincides with the spectrum of the Laplacian on an interval then the graph coincides with the interval and the potential is zero. This statement cannot be proven as a simple combination of the above mentioned results (Proposition 6.2 and Theorem 6.4). The main difficulty is to show that the graph coincides with the interval. Theorem 6.4 cannot be applied directly, since it requires $q \equiv 0$.

Theorem 6.5. *Let Γ be a finite compact metric graph and $q \in L_1(\Gamma)$. The spectrum of the standard Schrödinger operator $L_q^{\text{st}}(\Gamma)$ coincides with the spectrum of the standard (i.e. Neumann) Laplacian on an interval*

$$\lambda_j(L_q^{\text{st}}(\Gamma)) = \lambda_j(L_0^{\text{st}}(I)), \quad (6.2)$$

if and only if $\Gamma = I$ and $q \equiv 0$.

Proof. Equation (6.2) implies that the total length of the graph Γ coincides with the length of the interval I . To see this it is enough to compare the corresponding asymptotics. Then the spectrum of the Laplacian on I is

$$\lambda_n(L_0^{\text{st}}(I)) = \left(\frac{\pi n}{\mathcal{L}}\right)^2, \quad n \in \mathbb{N}.$$

The spectral estimate (4.1) implies that

$$\begin{aligned} & \left| \underbrace{\lambda_n(L_q^{\text{st}}(\Gamma))}_{= \left(\frac{\pi n}{\mathcal{L}}\right)^2} - \lambda_n(L_0^{\text{st}}(\Gamma)) \right| = O(1). \end{aligned}$$

Hence the square roots of the eigenvalues of $L_0^{\text{st}}(\Gamma)$ satisfy

$$\left| k_n(L_0^{\text{st}}(\Gamma)) - \frac{\pi n}{\mathcal{L}} \right| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{6.3}$$

But k_n are given as the zeros of a trigonometric polynomial (Theorem 5.1). Hence the spectrum of the Laplacian on Γ given by zeroes of the trigonometric polynomial is asymptotically close to a set of equidistant points and Theorem 5.2 can be applied. We conclude that in fact $k_n(L_0^{\text{st}}(\Gamma)) = \pi n/\mathcal{L}$, for all n . This means that the spectrum of the Laplacian on Γ coincides with the spectrum of the Laplacian on an interval. We may finally apply the geometric version of Ambartsumian theorem for Laplacians (Theorem 6.4) to conclude that the graph Γ coincides with the interval I . But then Ambartsumian’s classical theorem (Proposition 6.1) implies that the potential is identically equal to zero $q \equiv 0$. \square

Proving our main theorem we have shown that in order to ensure that the graph Γ coincides with the interval, it is enough to require that the spectrum of the Schrödinger operator is close to the spectrum of the Laplacian on an interval. In fact we proved the following theorem:

Theorem 6.6. *Let $L_q^{\text{st}}(\Gamma)$ be standard Schrödinger operator with L_1 -potential on a finite compact connected metric graph Γ . Then $\Gamma = I$ if*

$$\sqrt{\lambda_n(L_q^{\text{st}}(\Gamma))} - \sqrt{\lambda_n(L_0^{\text{st}}(I))} \rightarrow 0, \quad n \rightarrow \infty. \tag{6.4}$$

7. Euler Characteristic

The spectral estimate (4.1) allows us to extend a previous result of one of the authors regarding the Euler characteristic $\chi = M - N$.

Theorem 7.1. *Let Γ be a finite compact metric graph and $q \in L_1(\Gamma)$. Then the Euler characteristic $\chi(\Gamma)$ is uniquely determined by the spectrum $\sigma(L_q^{\text{st}}(\Gamma))$, and can be calculated as the limit*

$$\chi(\Gamma) = 2 \lim_{t \rightarrow \infty} \sum_{n=0}^{\infty} \cos \sqrt{\lambda_n(L_q^{\text{st}}(\Gamma))}/t \left(\frac{\sin \sqrt{\lambda_n(L_q^{\text{st}}(\Gamma))}/2t}{\sqrt{\lambda_n(L_q^{\text{st}}(\Gamma))}/2t} \right)^2, \tag{7.1}$$

with the convention that $\lambda_n = 0$ implies

$$\frac{\sin \sqrt{\lambda_n(L_q^{\text{st}})/2t}}{\sqrt{\lambda_n(L_q^{\text{st}})/2t}} = 1. \quad (7.2)$$

Proof. In [19] formula (7.1) was proven for the case of zero potential. In other words formula (7.1) gives the Euler characteristics if one substitutes the eigenvalues $\lambda_n(L_0^{\text{st}})$. We have shown that the difference between $\lambda_n(L_q^{\text{st}}(\Gamma))$ and $\lambda_n(L_0^{\text{st}}(\Gamma))$ is uniformly bounded and therefore

$$|k_n - k_n^0| = O\left(\frac{1}{n}\right). \quad (7.3)$$

Taking into account Weyl's asymptotics

$$k_n^0 = \frac{\pi}{\mathcal{L}}n + O(1), \quad (7.4)$$

one may apply Lemma 2 from [20] to conclude that formula (7.1) gives the same result independently of whether the eigenvalues of the standard Laplacian $L_0^{\text{st}}(\Gamma)$ or of the standard Schrödinger operator $L_q^{\text{st}}(\Gamma)$ are substituted. \square

Note that proving the theorem we assumed that the vertex conditions are standard; this assumption in general cannot be removed.

8. Discussion

It would be interesting to understand under which assumptions a similar result holds for other than the standard vertex conditions that are assumed throughout in this paper. The cases, where the vertex conditions are such that the coefficients of the trigonometric polynomial given by Theorem 5.1 are k independent, could potentially be dealt with in a similar way as in this paper. The bounds on the differences of the eigenvalues obtained in Theorem 4.3 can be established for other choices of vertex conditions, but Theorem 6.4 in general may fail.

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Appendix A: Elementary spectral properties

In this appendix we provide proofs for the elementary spectral properties of quantum graphs already mentioned in Sect. 2. Most of these results are well-known, but we need them for the sake of completeness.

Proof of formula (2.2). For a bounded from below self-adjoint operator A with discrete spectrum, define the *eigenvalue counting function* $E_A : \mathbb{R} \rightarrow \mathbb{N}$, by

$$E_A(\lambda) = \#\{\lambda_j \in \sigma(A) \mid \lambda_j \leq \lambda\}.$$

The standard Laplacian is positive (hence the lower estimate in (2.2) is interesting only if $n > M$) therefore when calculating the eigenvalue counting function we assume that $\lambda \geq 0$.

Consider first the Laplace operator L_0^D on a single interval I of length ℓ with Dirichlet conditions at the end-points. The eigenvalues are $\lambda_n = (\frac{\pi n}{\ell})^2$, $n = 1, 2, \dots$. So the eigenvalue counting function for $\lambda \geq 0$ is in fact given by

$$E_{L_0^D(I)}(\lambda) = \left\lfloor \frac{\sqrt{\lambda}}{\pi} \ell \right\rfloor,$$

where square-brackets denote the integer-part of the argument. Returning to Γ we note that if we impose *Dirichlet* conditions on the vertices of Γ —denote the operator by $L_0^D(\Gamma)$ —then we really just have a decoupled set of intervals and therefore the set of eigenvalues is just the union of the eigenvalues for each interval (counting multiplicities). Therefore the corresponding counting function $E_{L_0^D(\Gamma)}$ is given by

$$\begin{aligned} E_{L_0^D(\Gamma)}(\lambda) &= \sum_{n=1}^N E_{L_0^D(E_n)}(\lambda) \\ &= \left\lfloor \frac{\sqrt{\lambda}}{\pi} \ell_1 \right\rfloor + \left\lfloor \frac{\sqrt{\lambda}}{\pi} \ell_2 \right\rfloor + \dots + \left\lfloor \frac{\sqrt{\lambda}}{\pi} \ell_N \right\rfloor \leq \left\lfloor \frac{\sqrt{\lambda}}{\pi} \mathcal{L} \right\rfloor, \end{aligned} \tag{A.1}$$

since taking integer parts may only decrease the value. First adding and then taking integer parts may—compared to adding the integer parts—at most raise the value by the number of terms -1 , i.e. the number of edges -1 , so conversely we also have

$$\left\lfloor \frac{\sqrt{\lambda}}{\pi} \mathcal{L} \right\rfloor - N + 1 \leq E_{L_0^D(\Gamma)}(\lambda). \tag{A.2}$$

Formulas (A.1) and (A.2) give effective two-sided bounds for the eigenvalues of the Dirichlet Laplacian on Γ .

We now show that $(L_0^D(\Gamma) - \lambda)^{-1} - (L_0^{st}(\Gamma) - \lambda)^{-1}$ is of finite rank. Take $\exists \lambda \neq 0$, and suppose that

$$(L_0^D - \lambda)u_D = f, \quad (L_0^{st} - \lambda)u_{st} = f.$$

Then for the differential operator $-d^2/dx^2$ we have that

$$\left(-\frac{d^2}{dx^2} - \lambda \right) (u_D - u_{st}) = 0.$$

Since functions in the domain of L_0^D and L_0^{st} are continuous, to determine the rank of the resolvent difference, we need to determine $\dim \ker \left(-\frac{d^2}{dx^2} - \lambda\right)$ on continuous functions. Prescribing values u_j at each vertex V_j in Γ , we see that a unique solution to this boundary value problem is given as follows: on an edge $E_n = [x_{2n-1}, x_{2n}]$ between V_i and V_j , set

$$u(x) := u_i \frac{\sin k(x - x_{2n})}{\sin k(x_{2n-1} - x_{2n})} + u_j \frac{\sin k(x - x_{2n-1})}{\sin k(x_{2n} - x_{2n-1})}$$

for $k^2 = \lambda$. Then u is continuous on Γ and solves $(-d^2/dx^2 - \lambda)u = 0$, so $\dim \ker(-d^2/dx^2 - \lambda) \leq M$. Therefore we have

$$E_{L_0^{st}(\Gamma)}(\lambda) \leq E_{L_0^D(\Gamma)}(\lambda) + M \leq \left\lceil \frac{\sqrt{\lambda}}{\pi} \mathcal{L} \right\rceil + M. \tag{A.3}$$

The lower estimate estimate (A.2) can be modified in a similar way, but instead we shall take into account that $L_0^{st}(\Gamma) \leq L_0^D(\Gamma)$. Really Dirichlet conditions in particular imply the continuity of functions in the domain of the quadratic form. Passing to standard conditions means weakening the conditions on functions in the domain of the quadratic form, since now *only* continuity is required at the vertices. Therefore the domain of the quadratic form $Q_{L_0^{st}}$ is larger than that of $Q_{L_0^D}$, so by the Min-Max principle (see Proposition 4.1) eigenvalues can only go up when imposing Dirichlet conditions. In particular, the lower bound (A.2) on the eigenvalue counting function $E_{L_0^D(\Gamma)}$ is also valid for $E_{L_0^{st}(\Gamma)}$.

Putting the lower and upper estimates together we have

$$\left\lfloor \frac{\sqrt{\lambda}}{\pi} \mathcal{L} \right\rfloor - N + 1 \leq E_{L_0^{st}(\Gamma)}(\lambda) \leq \left\lceil \frac{\sqrt{\lambda}}{\pi} \mathcal{L} \right\rceil + M \tag{A.4}$$

Setting $\lambda = \frac{\pi^2}{\mathcal{L}^2} n^2$ we obtain

$$n - N + 1 \leq E_{L_0^{st}(\Gamma)}\left(\frac{\pi^2}{\mathcal{L}^2} n^2\right) \leq n + M,$$

so

$$\lambda_{n-N+1} \leq \frac{\pi^2}{\mathcal{L}^2} n^2 \leq \lambda_{n+M}.$$

Setting $n' = n + M$ we get $\lambda_{n'} \geq \frac{\pi^2}{\mathcal{L}^2} (n' - M)^2$ and similarly we find $\lambda_{n'} \leq \frac{\pi^2}{\mathcal{L}^2} (n' + N - 1)^2$, which proves the theorem. \square

Proof of formula (2.3). The Weyl asymptotics follow from the relation

$$\left(\frac{\pi}{\mathcal{L}}\right)^2 (n - M)^2 \leq \lambda_n(L_0^{st}) \leq \left(\frac{\pi}{\mathcal{L}}\right)^2 (n + N - 1)^2. \tag{A.5} \quad \square$$

Proof of formula (2.4). For ψ corresponding to $\lambda = k^2$ we have

$$\|\psi\|_{L_2(\Gamma)}^2 \geq \int_{E_n} |\psi(x)|^2 dx \geq \left(\max_{x \in E_n} |\psi(x)|\right)^2 \frac{1}{2} \left\lfloor \frac{\ell_n k}{2\pi} \right\rfloor \frac{2\pi}{k},$$

where $[\cdot]$ denotes the integer part of the argument. $\lfloor \ell_n k / 2\pi \rfloor$ may be equal to zero for only finitely many k since the eigenvalues satisfy Weyl asymptotics.

Since $[\ell_n k/2\pi]/k$ is bounded for $k \in \mathbb{R}$ this implies the existence of a k -independent bound $c(\Gamma)$. \square

Proof of formula (3.1). Let x_{\min} denote a global minimum for ψ . Then in particular $|\psi(x_{\min})|^2 \leq \|\psi\|_{L_2}^2/\ell$. We then have

$$\begin{aligned} |\psi(x)|^2 &= |\psi(x_{\min})|^2 + 2 \int_{x_{\min}}^x \overline{\psi(y)} \psi'(y) dy \\ &\leq |\psi(x_{\min})|^2 + 2 \int_0^\ell |\overline{\psi(y)} \psi'(y)| dy \\ &\leq |\psi(x_{\min})|^2 + \epsilon \int_0^\ell |\psi'(y)|^2 dy + \frac{1}{\epsilon} \int_0^\ell |\psi(y)|^2 dy \\ &\leq \|\psi\|_2^2/\ell + \epsilon \|\psi'\|_2^2 + \frac{1}{\epsilon} \|\psi\|_2^2 \\ &= \epsilon \|\psi'\|_2^2 + \left(\frac{1}{\epsilon} + \frac{1}{\ell} \right) \|\psi\|_2^2. \end{aligned}$$

For $\epsilon < \ell$ we have $1/\epsilon > 1/\ell$ and the claim follows. \square

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