



Compressions of Self-Adjoint Extensions of a Symmetric Operator and M.G. Krein's Resolvent Formula

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Dedicated to Professor Rien Kaashoek on the occasion of his 80th birthday.

Abstract. Let S be a symmetric operator with finite and equal defect numbers in the Hilbert space \mathfrak{H} . We study the compressions $P_{\mathfrak{H}}\tilde{A}|_{\mathfrak{H}}$ of the self-adjoint extensions \tilde{A} of S in some Hilbert space $\tilde{\mathfrak{H}} \supset \mathfrak{H}$. These compressions are symmetric extensions of S in \mathfrak{H} . We characterize properties of these compressions through the corresponding parameter of \tilde{A} in M.G. Krein's resolvent formula. If $\dim(\tilde{\mathfrak{H}} \ominus \mathfrak{H})$ is finite, according to Stenger's lemma the compression of \tilde{A} is self-adjoint. In this case we express the corresponding parameter for the compression of \tilde{A} in Krein's formula through the parameter of the self-adjoint extension \tilde{A} .

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1. Introduction

1.1. In this paper we study the compressions of self-adjoint extensions of a densely defined, closed symmetric operator S with equal and finite defect numbers $d > 0$. Recall that if \mathfrak{H} and $\tilde{\mathfrak{H}}$ are Hilbert spaces, \mathfrak{H} is a subspace of $\tilde{\mathfrak{H}}$ and \tilde{A} is an operator in $\tilde{\mathfrak{H}}$, the *compression* of \tilde{A} to \mathfrak{H} is the operator $C_{\mathfrak{H}}(\tilde{A}) := \tilde{P}_{\mathfrak{H}}\tilde{A}|_{\mathfrak{H}}$; here $\tilde{P}_{\mathfrak{H}}$ denotes the orthogonal projection onto \mathfrak{H} in $\tilde{\mathfrak{H}}$, and $C_{\mathfrak{H}}(\tilde{A})$ is an operator in \mathfrak{H} , defined on the intersection $(\text{dom } \tilde{A}) \cap \mathfrak{H}$. If \tilde{A} is self-adjoint and the extending space $\tilde{\mathfrak{H}} \ominus \mathfrak{H}$ is finite dimensional, then Stenger's lemma [24] yields that the compression $C_{\mathfrak{H}}(\tilde{A})$ is also self-adjoint; if $\tilde{\mathfrak{H}} \ominus \mathfrak{H}$ is infinite dimensional this is no longer true in general.

Compressions of linear operators or relations were recently studied in the papers [2–5, 23], and [11]. In the latter we gave a description—in terms

of certain parameters—of the compressions of the self-adjoint extensions \tilde{A} of a symmetric operator S with finite and equal defect numbers $d > 0$ under the assumption that $\dim(\tilde{\mathfrak{H}} \ominus \mathfrak{H}) < \infty$. According to Stenger’s lemma these compressions are self-adjoint extensions of S , and hence their resolvents can also be described by Krein’s resolvent formula. Such a description was given at the end of [11].

In the present paper Krein’s resolvent formula is the starting point. We consider the self-adjoint extensions $\tilde{A}_{\mathcal{T}}$ of a symmetric operator S with exit in a space $\tilde{\mathfrak{H}}$ such that $\dim(\tilde{\mathfrak{H}} \ominus \mathfrak{H})$ is not necessarily finite. Here \mathcal{T} is the parameter in Krein’s formula (see (1.12)) which characterizes the self-adjoint extension $\tilde{A}_{\mathcal{T}}$: It is a $d \times d$ relation valued Nevanlinna function. The compressions $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$ are in general symmetric and closed, but not self-adjoint extensions of S , acting in the space \mathfrak{H} . We describe these compressions and, in particular, we describe those parameters \mathcal{T} for which the compression $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$ coincides with S or with the self-adjoint extension A_0 of S which acts as basic operator in Krein’s formula (1.12). If $\dim(\tilde{\mathfrak{H}} \ominus \mathfrak{H}) < \infty$ and hence the compression is self-adjoint, we show that the corresponding parameter for $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$ in Krein’s formula is $\mathcal{T}(\infty) = \lim_{z \rightarrow \infty} \mathcal{T}(z)$, where the limit is understood in the sense of linear relations, see (2.5).

A short synopsis is as follows. In the next two subsections of this Introduction we recall some facts about matrix or relation valued Nevanlinna functions, and about Krein’s resolvent formula. In Sect. 2 we prove some statements connected with Krein’s formula which might be of general interest. In Subsect. 2.1 we derive a relation that connects the parameters for two Krein formulas with basic extensions A_0 and A_1 , in Subsect. 2.2 we prove a representation of the resolvent of the self-adjoint extension $\tilde{A}_{\mathcal{T}}$ using the operator or relation representation of the parameter \mathcal{T} (comp. [9]). It leads to a formula for the extension $\tilde{A}_{\mathcal{T}}$ which is the starting point for our study of the compressions of $\tilde{A}_{\mathcal{T}}$ in Sect. 3. There, in Theorem 3.2, we give sufficient conditions for the parameters \mathcal{T} which lead to extensions $\tilde{A}_{\mathcal{T}}$ such that

$$S \subset C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0.$$

Conditions under which in this relation the signs \subset become equalities are given in Subsect. 3.2. In Subsect. 3.3 we show that any symmetric extension \tilde{S} of S in \mathfrak{H} is the compression $C_{\mathfrak{H}}(\tilde{A})$ for some self-adjoint extension \tilde{A} of S . Clearly, because of Stenger’s lemma, if \tilde{S} is not self-adjoint the extending space $\tilde{\mathfrak{H}} \ominus \mathfrak{H}$ has to be infinite dimensional.

In Sect. 3 the parameter \mathcal{T} is assumed to be a matrix function. The main results there can easily be adapted to the case where \mathcal{T} is a relation valued function. This is indicated in Remark 4.2.

In Sect. 4 we consider extensions $\tilde{A}_{\mathcal{T}}$ with finite-dimensional exit space and hence with self-adjoint compressions. Such a self-adjoint compression corresponds to a constant parameter in Krein’s formula. As one of the main results of this paper we show in Theorem 4.6 that this parameter is the limit $\mathcal{T}(\infty)$.

Finally, in an Appendix we show that the dilation theory for dissipative operators as developed in [19, 20] and [21] leads in a natural way to self-adjoint extensions for which the compression is the original symmetry S .

This paper is dedicated to our colleague and dear friend Rien Kaashoek, to appreciate his leading role in operator theory and also to thank him for his personal support in establishing the contact of the second author to the colleagues in Groningen.

1.2. In this subsection we collect some facts about matrix and relation valued Nevanlinna functions. Let $d \in \mathbb{N}$. The $d \times d$ matrix valued function \mathcal{N} , defined on $\mathbb{C} \setminus \mathbb{R}$, is a *Nevanlinna function* if it has one of the following equivalent properties:

(a) \mathcal{N} is holomorphic and satisfies

$$\mathcal{N}(z^*) = \mathcal{N}(z)^* \quad \text{and} \quad \frac{\mathcal{N}(z) - \mathcal{N}(z)^*}{z - z^*} \geq 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

(b) \mathcal{N} admits the integral representation

$$\mathcal{N}(z) = \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma(t) + \mathcal{A} + z\mathcal{B}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (1.1)$$

where \mathcal{A} and \mathcal{B} are symmetric $d \times d$ matrices, $\mathcal{B} \geq 0$, and Σ is a symmetric non-decreasing $d \times d$ matrix function on \mathbb{R} such that

$$\int_{\mathbb{R}} (t^2 + 1)^{-1} d\Sigma(t) < \infty.$$

The properties (a) and (b) are also equivalent to the following:

(c) \mathcal{N} admits an operator or relation representation, that is, there exist a Hilbert space $\mathfrak{H}_{\mathcal{N}}$, a self-adjoint linear relation $B_{\mathcal{N}}$ in $\mathfrak{H}_{\mathcal{N}}$, and, after fixing a point $z_0 \in \mathbb{C} \setminus \mathbb{R}$, a linear mapping $\delta : \mathbb{C}^d \rightarrow \mathfrak{H}_{\mathcal{N}}$, such that

$$\mathcal{N}(z) = \mathcal{N}(z_0)^* + (z - z_0)^* \delta^* (I + (z - z_0)(B_{\mathcal{N}} - z)^{-1}) \delta, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (1.2)$$

We denote by $R_{\mathcal{N}}(z) := (B_{\mathcal{N}} - z)^{-1}$ the resolvent of $B_{\mathcal{N}}$, and set

$$\delta_z := (I + (z - z_0)R_{\mathcal{N}}(z))\delta, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (1.3)$$

It follows that

$$\frac{\mathcal{N}(z) - \mathcal{N}(w)^*}{z - w^*} = \delta_w^* \delta_z, \quad z, w \in \mathbb{C} \setminus \mathbb{R},$$

and for an arbitrary $\widehat{z}_0 \in \mathbb{C} \setminus \mathbb{R}$ the relation (1.2) becomes

$$\mathcal{N}(z) = \mathcal{N}(\widehat{z}_0)^* + (z - \widehat{z}_0)^* \delta_{\widehat{z}_0}^* (I + (z - \widehat{z}_0)(B_{\mathcal{N}} - z)^{-1}) \delta_{\widehat{z}_0}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

The operator representation (1.2) will always be chosen *minimal*, which means that

$$\mathfrak{H}_{\mathcal{N}} = \overline{\text{span}} \{ \delta_z \mathbf{x} : \mathbf{x} \in \mathbb{C}^d, z \in \mathbb{C} \setminus \mathbb{R} \}. \quad (1.4)$$

The triplet $(\mathfrak{H}_{\mathcal{N}}, B_{\mathcal{N}}, \delta)$ is sometimes called a *model* of the function \mathcal{N} . The above relations extend to points $z \in \mathbb{R}$ into which \mathcal{N} can be continued analytically or, equivalently, which belong to $\rho(B_{\mathcal{N}})$.

Besides matrix valued functions, we also need $d \times d$ relation valued Nevanlinna functions \mathcal{N} . Their values are linear relations

$$\mathcal{N}(z) = \mathcal{N}_{op}(z) \oplus \mathcal{N}_\infty, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.5}$$

This representation is with respect to a decomposition $\mathbb{C}^d = \mathbb{L}_{op} \oplus \mathbb{L}_\infty$ of the space \mathbb{C}^d , $\mathcal{N}_{op}(z)$ is a $d_{op} \times d_{op}$ matrix valued Nevanlinna function, where $d_{op} = \dim \mathbb{L}_{op}$, and $\mathcal{N}_\infty = \{0, \mathbb{L}_\infty\}$, or $\mathcal{N}(0) = \mathbb{L}_\infty$, also called the *multi-valued part* of \mathcal{N} . With the orthogonal projection P onto \mathbb{L}_{op} , the relation (1.5) can also be written as

$$\begin{aligned} \mathcal{N}(z) &= \{ \{P\mathbf{x}, P\mathcal{N}_{op}(z)P\mathbf{x}\} + \{0, (I - P)\mathbf{x}\} : \mathbf{x} \in \mathbb{C}^d \} \\ &= \{ \{\mathbf{x}, \mathcal{N}_{op}(z)\mathbf{x}\} : \mathbf{x} \in \mathbb{L}_{op} \} \oplus \{ \{0, \mathbf{x}\} : \mathbf{x} \in \mathbb{L}_\infty \}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \end{aligned} \tag{1.6}$$

Clearly, \mathcal{N} is a matrix valued function if and only if $P = I$. The first summand on the right-hand side of (1.6) can be decomposed further as

$$\{ \{\mathbf{x}, \mathcal{N}_{op}(z)\mathbf{x}\} : \mathbf{x} \in \mathbb{L}_{op} \} = \{ \{\mathbf{x}, 0\} : \mathbf{x} \in \mathbb{L}_0 \} \oplus \{ \{\mathbf{x}, \widehat{\mathcal{N}}_{op}(z)\mathbf{x}\} : \mathbf{x} \in \widehat{\mathbb{L}}_{op} \},$$

where $\mathbb{L}_{op} = \mathbb{L}_0 \oplus \widehat{\mathbb{L}}_{op}$ and $\widehat{\mathcal{N}}_{op}(z)$ has no kernel.

An intrinsic definition of $d \times d$ relation valued Nevanlinna functions, with the above decompositions as consequences, was given in [22].

If the $d \times d$ matrix valued Nevanlinna function \mathcal{N} is rational its representation (1.1) becomes

$$\mathcal{N}(z) = \sum_{j=1}^{\ell} \frac{\mathcal{A}_j}{\alpha_j - z} + \mathcal{A} + z\mathcal{B}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1.7}$$

with mutually distinct points $\alpha_j \in \mathbb{R}$ and nonzero $d \times d$ matrices $\mathcal{A}_j \geq 0$, $j = 1, 2, \dots, \ell$, a symmetric $d \times d$ matrix \mathcal{A} , and a $d \times d$ matrix $\mathcal{B} \geq 0$.

In the following, the asymptotic behavior of the $d \times d$ matrix Nevanlinna function \mathcal{N} with the representation (1.1) or (1.7) plays an essential role. We mention the following relations:

$$\begin{aligned} \mathcal{B} &= \lim_{y \uparrow \infty} \frac{\mathcal{N}(iy)}{iy} = \lim_{y \uparrow \infty} \frac{\text{Im } \mathcal{N}(iy)}{y}, \\ \lim_{y \uparrow \infty} y \text{Im } \langle \mathcal{N}(iy)\mathbf{x}, \mathbf{x} \rangle &= \infty \text{ for all } \mathbf{x} \in \mathbb{C}^d \setminus \{0\} \\ \iff \begin{cases} \mathcal{B} > 0 & \text{if } \mathcal{N} \text{ is rational,} \\ \left\langle \int_{\mathbb{R}} d\Sigma(t)\mathbf{x}, \mathbf{x} \right\rangle = \infty \text{ for } \mathbf{x} \in \ker \mathcal{B} \setminus \{0\} & \text{otherwise,} \end{cases} \end{aligned} \tag{1.8}$$

and, if $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\text{Im } \mathcal{N}(z) / \text{Im } z > 0 \iff \begin{cases} \mathcal{B} + \sum_{j=1}^{\ell} \mathcal{A}_j > 0 & \text{if } \mathcal{N} \text{ is rational,} \\ \mathcal{B} + \int_{\mathbb{R}} \frac{d\Sigma(t)}{|t - z|^2} > 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

In the sequel, using the language of linear relations we often make no distinction between operators and their graphs (as, for example in [7, 10] and [9]).

1.3. Here we recall M.G. Krein's resolvent formula. In the following, S denotes a densely defined, closed symmetric operator in a Hilbert space \mathfrak{H} with finite and equal defect numbers $d > 0$. We choose a canonical self-adjoint extension A_0 of S (*canonical* means that A_0 acts in \mathfrak{H}), a point $z_0 \in \mathbb{C} \setminus \mathbb{R}$, and a bijective mapping $\gamma : \mathbb{C}^d \rightarrow \ker(S^* - z_0)$. With γ and the canonical self-adjoint extension A_0 we define a so-called γ -field

$$\gamma_z : \mathbb{C}^d \rightarrow \ker(S^* - z), \quad \gamma_z := (A_0 - z_0)(A_0 - z)^{-1}\gamma, \quad z \in \rho(A_0).$$

Evidently, γ_z is a bijection, and $\gamma_{z_0} = \gamma$. Note that for each $z \in \mathbb{C} \setminus \mathbb{R}$

$$S = \{ \{f, g\} \in A_0 : \gamma_z^*(g - zf) = 0 \}. \quad (1.10)$$

With the γ -field γ_z there is defined a corresponding Q -function \mathcal{Q}_0 by the relation

$$\frac{\mathcal{Q}_0(z) - \mathcal{Q}_0(w)^*}{z - w^*} = \gamma_w^* \gamma_z, \quad z, w \in \rho(A_0), \quad (1.11)$$

see [18]. It is a $d \times d$ matrix valued function, which is determined by (1.11) up to a constant symmetric $d \times d$ matrix summand. Evidently,

$$\operatorname{Im} \mathcal{Q}_0(z) / \operatorname{Im} z = \gamma_z^* \gamma_z > 0, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

hence \mathcal{Q}_0 is a Nevanlinna function.

If $\hat{\gamma}_z : \mathbb{C}^d \rightarrow \ker(S^* - z)$ is another γ -field with corresponding Q -function $\hat{\mathcal{Q}}_0(z)$, then there are an invertible $d \times d$ matrix \mathcal{C} and a symmetric $d \times d$ matrix \mathcal{D} such that

$$\hat{\gamma}_z = \gamma_z \mathcal{C} \text{ and } \hat{\mathcal{Q}}_0(z) = \mathcal{C}^* \mathcal{Q}_0(z) \mathcal{C} + \mathcal{D}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

(comp. [8, Lemma 2 and Corollary 3]).

The Q -function plays an essential role in M.G. Krein's resolvent formula. If \tilde{A} is any self-adjoint extension of S , acting in \mathfrak{H} or in some larger Hilbert space $\tilde{\mathfrak{H}}$, the compressed resolvent of \tilde{A} : $P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{H}}$ is called a *generalized resolvent of S* , corresponding to the extension \tilde{A} . The set of all generalized resolvents of S can be described as follows (see [17, Theorem 5.1], [22, Theorem 3.2] and [9, Theorem 6.2]):

There is a bijective correspondence between all generalized resolvents of S and all $d \times d$ relation valued Nevanlinna functions \mathcal{T} such that

$$P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{H}} = (A_0 - z)^{-1} - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \gamma_z^*, \quad z \in \rho(A_0) \cap \rho(\tilde{A}). \quad (1.12)$$

We call (1.12) *Krein's resolvent formula*. It depends on the chosen canonical self-adjoint extension A_0 of S , which determines the γ -field and the Q -function. To express this dependence on A_0 we call (1.12) sometimes *Krein's formula based on A_0* . The operator \tilde{A} on the left-hand side of (1.12) corresponding to \mathcal{T} is denoted by $\tilde{A}_{\mathcal{T}}$. If \mathcal{T} is relation valued the inverse on the right-hand side of (1.12) reads as

$$(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} = P(P\mathcal{Q}_0(z)P + \mathcal{T}_{op}(z))^{-1}P, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where the operator part $\mathcal{T}_{op}(z)$ of $\mathcal{T}(z)$ and the projection P are as in (1.6), see also [17, Theorem 5.1] and [22, (1.8)].

In Krein’s resolvent formula, the parameter $\mathcal{T}(z)$ is a z -independent self-adjoint relation in \mathbb{C}^d if and only if $\tilde{A}_{\mathcal{T}}$ is a canonical self-adjoint extension of S . If \mathcal{T} is a rational $d \times d$ relation valued function then the extending space $\tilde{\mathfrak{H}} \ominus \mathfrak{H}$ is finite-dimensional, its dimension being the total multiplicity of the poles (including ∞) of \mathcal{T}_{op} . The parameter \mathcal{T} is a matrix valued function if and only if $\tilde{A}_{\mathcal{T}} \cap A_0 = S$ (comp. Proposition 3.4 below).

2. Auxiliary Statements

2.1. In this subsection we compare the parameters in two Krein formulas based on two different canonical self-adjoint extensions. Let S be a densely defined, closed symmetric operator in a Hilbert space \mathfrak{H} with finite and equal defect numbers $d > 0$. Let A_0 and A_1 be two canonical self-adjoint extensions of S , denote by $\mathcal{Q}_0(z)$ and $\mathcal{Q}_1(z)$ corresponding Q -functions and by $\gamma_{0,z}$ and $\gamma_{1,z}$ corresponding γ -fields. The latter means that for $j = 0, 1$ and $z, w \in \mathbb{C} \setminus \mathbb{R}$

$$\frac{Q_j(z) - Q_j(w)^*}{z - w^*} = \gamma_{j,w}^* \gamma_{j,z}, \quad \gamma_{j,z} = \gamma_{j,w} + (z - w)(A_j - z)^{-1} \gamma_{j,w}.$$

Then, by Krein’s formulas based on A_0 , there is a self-adjoint relation \mathcal{T}_0 in \mathbb{C}^d such that

$$(A_1 - z)^{-1} = (A_0 - z)^{-1} - \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \gamma_{0,z}^*, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.1)$$

Let \tilde{A} be a self-adjoint extension of S in a possibly larger Hilbert space $\tilde{\mathfrak{H}} \supset \mathfrak{H}$. Then, by Krein’s formula, there exist $d \times d$ matrix or relation valued Nevanlinna functions $\mathcal{S}_0, \mathcal{S}_1$ such that

$$\begin{aligned} P_{\tilde{\mathfrak{H}}}(\tilde{A} - z)^{-1}|_{\mathfrak{H}} &= (A_0 - z)^{-1} - \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{S}_0(z))^{-1} \gamma_{0,z}^* \\ &= (A_1 - z)^{-1} - \gamma_{1,z}(\mathcal{Q}_1(z) + \mathcal{S}_1(z))^{-1} \gamma_{1,z}^*. \end{aligned} \quad (2.2)$$

In the present subsection we prove a formula connecting $\mathcal{S}_0(z)$ and $\mathcal{S}_1(z)$.

To this end we ‘normalize’ the Q -functions and the γ -fields as follows. We fix $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and a bijection $\gamma_{z_0} : \mathbb{C}^d \rightarrow \ker(S^* - z_0^*)$, and then choose $\gamma_{j,z}$ and $\mathcal{Q}_j(z)$, $j = 0, 1$, such that

$$\gamma_{0,z_0} = \gamma_{1,z_0} = \gamma_{z_0}, \quad \mathcal{Q}_0(z_0) = \mathcal{Q}_1(z_0) = i \operatorname{Im} z_0 (\gamma_{z_0}^* \gamma_{z_0}) =: \mathcal{Q}. \quad (2.3)$$

The latter normalization can be made since a Q -function is determined up to a constant symmetric $d \times d$ matrix summand. Then \mathcal{Q} is an invertible skew symmetric $d \times d$ matrix, $\mathcal{Q}_j(z) - \mathcal{Q} = (z - z_0) \gamma_{j,z_0}^* \gamma_{z_0}$ and this matrix is invertible in a neighborhood of $z = z_0^*$.

Theorem 2.1. *With the normalization (2.3) and under the assumption that $\mathcal{S}_0(z)$ is a matrix function we have*

$$\mathcal{S}_1(z) = \mathcal{S}_0(z) + (\mathcal{S}_0(z) - \mathcal{Q})(\mathcal{T}_0 - \mathcal{S}_0(z))^{-1}(\mathcal{S}_0(z) + \mathcal{Q}); \quad (2.4)$$

the equality (2.4) holds in terms of linear relations.

Recall that if \mathcal{F} and \mathcal{G} are linear relations in \mathbb{C}^d then

$$\begin{aligned}\mathcal{F}^{-1} &= \{\{\mathbf{y}, \mathbf{x}\} : \{\mathbf{x}, \mathbf{y}\} \in \mathcal{F}\}, \\ \mathcal{F} \pm \mathcal{G} &= \{\{\mathbf{x}, \mathbf{y} \pm \mathbf{z}\} : \{\mathbf{x}, \mathbf{y}\} \in \mathcal{F}, \{\mathbf{x}, \mathbf{z}\} \in \mathcal{G}\}, \\ \mathcal{G}\mathcal{F} &= \{\{\mathbf{x}, \mathbf{z}\} : \{\mathbf{x}, \mathbf{y}\} \in \mathcal{F}, \{\mathbf{y}, \mathbf{z}\} \in \mathcal{G}\}.\end{aligned}$$

On the right-hand side of (2.4), \mathcal{T}_0 and $(\mathcal{T}_0 - \mathcal{S}_0(z))^{-1}$ can be relations. If

$$\mathcal{T}_0 = \{\{\mathcal{P}\mathbf{y}, \mathcal{P}\mathcal{T}_{0,op}\mathcal{P}\mathbf{y} + (I - \mathcal{P})\mathbf{z}\} : \mathbf{y}, \mathbf{z} \in \mathbb{C}^d\},$$

then

$$(\mathcal{T}_0 - \mathcal{S}_0(z))^{-1} = \{\{\mathcal{P}(\mathcal{T}_{0,op} - \mathcal{S}_0(z))\mathcal{P}\mathbf{y} + (I - \mathcal{P})\mathbf{z}, \mathcal{P}\mathbf{y}\} : \mathbf{y}, \mathbf{z} \in \mathbb{C}^d\}$$

and $\mathcal{S}_1(z)$ has a multi-valued part if and only if $\ker(\mathcal{T}_{0,op} - \mathcal{P}\mathcal{S}_0(z)\mathcal{P}) \neq \{0\}$. In this case the multi-valued part is given by

$$\mathcal{S}_1(z)(0) = (\mathcal{S}_0(z) - \mathcal{Q})\ker(\mathcal{T}_{0,op} - \mathcal{P}\mathcal{S}_0(z)\mathcal{P}).$$

To see that it is independent of z , we use that the subspace $\ker(\operatorname{Im}\mathcal{S}_0(z))$ of \mathbb{C}^d is independent of z and that $\mathcal{S}_0(z)$ restricted to this subspace is identically equal to a constant matrix \mathcal{C} , say (see [13, Lemma 5.3] and [6, Step 1 in the proof of Theorem 3.2]). Let $\mathbf{x} \in \ker(\mathcal{T}_{0,op} - \mathcal{P}\mathcal{S}_0(z)\mathcal{P})$. Then $\mathbf{x} = \mathcal{P}\mathbf{x}$, $\operatorname{Im}\langle \mathcal{S}_0(z)\mathbf{x}, \mathbf{x} \rangle = \operatorname{Im}\langle \mathcal{T}_{0,op}\mathbf{x}, \mathbf{x} \rangle = 0$ and hence $\mathbf{x} \in \ker(\operatorname{Im}\mathcal{S}_0(z))$. It follows that $\mathcal{S}_0(z)\mathbf{x} = \mathcal{C}\mathbf{x}$, and we find that

$$(\mathcal{S}_0(z) - \mathcal{Q})\ker(\mathcal{T}_{0,op} - \mathcal{P}\mathcal{S}_0(z)\mathcal{P}) = (\mathcal{C} - \mathcal{Q})\ker(\mathcal{T}_{0,op} - \mathcal{P}\mathcal{C}\mathcal{P}) = \mathbb{L}_\infty(\mathcal{S}_1).$$

In the proof of Theorem 2.1 we use properties of the convergence of linear relations. Let \mathcal{T} and \mathcal{T}_n , $n \in \mathbb{N}$, be linear relations in \mathbb{C}^d . We say that \mathcal{T}_n converges to \mathcal{T} as $n \rightarrow \infty$, in symbols $\mathcal{T}_n \rightarrow \mathcal{T}$ if

$$\mathcal{T} = \{\{\mathbf{u}, \mathbf{v}\} : \exists \{\mathbf{u}_n, \mathbf{v}_n\} \in \mathcal{T}_n : \mathbf{u}_n \rightarrow \mathbf{u}, \mathbf{v}_n \rightarrow \mathbf{v}\}. \quad (2.5)$$

For example, if $a \in \mathbb{C}$ and $n \rightarrow \infty$, then in \mathbb{C}^2

$$\begin{pmatrix} n & 0 \\ 0 & a \end{pmatrix} \rightarrow \left\{ \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix}, \begin{pmatrix} s \\ at \end{pmatrix} \right\} : s, t \in \mathbb{C} \right\}.$$

Lemma 2.2. *Let \mathcal{T} and \mathcal{T}_n , $n \in \mathbb{N}$, be linear relations in \mathbb{C}^d and assume $\mathcal{T}_n \rightarrow \mathcal{T}$ if $n \rightarrow \infty$. Then:*

- (i) $\mathcal{T}_n^{-1} \rightarrow \mathcal{T}^{-1}$.
- (ii) If \mathcal{A} is an invertible $d \times d$ matrix, then $\mathcal{T}_n\mathcal{A} \rightarrow \mathcal{T}\mathcal{A}$ and $\mathcal{A}\mathcal{T}_n \rightarrow \mathcal{A}\mathcal{T}$.
- (iii) If \mathcal{C} and \mathcal{C}_n , $n \in \mathbb{N}$, are $d \times d$ matrices such that $\mathcal{C}_n \rightarrow \mathcal{C}$, then

$$\mathcal{C}_n + \mathcal{T}_n \rightarrow \mathcal{C} + \mathcal{T}.$$

- (iv) If in addition \mathcal{T}_n and \mathcal{T} are matrices and the \mathcal{T}_n 's are uniformly bounded, then $\mathcal{T}_n \rightarrow \mathcal{T}$.

Proof. We only prove the first statement in (ii) and (iv). Let \mathcal{L} be the limit of $\mathcal{T}_n\mathcal{A}$ and let $\{\mathbf{u}, \mathbf{w}\} \in \mathcal{T}\mathcal{A}$. Then $\{\mathcal{A}\mathbf{u}, \mathbf{w}\} \in \mathcal{T}$ and hence there is a sequence $\{\mathbf{v}_n, \mathbf{w}_n\} \in \mathcal{T}_n$ converging to $\{\mathcal{A}\mathbf{u}, \mathbf{w}\}$. Set $\mathbf{u}_n = \mathcal{A}^{-1}\mathbf{v}_n$. Then $\{\mathbf{u}_n, \mathbf{w}_n\} \in \mathcal{T}_n\mathcal{A}$ and this sequence converges to $\{\mathbf{u}, \mathbf{w}\}$. Hence $\{\mathbf{u}, \mathbf{w}\} \in \mathcal{L}$ and $\mathcal{T}\mathcal{A} \subset \mathcal{L}$.

Conversely, let $\{\mathbf{x}, \mathbf{y}\} \in \mathcal{L}$ and assume that $\{\mathbf{v}_n, \mathbf{w}_n\} \in \mathcal{T}_n \mathcal{A}$ converges to $\{\mathbf{x}, \mathbf{y}\}$. Then $\{\mathcal{A}\mathbf{v}_n, \mathbf{w}_n\} \in \mathcal{T}_n$ converges to $\{\mathcal{A}\mathbf{x}, \mathbf{y}\}$. Hence $\{\mathcal{A}\mathbf{x}, \mathbf{y}\} \in \mathcal{T}$, that is $\{\mathbf{x}, \mathbf{y}\} \in \mathcal{TA}$. Thus $\mathcal{L} = \mathcal{TA}$.

To prove (iv), let $\{\mathbf{x}, \mathcal{T}\mathbf{x}\} \in \mathcal{T}$. Then there are $\{\mathbf{u}_n, \mathbf{v}_n\} \in \mathcal{T}_n$ converging to $\{\mathbf{x}, \mathcal{T}\mathbf{x}\}$. Hence, if $\|\cdot\|$ denotes the norm in \mathbb{C}^d ,

$$\|\mathcal{T}_n \mathbf{x} - \mathcal{T}\mathbf{x}\| = \|\mathcal{T}_n \mathbf{x} - \mathcal{T}_n \mathbf{u}_n + \mathbf{v}_n - \mathcal{T}\mathbf{x}\| \leq \|\mathcal{T}_n\| \|\mathbf{x} - \mathbf{u}_n\| + \|\mathbf{v}_n - \mathcal{T}\mathbf{x}\| \rightarrow 0. \quad \square$$

Proof of Theorem 2.1. The proof is split into two parts. In the first part we additionally assume that \mathcal{T}_0 is a matrix, in the second part \mathcal{T}_0 is a relation.

(i) Assume \mathcal{T}_0 is a matrix. We set

$$\begin{aligned} \Delta_0(z) &= (\mathcal{Q}_0(z) + \mathcal{S}_0(z))^{-1} - (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \\ &= (\mathcal{Q}_0(z) + \mathcal{S}_0(z))^{-1} (\mathcal{T}_0 - \mathcal{S}_0(z)) (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}, \end{aligned}$$

where all the inverses exist as matrices. Via Krein’s formula based on A_1 the generalized resolvent $P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{H}}$ determines and is determined by the parameter $\mathcal{S}_1(z)$. Thus, if we assume that $\mathcal{S}_1(z)$ is given by (2.4), (2.2) implies that the theorem is proved by showing that

$$\begin{aligned} \gamma_{1,z}(\mathcal{Q}_1(z) + \mathcal{S}_1(z))^{-1} \gamma_{1,z}^* &= (A_1 - z)^{-1} - (A_0 - z)^{-1} + \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{S}_0(z))^{-1} \gamma_{0,z}^* \\ &= -\gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \gamma_{0,z}^* + \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{S}_0(z))^{-1} \gamma_{0,z}^* \\ &= \gamma_{0,z} \Delta_0(z) \gamma_{0,z}^*. \end{aligned} \tag{2.6}$$

We set $\mathcal{D} := \mathcal{T}_0 + \mathcal{Q}$ and obtain

$$\begin{aligned} \gamma_{1,z} &= \gamma_{z_0} + (z - z_0)(A_1 - z)^{-1} \gamma_{z_0} \\ &= \gamma_{z_0} + (z - z_0)(A_0 - z)^{-1} \gamma_{z_0} - \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \gamma_{0,z}^* \gamma_{z_0} \\ &= \gamma_{0,z} (I - (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} (\mathcal{Q}_0(z) - \mathcal{Q})) \\ &= \gamma_{0,z} (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \mathcal{D}. \end{aligned} \tag{2.7}$$

Using $\mathcal{D}^* = \mathcal{T}_0 - \mathcal{Q}$ and (2.7) it follows that

$$\begin{aligned} \mathcal{Q}_1(z) &= \mathcal{Q} + (z - z_0) \gamma_{1,z}^* \gamma_{z_0} \\ &= \mathcal{Q} + (z - z_0) \mathcal{D}^* (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \gamma_{0,z}^* \gamma_{z_0} \\ &= \mathcal{Q} + \mathcal{D}^* (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} (\mathcal{Q}_0(z) - \mathcal{Q}) \\ &= \mathcal{Q} + \mathcal{D}^* (I - (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} (\mathcal{Q} + \mathcal{T}_0)) \\ &= \mathcal{T}_0 - \mathcal{D}^* (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} \mathcal{D}. \end{aligned} \tag{2.8}$$

Now assume that $\mathcal{S}_1(z)$ is given as in the theorem. Then

$$\begin{aligned} \mathcal{S}_1(z) &= \mathcal{S}_0(z) + (\mathcal{S}_0(z) - \mathcal{Q})(\mathcal{T}_0 - \mathcal{S}_0(z))^{-1} (\mathcal{S}_0(z) + \mathcal{Q}) \\ &= \{ \{\mathbf{h}, \mathcal{S}_0(z)\mathbf{h} + (\mathcal{S}_0(z) - \mathcal{Q})\mathbf{k}\} : \{(\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h}, \mathbf{k}\} \in (\mathcal{T}_0 - \mathcal{S}_0(z))^{-1} \} \\ &= \{ \{\mathbf{h}, \mathcal{S}_0(z)\mathbf{h} + (\mathcal{S}_0(z) - \mathcal{Q})\mathbf{k}\} : (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} \}. \end{aligned}$$

Hence, by (2.8),

$$\begin{aligned} \mathcal{Q}_1(z) + \mathcal{S}_1(z) = & \{ \mathbf{h}, (\mathcal{S}_0(z) + \mathcal{T}_0)\mathbf{h} + (\mathcal{S}_0(z) - \mathcal{Q})\mathbf{k} \\ & - \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\mathcal{D}\mathbf{h} \} : (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} \}. \end{aligned}$$

Since

$$\begin{aligned} & (\mathcal{S}_0(z) + \mathcal{T}_0)\mathbf{h} - \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\mathcal{D}\mathbf{h} \\ & = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} + \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h} \end{aligned}$$

we have

$$\begin{aligned} \mathcal{Q}_1(z) + \mathcal{S}_1(z) = & \{ \mathbf{h}, (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} + (\mathcal{S}_0(z) - \mathcal{Q})\mathbf{k} \\ & + \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h} \} : (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} \} \\ = & \{ \mathbf{h}, \mathcal{D}^*\mathbf{k} + \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h} \} : \\ & (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} \}, \end{aligned}$$

and therefore

$$\begin{aligned} (\mathcal{Q}_1(z) + \mathcal{S}_1(z))^{-1} = & \{ \{ \mathcal{D}^*\mathbf{k} + \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h}, \mathbf{h} \} : \\ & (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} \}. \end{aligned}$$

This implies

$$\begin{aligned} \gamma_{1,z}(\mathcal{Q}_1(z) + \mathcal{S}_1(z))^{-1}\gamma_{1,z}^* = & \{ \{ u, \gamma_{1,z}\mathbf{h} \} : (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} \\ & \text{and } \gamma_{1,z}^*u = \mathcal{D}^*\mathbf{k} + \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h} \}. \end{aligned}$$

We show that the two defining equalities in the set on the right-hand side imply that

$$\gamma_{1,z}\mathbf{h} = \gamma_{0,z}\Delta_0(z)\gamma_{0,z}^*u. \quad (2.9)$$

Then (2.6) and hence the claim in the theorem are proved.

From

$$\gamma_{1,z}^*u = \mathcal{D}^*\mathbf{k} + \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h}$$

and

$$\gamma_{1,z}^*u = \mathcal{D}^*(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\gamma_{0,z}^*u$$

(see (2.7)) we obtain

$$\begin{aligned} (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\gamma_{0,z}^*u = & \mathbf{k} + (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}(\mathcal{Q}_0(z) - \mathcal{Q})\mathbf{h} \\ = & \mathbf{k} + \mathbf{h} - (\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\mathcal{D}\mathbf{h}. \end{aligned}$$

We apply $\mathcal{T}_0 - \mathcal{S}_0(z)$ to both sides of this equality and use the relation

$$(\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{k} = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h}$$

to get

$$\begin{aligned} & (\mathcal{T}_0 - \mathcal{S}_0(z))(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\gamma_{0,z}^*u \\ & = (\mathcal{S}_0(z) + \mathcal{Q})\mathbf{h} + (\mathcal{T}_0 - \mathcal{S}_0(z))\mathbf{h} - (\mathcal{T}_0 - \mathcal{S}_0(z))(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\mathcal{D}\mathbf{h} \\ & = \mathcal{D}\mathbf{h} - (\mathcal{T}_0 - \mathcal{S}_0(z))(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\mathcal{D}\mathbf{h} \\ & = (\mathcal{Q}_0(z) + \mathcal{S}_0(z))(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\mathcal{D}\mathbf{h}. \end{aligned}$$

This and (2.7) imply the asserted equality (2.9).

(ii) Now we drop the assumption that \mathcal{T}_0 is a matrix. Then it is a relation

$$\mathcal{T}_0 = \{P_0\mathbf{x}, P_0\mathcal{T}_{0,op}P_0\mathbf{x} + (I - P_0)\mathbf{x}\} : \mathbf{x} \in \mathbb{C}^d\},$$

where P_0 is an orthogonal projection in \mathbb{C}^d and $\mathcal{T}_{0,op}$ is the operator part of \mathcal{T}_0 . Let $(\mathcal{T}_n)_{n \in \mathbb{N}}$ be a sequence of matrices such that $\mathcal{T}_n \rightsquigarrow \mathcal{T}_0$ if $n \rightarrow \infty$. For example, relative to the decomposition $\mathbb{C}^d = \ker P_0 \oplus \text{ran } P_0$ we choose

$$\mathcal{T}_n = \begin{pmatrix} n\mathcal{I}_{\ker P_0} & 0 \\ 0 & \mathcal{T}_{0,op} \end{pmatrix}.$$

Let $A_{1,n}$ be the canonical self-adjoint extension of S which corresponds to the parameter \mathcal{T}_n in Krein’s formula based on A_0 :

$$(A_{1,n} - z)^{-1} = (A_0 - z)^{-1} - \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{T}_n)^{-1}\gamma_{0,z}^* \tag{2.10}$$

In what follows we fix $z \in \mathbb{C} \setminus \mathbb{R}$. Then there exist a $c > 0$ such that

$$\text{Im } \mathcal{Q}_0(z) / \text{Im } z = \gamma_{0,z}^* \gamma_{0,z} > c$$

and hence the matrices $(\mathcal{Q}_0(z) + \mathcal{T}_n)^{-1}$ are uniformly bounded:

$$\|(\mathcal{Q}_0(z) + \mathcal{T}_n)^{-1}\mathbf{x}\| \leq (c |\text{Im } z|)^{-1} \|\mathbf{x}\|, \quad \mathbf{x} \in \mathbb{C}^d.$$

From Lemma 2.2 it follows that for $n \rightarrow \infty$ they converge to the block matrix

$$(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & (P_0\mathcal{Q}_0(z)P_0 + \mathcal{T}_{0,op})^{-1} \end{pmatrix}$$

relative to the decomposition $\mathbb{C}^d = \ker P_0 \oplus \text{ran } P_0$. The equality (2.10) implies

$$(A_{1,n} - z)^{-1} \rightarrow (A_0 - z)^{-1} - \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\gamma_{0,z}^* =: (A_1 - z)^{-1} \tag{2.11}$$

(strongly in \mathfrak{H}), where, by Krein’s formula, A_1 is a canonical self-adjoint extension of S . Denote by $\gamma_{1,n;z}$ and $\mathcal{Q}_{1,n}(z)$ the γ -field and \mathcal{Q} -function associated with $A_{1,n}$ and S , normalized so that, in accordance with (2.3),

$$\gamma_{1,n;z_0} = \gamma_{z_0}, \quad \mathcal{Q}_{1,n}(z_0) = i(\text{Im } z_0) \gamma_{z_0}^* \gamma_{z_0} = \mathcal{Q}.$$

Then there exist parameters $\mathcal{S}_{1,n}(z)$ of the form

$$\mathcal{S}_{1,n}(z) = \{P_n\mathbf{x}, P_n\mathcal{S}_{1,n;op}(z)P_n\mathbf{x} + (I - P_n)\mathbf{x}\} : \mathbf{x} \in \mathbb{C}^d\},$$

where P_n is an orthogonal projection in \mathbb{C}^d and $\mathcal{S}_{1,n;op}(z)$ is the operator part of $\mathcal{S}_{1,n}(z)$, such that

$$P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{S}} = (A_{1,n} - z)^{-1} - \gamma_{1,n;z}(\mathcal{Q}_{1,n}(z) + \mathcal{S}_{1,n}(z))^{-1}\gamma_{1,n;z}^* \tag{2.12}$$

By part (i) they are given by

$$\mathcal{S}_{1,n}(z) = \mathcal{S}_0(z) + (\mathcal{S}_0(z) - \mathcal{Q})(\mathcal{T}_n - \mathcal{S}_0(z))^{-1}(\mathcal{S}_0(z) + \mathcal{Q}).$$

By Lemma 2.2 (i)-(iii) we have

$$\mathcal{S}_{1,n}(z) \rightsquigarrow \mathcal{S}_0(z) + (\mathcal{S}_0(z) - \mathcal{Q})(\mathcal{T}_0 - \mathcal{S}_0(z))^{-1}(\mathcal{S}_0(z) + \mathcal{Q}) =: \mathcal{S}_1(z).$$

It remains to show that $\mathcal{S}_1(z)$ is the parameter associated with \tilde{A} in Krein’s formula based on A_1 . This follows from the equality (2.12) by letting $n \rightarrow \infty$. Indeed from (2.11) and the equalities

$$\gamma_{1,n;z} = (I + (z - z_0)(A_{1,n} - z)^{-1})\gamma_{z_0}$$

and

$$\mathcal{Q}_{1,n}(z) = \mathcal{Q}^* + (z - z_0^*)\gamma_{z_0}^*\gamma_{1,n;z}$$

it follows that $\gamma_{1,n;z} \rightarrow \gamma_{1,z}$ and $\mathcal{Q}_{1,n}(z) \rightarrow \mathcal{Q}_1(z)$. The latter convergence implies (as in the beginning of the proof of part (ii)) that the matrices $(\mathcal{Q}_{1,n}(z) + \mathcal{S}_{1,n}(z))^{-1}$ are uniformly bounded. Hence, by Lemma 2.2,

$$(\mathcal{Q}_{1,n}(z) + \mathcal{S}_{1,n}(z))^{-1} = \begin{pmatrix} 0 & 0 \\ 0(P_n\mathcal{Q}_{1,n}(z)P_n + \mathcal{S}_{1,n;op}(z))^{-1} & 0 \end{pmatrix} \rightarrow (\mathcal{Q}_1(z) + \mathcal{S}_1(z))^{-1}.$$

It follows that

$$P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{S}} = (A_1 - z)^{-1} - \gamma_{1,z}(\mathcal{Q}_1(z) + \mathcal{S}_1(z))^{-1}\gamma_{1,z}^*. \quad \square$$

Remark 2.3. If in Theorem 2.1 \mathcal{T}_0 is a matrix, then (2.4) can be written as

$$\mathcal{S}_1(z) = -\mathcal{T}_0 + (\mathcal{T}_0 - \mathcal{Q})(\mathcal{T}_0 - \mathcal{S}_0(z))^{-1}(\mathcal{T}_0 + \mathcal{Q}).$$

In this formula, for $\mathcal{S}_0(z)$ we insert the elements of a sequence $(\mathcal{S}_{0,n}(z))$ of $d \times d$ matrix Nevanlinna functions which tend to the relation $\{0, \mathbb{C}^d\}$ if $n \rightarrow \infty$. Then the corresponding relations $\mathcal{S}_{1,n}(z)$ tend to $-\mathcal{T}_0$. According to (2.2), to $\mathcal{S}_{0,n}(z)$ there correspond generalized resolvents $P_{\mathfrak{H}}(\tilde{A}_n - z)^{-1}|_{\mathfrak{S}}$ of S , which converge for $n \rightarrow \infty$ strongly to $(A_0 - z)^{-1}$, and from the second equality in (2.2) we obtain

$$(A_0 - z)^{-1} = (A_1 - z)^{-1} - \gamma_{1,z}(\mathcal{Q}_1(z) - \mathcal{T}_0)^{-1}\gamma_{1,z}^*, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

This relation should be compared with (2.1):

$$(A_1 - z)^{-1} = (A_0 - z)^{-1} - \gamma_{0,z}(\mathcal{Q}_0(z) + \mathcal{T}_0)^{-1}\gamma_{0,z}^*, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Hence, for two canonical self-adjoint extensions A_0 and A_1 of S the parameters in Krein’s formula for A_0 , based on A_1 , and in Krein’s formula for A_1 , based on A_0 , differ just in their sign.

2.2. In this subsection we assume that the parameter \mathcal{T} in Krein’s formula (1.12) is a $d \times d$ matrix Nevanlinna function with minimal operator or relation representation as in (1.2): $B_{\mathcal{T}}$ and $R_{\mathcal{T}}(z)$, $z \in \rho(B_{\mathcal{T}})$, denote the representing relation for \mathcal{T} in $\mathfrak{H}_{\mathcal{T}}$ and its resolvent, respectively, and we define δ_z , $z \in \mathbb{C} \setminus \mathbb{R}$, as in (1.3). Since \mathcal{T} is a matrix function, for the self-adjoint extension $\tilde{A}_{\mathcal{T}}$, corresponding to \mathcal{T} , we have $\tilde{A}_{\mathcal{T}} \cap A_0 = S$.

The following theorem was proved in [9, (1.10)] by means of boundary triplets. For the convenience of the reader we give a proof, using the minimal model for the function \mathcal{T} .

Theorem 2.4. *The operator function \tilde{R}_T :*

$$\begin{aligned} \tilde{R}_T(z) &:= \begin{pmatrix} R_0(z) - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \gamma_{z^*}^* & -\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* \\ -\delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \gamma_{z^*}^* & R_T(z) - \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* \end{pmatrix} \\ &= \begin{pmatrix} R_0(z) & 0 \\ 0 & R_T(z) \end{pmatrix} - \begin{pmatrix} \gamma_z \\ \delta_z \end{pmatrix} (\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \begin{pmatrix} \gamma_{z^*}^* & \delta_{z^*}^* \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned} \tag{2.13}$$

whose values are bounded operators in $\mathfrak{H} \oplus \mathfrak{H}_T$, is the resolvent of a self-adjoint operator \tilde{A} in the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}_T$; \tilde{A} is a minimal self-adjoint extension of the symmetric operator S .

Clearly,

$$\tilde{A} = \left\{ \left\{ \tilde{R}_T(z) \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} + z \tilde{R}_T(z) \begin{pmatrix} f \\ g \end{pmatrix} \right\} : f \in \mathfrak{H}, g \in \mathfrak{H}_T \right\} \tag{2.14}$$

and the set on the right-hand side of (2.14) is independent of $z \in \mathbb{C} \setminus \mathbb{R}$. Indeed, if we replace

$$\begin{pmatrix} f \\ g \end{pmatrix} \text{ by } \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + (w - z) \tilde{R}_T(w) \begin{pmatrix} f \\ g \end{pmatrix}$$

then, by the resolvent identity,

$$\tilde{R}_T(z) \begin{pmatrix} f \\ g \end{pmatrix} = \tilde{R}_T(w) \begin{pmatrix} h \\ k \end{pmatrix} \text{ and } \begin{pmatrix} f \\ g \end{pmatrix} + z \tilde{R}_T(z) \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + w \tilde{R}_T(w) \begin{pmatrix} h \\ k \end{pmatrix}.$$

The entry in the left upper corner of the matrix in the middle term of (2.13) is the generalized resolvent of S generated by $\tilde{A} = \tilde{A}_T$.

Proof of Theorem 2.4. We first observe that

$$\tilde{R}_T(z)^* = \tilde{R}_T(z^*), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Using the equalities

$$\gamma_z = (I + (z - w)R_0(z))\gamma_w, \quad \delta_z = (I + (z - w)R_T(z))\delta_w$$

and

$$\frac{(\mathcal{Q}_0(z) + \mathcal{T}(z)) - (\mathcal{Q}_0(w) + \mathcal{T}(w))^*}{z - w^*} = \gamma_w^* \gamma_z + \delta_w^* \delta_z$$

we find that $\tilde{R}_T(z)$ also satisfies the resolvent identity

$$\tilde{R}_T(z) - \tilde{R}_T(w) = (z - w) \tilde{R}_T(z) \tilde{R}_T(w), \quad z, w \in \mathbb{C} \setminus \mathbb{R}.$$

Hence it is the resolvent of the self-adjoint relation (2.14).

To prove that \tilde{A} is an operator consider $\begin{pmatrix} f \\ g \end{pmatrix} \in \tilde{A}(0)$. Then, since $S \subset \tilde{A}$ and \tilde{A} is self-adjoint,

$$(f, \text{dom } S)_{\mathfrak{H}} = \left(\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \text{dom } S \\ \{0\} \end{pmatrix} \right)_{\mathfrak{H} \oplus \mathfrak{H}_T} = \{0\}.$$

It follows that $f = 0$, because S is densely defined. Thus, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$0 = \tilde{R}_T(z) \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} -\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* g \\ R_T(z)g - \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* g \end{pmatrix}.$$

The top component on the right-hand side being zero and the relation

$$\gamma_z^* \gamma_z = \text{Im } \mathcal{Q}_0(z) / \text{Im } z > 0 \tag{2.15}$$

imply that $(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* g = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. The relation (2.15) also implies that the matrix $\mathcal{Q}_0(z) + \mathcal{T}(z)$ is invertible, hence $\delta_{z^*}^* g = 0$. Thus

$$(g, \delta_{z^*} \mathbf{x})_{\mathfrak{H}_{\mathcal{T}}} = 0, \quad \mathbf{x} \in \mathbb{C}^d, z \in \mathbb{C} \setminus \mathbb{R}.$$

From the minimality of the operator $B_{\mathcal{T}}$ in the representation model (1.2) for \mathcal{T} it follows that also $g = 0$. Hence $\tilde{A}(0) = \{0\}$, that is, \tilde{A} is an operator.

It remains to show that the extension \tilde{A} is minimal:

$$\overline{\text{span}} \left\{ (I + (z - w)\tilde{R}_{\mathcal{T}}(z)) \begin{pmatrix} f \\ 0 \end{pmatrix} : f \in \mathfrak{H}, z \in \mathbb{C} \setminus \mathbb{R} \right\} = \mathfrak{H} \oplus \mathfrak{H}_{\mathcal{T}}.$$

To this end, since \mathfrak{H} is contained in the set on the left-hand side (choose $z = w$), it suffices to prove the implication

$$g \in \mathfrak{H}_{\mathcal{T}}, \left(\tilde{R}_{\mathcal{T}}(z) \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right)_{\mathfrak{H} \oplus \mathfrak{H}_{\mathcal{T}}} = 0 \text{ for all } f \in \mathfrak{H}, z \in \mathbb{C} \setminus \mathbb{R} \implies g = 0.$$

Rewriting the first equality we get

$$\left(f, \gamma_{z^*} (\mathcal{Q}_0(z^*) + \mathcal{T}(z^*))^{-1} \delta_{z^*}^* g \right)_{\mathfrak{H}} = 0.$$

Thus \tilde{A} is minimal if

$$\gamma_{z^*} (\mathcal{Q}_0(z^*) + \mathcal{T}(z^*))^{-1} \delta_{z^*}^* g = 0 \text{ for all } z \in \mathbb{C} \setminus \mathbb{R} \implies g = 0.$$

This implication follows from the same arguments used above to show that \tilde{A} is an operator. □

2.3. Let \mathcal{T} be a $d \times d$ matrix Nevanlinna function with integral representation (1.1) and operator representation (1.2). In the next lemma the multi-valued part $B_{\mathcal{T}}(0)$ of the self-adjoint relation $B_{\mathcal{T}}$ in (1.2) is related to the matrix \mathcal{B} in (1.1) (comp. [1, Theorem 3]). We denote by $P_{B_{\mathcal{T}}(0)}$ the orthogonal projection in $\mathfrak{H}_{\mathcal{T}}$ onto $B_{\mathcal{T}}(0)$.

Lemma 2.5. *Let \mathcal{T} be a $d \times d$ matrix Nevanlinna function \mathcal{T} with integral and operator representations (1.1) and (1.2). Then:*

- (i) $\mathcal{B} = \lim_{y \uparrow \infty} \mathcal{T}(iy) / iy = \delta^* P_{B_{\mathcal{T}}(0)} \delta \geq 0$.
- (ii) $\delta^* : B_{\mathcal{T}}(0) \rightarrow \text{ran } \mathcal{B}$ is a bijection.

In particular,

- (iii) $\dim B_{\mathcal{T}}(0) = \text{rank } \mathcal{B}$, and
- (iv) $\mathcal{B} > 0 \iff P_{B_{\mathcal{T}}(0)} \delta : \mathbb{C}^d \rightarrow B_{\mathcal{T}}(0)$ is a bijection.

Proof. (i) We only need to prove the second equality. If $B_{\mathcal{T},op}$ is the operator part of $B_{\mathcal{T}}$ then this follows from (1.2) and the equality

$$\lim_{y \uparrow \infty} iy(B_{\mathcal{T}} - iy)^{-1} = \lim_{y \uparrow \infty} iy(B_{\mathcal{T},op} - iy)^{-1} (I_{\mathfrak{H}_{\mathcal{T}}} - P_{B_{\mathcal{T}}(0)}) = P_{B_{\mathcal{T}}(0)} - I_{\mathfrak{H}_{\mathcal{T}}}.$$

(ii) That $\delta^* B_{\mathcal{T}}(0) = \text{ran } \mathcal{B}$ follows from (i), the if and only if statements

$$\begin{aligned} \mathbf{x} \in (\delta^* B_{\mathcal{T}}(0))^{\perp} &\iff (\delta \mathbf{x}, B_{\mathcal{T}}(0))_{\mathfrak{H}_{\mathcal{T}}} = \{0\} \\ &\iff P_{B_{\mathcal{T}}(0)} \delta \mathbf{x} = 0 \\ &\iff \|P_{B_{\mathcal{T}}(0)} \delta \mathbf{x}\|_{\mathfrak{H}_{\mathcal{T}}} = 0 \\ &\iff \delta^* P_{B_{\mathcal{T}}(0)} \delta \mathbf{x} = 0 \\ &\iff \mathcal{B} \mathbf{x} = 0 \end{aligned}$$

and the equality $\ker \mathcal{B} = (\text{ran } \mathcal{B})^{\perp}$. To show that $\delta^*|_{B_{\mathcal{T}}(0)}$ is injective we assume that $\delta^* f = 0$ for some $f \in B_{\mathcal{T}}(0)$. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$: $R_{\mathcal{T}}(z^*) f = 0$ and hence

$$\delta_z^* f = \delta^*(I + (z^* - z_0^*)R_{\mathcal{T}}(z^*))f = 0.$$

The minimality of the operator representation of \mathcal{T} (see (1.4)) implies that $f = 0$. Hence $\delta^*|_{B_{\mathcal{T}}(0)}$ is a bijection onto $\text{ran } \mathcal{B}$.

The claim (iii) follows from (ii), and (iv) follows from (ii) and (iii). \square

3. Compressions of self-adjoint extensions: $S \subset C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0$

3.1. The operators of interest in this paper are the compressions

$$C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) := P_{\mathfrak{H}} \tilde{A}_{\mathcal{T}}|_{\mathfrak{H}}$$

of the self-adjoint extensions $\tilde{A}_{\mathcal{T}}$ in (2.14) to the space \mathfrak{H} . They are symmetric extensions of S in \mathfrak{H} , and they are closed, because S is closed and

$$\dim (C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})/S) \leq d < \infty.$$

In this subsection we formulate conditions on the parameter \mathcal{T} under which

$$S \subset C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0, \tag{3.1}$$

where A_0 is the basic canonical self-adjoint extension of S in Krein’s formula (1.12). In Subsect. 3.2 we are interested in the extreme cases $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) = S$ and $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) = A_0$.

Writing formula (2.14) in full detail we get

$$\tilde{A}_{\mathcal{T}} = \left\{ \left\{ \begin{aligned} &\left(\begin{array}{l} R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ R_{\mathcal{T}}(z)g - \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \end{array} \right), \\ &\left(\begin{array}{l} f \\ g \end{array} \right) + z \left(\begin{array}{l} R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ R_{\mathcal{T}}(z)g - \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \end{array} \right) \right\} : f \in \mathfrak{H}, g \in \mathfrak{H}_{\mathcal{T}} \right\}.$$

Hence the restriction $\tilde{A}_{\mathcal{T}}|_{\mathfrak{H}}$ becomes

$$\tilde{A}_{\mathcal{T}}|_{\mathfrak{H}} = \left\{ \left\{ \begin{pmatrix} R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ 0 \end{pmatrix}, \right. \right. \\ \left. \left. \begin{pmatrix} f \\ g \end{pmatrix} + z \begin{pmatrix} R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ 0 \end{pmatrix} \right\} : \right. \\ \left. f \in \mathfrak{H}, g \in \mathfrak{H}_{\mathcal{T}}, R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \right\}, \quad (3.2)$$

and we obtain for the compression

$$C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) = \left\{ \begin{aligned} & \{ R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g), \\ & f + z(R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g)) \} : \\ & f \in \mathfrak{H}, g \in \mathfrak{H}_{\mathcal{T}}, R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \end{aligned} \right\}. \quad (3.3)$$

Proposition 3.1. *The inclusion $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0$ holds if and only if*

$$\begin{aligned} f \in \mathfrak{H}, g \in \mathfrak{H}_{\mathcal{T}}, R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \\ \implies \gamma_{z^*}^* f + \delta_{z^*}^* g = 0, g \in B_{\mathcal{T}}(0). \end{aligned} \quad (3.4)$$

Proof. The if part of the statement follows from (3.3) and from the equality

$$A_0 = \{ \{ R_0(z)f, f + zR_0(z)f \} : f \in \mathfrak{H} \}. \quad (3.5)$$

As to the only if part, assume $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0$ and consider $f \in \mathfrak{H}$ and $g \in \mathfrak{H}_{\mathcal{T}}$ satisfying

$$R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g).$$

Then there exists an $h \in \mathfrak{H}$ such that

$$\begin{aligned} R_0(z)h &= R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g), \\ h + zR_0(z)h &= f + z(R_0(z)f - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g)). \end{aligned}$$

It follows that $h = f$ and $\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) = 0$. By (2.15) we have $\gamma_{z^*}^* f + \delta_{z^*}^* g = 0$ and then also $R_{\mathcal{T}}(z)g = 0$, that is $g \in B_{\mathcal{T}}(0)$. \square

The following theorem gives a sufficient condition on the matrix function \mathcal{T} for the implication (3.4) to hold.

Theorem 3.2. *If the $d \times d$ matrix Nevanlinna function \mathcal{T} satisfies the condition*

$$\lim_{y \uparrow \infty} y \operatorname{Im} \langle \mathcal{T}(iy)\mathbf{x}, \mathbf{x} \rangle = \infty \text{ for all } \mathbf{x} \in \mathbb{C}^d \setminus \{0\}, \quad (3.6)$$

then

$$S \subset C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0.$$

Remark 3.3. In the proof below we shall show that under the assumption of Theorem 3.2 we have, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\tilde{A}_{\mathfrak{H}}|_{\mathfrak{H}} = \left\{ \left\{ \begin{pmatrix} R_0(z)f \\ 0 \end{pmatrix}, \begin{pmatrix} f + zR_0(z)f \\ g \end{pmatrix} \right\} : f \in \mathfrak{H}, g \in B_{\mathcal{T}}(0), \gamma_{z^*}^* f + \delta_{z^*}^* g = 0 \right\}, \tag{3.7}$$

and hence

$$C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) = \{ \{R_0(z)f, f + zR_0(z)f\} : f \in \mathfrak{H}, \exists g \in B_{\mathcal{T}}(0) : \gamma_{z^*}^* f + \delta_{z^*}^* g = 0 \}. \tag{3.8}$$

The relation (3.6) means that in the integral representation (1.1) for \mathcal{T} we have

$$\left\langle \int_{\mathbb{R}} d\Sigma(t) \mathbf{x}, \mathbf{x} \right\rangle = \infty \text{ for all } \mathbf{x} \in (\ker \mathcal{B}) \setminus \{0\}.$$

If \mathcal{T} is rational or, equivalently, $\dim \mathfrak{H}_{\mathcal{T}} < \infty$, this simply means that $\mathcal{B} > 0$ (see (1.8)).

Proof of Theorem 3.2. Fix $z \in \mathbb{C} \setminus \mathbb{R}$. According to [18, Theorem 3.2] or [22, Theorem 2.4 (2)], the relation (3.6) is valid if and only if

$$\text{ran } \delta \cap \text{dom } B_{\mathcal{T}} = \{0\}.$$

By (1.3) and the relation $\text{ran } R_{\mathcal{T}}(z) = \text{dom } B_{\mathcal{T}}$, this equality holds if and only if $\text{ran } \delta_z \cap \text{dom } B_{\mathcal{T}} = \{0\}$, $z \in \mathbb{C} \setminus \mathbb{R}$. Thus (3.6) implies that the defining relation in (3.2) and (3.3),

$$R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g),$$

breaks down into the two equalities $R_{\mathcal{T}}(z)g = 0$ and

$$\delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) = 0. \tag{3.9}$$

The first equality is equivalent to $g \in B_{\mathcal{T}}(0)$, the second equality holds if and only if

$$\gamma_{z^*}^* f + \delta_{z^*}^* g = 0. \tag{3.10}$$

Here the ‘if’ part is evident, we prove the ‘only if’ part. Multiply both sides of (3.9) by δ_z^* and use $\delta_z^* \delta_z = \text{Im } \mathcal{T}(z) / \text{Im } z$. Since $\ker \text{Im } \mathcal{T}(w)$ is independent of $w \in \mathbb{C} \setminus \mathbb{R}$ (see [13, Lemma 5.3]), (3.6) implies

$$\text{Im } \mathcal{T}(w) / \text{Im } w > 0, \quad w \in \mathbb{C} \setminus \mathbb{R}. \tag{3.11}$$

Thus we obtain

$$(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) = 0.$$

By (2.15), $\mathcal{Q}_0(z) + \mathcal{T}(z)$ is an invertible matrix, whence (3.10) holds.

Now (3.7) follows from (3.2) and the inclusion $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \subset A_0$ follows from Proposition 3.1. As observed before, the inclusion $S \subset C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$ follows from the definition of the compression. \square

3.2. To prove statements about the equality signs in (3.1), in the following proposition we collect some facts about symmetric extensions of S (comp. also [16, Section 3]).

Proposition 3.4. (i) *The relation*

$$\tilde{S} \equiv \tilde{S}_{\mathbb{L}} = \{ \{R_0(z)f, f + zR_0(z)f\} : f \in \mathfrak{H}, \gamma_{z^*}^* f \in \mathbb{L} \} \quad (3.12)$$

defines a bijective correspondence between all subspaces \mathbb{L} of \mathbb{C}^d and all symmetric extensions \tilde{S} of S in \mathfrak{H} such that

$$S \subset \tilde{S} \subset A_0.$$

(ii) $\tilde{A}_{\mathcal{T}} \cap A_0 = \tilde{S}_{\mathbb{L}}$ *if and only if \mathbb{L} equals the multi-valued part of the parameter \mathcal{T} in Krein's formula based on A_0 .*

(iii) *If \mathcal{B} denotes the matrix in the representation (1.1) for \mathcal{T} , then*

$$C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \cap A_0 \subset \tilde{S}_{\text{ran } \mathcal{B}};$$

under the assumption (3.6) the inclusion is an equality.

The set on the right-hand side of (3.12) is independent of z :

$$\tilde{S}_{\mathbb{L}} = \{ \{u, S^*u\} : u \in \text{dom } A_0, \gamma^*(A_0 - z_0^*)u \in \mathbb{L} \},$$

and, by (3.5), $A_0 = \tilde{S}_{\mathbb{C}^d}$, and with (1.10) we obtain $S = \tilde{S}_{\{0\}}$.

Proof of Proposition 3.4. (i) Fix $z \in \mathbb{C} \setminus \mathbb{R}$. From the definition of $\tilde{S}_{\mathbb{L}}$ it follows that for every subspace \mathbb{L} of \mathbb{C}^d we have $S \subset \tilde{S}_{\mathbb{L}} \subset A_0$, and that $\tilde{S}_{\mathbb{L}}$ is a closed densely defined symmetric operator.

Conversely, let \tilde{S} be a symmetric operator with $S \subset \tilde{S} \subset A_0$. Then $\tilde{S} = \tilde{S}_{\mathbb{L}}$ with

$$\mathbb{L}^\perp = \gamma_{z^*}^{-1} \ker(\tilde{S}^* - z^*).$$

This follows from the inclusion $\tilde{S} \subset \tilde{S}_{\mathbb{C}^d}$ and the following equivalent statements for $f \in \mathfrak{H}$:

$$\begin{aligned} \gamma_{z^*}^* f \in \mathbb{L} &\iff (f, \gamma_{z^*} \mathbb{L}^\perp)_{\mathfrak{H}} = \{0\} \\ &\iff (f, \ker(\tilde{S}^* - z^*))_{\mathfrak{H}} = \{0\} \\ &\iff f \in \text{ran}(\tilde{S} - z) \\ &\iff \{R_0(z)f, f + zR_0(z)f\} \in \tilde{S}. \end{aligned}$$

Thus the set of all \tilde{S} with $S \subset \tilde{S} \subset A_0$ coincides with the set of all $\tilde{S}_{\mathbb{L}}$ where \mathbb{L} runs through the set of subspaces of \mathbb{C}^d .

As to the bijective correspondence: $\tilde{S}_{\mathbb{L}} = \tilde{S}_{\mathbb{M}}$ if and only if for all $f \in \mathfrak{H}$

$$\gamma_{z^*}^* f \in \mathbb{L} \iff \gamma_{z^*}^* f \in \mathbb{M}.$$

Since $\gamma_{z^*}^* : \mathfrak{H} \rightarrow \mathbb{C}^d$ is surjective, we have $\tilde{S}_{\mathbb{L}} = \tilde{S}_{\mathbb{M}}$ if and only if $\mathbb{L} = \mathbb{M}$.

(ii) With

$$\mathcal{T}(z) = \{ \{P\mathbf{y}, P\mathcal{T}_{op}(z)P\mathbf{y} + (I - P)\mathbf{y}\} : \mathbf{y} \in \mathbb{C}^d \}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where P is a projection in \mathbb{C}^d and \mathcal{T}_{op} is the operator part of \mathcal{T} acting in $\text{ran } P$, Krein's formula for $\tilde{A}_{\mathcal{T}}$ acting in the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}_{\mathcal{T}}$ becomes

$$P_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}} - z)^{-1} \Big|_{\mathfrak{H}} = R_0(z) - \gamma_z P (P\mathcal{Q}_0(z)P + \mathcal{T}_{op}(z))^{-1} P\gamma_{z^*}^*, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.13)$$

By (i), to prove the ‘if and only if’ statement it suffices to show that $\widetilde{A}_T \cap A_0 = \widetilde{S}_{\ker P}$. Consider $f \in \mathfrak{H}$ such that

$$\{R_0(z)f, f + zR_0(z)f\} \in \widetilde{S}_{\ker P}.$$

Then $P\gamma_{z^*}^* f = 0$ and $P_{\mathfrak{H}}(\widetilde{A}_T - z)^{-1} f = R_0(z)f$. Hence for some $g_z \in \mathfrak{H}_T$

$$\{R_0(z)f - g_z, f + zR_0(z)f - zg_z\} \in \widetilde{A}_T.$$

Since \widetilde{A}_T is self-adjoint, we find $(z - z^*)(g_z, g_z)_{\mathfrak{H}_T} = 0$, that is $g_z = 0$. Thus

$$\{R_0(z)f, f + zR_0(z)f\} \in \widetilde{A}_T \cap A_0.$$

This proves $\widetilde{S}_{\ker P} \subset \widetilde{A}_T \cap A_0$.

To prove the reverse inclusion assume $\{R_0(z)f, f + zR_0(z)f\} \in \widetilde{A}_T \cap A_0$. Then, by Krein’s formula,

$$\gamma_z P(PQ_0(z)P + T_{op}(z))^{-1} P\gamma_{z^*}^* f = 0$$

and this implies that $P\gamma_{z^*}^* f = 0$, whence $\{R_0(z)f, f + zR_0(z)f\} \in \widetilde{S}_{\ker P}$. Thus $\widetilde{A}_T \cap A_0 \subset \widetilde{S}_{\ker P}$ and equality prevails.

(iii) Assume $\{u, v\} \in C_{\mathfrak{H}}(\widetilde{A}_T) \cap A_0$. Then for some $f, h \in \mathfrak{H}$ and $g \in \mathfrak{H}_T$ with

$$R_T(z)g = \delta_z(Q_0(z) + T(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g)$$

we have that

$$u = R_0(z)f - \gamma_z(Q_0(z) + T(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) \stackrel{1}{=} R_0(z)h$$

and $v = f + zu \stackrel{2}{=} h + zu$. The equalities $\stackrel{2}{=}$ and $\stackrel{1}{=}$ are valid if and only if $f = h$ and

$$\gamma_z(Q_0(z) + T(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) = 0,$$

that is $\gamma_{z^*}^* f + \delta_{z^*}^* g = 0$, and $R_T(z)g = 0$, that is $g \in B_T(0)$. It follows that $u = R_0(z)f, v = f + zR_0(z)f$ for some $f \in \mathfrak{H}$ for which

$$\gamma_{z^*}^* f \in \delta_{z^*}^* B_T(0) = \delta^* B_T(0) = \text{ran } \mathcal{B},$$

by Lemma 2.5 (ii). Thus $\{u, v\} \in \widetilde{S}_{\text{ran } \mathcal{B}}$ and hence $C_{\mathfrak{H}}(\widetilde{A}_T) \cap A_0 \subset \widetilde{S}_{\text{ran } \mathcal{B}}$. If (3.6) holds then, by (3.8), equality prevails. \square

Theorem 3.5. *Let T be a $d \times d$ matrix Nevanlinna function and suppose that it satisfies at ∞ the asymptotic relation (3.6):*

$$\lim_{y \uparrow \infty} y \text{Im} \langle T(iy)\mathbf{x}, \mathbf{x} \rangle = \infty \text{ for all } \mathbf{x} \in \mathbb{C}^d \setminus \{0\}. \tag{3.14}$$

Then

$$\lim_{y \uparrow \infty} T(iy)/y = 0 \tag{3.15}$$

if and only if

$$C_{\mathfrak{H}}(\widetilde{A}_T) = S = \widetilde{A}_T|_{\mathfrak{H}}. \tag{3.16}$$

The assumption (3.15) means that in the integral representation (1.1) for \mathcal{T} we have $\mathcal{B} = 0$, and then the formula (3.14) means that

$$\left\langle \int_{\mathbb{R}} d\Sigma(t)\mathbf{x}, \mathbf{x} \right\rangle = \infty \text{ for all } \mathbf{x} \in \mathbb{C}^d \setminus \{0\}.$$

Clearly, this can only hold if $\dim \mathfrak{H}_{\mathcal{T}} = \infty$.

Proof of Theorem 3.5. As observed in the proof of Theorem 3.2, (3.6) implies (3.11):

$$\operatorname{Im} \mathcal{T}(z) / \operatorname{Im} z > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

According to [22, Corollary 2.5], if this inequality holds, then (3.15) is valid if and only if $B_{\mathcal{T}}$ is an operator or, equivalently, $B_{\mathcal{T}}(0) = \{0\}$. The implication (3.15) \Rightarrow (3.16) follows from (3.7), (3.8) and the equality (in the notation of Proposition 3.4) $S = \tilde{S}_{\{0\}}$. Now assume (3.16). Then, by Proposition 3.4,

$$\tilde{S}_{\{0\}} = S = C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) \cap A_0 = \tilde{S}_{\operatorname{ran} \mathcal{B}}$$

which implies $\operatorname{ran} \mathcal{B} = \{0\}$, that is $\mathcal{B} = 0$, whence (3.15). □

Theorem 3.6. *If \mathcal{T} is a $d \times d$ matrix Nevanlinna function that satisfies the condition*

$$\lim_{y \uparrow \infty} \mathcal{T}(iy) / iy > 0 \tag{3.17}$$

at ∞ , then

$$C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) = A_0. \tag{3.18}$$

If the condition (3.6) holds, then, also conversely, (3.18) implies (3.17).

Proof. To prove (3.18) we show that $A_0 \subset C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$, the converse inclusion follows from the self-adjointness of A_0 and the symmetry of $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$.

For $f \in \mathfrak{H}$, consider the element

$$\{R_0(z)f, f + R_0(z)f\} \in A_0. \tag{3.19}$$

By (3.17) and Lemma 2.5 the mapping δ^* and therefore also the mapping $\delta_{z^*}^* : B_{\mathcal{T}}(0) \rightarrow \mathbb{C}^d$ is a bijection, and hence there exists a $g \in B_{\mathcal{T}}(0)$ such that

$$\gamma_{z^*}^* f + \delta_{z^*}^* g = 0.$$

It follows that

$$R_{\mathcal{T}}(z)g = \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1}(\gamma_{z^*}^* f + \delta_{z^*}^* g) = 0$$

and therefore, according to (3.3), the element in (3.19) belongs to $C_{\mathfrak{H}}(\tilde{A})$. Thus $A_0 \subset C_{\mathfrak{H}}(\tilde{A})$.

Now assume that (3.6) holds. Then, by (3.5) and (3.8), the equality $C_{\mathfrak{H}}(\tilde{A}) = A_0$ is equivalent to the implication:

$$f \in \mathfrak{H} \implies \gamma_{z^*}^* f + \delta_{z^*}^* g = 0 \text{ for some } g \in B_{\mathcal{T}}(0).$$

Since $\gamma_{z^*}^* : \mathfrak{H} \rightarrow \mathbb{C}^d$ is surjective, the implication yields that

$$\delta^* B_{\mathcal{T}}(0) = \delta_{z^*}^* B_{\mathcal{T}}(0) = \mathbb{C}^d$$

and this readily implies that the map $P_{B_{\mathcal{T}}(0)}\delta : \mathbb{C}^d \rightarrow B_{\mathcal{T}}(0)$ is injective. The relation (3.17) now follows from Lemma 2.5. \square

3.3. The following theorem implies that every symmetric operator between S and A_0 is the compression $C_{\mathfrak{H}}(\tilde{A})$ of some self-adjoint extension \tilde{A} of S .

Theorem 3.7. *For every symmetric operator \tilde{S} with $S \subset \tilde{S} \subset A_0$ there exists a self-adjoint extension \tilde{A} of S such that $\tilde{A} \cap A_0 = S$ and $C_{\mathfrak{H}}(\tilde{A}) = \tilde{S}$.*

Proof. For a given extension \tilde{S} we choose the subspace \mathbb{L} such that $\tilde{S} = \tilde{S}_{\mathbb{L}}$ as in (3.12). Consider a $d \times d$ matrix Nevanlinna function \mathcal{T} with operator representation (1.2). The defining relations $g \in B_{\mathcal{T}}(0)$ and $\gamma_{z^*}^* f + \delta_{z^*}^* g = 0$ for $\tilde{A} = \tilde{A}_{\mathcal{T}}$ in (3.8) mean

$$\gamma_{z^*}^* f \in \delta_{z^*}^* B_{\mathcal{T}}(0) = \delta^* B_{\mathcal{T}}(0). \tag{3.20}$$

Hence if we choose \mathcal{T} such that it satisfies (3.6) and

$$\mathbb{L} = \delta^* B_{\mathcal{T}}(0), \tag{3.21}$$

then, by Remark 3.3, (3.20) means $C_{\mathfrak{H}}(\tilde{A}) = \tilde{S}_{\mathbb{L}} = \tilde{S}$. The example below shows that such a choice of \mathcal{T} is possible. \square

Example 3.8. Let \mathbb{L} be any subspace of \mathbb{C}^d . We construct a model as in (1.2) of a $d \times d$ matrix Nevanlinna function \mathcal{T} satisfying (3.6) and (3.21). We choose

- (1) $\mathfrak{H}_{\mathcal{T}} = \ell_2$ and denote by $(e_j)_{j=1}^{\infty}$ an orthonormal basis in this space; elements of ℓ_2 have the form

$$\sum_{j=1}^{\infty} \alpha_j e_j \text{ with } \alpha_j \in \mathbb{C} \text{ and } \sum_{j=1}^{\infty} |\alpha_j|^2 < \infty.$$

- (2) $B_{\mathcal{T}} = B_{\mathcal{T},op} \oplus B_{\mathcal{T},\infty}$ where for some fixed $m \in \mathbb{N}_0$

$$B_{\mathcal{T},op} = \left\{ \left\{ \sum_{j=m+1}^{\infty} \alpha_j e_j, \sum_{j=m+1}^{\infty} j \alpha_j e_j \right\} : \sum_{j=1}^{\infty} |j \alpha_j|^2 < \infty \right\},$$

$$B_{\mathcal{T},\infty} = \left\{ \left\{ 0, \sum_{j=1}^m \beta_j e_j \right\} : \beta_j \in \mathbb{C} \right\}$$

(if $m = 0$, then $B_{\mathcal{T},\infty} = \{0, 0\}$ and $B_{\mathcal{T}} = B_{\mathcal{T},op}$ is an operator).

- (3) $\delta : \mathbb{C}^d \rightarrow \ell_2$ such that for some fixed basis $\vec{x} := (\mathbf{x}_1 \cdots \mathbf{x}_d)$ of \mathbb{C}^d

$$\delta \mathbf{x}_k = \sum_{j=1}^{\infty} j^{-p_k} e_j, \quad k = 1, \dots, d,$$

where $p_k \in (1/2, 3/2]$, $p_k \neq p_\ell$, $k, \ell = 1, 2, \dots, d$.

Then $B_{\mathcal{T}}$ is a self-adjoint relation in ℓ_2 , $B_{\mathcal{T}}(0) = \text{span}\{e_j : j = 1, \dots, m\}$,

$$(B_{\mathcal{T}} - z)^{-1} \sum_{j=1}^{\infty} \alpha_j e_j = \sum_{j=m+1}^{\infty} \frac{\alpha_j}{j - z} e_j$$

and with $\delta_z = (I + (z - z_0)(B_{\mathcal{T}} - z)^{-1})\delta$

$$\delta_z \mathbf{x}_k = \sum_{j=1}^m j^{-pk} e_j + \sum_{j=m+1}^{\infty} \frac{j - z_0}{j - z} j^{-pk} e_j, \quad k = 1, \dots, d.$$

Note that

$$\sum_{j=1}^{\infty} |j^{-p}|^2 < \infty \text{ and } \sum_{j=1}^{\infty} |j^{1-p}|^2 = \infty$$

if and only if $p \in (1/2, 3/2]$, hence $\text{ran } \delta \cap \text{dom } B_{\mathcal{T}} = \text{ran } \delta_z \cap \text{dom } B_{\mathcal{T}} = \{0\}$. We define the matrix Nevanlinna function \mathcal{T} as in (1.2) with $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and $\mathcal{T}(z_0) = z_0 I$.

Then, by the references at the beginning of the proof of Theorem 3.2, \mathcal{T} satisfies (3.6). Moreover, the self-adjoint operator $\tilde{A}_{\mathcal{T}}$ corresponding to \mathcal{T} in Krein’s formula (3.13) satisfies $\tilde{A}_{\mathcal{T}} \cap A_0 = S$ and (3.7).

We now show that \mathcal{T} , that is m and the basis $\vec{\mathbf{x}}$ of \mathbb{C}^d , can be chosen such that (3.21) is satisfied: $\mathbb{L} = \delta^* B_{\mathcal{T}}(0)$. We have

$$\begin{aligned} \delta^* B_{\mathcal{T}}(0) &= \text{span} \{ \delta_{z^*}^* e_j : j = 1, \dots, m \} \\ &= \text{span} \left\{ \vec{\mathbf{x}} \langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle^{-1} \begin{pmatrix} j^{-p_1} \\ \vdots \\ j^{-p_d} \end{pmatrix} : j = 1, \dots, m \right\}. \end{aligned} \tag{3.22}$$

Denote by $\vec{\mathbf{y}} = (\mathbf{y}_1 \cdots \mathbf{y}_r)$ a basis for \mathbb{L} , $r := \dim \mathbb{L}$. We choose $m = r$ and claim that there is a basis $\vec{\mathbf{x}}$ for \mathbb{C}^d such that

$$\vec{\mathbf{y}} = \vec{\mathbf{x}} \langle \vec{\mathbf{x}}, \vec{\mathbf{x}} \rangle^{-1} V_r, \quad V_r := \begin{pmatrix} 1 & 2^{-p_1} & \dots & r^{-p_1} \\ \vdots & \vdots & & \vdots \\ 1 & 2^{p_d} & \dots & r^{-p_d} \end{pmatrix}. \tag{3.23}$$

The claim and (3.22) imply $\mathbb{L} = \delta_{z^*}^* B_{\mathcal{T}}(0)$ and hence (3.21).

To prove the claim, note that the $d \times d$ Vandermonde matrix

$$V_d = \begin{pmatrix} 1 & 2^{-p_1} & \dots & d^{-p_1} \\ \vdots & \vdots & & \vdots \\ 1 & 2^{-p_d} & \dots & d^{-p_d} \end{pmatrix}$$

is invertible. Extend the basis $\vec{\mathbf{y}}$ of \mathbb{L} by $\vec{\mathbf{y}}_0$ to a basis $(\vec{\mathbf{y}} \ \vec{\mathbf{y}}_0)$ of \mathbb{C}^d . Then $(\vec{\mathbf{y}} \ \vec{\mathbf{y}}_0) V_d^{-1}$ is also a basis of \mathbb{C}^d . Let $\vec{\mathbf{x}}$ be the basis of \mathbb{C}^d dual to it:

$$\langle (\vec{\mathbf{y}} \ \vec{\mathbf{y}}_0) V_d^{-1}, \vec{\mathbf{x}} \rangle = I_d.$$

Then

$$\langle (\vec{\mathbf{y}} \ \vec{\mathbf{y}}_0), \vec{\mathbf{x}} \rangle = V_d.$$

If we multiply both sides of this equality from the right by the $d \times r$ matrix $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$ we obtain $\langle \vec{\mathbf{y}}, \vec{\mathbf{x}} \rangle = V_r$ and then (3.23) follows and the claim is proved.

Remark 3.9. It follows immediately from Theorem 3.7 (by varying A_0) that every densely defined, closed symmetric extension \tilde{S} of S in \mathfrak{H} is the compression of a self-adjoint extension of S . A proof of this fact can also be given using the theory of dilations (see Theorem 5.2 in the Appendix). If \tilde{S} is not self-adjoint, according to Stenger’s lemma ([24]) the extending space has to be infinite dimensional.

4. Finite-dimensional extensions

4.1. Let S be again a symmetric operator with finite and equal defect numbers $d > 0$ and consider a self-adjoint extension $A_{\mathcal{T}}$. In this section we suppose that the dimension of the extending space $\mathfrak{H}_{\mathcal{T}}$ is finite, say equal to $m \in \mathbb{N}$. This is equivalent to the fact that the corresponding parameter function \mathcal{T} is rational with poles of total multiplicity m :

$$\mathcal{T}(z) = \sum_{j=1}^{\ell} \frac{\mathcal{A}_j}{\alpha_j - z} + \mathcal{A} + z\mathcal{B} \tag{4.1}$$

with entries as in (1.7) and

$$\sum_{j=1}^{\ell} \text{rank } \mathcal{A}_j + \text{rank } \mathcal{B} = m.$$

In this situation the assumption (3.6) is equivalent to $\mathcal{B} > 0$.

Recall that for a self-adjoint extension \tilde{A} with finite-dimensional exit space Stenger’s lemma assures that the compression $C_{\mathfrak{H}}(\tilde{A})$ is a canonical self-adjoint extension of S .

Theorem 4.1. *If in Krein’s resolvent formula (3.13) the parameter \mathcal{T} is a rational $d \times d$ matrix Nevanlinna function, then the following statements are equivalent:*

- (a) $\lim_{y \uparrow \infty} \mathcal{T}(iy)/iy > 0$, that is $\mathcal{B} > 0$ in (4.1).
- (b) $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}}) = A_0$.

Proof. Theorem 3.6 implies (a) \implies (b). We prove (b) \implies (a). Fix $z \in \mathbb{C} \setminus \mathbb{R}$. Assume that (b) holds and consider the operators

$$M_1(z) := \left(-\gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \gamma_{z^*}^* - \gamma_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* \right) : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H}_{\mathcal{T}} \end{pmatrix} \rightarrow \mathfrak{H},$$

$$M_2(z) := \left(-\delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \gamma_{z^*}^* R_{\mathcal{T}}(z) - \delta_z(\mathcal{Q}_0(z) + \mathcal{T}(z))^{-1} \delta_{z^*}^* \right) : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H}_{\mathcal{T}} \end{pmatrix} \rightarrow \mathfrak{H}_{\mathcal{T}}.$$

Then, by (3.4), $\ker M_2(z) \subset \ker M_1(z)$ and hence $\overline{\text{ran}} M_1(z)^* \subset \overline{\text{ran}} M_2(z)^*$. The rationality of $\mathcal{T}(z)$ implies $\dim \mathfrak{H}_{\mathcal{T}} < \infty$ and therefore $\text{ran } M_1(z)^*$ and $\text{ran } M_2(z)^*$ are finite-dimensional subspaces and closed. By the Douglas-Halmos theorem (see [12, 15]), there is a bounded operator $G' : \mathfrak{H}_{\mathcal{T}} \rightarrow \mathfrak{H}$ such that $G' M_2(z) = M_1(z)$. If we apply both sides of the equality to $\begin{pmatrix} f \\ 0 \end{pmatrix}$

with arbitrary $f \in \mathfrak{H}$ and use that $(Q_0(z) + T(z))^{-1} \gamma_z^* : \mathfrak{H} \rightarrow \mathbb{C}^d$ is surjective we find that $\gamma_z = G' \delta_z$. Now apply both sides to $\begin{pmatrix} 0 \\ g \end{pmatrix}$ with arbitrary $g \in \mathfrak{H}_T$ and find that $G' R_T(z) = 0$. It follows that $G := (\gamma_z^* \gamma_z)^{-1} \gamma_z^* G'$ has the properties

$$G(\mathfrak{H}_T \ominus B_T(0)) = \{0\}, \quad G\delta = G\delta_z = I_{\mathbb{C}^d}.$$

We claim that $P_{B_T(0)}\delta$ is injective. Indeed, if $P_{B_T(0)}\delta \mathbf{x} = 0$ for some $\mathbf{x} \in \mathbb{C}^d$, then

$$\mathbf{x} = G\delta \mathbf{x} = G(I - P_{B_T(0)})\delta \mathbf{x} + GP_{B_T(0)}\delta \mathbf{x} = 0.$$

The claim and Lemma 2.5 imply (a). □

Remark 4.2. In Theorem 3.2, Theorem 3.5, Theorem 3.6, and in Theorem 4.1 we can replace the assumption that T is a matrix function by the assumption that it is relation valued:

$$T(z) = \{P\mathbf{y}, P\mathcal{T}_{op}(z)P\mathbf{y} + (I - P)\mathbf{y} : \mathbf{y} \in \mathbb{C}^d\}. \tag{4.2}$$

Then the results concerning the inclusions $S \subset C_{\mathfrak{H}}(\tilde{A}_T) \subset A_0$ and their extreme cases are still valid if we also replace \mathbb{C}^d by $\text{ran } P$, $T(z)$ by its operator part $\mathcal{T}_{op}(z)$ and S by its symmetric extension $\tilde{S}_{\ker P}$ as defined in Proposition 3.4.

As a consequence, we have the following corollary to Theorem 4.1, which will be applied below.

Corollary 4.3. *Suppose that in Krein’s formula based on A_0 the parameter $T(z)$ is relation valued as in (4.2) and that the operator part $\mathcal{T}_{op}(z)$, acting in $\text{ran } P$, is rational. Then*

$$C_{\mathfrak{H}}(\tilde{A}_T) = A_0$$

if and only if $\mathcal{T}_{op}(z)$ satisfies at ∞ the condition

$$\lim_{y \uparrow \infty} \mathcal{T}_{op}(iy)/iy > 0.$$

In [11, Theorem 5.5] it was proved that statements (a) and (b) in Theorem 4.1 are equivalent to the fact that T is the Schur complement of a z -linear matrix pencil $\mathcal{L}(z) = \mathcal{X} + z\mathcal{Y}$ with $\mathcal{Y} > 0$. This is also a consequence of Theorem 4.1 and the following proposition.

Proposition 4.4. *The $d \times d$ matrix Nevanlinna function T of the form (4.1) is the first Schur complement of a z -linear $k \times k$ matrix pencil*

$$\mathcal{L}(z) = \mathcal{X} + z\mathcal{Y} \quad \text{with } \mathcal{X} = \mathcal{X}^*, \mathcal{Y} = \mathcal{Y}^* \geq 0,$$

with k equal to d plus the sum of the multiplicities of the poles $\alpha_j, j = 1, \dots, \ell$. The pencil $\mathcal{L}(z)$ can be chosen such that $\mathcal{Y} > 0$ if and only if

$$\lim_{y \uparrow \infty} \mathcal{T}(iy)/iy > 0.$$

Proof. Set $r_j = \text{rank } \mathcal{A}_j$ and factorize \mathcal{A}_j into the product

$$\mathcal{A}_j = \mathcal{F}_j \mathcal{F}_j^*,$$

where \mathcal{F}_j is a $d \times r_j$ matrix, $j = 1, \dots, \ell$. Then $\mathcal{T}(z)$ is the $d \times d$ first Schur complement of the following z -linear pencil, which has the asserted properties:

$$\mathcal{X} + z\mathcal{Y} := \begin{pmatrix} \mathcal{A} & \mathcal{F}_1 & \cdots & \mathcal{F}_\ell \\ \mathcal{F}_1^* & -\alpha_1 \mathcal{I}_{r_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{F}_\ell^* & 0 & \cdots & -\alpha_\ell \mathcal{I}_{r_\ell} \end{pmatrix} + z \begin{pmatrix} \mathcal{B} & 0 & \cdots & 0 \\ 0 & \mathcal{I}_{r_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{I}_{r_\ell} \end{pmatrix}.$$

Clearly, if $\mathcal{B} = \lim_{y \uparrow \infty} \mathcal{T}(iy)/iy$ is positive, then $\mathcal{Y} > 0$.

Conversely, assume that

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{12}^* & \mathcal{X}_{22} \end{pmatrix} = \mathcal{X}^*, \quad \mathcal{Y} = \begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{12}^* & \mathcal{Y}_{22} \end{pmatrix} = \mathcal{Y}^* > 0$$

are block matrices such that $\mathcal{T}(z)$ is the first Schur complement of $\mathcal{X} + z\mathcal{Y}$, that is

$$\mathcal{T}(z) = \mathcal{X}_{11} + z\mathcal{Y}_{11} - (\mathcal{X}_{12} + z\mathcal{Y}_{12})(\mathcal{X}_{22} + z\mathcal{Y}_{22})^{-1}(\mathcal{X}_{12} + z^*\mathcal{Y}_{12})^*.$$

Then

$$\lim_{y \uparrow \infty} \mathcal{T}(iy)/iy = \mathcal{Y}_{11} - \mathcal{Y}_{12}\mathcal{Y}_{22}^{-1}\mathcal{Y}_{12}^*$$

and, since $\mathcal{Y} > 0$, the Schur-Frobenius factorization of \mathcal{Y} (see [25, Proposition 1.6.2]) implies that this limit is positive. \square

Remark 4.5. That every $d \times d$ matrix Nevanlinna function \mathcal{T} is a first Schur complement follows from the formula (see [22, (2.4)])

$$\mathcal{T}(z) = \text{Re } \mathcal{T}(z_0) - \text{Re } z_0 \delta^* \delta + z \delta^* \delta - (z - z_0^*) \delta^* (z - B_{\mathcal{T}})^{-1} (z - z_0) \delta$$

which can be obtained from (1.2) by using $\frac{1}{2}\mathcal{T}(z_0)^* = \frac{1}{2}\mathcal{T}(z_0) - \frac{1}{2}(z_0 - z_0^*)\delta^*\delta$.

4.2. In this subsection we show that for a finite-dimensional exit space, that is for a rational parameter $\mathcal{T}(z)$, the parameter in Krein’s formula for the compression $C_{\mathfrak{H}}(\tilde{A}_{\mathcal{T}})$ is $\mathcal{T}(\infty)$. Here we use the relation (2.4) for a transformation of the parameter, and Corollary 4.3.

Theorem 4.6. *Let S be a densely defined, closed symmetric operator with finite and equal defect numbers $d > 0$, and let A_0 be a canonical self-adjoint extension of S . Consider a self-adjoint extension \tilde{A} of S with finite-dimensional exit space such that $A_0 \cap \tilde{A} = S$, and denote the corresponding parameter in Krein’s formula based on A_0 by $\mathcal{T} : \tilde{A} = \tilde{A}_{\mathcal{T}}$. Then the compression $C_{\mathfrak{H}}(\tilde{A})$ corresponds in Krein’s formula based on A_0 to the parameter $\mathcal{T}(\infty) = \lim_{z \rightarrow \infty} \mathcal{T}(z)$.*

In formulas: if the extension \tilde{A} satisfies Krein’s formula

$$P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{H}} = (A_0 - z)^{-1} - \gamma_z(Q_0(z) + \mathcal{T}(z))^{-1}\gamma_{z^*}^*, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

then for the compression $C_{\mathfrak{H}}(\tilde{A})$ it holds

$$(C_{\mathfrak{H}}(\tilde{A}) - z)^{-1} = (A_0 - z)^{-1} - \gamma_z(Q_0(z) + \mathcal{T}(\infty))^{-1}\gamma_z^*, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, with the representation (4.1) for $\mathcal{T}(z)$ the limit $\mathcal{T}(\infty)$ is given by

$$\mathcal{T}(\infty) = \mathcal{A} + \{\ker \mathcal{B}, \text{ran } \mathcal{B}\} = P_{\ker \mathcal{B}} \mathcal{A} P_{\ker \mathcal{B}} \oplus \{0, \text{ran } \mathcal{B}\}.$$

In particular, if $\mathcal{B} = 0$ then $\mathcal{T}(\infty) = \mathcal{A}$, and if $\mathcal{B} > 0$ then $\mathcal{T}(\infty) = \{0, \mathbb{C}^d\}$ and $C_{\mathfrak{H}}(\tilde{A}) = A_0$.

Proof of Theorem 4.6. To apply Theorem 2.1 we set $\gamma_{0,z} = \gamma_z$, $\mathcal{S}_0(z) = \mathcal{T}(z)$, and $\mathcal{T}_0 = \mathcal{T}(\infty)$. Further, we define the canonical self-adjoint extension A_1 of S by

$$(A_1 - z)^{-1} = (A_0 - z)^{-1} - \gamma_{0,z}(Q_0(z) + \mathcal{T}(\infty))^{-1}\gamma_{0,z}^* \tag{4.3}$$

and the parameter $\mathcal{S}_1(z) = \mathcal{T}_1(z)$ by

$$P_{\mathfrak{H}}(\tilde{A} - z)^{-1}|_{\mathfrak{H}} = (A_1 - z)^{-1} - \gamma_{1,z}(Q_1(z) + \mathcal{T}_1(z))^{-1}\gamma_{1,z}^*. \tag{4.4}$$

Without loss of generality we can suppose that $Q_0(z)$, $\gamma_{0,z}$, $Q_1(z)$, and $\gamma_{1,z}$ are normalized to satisfy (2.3). Then $\mathcal{T}_{1,op}(z)$ is a rational Nevanlinna function and

$$\lim_{y \uparrow \infty} \mathcal{T}_{1,op}(iy)/iy = \lim_{x \rightarrow \infty} \mathcal{T}_{1,op}(x)/x.$$

To prove the theorem it suffices to show that

$$\lim_{x \rightarrow \infty} \mathcal{T}_{1,op}(x)/x > 0. \tag{4.5}$$

Indeed, then Corollary 4.3 and (4.4) imply $C_{\mathfrak{H}}(\tilde{A}) = A_1$, and (4.3) yields the claim.

To prove (4.5), we write again (see (4.1))

$$\mathcal{T}(z) = \mathcal{S}_0(z) = \sum_{j=1}^{\ell} \frac{\mathcal{A}_j}{\alpha_j - z} + \mathcal{A} + z\mathcal{B}, \tag{4.6}$$

and introduce the following decomposition of \mathbb{C}^d :

$$\mathbb{C}^d = \text{ran } \mathcal{B} \oplus \mathbb{L}' \oplus \mathbb{L}'', \tag{4.7}$$

where

$$\mathbb{L}'' = \ker \mathcal{B} \cap \left(\bigcap_{j=1}^{\ell} \ker \mathcal{A}_j \right), \quad \mathbb{L}' = \ker \mathcal{B} \ominus \mathbb{L}''.$$

With respect to the decomposition (4.7) the equation (4.6) becomes

$$\mathcal{T}(z) = \begin{pmatrix} \mathcal{R}_{11}(z) & \mathcal{R}_{12}(z) & 0 \\ \mathcal{R}_{21}(z) & \mathcal{R}_{22}(z) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix} + \begin{pmatrix} z\mathcal{B}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\mathcal{R}_{rs}(z) = \sum_{j=1}^{\ell} \frac{\mathcal{A}_{j,rs}}{\alpha_j - z}$, $r, s = 1, 2$. Note that $\mathcal{R}_{22}(z)$ is invertible.

Then

$$\mathcal{T}_0 = \mathcal{T}(\infty) = \left\{ \left\{ \begin{pmatrix} 0 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{u}_1 \\ 0 \\ 0 \end{pmatrix} \right\} : \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} \in \mathbb{C}^d \right\}$$

and hence

$$\mathcal{T}_0 - \mathcal{T}(z) = \left\{ \left\{ \begin{pmatrix} 0 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{R}_{22}(z) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{u}_1 \\ 0 \\ 0 \end{pmatrix} \right\} : \mathbf{u} \in \mathbb{C}^d \right\}.$$

This implies

$$\begin{aligned} (\mathcal{T}_0 - \mathcal{T}(z))^{-1} &= \left\{ \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{R}_{22}(z) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{u}_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} \right\} : \mathbf{u} \in \mathbb{C}^d \right\} \\ &= \left\{ \left\{ \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{R}_{22}(z)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{v}_3 \end{pmatrix} \right\} : \mathbf{v} \in \mathbb{C}^d \right\}. \end{aligned}$$

Now the relation (2.4) reads as

$$\mathcal{T}_1(z) = \mathcal{T}(z) + (\mathcal{T}(z) - \mathcal{Q})(\mathcal{T}_0 - \mathcal{T}(z))^{-1}(\mathcal{T}(z) + \mathcal{Q}),$$

and we obtain

$$\begin{aligned} \mathcal{T}_1(z)(0) &= (\mathcal{T}(z) - \mathcal{Q})(\mathcal{T}_0 - \mathcal{T}(z))^{-1}(0) \\ &= \left\{ (\mathcal{T}(z) - \mathcal{Q}) \begin{pmatrix} 0 \\ 0 \\ \mathbf{v}_3 \end{pmatrix} : \mathbf{v}_3 \in \mathbb{L}'' \right\} \\ &= (\mathcal{T}(z) - \mathcal{Q}) \mathbb{L}'' \end{aligned}$$

which is independent of z . Then so is its orthogonal complement:

$$\mathcal{T}_1(z)(0)^\perp = (\mathcal{T}(z^*) + \mathcal{Q})^{-1}(\text{ran } \mathcal{B} \oplus \mathbb{L}'),$$

and the operator $\mathcal{T}_{1,op}(z)$ acts in this space.

From now on we consider $z = x \in \mathbb{R}$, $z \neq \alpha_j$, $j = 1, 2, \dots, \ell$. Choose $\mathbf{x} \in \mathcal{T}_1(x)(0)^\perp$. It is of the form

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = (\mathcal{T}(x) + \mathcal{Q})^{-1} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ 0 \end{pmatrix} \tag{4.8}$$

and hence $(\mathcal{T}(x) + \mathcal{Q})\mathbf{x} \in \text{dom}(\mathcal{T}_0 - \mathcal{T}(x))^{-1}$. It follows that

$$\begin{aligned} \frac{1}{x} \langle \mathcal{T}_{1,op}(x)\mathbf{x}, \mathbf{x} \rangle &= \frac{1}{x} \langle \mathcal{T}(x)\mathbf{x}, \mathbf{x} \rangle \\ &+ \left\langle \frac{1}{x} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathcal{R}_{22}(x)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} (\mathcal{T}(x) + \mathcal{Q})\mathbf{x}, (\mathcal{T}(x) + \mathcal{Q})\mathbf{x} \right\rangle \end{aligned}$$

and this, if $x \rightarrow \infty$, converges to

$$\langle \mathcal{B}_{11}\mathbf{x}_1, \mathbf{x}_1 \rangle + \left\langle \left(\sum_{j=1}^{\ell} \mathcal{A}_{j,22} \right)^{-1} \sum_{k=1}^3 (\mathcal{A}_{2k} + \mathcal{Q}_{2k})\mathbf{x}_k, \sum_{k=1}^3 (\mathcal{A}_{2k} + \mathcal{Q}_{2k})\mathbf{x}_k \right\rangle.$$

Thus, since \mathcal{B}_{11} and $\sum_{j=1}^{\ell} \mathcal{A}_{j,22}$ are positive,

$$\lim_{x \rightarrow \infty} \langle \mathcal{T}_{1,op}(x)\mathbf{x}, \mathbf{x} \rangle / x \geq 0$$

and the limit equals 0 if and only if

$$\mathbf{x}_1 = 0 \quad \text{and} \quad \sum_{k=1}^3 (\mathcal{A}_{2k} + \mathcal{Q}_{2k})\mathbf{x}_k = 0.$$

From (4.8) it follows that also

$$\sum_{k=1}^3 (\mathcal{A}_{3k} + \mathcal{Q}_{3k})\mathbf{x}_k = 0.$$

The three equalities imply $\mathbf{x}_1 = 0$ and

$$(\widehat{\mathcal{A}} + \widehat{\mathcal{Q}}) \begin{pmatrix} \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = 0, \quad \text{where} \quad \widehat{\mathcal{A}} := \begin{pmatrix} \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix}, \quad \widehat{\mathcal{Q}} := \begin{pmatrix} \mathcal{Q}_{22} & \mathcal{Q}_{23} \\ \mathcal{Q}_{32} & \mathcal{Q}_{33} \end{pmatrix}.$$

Since $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^*$ and, by the normalization (2.3), $\text{Im } \widehat{\mathcal{Q}} > 0$, it follows that $\mathbf{x}_2 = 0$ and $\mathbf{x}_3 = 0$. Thus $\mathbf{x} = 0$ and this implies (4.5). \square

5. Appendix

Let S be again a densely defined, closed symmetric operator with finite and equal defect numbers $d > 0$. In this Appendix we show that self-adjoint extensions \widetilde{A} of S with the property $C_{\mathfrak{H}}(\widetilde{A}) = S$ arise in a natural way from dilation theory. In fact, \widetilde{A} can be chosen as the self-adjoint dilation of any maximal dissipative extension T of S in \mathfrak{H} such that $S = T \cap T^*$.

Recall that the densely defined operator T in \mathfrak{H} is called *dissipative* if $\text{Im}(Tf, f) \geq 0$, $f \in \text{dom } T$, and *maximal dissipative* if it is dissipative and does not have a proper dissipative extension in \mathfrak{H} . In the proof of the proposition below we use boundary triplets: $(\mathbb{C}^d, \Gamma_1, \Gamma_2)$ is a *boundary triplet* for S if Γ_1 and Γ_2 are linear mappings from $\text{dom } S^*$ to \mathbb{C}^d such that

$$(S^*f, g) - (f, S^*g) = \langle \Gamma_1 f, \Gamma_2 g \rangle - \langle \Gamma_2 f, \Gamma_1 g \rangle, \quad f, g \in \text{dom } S^*,$$

and the mapping

$$x \mapsto \begin{pmatrix} \Gamma_1 x \\ \Gamma_2 x \end{pmatrix} \text{ from } \text{dom } S^* \text{ to } \mathbb{C}^{2d} \text{ is surjective.}$$

Proposition 5.1. *Let S be a densely defined, closed symmetric operator in a Hilbert space \mathfrak{H} with finite and equal defect numbers $d > 0$. Then there exists a maximal dissipative extension T of S in \mathfrak{H} such that S is the hermitian part of T : $S = T \cap T^*$.*

Proof. Let T be a dissipative extension of S in \mathfrak{H} . Then, by Phillips' theorem [14, Theorem 3.1.3], T is a restriction of S^* : $S \subset T \subset S^*$. By [14, Theorem 3.1.6] T is a maximal dissipative extension of S if and only if there is a contractive $d \times d$ matrix \mathcal{K} such that

$$T = \{ \{f, S^*f\} : f \in \text{dom } S^*, (\mathcal{K} - I)\Gamma_1 f + i(\mathcal{K} + I)\Gamma_2 f = 0 \}.$$

In this case $-T^*$ is a maximal dissipative operator and hence, according to [14, Theorem 3.1.6], T^* is given by

$$T^* = \{ \{f, S^*f\} : f \in \text{dom } S^*, (\mathcal{L} - I)\Gamma_1 f - i(\mathcal{L} + I)\Gamma_2 f = 0 \}$$

for some contractive $d \times d$ matrix \mathcal{L} . It can be shown that $\mathcal{L} = \mathcal{K}^*$.

We choose \mathcal{K} such that $I - \mathcal{K}^* \mathcal{K} > 0$ and show that in this case $T \cap T^* = S$. Clearly $S \subset T \cap T^*$. To prove the converse inclusion let $f \in \text{dom } T \cap T^*$. Then

$$\begin{pmatrix} \mathcal{K} - I & i(\mathcal{K} + I) \\ \mathcal{K}^* - I & -i(\mathcal{K}^* + I) \end{pmatrix} \begin{pmatrix} \Gamma_1 f \\ \Gamma_2 f \end{pmatrix} = 0.$$

The matrix on the left is invertible since

$$\begin{pmatrix} \mathcal{K} - I & i(\mathcal{K} + I) \\ \mathcal{K}^* - I & -i(\mathcal{K}^* + I) \end{pmatrix} \begin{pmatrix} \mathcal{K} + I & \mathcal{K}^* + I \\ i(\mathcal{K} - I) & -i(\mathcal{K}^* - I) \end{pmatrix} = 2 \begin{pmatrix} 0 & \mathcal{K} \mathcal{K}^* - I \\ \mathcal{K}^* \mathcal{K} - I & 0 \end{pmatrix}$$

and with $\mathcal{K}^* \mathcal{K} - I$ also $\mathcal{K} \mathcal{K}^* - I$ is invertible. Therefore $\Gamma_1 f = \Gamma_2 f = 0$, that is $f \in \text{dom } S$. Hence $T \cap T^* \subset S$. □

In the following theorem we consider the self-adjoint dilation \tilde{A} of a maximal dissipative operator T as defined by Kudryashov in [19, 21], see also [20, Theorem 4.3.2.].

Theorem 5.2. *Let S be a densely defined, closed symmetric operator in a Hilbert space \mathfrak{H} with finite and equal defect numbers $d > 0$, and let T be a maximal dissipative extension of S in $\tilde{\mathfrak{H}}$ such that $T \cap T^* = S$. Then the compression of the self-adjoint dilation \tilde{A} of T equals $S : C_{\mathfrak{H}}(\tilde{A}) = S$.*

Proof. According to Proposition 5.1 maximal dissipative operators T with $T \cap T^* = S$ exist. Let \tilde{A} be the self-adjoint dilation of T in a Hilbert space $\tilde{\mathfrak{H}}$ as constructed in loc. cit. We will not describe this construction in detail. We just mention that the definition of \tilde{A} readily implies that $C_{\mathfrak{H}}(\tilde{A}) = \tilde{A}|_{\mathfrak{H}}$, and that

$$\tilde{A}|_{\mathfrak{H}} = \left\{ \{f, S^* f\} : f \in \text{dom } T, (D_{-i})^{1/2} (T + i)f = 0 \right\};$$

here

$$D_{-i} := iR(-i) - iR(-i)^* - 2R(-i)^* R(-i),$$

with the resolvent $R(-i) := (T + i)^{-1}$ (note: $-i \in \rho(T)$, since T is maximal dissipative). The operator D_{-i} is non-negative as T is dissipative:

$$(D_{-i} f, f) = \text{Im} (TR(-i)f, R(-i)f) \geq 0.$$

It follows that $\ker (D_{-i})^{1/2} = \ker D_{-i}$ and, since by [20, Theorem 1.2.5]

$$\ker D_{-i} = \text{ran}(T \cap T^* + i),$$

we have

$$C_{\mathfrak{H}}(\tilde{A}) = \left\{ \{f, S^* f\} : f \in \text{dom } T \cap T^* \right\} = T \cap T^* = S. \quad \square$$

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