**Integral Equations
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Born–Jordan Pseudodifferential Calculus, Bopp Operators and Deformation Quantization

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Abstract. There has recently been a resurgence of interest in Born–Jordan quantization, which historically preceded Weyl's prescription. Both mathematicians and physicists have found that this forgotten quantization scheme is actually not only of great mathematical interest, but also has unexpected application in operator theory, signal processing, and time-frequency analysis. In the present paper we discuss the applications to deformation quantization, which in its traditional form relies on Weyl quantization. Introducing the notion of "Bopp operator" which we have used in previous work, this allows us to obtain interesting new results in the spectral theory of deformation quantization.

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1. Introduction

Deformation quantization is a popular framework for quantum mechanics among mathematical physicists. It was suggested by Moyal [33] and Groenewold [26], and put on a firm mathematical ground by Bayen et al. [1,2]; later Kontsevich [29,30] extended the theory to Poisson manifolds. Roughly speaking, the idea is to "deform" classical (Hamiltonian) mechanics into quantum mechanics using a parameter (Planck's constant); this is achieved using the notion of "star product" or "Moyal product" $\star_\hbar$ of two functions on $\mathbb{R}^{2n}$. The star product is defined in physics by the suggestive formula

$$a \star_\hbar b = a \exp\left(\frac{i\hbar}{2}\left[\overleftarrow{\partial_x} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_x}\right]\right) b; \qquad (1)$$

the exponential in the right hand side (the "Janus operator") is understood as a power series, the arrows indicating the direction in which the derivatives act.

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This formula was proposed for the first time by Groenewold in his seminal work [26] in 1946. A rigorous definition is the following: denoting by $\overset{\text{Weyl}}{\longleftrightarrow}$ the Weyl correspondence between operators and symbols, assume that $a, b \in \mathcal{S}'(\mathbb{R}^{2n})$ and let $A \overset{\text{Weyl}}{\longleftrightarrow} a$ and $B \overset{\text{Weyl}}{\longleftrightarrow} b$ (A is sometimes called the "Weyl transform" of a). If the product $C = AB$ is defined and $C \overset{\text{Weyl}}{\longleftrightarrow} c$ then, by definition, $c = a \star_\hbar b$.

We have shown in previous work [17,18] that we have

$$a \star_\hbar b = \widetilde{A} b \tag{2}$$

where $\widetilde{A}$ is a pseudodifferential operator acting on distributions defined on $\mathbb{R}^{2n}$; formally

$$\widetilde{A} = a\left(x + \tfrac{1}{2} i\hbar\partial_p, p - \tfrac{1}{2} i\hbar\partial_x\right). \tag{3}$$

We have called $\widetilde{A}$ the "Bopp pseudodifferential operator" with symbol a; it is the Weyl operator on $T^*\mathbb{R}^{2n} \equiv \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with symbol

$$\widetilde{a}(z, \zeta) = a\left(x - \tfrac{1}{2}\zeta_p, p + \tfrac{1}{2}\zeta_x\right) \tag{4}$$

where (ζ_x, ζ_p) are viewed as the dual variables of (x, p). This reformulation of the star product in terms of pseudodifferential operators is very fruitful; not only does it allow the study of the generalized eigenvalues and eigenfunctions of "stargenvalue" problems using standard pseudodifferential techniques, but it also leads to interesting regularity results in various functional spaces. The main observation, which leads to the theme of the present paper, is that the whole procedure heavily relies on the Weyl pseudodifferential calculus. From a physical point of view, this means that we are privileging Weyl quantization; technically this choice has many advantages because Weyl quantization is the simplest and most austere of all quantizations: using Schwartz's kernel theorem one shows that every continuous linear operator $A\colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ can be viewed as a Weyl operator, and the Weyl correspondence is uniquely characterized by the property of symplectic covariance: if $A \overset{\text{Weyl}}{\longleftrightarrow} a$ then $\widehat{S} A \widehat{S}^{-1} \overset{\text{Weyl}}{\longleftrightarrow} a \circ S^{-1}$ for every metaplectic operator $\widehat{S} \in \mathrm{Mp}(n)$ with projection $S \in \mathrm{Sp}(n)$ (the symplectic group). However, in real life things are not always that simple. Just a couple of years before Weyl [38] defined the eponymous correspondence, Born and Jordan [7], elaborating on Heisenberg's 1925 "matrix mechanics" [27], proposed a quantization procedure having a firm physical motivation (conservation of energy); their approach culminated one year later in their famous "drei Männer Arbeit" [8] with Heisenberg. There are many good reasons to believe that the Born and Jordan quantization scheme is the right one in physics (Kauffmann [28]); in addition, some very recent work of Boggiatto and his collaborators [3–5] shows that the Wigner formalism corresponding to Born–Jordan quantization is much more adequate in signal analysis than the traditional Weyl–Wigner approach. It allows to damp the appearance of unwanted "ghost" frequencies in spectrograms; numerical experiments confirm these theoretical facts.

In [16] the first of the authors has studied the properties of Born–Jordan pseudodifferential calculus; in the present paper we go one step further, and reformulate deformation quantization in terms of this calculus.

Notation. We will write $z = (x, p)$ where $x \in \mathbb{R}^n$ and $p \in (\mathbb{R}^n)^* \equiv \mathbb{R}^n$. Operators $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ are usually denoted by $A, B, \ldots$ while operators $\mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ are denoted by $\widetilde{A}, \widetilde{B}, \ldots$ The lower-case Greek letters $\psi, \phi, \ldots$ stand for functions (or distributions) defined on $\mathbb{R}^n$ while their upper-case counterparts $\psi, \phi, \ldots$ denote functions (or distributions) defined on $\mathbb{R}^{2n}$. The distributional bracket on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$ and that on $\mathbb{R}^{2n}$ by $\langle\langle \cdot, \cdot \rangle\rangle$. We denote by

$$\sigma = dp_1 \wedge dx_1 + \cdots + dp_n \wedge dx_n$$

the standard symplectic form on $T^*\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$; in coordinates: $\sigma(z, z') = Jz \cdot z'$ where $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ is the standard symplectic matrix.

2. Bopp Operators and Born–Jordan Quantization

2.1. Born–Jordan Versus Weyl

Let us quickly review the Born–Jordan and Weyl quantizations of monomials $x_j^m p_j^\ell$. In what follows, the capital letters X_j and P_j denote operators acting on some space of functions or distributions on $\mathbb{R}^d$, and satisfying Born's commutation relations

$$[X_j, P_j] = X_j P_j - P_j X_j = i\hbar. \tag{5}$$

For instance, in traditional quantum mechanics $d = n$ and X_j is the operator of multiplication by x_j while $P_j = -i\hbar \partial_{x_j}$, but there is no compelling reason for limiting ourselves to these operators. Keeping this in mind, the Weyl quantization of monomials is given by the rule

$$x_j^m p_j^\ell \xrightarrow{\text{Weyl}} \frac{1}{2^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} P_j^{\ell-k} X_j^m P_j^k \tag{6}$$

while Born–Jordan quantization is given by

$$x_j^m p_j^\ell \xrightarrow{\text{BJ}} \frac{1}{\ell+1} \sum_{k=0}^{\ell} P_j^{\ell-k} X_j^m P_j^k. \tag{7}$$

The Weyl and Born–Jordan correspondences agree for all monomials which are at most quadratic, as well as for monomials of the type $p_j x_j^m$ or $p_j^\ell x_j$. They are however different as soon as we have $l \geq 2$ and $m \geq 2$ (Turunen [37]). It turns out that both rules can be obtained from the τ-correspondence, defined by

$$x_j^m p_j^\ell \xrightarrow{\tau} \sum_{k=0}^{\ell} \binom{\ell}{k} (1-\tau)^k \tau^{\ell-k} P_j^{\ell-k} X_j^m P_j^k \tag{8}$$

where τ is a real number. The case $\tau = \frac{1}{2}$ yields the Weyl correspondence (6). Integrating the τ-correspondence over the interval $[0,1]$ and using the formula

$$\int_0^1 (1-\tau)^k \tau^{\ell-k} d\tau = \frac{k!(\ell-k)!}{(\ell+1)!}$$

we get the Born–Jordan correspondence (7). Historically, things evolved the other way round: in [7] Born and Jordan were led to the eponymous correspondence (7) by a strict analysis of Heisenberg's [27] ideas. In their subsequent publication [8] with Heisenberg they showed that their constructions extend *mutatis mutandis* to systems with an arbitrary number of degrees of freedom.

In the general case one proceeds as follows (de Gosson [16]): let τ be a real parameter, and define the τ-pseudodifferential operator $A_\tau = \mathrm{Op}^\tau(a)$ with symbol $a \in \mathcal{S}'(\mathbb{R}^{2n})$ as being the operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ with distributional kernel

$$K_\tau(x,y) = F_2^{-1}[a(\tau x + (1-\tau)y, \cdot)](x-y)$$

where F_2^{-1} is the inverse Fourier transform in the second set of variables. This defines the so-called Shubin τ-correspondence [34] $A \xleftrightarrow{\tau} a$ by; it is easy to check that one recovers the correspondence (8) for monomials. For $a \in \mathcal{S}(\mathbb{R}^{2n})$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ the more suggestive formula

$$A_\tau \psi(x) = \iint_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p(x-y)} a(\tau x + (1-\tau)y, p)\psi(y)dpdy \qquad (9)$$

holds (Shubin [34], §23), which can also be extended to more general settings. The choice $\tau = \frac{1}{2}$ leads to the usual Weyl operators: $A = \mathrm{Op_{BJ}}(a)$ with

$$A\psi(x) = \iint_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p(x-y)} a(\tfrac{1}{2}(x+y), p)\psi(y)dpdy. \qquad (10)$$

The Born–Jordan pseudodifferential operator $A = \mathrm{Op_{BJ}}(a)$ is obtained by averaging the Shubin operators A_τ over $\tau \in [0,1]$:

$$A = \mathrm{Op_{BJ}}(a) = \int_0^1 A_\tau d\tau; \qquad (11)$$

it is thus the operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ with kernel

$$K_{\mathrm{BJ}}(x,y) = F_2^{-1}[a_{\mathrm{BJ}}(x,y,p), \cdot)](x-y)$$

where the symbol a_{BJ} is defined by

$$a_{\mathrm{BJ}}(x,y,p) = \int_0^1 a(\tau x + (1-\tau)y, p)d\tau. \qquad (12)$$

(Heuristically, the Weyl operator (10) is obtained by approximating the integral in (12) using the midpoint rule). One verifies by a direct calculation, that the correspondence $A \xleftrightarrow{\mathrm{BJ}} a$ reduces to the Born–Jordan rules (7) for polynomials $x_j^m p_j^\ell$.

As already mentioned, the Born–Jordan and Weyl correspondences agree for all quadratic polynomials in the variables x_j, p_j. More generally

(de Gosson [16]) both quantizations are also identical for symbols arising from physical Hamiltonians of the type

$$H(z) = \sum_{j=1}^{n} \frac{1}{2m_j} (p_j - A_j(x))^2 + V(x) \tag{13}$$

where A_j and V are real C^∞ functions.

2.2. Harmonic Analysis of A_{BJ}

It is usual to write Weyl operators $A = \mathrm{Op}^{\mathrm{W}}(a)$ in the form of operator valued integrals

$$A = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} a_\sigma(z_0)\widehat{T}(z_0)dz_0 \tag{14}$$

where a_σ is the "twisted symbol" of A:

$$a_\sigma(z) = F_\sigma a(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z,z')} a(z')dz' \tag{15}$$

(F_σ is called the "symplectic Fourier transform") and $\widehat{T}(z_0)$ is the Heisenberg–Weyl operator defined, for $z_0 \in \mathbb{R}^{2n}$ by

$$\widehat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0 x - \frac{1}{2}p_0 x_0)}\psi(x - x_0) \tag{16}$$

for a function (or distribution) ψ on $\mathbb{R}^n$. Similarly, the Shubin operator $A_\tau = \mathrm{Op}^\tau(a)$ can be written (de Gosson [16])

$$A_\tau = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} a_\sigma(z_0)\widehat{T}_\tau(z_0)dz_0 \tag{17}$$

where $\widehat{T}_\tau(z_0)$ is the modified Heisenberg–Weyl operator defined by

$$\widehat{T}_\tau(z_0) = e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0}\widehat{T}(z_0). \tag{18}$$

Proposition 1.   (i) *The Born–Jordan operator $A_{BJ} = \mathrm{Op}_{\mathrm{BJ}}(a)$ is given by*

$$A_{BJ} = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} a_\sigma(z_0)\Theta(z_0)\widehat{T}(z_0)dz_0 \tag{19}$$

where Θ is defined by

$$\Theta(z_0) = \frac{\sin(p_0 x_0/2\hbar)}{p_0 x_0/2\hbar}. \tag{20}$$

(ii) *The twisted Weyl symbol a_σ^W of A_{BJ} is given by the explicit formula*

$$a_\sigma^W(z_0) = a_\sigma(z_0)\Theta(z_0). \tag{21}$$

(iii) *The operator A_{BJ} is hence a continuous operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ for every $a \in \mathcal{S}'(\mathbb{R}^{2n})$.*

Remark 2. Notice that $\Theta(z) = \mathrm{sinc}(px/2\hbar)$ where $\mathrm{sinc}(t) = (\sin t)/t$ is the cardinal sine function familiar from signal analysis.

Proof. The statement (ii) immediately follows from formula (19) taking the representation (14) of Weyl operators into account. The proof of formula (19) goes as follows (cf. [16], Proposition 11): integrating both sides of the equality (17) with respect to the parameter $\tau \in [0, 1]$ one gets

$$A_{\mathrm{BJ}} = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} a_\sigma(z_0) \left(\int_0^1 \widehat{T}_\tau(z_0)d\tau\right) dz_0.$$

Now, in view of definition (18), for $p_0 x_0 \neq 0$

$$\int_0^1 \widehat{T}_\tau(z_0)d\tau = \left(\int_0^1 e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0}d\tau\right) \widehat{T}(z_0)$$

$$= \frac{\sin(p_0 x_0/2\hbar)}{p_0 x_0/2\hbar}\widehat{T}(z_0)$$

hence formula (19) (the formula holds by continuity for $p_0 x_0 = 0$). (iii) Formula (21) implies that $a_\sigma^W \in \mathcal{S}'(\mathbb{R}^n)$ if $a_\sigma \in \mathcal{S}'(\mathbb{R}^n)$ because $\Theta \in L^\infty(\mathbb{R}^{2n}) \cap C^\infty(\mathbb{R}^{2n})$. □

It immediately follows from formula (20) that since $\Theta(z_0) = 0$ for all $z_0 = (x_0, p_0)$ such that $p_0 x_0 = 2N\pi\hbar$ for some integer $N \in \mathbb{Z}$ we see that an arbitrary continuous operator $A \colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is not in general a Born–Jordan operator: every such operator A has indeed a twisted Weyl symbol a_σ^W in view of Schwartz's kernel theorem, but because of zeroes of we cannot in general expect the Eq. (21) to be solved for a_σ. This property of Born–Jordan operators really distinguishes them among all traditional pseudodifferential operators: the Born–Jordan "correspondence" is neither surjective, nor injective. Keeping this caveat in mind, we will still write symbolically $a \xrightarrow{\mathrm{BJ}} A$ or $A = \mathrm{Op}^{\mathrm{BJ}}(a)$.

2.2.1. Composition and Adjoints of Born–Jordan Operators. Let $A \colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $B \colon \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ be two continuous operators; their product AB is well-defined, and its Weyl symbol can be explicitly determined in terms of those of A and B. In fact if $A = \mathrm{Op}_W(a)$ and $B = \mathrm{Op}_W(b)$ then $AB = \mathrm{Op}_W(a \star_\hbar b)$ where $a \star_\hbar b$ is the Moyal product:

$$(a \star_\hbar b)(z) = \left(\tfrac{1}{4\pi\hbar}\right)^{2n} \iint_{\mathbb{R}^{4n}} e^{\frac{i}{2\hbar}\sigma(u,v)} a(z + \tfrac{1}{2}u)b(z - \tfrac{1}{2}v)du\,dv. \qquad (22)$$

There are several ways to rewrite this formula; performing elementary changes of variables we have

$$(a \star_\hbar b)(z) = \left(\tfrac{1}{\pi\hbar}\right)^{2n} \iint_{\mathbb{R}^{4n}} e^{-\frac{2i}{\hbar}\sigma(z-z',z-z'')} a(z')b(z'')dz'\,dz'' \qquad (23)$$

which is well-known in the literature. For our purposes, it will be more tractable to use the following formula, which gives the twisted symbol of the compose in terms of the twisted symbols of the factors:

$$(a \star_\hbar b)_\sigma(z) = \int_{\mathbb{R}^{2n}} e^{\frac{i}{2\hbar}\sigma(z,z')} a_\sigma(z - z')b_\sigma(z')dz'. \qquad (24)$$

Proposition 3. *Let $A = \mathrm{Op}_{\mathrm{BJ}}(a)$ and $B = \mathrm{Op}_{\mathrm{BJ}}(b)$ be two Born–Jordan pseudodifferential operators; we suppose that $C = AB$ is defined as an operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$. (i) The Weyl symbol of $C = AB$ is given by the formula*

$$c_\sigma^{\mathrm{W}}(z) = \int_{\mathbb{R}^{2n}} e^{\frac{i}{2\hbar}\sigma(z,z')} a_\sigma(z - z') b_\sigma(z') \Theta(z - z')\Theta(z')dz' \qquad (25)$$

where Θ is defined by (20). (ii) If we can factorize c_σ^{W} as $c_\sigma^{\mathrm{W}}(z) = \chi(z)\Theta(z)$ where $\chi \in \mathcal{S}'(\mathbb{R}^n)$ then $\chi = c_\sigma$ with $C = \mathrm{Op}_{\mathrm{BJ}}(c)$. (iii) The adjoint of $A = \mathrm{Op}_{\mathrm{BJ}}(a)$ is $A^* = \mathrm{Op}_{\mathrm{BJ}}(\overline{a})$. In particular, the Born–Jordan operator A is formally self-adjoint if and only its symbol is real.

Proof. (i) Formula (25) is an immediate consequence of formulas (24) and of (21) since $c_\sigma = a \star_\hbar b$. The statement (ii) follows, using again (21). (iii) The adjoint of the τ-pseudodifferential operator $A_\tau = \mathrm{Op}_\tau(a)$ is $A_\tau^* = \mathrm{Op}_{1-\tau}(\overline{a})$ (see [15,34]); it follows that

$$A^* = \int_0^1 \mathrm{Op}_{1-\tau}(\overline{a})d\tau = \int_0^1 \mathrm{Op}_\tau(\overline{a})d\tau = \mathrm{Op}_{\mathrm{BJ}}(\overline{a}). \qquad \square$$

Remark 4. Note that χ, and hence c, are not uniquely defined by the relation $c_\sigma^{\mathrm{W}}(z) = \chi(z)\Theta(z)$ since $\Theta(z) = 0$ for infinitely many values of z. On the other hand, it is not obvious that an arbitrary Weyl operator can be written as a Born–Jordan operator. That this is however the case has been proven recently in Cordero et al. [11] using techniques from distribution theory (the Paley–Wiener theorem).

3. Bopp Quantization of Born–Jordan Operators

3.1. Bopp Calculus

Setting $v = z_0$, $z + \frac{1}{2}u = z'$ in the formula (23) and introducing the notation

$$\widetilde{T}(z_0)b(z) = e^{-\frac{i}{\hbar}\sigma(z,z_0)}b(z - \tfrac{1}{2}z_0) \qquad (26)$$

we can rewrite formula (22) as

$$a \star_\hbar b(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0)\widetilde{T}(z_0)b(z)dz_0. \qquad (27)$$

The restrictions of the operators $\widetilde{T}(z_0)\colon \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^{2n})$ are unitary, and satisfy the same commutation relations

$$\widetilde{T}(z_0)\widetilde{T}(z_1) = e^{\frac{i}{\hbar}\sigma(z_0,z_1)}\widetilde{T}(z_1)\widetilde{T}(z_0) \qquad (28)$$

as the Heisenberg–Weyl operators. In [17] we have proven the following result:

Proposition 5. *The Weyl symbol of the operator $\widetilde{A}\colon b \longmapsto a \star_\hbar b$ is the distribution $\widetilde{a} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ defined by*

$$\widetilde{a}(z,\zeta) = a\big(z - \tfrac{1}{2}J\zeta\big) = a\big(x - \tfrac{1}{2}\zeta_p, p + \tfrac{1}{2}\zeta_x\big) \qquad (29)$$

where $(z,\zeta) \in T^*\mathbb{R}^{2n}$, $\zeta = (\zeta_x, \zeta_p)$.

We now introduce the following elementary operators (called "Bopp shifts" following Bopp [6]; also see Kubo [31]) acting on phase space functions and distributions:

$$\widetilde{X}_j = x_j + \frac{i\hbar}{2}\partial_{p_j}, \quad \widetilde{P}_j = p_j - \frac{i\hbar}{2}\partial_{x_j}. \tag{30}$$

These operators satisfy Born's commutation relations (5), and we can thus define the extended quantization rule

$$x_j^m p_j^\ell \xrightarrow{\text{Weyl}} \frac{1}{2^\ell} \sum_{k=0}^{\ell} \binom{\ell}{k} \widetilde{P}_j^{\ell-k} \widetilde{X}_j^m \widetilde{P}_j^k \tag{31}$$

corresponding to (6)–(7), respectively. The Weyl and Born–Jordan symbols of $\widetilde{X}_j$ and $\widetilde{P}_j$ being, respectively, $x_j - \frac{1}{2}\zeta_{p,j}$ and $p_j + \frac{1}{2}\zeta_{x,j}$ formula (29) suggests the notation

$$\widetilde{A} = a\big(x + \tfrac{1}{2}i\hbar\partial_p, p - \tfrac{1}{2}i\hbar\partial_x\big)$$

used in the Sect. 1.

3.2. The Born–Jordan Starproduct

In the Born–Jordan case we would like to define Bopp quantization using a procedure extending the natural correspondence

$$x_j^m p_j^\ell \xrightarrow{\text{BJ}} \frac{1}{\ell+1} \sum_{k=0}^{\ell} \widetilde{P}_j^{\ell-k} \widetilde{X}_j^m \widetilde{P}_j^k.$$

induced by the monomial rule (7). We will proceed as follows: returning to formula (17) we define the phase-space τ-operator by

$$\widetilde{A}_\tau = \big(\tfrac{1}{2\pi\hbar}\big)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0)\widetilde{T}_\tau(z_0)dz_0 \tag{32}$$

where $\widetilde{T}_\tau(z_0)$ is defined in terms of the operator (26) by

$$\widetilde{T}_\tau(z_0) = e^{\frac{i}{2\hbar}(2\tau-1)p_0 x_0}\widetilde{T}(z_0). \tag{33}$$

In analogy with formula (2) we now define the "Born–Jordan starproduct" $\star_{\hbar,\text{BJ}}$:

Definition 6. Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$. The Bopp–Born–Jordan (BBJ) operator with symbol a is the operator

$$\widetilde{A}_{\text{BJ}} = \widetilde{\text{Op}_{\text{BJ}}}(a)\colon \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

defined by the integral

$$\widetilde{A}_{\text{BJ}} = \int_0^1 \widetilde{A}_\tau d\tau \tag{34}$$

where $\widetilde{A}_\tau$ is the pseudodifferential operator (32). Let $b \in \mathcal{S}(\mathbb{R}^{2n})$. We set

$$a \star_{\hbar,\text{BJ}} b = \widetilde{A}_{\text{BJ}}b. \tag{35}$$

In view of formula (19) the BBJ operator has the explicit expression

$$\widetilde{A}_{\mathrm{BJ}} = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z)\Theta(z)\widetilde{T}(z)dz \qquad (36)$$

where $\Theta \in L^\infty(\mathbb{R}^{2n}) \cap C^\infty(\mathbb{R}^{2n})$ is given by (20).

3.3. The Functions Amb$_{\mathrm{BJ}}$ and Wig$_{\mathrm{BJ}}$

In what follows $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the distributional bracket on $\mathbb{R}^{2n}$.

The Weyl correspondence between symbols and operators can be defined using the Wigner formalism. In fact, given a symbol $a \in \mathcal{S}(\mathbb{R}^{2n})$ one can show (see e.g. [15], §10.1) that the operator $A \overset{\mathrm{Weyl}}{\longleftrightarrow} a$ is the only operator such that

$$(A\psi|\phi)_{L^2} = \langle\langle a, \mathrm{Wig}(\psi, \phi)\rangle\rangle \qquad (37)$$

where $\mathrm{Wig}(\psi, \phi)$ is the cross-Wigner distribution (or function) of $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$:

$$\mathrm{Wig}(\psi, \phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}py}\psi(x + \tfrac{1}{2}y)\overline{\phi(x - \tfrac{1}{2}y)}dy. \qquad (38)$$

Noting that $(A\psi|\phi)_{L^2} = \langle A\psi, \overline{\phi}\rangle$ and that $\mathrm{Wig}(\psi, \phi) \in \mathcal{S}(\mathbb{R}^{2n})$ formula (37) allows to extend the definition of the operator A to the case where $a \in \mathcal{S}'(\mathbb{R}^{2n})$. In view of Plancherel's theorem we can rewrite (37) as

$$(A\psi|\phi)_{L^2} = \langle\langle a_\sigma, F_\sigma \mathrm{Wig}(\psi, \phi)^\vee\rangle\rangle \qquad (39)$$

where $f^\vee(z) = f(-z)$. Since the symplectic Fourier transform of the cross-Wigner transform is the cross-ambiguity function [14,15,23]

$$\mathrm{Amb}(\psi, \phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}py}\psi(y + \tfrac{1}{2}x)\overline{\phi(y - \tfrac{1}{2}x)}dy$$

we have

$$(A\psi|\phi)_{L^2} = \langle\langle a_\sigma, \mathrm{Amb}(\psi, \phi)^\vee\rangle\rangle. \qquad (40)$$

It turns out that we have similar formulas for Born–Jordan operators. We first recall [15, §8.3.1] that the symplectic Fourier transform defined by (15) is involutive: $F_\sigma^2 = I_\mathrm{d}$ and satisfies the following variant of the Plancherel identity where $\langle\langle ., . \rangle\rangle$ denotes the scalar product on $\mathbb{R}^{2n}$:

$$\langle\langle a, F_\sigma b\rangle\rangle = \langle\langle (F_\sigma a)^\vee, b\rangle\rangle = \langle\langle F_\sigma a, b^\vee\rangle\rangle. \qquad (41)$$

Proposition 7. *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$; we have*

$$(A_{BJ}\psi|\phi)_{L^2} = \langle\langle a_\sigma, \mathrm{Amb}_{BJ}(\psi, \phi)^\vee\rangle\rangle \qquad (42)$$

$$(A_{BJ}\psi|\phi)_{L^2} = \langle\langle a, \mathrm{Wig}_{BJ}(\psi, \phi)\rangle\rangle \qquad (43)$$

where $\mathrm{Amb}_{\mathrm{BJ}}(\psi, \phi)$ and $\mathrm{Wig}_{\mathrm{BJ}}(\psi, \phi)$ are defined by

$$\mathrm{Amb}_{\mathrm{BJ}}(\psi, \phi) = \mathrm{Amb}(\psi, \phi)\Theta \qquad (44)$$

$$\mathrm{Wig}_{\mathrm{BJ}}(\psi, \phi) = F_\sigma \mathrm{Amb}_{\mathrm{BJ}}(\psi, \phi) \qquad (45)$$

where $\Theta_\sigma = F_\sigma\Theta$ is the symplectic Fourier transform of Θ.

Proof. In view of formula (19) we have

$$(A_{\mathrm{BJ}}\psi|\phi)_{L^2} = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} a_\sigma(z)\Theta(z)(\widehat{T}(z)\psi|\phi)_{L^2}dz.$$

Now, a straightforward calculation [15, §9.1.1] using the explicit expression for the Heisenberg–Weyl operator $\widehat{T}(z_0)$ shows that

$$(\widehat{T}(z)\psi|\phi)_{L^2} = (2\pi\hbar)^n \operatorname{Amb}(\psi,\phi)(-z)$$

so that

$$(A_{\mathrm{BJ}}\psi|\phi)_{L^2} = \int_{\mathbb{R}^{2n}} a_\sigma(z)\Theta(z)\operatorname{Amb}(\psi,\phi)(-z)dz$$

hence formula (42) since $\Theta(-z) = \Theta(z)$. By the second equality in Plancherel's formula (41), we have, since F_σ is involutive,

$$\begin{aligned}
\langle\langle a_\sigma, \operatorname{Amb}_{\mathrm{BJ}}(\phi,\psi)^\vee\rangle\rangle &= \langle\langle a, F_\sigma(\operatorname{Amb}_{\mathrm{BJ}}(\psi,\phi)^\vee))^\vee\rangle\rangle \\
&= \langle\langle a, F_\sigma(\operatorname{Amb}_{\mathrm{BJ}}(\psi,\phi)))\rangle\rangle \\
&= \langle\langle a, \operatorname{Wig}_{\mathrm{BJ}}(\psi,\phi)\rangle\rangle
\end{aligned}$$

which is formula (43). $\square$

The symplectic Fourier transform satisfying the convolution formula $F_\sigma u * F_\sigma v = (2\pi\hbar)^n F_\sigma(uv)$ we have

$$\operatorname{Wig}_{\mathrm{BJ}}(\psi,\phi) = \left(\tfrac{1}{2\pi\hbar}\right)^n F_\sigma \operatorname{Amb}(\psi,\phi) * F_\sigma\Theta$$

hence the explicit formula

$$\operatorname{Wig}_{\mathrm{BJ}}(\psi,\phi) = \left(\tfrac{1}{2\pi\hbar}\right)^n \operatorname{Wig}(\psi,\phi) * \Theta_\sigma. \tag{46}$$

Remark 8. Due to formula (46) the modified Wigner function $\operatorname{Wig}_{\mathrm{BJ}}\psi$ is an element of the Cohen class (Cohen [12,13], Gröchenig [24]); as such it can be viewed as a probability quasi-distribution having a similar status as that of the usual Wigner function (it has, for instance the "right" marginal properties): for $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \operatorname{Wig}_{\mathrm{BJ}}\psi(x,p)dp = |\psi(x)|^2$$

$$\int_{\mathbb{R}^n} \operatorname{Wig}_{\mathrm{BJ}}\psi(x,p)dx = |\widehat{\psi}(p)|^2.$$

The symbol of Born–Jordan operators are obtained by averaging the τ-symbol over $[0,1]$ (formula (12)). A similar procedure holds for $\operatorname{Wig}_{\mathrm{BJ}}$: for $\tau \in \mathbb{R}$ and $\psi,\phi \in \mathcal{S}(\mathbb{R}^n)$ define the τ-cross-Wigner transform

$$\operatorname{Wig}_\tau(\psi,\phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}py}\psi(x+\tau y)\overline{\phi(x-(1-\tau)y)}dy. \tag{47}$$

One proves (Boggiatto et al. [3]) that Wig_τ belongs to the Cohen class; in fact:

$$\operatorname{Wig}_\tau(\psi,\phi) = \operatorname{Wig}(\psi,\phi) * F_\sigma\Theta_{(\tau)} \tag{48}$$

where $\Theta_{(\tau)}$ is the function

$$\Theta_{(\tau)}(z) = \frac{2^n}{|2\tau-1|^n} \exp\left(\frac{i}{\hbar}\frac{2px}{2\tau-1}\right). \tag{49}$$

We have [5, 16]

$$\mathrm{Wig}_{\mathrm{BJ}}(\psi, \phi) = \int_0^1 \mathrm{Wig}_\tau(\psi, \phi)d\tau. \tag{50}$$

As the usual cross-Wigner transform, Wig_τ satisfies a Moyal identity (or "orthogonality relation" as it is sometimes called): Boggiatto et al. [5] have shown that

$$((\mathrm{Wig}_\tau(\psi, \phi)|\,\mathrm{Wig}_\tau(\psi', \phi')))_{L^2} = \left(\tfrac{1}{2\pi\hbar}\right)^n (\psi|\psi')_{L^2}(\phi|\phi')_{L^2} \tag{51}$$

for every $\tau \in \mathbb{R}$ and for all functions ψ, ψ', ϕ, ϕ' in $L^2(\mathbb{R}^n)$. However, the Moyal identity does *not* hold for $\mathrm{Wig}_{\mathrm{BJ}}$. Here is why: let $Q(\psi, \phi) = \mathrm{Wig}(\psi, \phi) * \theta$ ($\theta \in \mathcal{S}'(\mathbb{R}^n)$) be an element of the Cohen class. The Moyal identity is satisfied if and only if the Fourier transform $\widehat{\theta}$ of the Cohen kernel θ satisfies $|\widehat{\theta}(z)| = (2\pi\hbar)^n$ (Cohen [12, 13]). In the Born–Jordan case the Fourier transform of the Cohen kernel is the function $\Theta(z) = \mathrm{sinc}(px/2\pi\hbar)$ which does not satisfy this condition.

4. Intertwiners

We are going to show that the usual Born–Jordan operator $A_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$ and the corresponding BBJ operator $\widetilde{A}_{\mathrm{BJ}} = \widetilde{\mathrm{Op}_{\mathrm{BJ}}}(a)$ are intertwined by a family of linear mappings $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$. This important result will allow us to study the regularity and spectral properties of the BBJ operators.

Definition 9. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $||\phi||_{L^2} = 1$ we denote by U_ϕ and $U_{\phi,(\tau)}$ the linear operators $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ defined, by

$$U_\phi \psi = (2\pi\hbar)^{n/2} \mathrm{Wig}_{\mathrm{BJ}}(\psi, \phi) \tag{52}$$

and

$$U_{\phi,(\tau)} \psi = (2\pi\hbar)^{n/2} \mathrm{Wig}_\tau(\psi, \phi).$$

We will call U_ϕ and $U_{\phi,(\tau)}$ the Born–Jordan and τ-intertwiner, respectively

The reason for this terminology will become clear in a moment.

4.1. The Intertwining Property

Recall that we defined (formula (26)) the unitary operator $\widetilde{T}(z_0) : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^{2n})$ by

$$\widetilde{T}(z_0)\psi(z) = e^{-\frac{i}{\hbar}\sigma(z,z_0)}\psi\big(z - \tfrac{1}{2}z_0\big).$$

We will need the following property of the cross-Wigner transform:

Lemma 10. *We have*

$$\mathrm{Wig}(\widehat{T}(z_0)\psi, \phi) = \widetilde{T}(z_0)\,\mathrm{Wig}(\psi, \phi). \tag{53}$$

Proof. The cross-Wigner transform has the following well-known translational property ([23], [15], §9.2.2): for all $z_0, z_1 \in \mathbb{R}^{2n}$

$$\mathrm{Wig}(\widehat{T}(z_0)\psi, \widehat{T}(z_1)\phi)(z) = e^{-\frac{i}{\hbar}\gamma(z, z_0, z_1)} \mathrm{Wig}(\psi, \phi)\big(z - \tfrac{1}{2}(z_0 + z_1)z\big)$$

where the phase γ is given by

$$\gamma(z, z_0, z_1) = \sigma(z, z_0 - z_1) + \tfrac{1}{2}\sigma(z_0, z_1).$$

Taking $z_1 = 0$ yields

$$\mathrm{Wig}(\widehat{T}(z_0)\psi, \phi)(z) = e^{-\frac{i}{\hbar}\sigma(z, z_0)} \mathrm{Wig}(\psi, \phi)\big(z - \tfrac{1}{2}z_0\big)$$

which is (53). □

The interest of the definition of the mapping U_ϕ comes from their intertwining properties:

Proposition 11. *Let* $\widetilde{A}_{\mathrm{BJ}} = \widetilde{\mathrm{Op}_{\mathrm{BJ}}(a)}$ *and* $A_{\mathrm{BJ}} = \mathrm{Op}_{\mathrm{BJ}}(a)$. *The following intertwining properties*

$$\widetilde{A}_{\mathrm{BJ}} U_\phi = U_\phi A_{\mathrm{BJ}}, \quad U_\phi^* \widetilde{A}_{\mathrm{BJ}} = A_{\mathrm{BJ}} U_\phi^* \tag{54}$$

hold for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$. In view of formula (36) we have

$$\widetilde{A}_{\mathrm{BJ}}\Psi(z) = \big(\tfrac{1}{2\pi\hbar}\big)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0)\Theta(z_0)\widetilde{T}(z_0)\Psi(z)dz$$

and hence, for $\Psi = U_\phi\psi$,

$$\widetilde{A}_{\mathrm{BJ}} U_\phi\psi(z) = \big(\tfrac{1}{2\pi\hbar}\big)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0)\Theta(z_0)\widetilde{T}(z_0)(U_\phi\psi)(z)dz.$$

We have

$$\widetilde{T}(z_0)(U_\phi\psi)(z) = (2\pi\hbar)^{n/2}\widetilde{T}(z_0)\,\mathrm{Wig}_{\mathrm{BJ}}(\psi, \phi)(z)$$
$$= (2\pi\hbar)^{n/2}\widetilde{T}(z_0)(\mathrm{Wig}(\psi, \phi) * \Theta_\sigma)(z).$$

In view of formula (53) we have

$$\widetilde{T}(z_0)(\mathrm{Wig}(\psi, \phi) * \Theta_\sigma)(z) = \int_{\mathbb{R}^{2n}} \widetilde{T}(z_0)\,\mathrm{Wig}(\psi, \phi)(z - u)\Theta_\sigma(u)du$$
$$= \int_{\mathbb{R}^{2n}} \mathrm{Wig}(\widehat{T}(z_0)\psi, \phi)(z - u)\Theta_\sigma(u)du$$
$$= \mathrm{Wig}_{\mathrm{BJ}}(\widehat{T}(z_0)\psi, \phi)$$

and hence

$$\widetilde{A}_{\mathrm{BJ}} U_\phi\psi(z) = \big(\tfrac{1}{2\pi\hbar}\big)^n \int_{\mathbb{R}^{2n}} a_\sigma(z_0)\Theta(z_0)(U_\phi\widehat{T}(z_0)\psi)(z)dz$$
$$= U_\phi A_{\mathrm{BJ}}\psi(z)$$

which proves the first formula (54). The second formula follows from the equalities

$$U_\phi^* \widetilde{A}_{\mathrm{BJ}} = (\widetilde{A}_{\mathrm{BJ}}^* U_\phi)^* = (U_\phi A_{\mathrm{BJ}}^*)^* = A_{\mathrm{BJ}} U_\phi^*. \quad □$$

4.2. Properties of Intertwiners

We begin by considering the τ-intertwiners.

Proposition 12. (i) *The τ-intertwiner $U_{\phi,(\tau)}$ is a linear isometry of $L^2(\mathbb{R}^n)$ on a closed subspace $\mathcal{H}_{\phi,(\tau)}$ of $L^2(\mathbb{R}^{2n})$. (ii) The adjoint $U^*_{\phi,(\tau)}$ is given by the formula*

$$U^*_{\phi,(\tau)}\psi(y) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2}\int_{\mathbb{R}^{2n}} e^{-\frac{2i}{\hbar}p(y-x)}\overline{\phi(2x-y)}(\psi * F_\sigma\Theta_{(\tau)})(x,p)dpdx \quad (55)$$

where $\Theta_{(\tau)}$ is defined by (49).

Proof. (i) Taking $\phi = \phi'$ with $||\phi|| = 1$ in Moyal's formula (51) we have

$$((U_{\phi,(\tau)}\psi|U_{\phi,(\tau)}\psi'))_{L^2} = (\psi|\psi') \quad (56)$$

hence $U_{\phi,(\tau)}$ is an isometry. By definition of the adjoint we have

$$((U_{\phi,(\tau)}\psi|\psi))_{L^2} = (\psi|U^*_{\phi,(\tau)}\psi)_{L^2}.$$

Set $P_{\phi,(\tau)} = U_{\phi,(\tau)}U^*_{\phi,(\tau)}$; we have $P^*_{\phi,(\tau)} = P_{\phi,(\tau)}$ and $P_\phi P^*_{\phi,(\tau)} = U_{\phi,(\tau)}U^*_{\phi,(\tau)} = P_{\phi,(\tau)}$ because $U^*_{\phi,(\tau)}U_{\phi,(\tau)}$ is the identity on $L^2(\mathbb{R}^n)$. It follows that $P_{\phi,(\tau)}$ is the orthogonal projection on $\mathcal{H}_{\phi,(\tau)}$; since the range of a projection is closed, so is $\mathcal{H}_{\phi,(\tau)}$. (ii) By definition of $U_{\phi,(\tau)}$ we have

$$((U_{\phi,(\tau)}\psi|\Psi))_{L^2} = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2}((\text{Wig}(\psi,\phi) * F_\sigma\Theta_{(\tau)}|\Psi))_{L^2}. \quad (57)$$

(cf. formula (46)). Recalling the classical formula $(f * g|h) = (f|g^\vee * h)$ and noting that $\overline{F_\sigma\Theta^\vee_{(\tau)}} = F_\sigma\Theta_{(\tau)}$, the formula above becomes

$$((U_{\phi,(\tau)}\psi|\Psi))_{L^2} = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2}((\text{Wig}(\psi,\phi)|\Psi * F_\sigma\Theta_{(\tau)}))_{L^2}.$$

Taking the definition (38) of $\text{Wig}(\psi,\phi)$ into account, we get

$$((U_{\phi,(\tau)}\psi|\Psi))_{L^2} = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2}\int_{\mathbb{R}^n}\psi(x+\tfrac{1}{2}y)$$
$$\times\left(\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}py}\overline{\phi(x-\tfrac{1}{2}y)}(\Psi * F_\sigma\Theta_{(\tau)})(x,p)dpdx\right)dy$$

that is, setting $u = x + \tfrac{1}{2}y$,

$$((U_{\phi,(\tau)}\psi|\Psi))_{L^2} = \left(\tfrac{2}{\pi\hbar}\right)^{n/2}\int_{\mathbb{R}^n}\psi(u)$$
$$\times\left(\int_{\mathbb{R}^{2n}} e^{-\frac{2i}{\hbar}p(u-x)}\overline{\phi(2x-u)}(\Psi * F_\sigma\Theta_{(\tau)})(x,p)dpdx\right)du$$

hence

$$U^*_{\phi,(\tau)}\Psi(u) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2}\int_{\mathbb{R}^{2n}} e^{-\frac{2i}{\hbar}p(u-x)}\overline{\phi(2x-u)}(\Psi * F_\sigma\Theta_{(\tau)})(x,p)dpdx$$

which is formula (55). $\qquad\qquad\square$

We would now like to extend this result to the intertwiners U_ϕ. However, the proof of part (i) of Proposition 12 relies on the Moyal identity (51), since the latter allows to derive (56). However, as we have remarked above, the Moyal identity does not hold for the transform $\mathrm{Wig}_{\mathrm{BJ}}(\psi, \phi)$. We must thus expect a somewhat weaker result. We will need the following lemma, which is a kind of interpolation result:

Lemma 13. *Let τ and τ' be two real numbers and two windows ϕ and ϕ'. There exists a constant $C_{\phi,\phi'} > 0$ such that*

$$|((U_{\phi,(\tau)}\psi|U_{\phi',(\tau')}\psi'))_{L^2}| \le C_{\phi,\phi'}||\psi|| \, ||\psi'|| \tag{58}$$

for all $(\psi, \psi') \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Proof. This amounts to establishing the existence of a constant $C_{\phi,\phi'} > 0$ such that

$$|((\mathrm{Wig}_\tau(\psi,\phi)|\, \mathrm{Wig}_{\tau'}(\psi',\phi')))_{L^2}| \le (2\pi\hbar)^{-n}C_{\phi,\phi'}||\psi|| \, ||\psi'||.$$

Using Cauchy–Schwarz's inequality we have

$$|((\mathrm{Wig}_\tau(\psi,\phi)|\, \mathrm{Wig}_{\tau'}(\psi',\phi')))_{L^2}| \le || \, \mathrm{Wig}_\tau(\psi,\phi)|| \, || \, \mathrm{Wig}_{\tau'}(\psi',\phi')||.$$

Applying Moyal's identity to the terms in the right-hand side we have

$$|| \, \mathrm{Wig}_\tau(\psi,\phi)|| = \left(\tfrac{1}{2\pi\hbar}\right)^n ||\psi|| \, ||\phi||$$
$$|| \, \mathrm{Wig}_{\tau'}(\psi',\phi')|| = \left(\tfrac{1}{2\pi\hbar}\right)^n ||\psi'|| \, ||\phi'||$$

hence the inequality (58) with $C_{\phi,\phi'} = ||\phi|| \, ||\phi'||$. $\qquad\square$

Let us now prove the analogue of Proposition 12 for the Born–Jordan intertwiners:

Proposition 14. (i) *The Born–Jordan intertwiner U_ϕ is a continuous linear mapping $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$. (ii) The adjoint U_ϕ^* is given by the formula*

$$U_\phi^*\Psi(y) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^{2n}} e^{-\frac{2i}{\hbar}p(y-x)}\overline{\phi(2x-y)}(\Psi * F_\sigma\Theta)(x,p)dpdx. \tag{59}$$

(iii) *Let $(\phi_j)_{j\in F}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ and set $\Phi_{jk} = U_{\phi_j}\phi_k$. The system $(\Phi_{jk})_{(j,k)\in F\times F}$ spans $L^2(\mathbb{R}^{2n})$.*

Proof. (i) We have

$$((U_\phi\psi|U_\phi\psi'))_{L^2} = \left(\left(\int_0^1 U_{\phi,(\tau)}\psi d\tau \middle| \int_0^1 U_{\phi,(\tau')}\psi d\tau'\right)\right)_{L^2}$$
$$= \int_{[0,1]\times[0,1]} ((U_{\phi,(\tau)}\psi|U_{\phi,(\tau')}\psi))_{L^2}.$$

In view of formula (58) we thus have

$$|((U_\phi\psi|U_\phi\psi'))_{L^2}| \le C_\phi||\psi|| \, ||\psi'||$$

where $C_\phi = C_{\phi,\phi}$, which proves the continuity of U_ϕ. (ii) The proof of formula (59) is similar to the proof of (55) in Proposition 12. (iii) We have to show that if $((\Phi_{jk}|\Psi))_{L^2} = 0$ for all $(j,k) \in F \times F$ then $\Psi = 0$ almost everywhere.

We have $((\Psi|\Phi))_{L^2} = (U^*_{\phi_j}\Psi|\Phi_k)_{L^2}$ hence $((\Psi|\Phi_{jk}))_{L^2} = 0$ is equivalent to $(U^*_{\phi_j}\Psi|\phi_k)_{L^2} = 0$; since this equality holds for all $k \in F$ it follows that $U^*_{\phi_j}\Psi = 0$ (for all $j \in F$). We have

$$((\Phi_{jk}|\Psi))_{L^2} = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2} ((\mathrm{Wig}(\psi_k,\phi_j) * \Theta_\sigma|\Psi))_{L^2}$$
$$= \left(\tfrac{1}{2\pi\hbar}\right)^{n/2} ((\mathrm{Wig}(\psi_k,\phi_j)|\Psi * \Theta_\sigma))_{L^2}.$$

The family of functions $(\mathrm{Wig}(\psi_k,\phi_j))_{(j,k)\in F\times F}$ being an orthonormal basis of $L^2(\mathbb{R}^{2n})$ (de Gosson and Luef [17], Lemma 3), it follows that $\Psi * \Theta_\sigma = 0$, and hence

$$F_\sigma(\Psi * \Theta_\sigma) = (2\pi\hbar)^{-n} (F_\sigma\Psi)\Theta = 0.$$

Since the set of zeroes of the function Θ is the union of the null sets $\{z\colon px = 2N\pi\hbar\}$ $(N \in \mathbb{Z})$ we have $F_\sigma\Psi = 0$ a.e. and hence $\Psi = 0$ a.e., which was to be proven. $\qquad\square$

5. Functional and Symbol Spaces

5.1. The Modulation Spaces $M^q_s(\mathbb{R}^n)$

The theory of modulation spaces goes back to Feichtinger [20,21]; for a detailed exposition see Gröchenig [24]. The traditional definition of these functional spaces makes use of the short-time Fourier transform (or Gabor transform) familiar from time-frequency analysis; we will replace the latter by the cross-Wigner transform whose symplectic symmetries are more visible; that both definitions are equivalent was proven in de Gosson and Luef [18] and de Gosson [15].

We will use the notation $\langle z \rangle^s = (1 + |z|^2)^{s/2}$ for $z \in \mathbb{R}^{2n}$; here s is any nonnegative real number. It follows from Peetre's inequality that the function $z \longmapsto \langle z \rangle^s$ is submultiplicative:

$$\langle z + z\prime \rangle^s \leq 2^s \langle z \rangle^s \langle z' \rangle^s. \tag{60}$$

Let q be a real number ≥ 1, or ∞. We denote by $L^q_s(\mathbb{R}^{2n})$ the space of all Lebesgue-measurable functions Ψ on $\mathbb{R}^{2n}$ such that $\langle\cdot\rangle^s\Psi \in L^q_s(\mathbb{R}^{2n})$. When $q < \infty$ the formula

$$||\psi||_{L^q_s} = \left(\int_{\mathbb{R}^{2n}} |\langle z \rangle\Psi(z)|^q dz\right)^{1/q}$$

defines a norm on $L^q_s(\mathbb{R}^{2n})$; when $q = \infty$ we set

$$||\Psi||_{L^\infty_s} = \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2n}}|\langle z\rangle^s\Psi(z)|.$$

Let now ϕ be a fixed element of $\mathcal{S}(\mathbb{R}^n)$, hereafter to be called a "window". For $q < \infty$ the modulation space $M^q_s(\mathbb{R}^n)$ is the vector space consisting of all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathrm{Wig}(\psi,\phi) \in L^q_s(\mathbb{R}^{2n})$ where by (38) is the cross-Wigner transform [14,15]; equivalently

$$||\psi||^\phi_{M^q_s} = \left(\int_{\mathbb{R}^{2n}} |\langle z \rangle^s \,\mathrm{Wig}(\psi,\phi)(z)|^q dz\right)^{1/q} < \infty. \tag{61}$$

The space $M_s^\infty(\mathbb{R}^n)$ is similarly defined by

$$||\psi||_{M_s^\infty}^\phi = \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2n}} |\langle z\rangle^s \operatorname{Wig}(\psi,\phi)(z)| < \infty. \tag{62}$$

One shows that in both cases the definitions are independent of the choice of the window ϕ, and that the $||\cdot||_{M_s^q}^\phi$ $(1\leq q\leq\infty)$ form a family of equivalent norms on $M_s^q(\mathbb{R}^n)$, which becomes a Banach space for the topology thus defined; in addition $M_s^q(\mathbb{R}^n)$ contains $\mathcal{S}(\mathbb{R}^n)$ as dense subspace.

The class of modulation spaces $M_s^q(\mathbb{R}^n)$ contain as particular cases many of the classical function spaces. For instance

$$M_s^2(\mathbb{R}^n) = L_s^2(\mathbb{R}^{2n}) \cap H^s(\mathbb{R}^{2n})$$

which is the Sobolev-like space $Q^s(\mathbb{R}^{2n})$ studied by Shubin [34], p. 45. We also have

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s\geq 0} M_s^2(\mathbb{R}^n).$$

A particularly interesting example of modulation space is obtained by choosing $q=1$ and $s=0$; the corresponding space $M_0^1(\mathbb{R}^n)$ is often denoted by $S_0(\mathbb{R}^n)$, and is called the *Feichtinger algebra* [21] (it is an algebra both for pointwise product and for convolution). We have the inclusions

$$\mathcal{S}(\mathbb{R}^n) \subset S_0(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \tag{63}$$

5.2. Metaplectic and Heisenberg–Weyl Invariance Properties

Recall that the Wigner transform and the Heisenberg–Weyl operators satisfy

$$\operatorname{Wig}(\widehat{T}(z_0)\psi, \widehat{T}(z_0)\phi)(z) = \operatorname{Wig}(\psi,\phi)(z-z_0). \tag{64}$$

for all $\psi,\phi \in \mathcal{S}'(\mathbb{R}^n)$. Let $(\mathbb{R}^{2n},\sigma)$ be the standard symplectic space. We denote by $\operatorname{Sp}(n)$ be the symplectic group of $(\mathbb{R}^{2n},\sigma)$: we have $S\in\operatorname{Sp}(n)$ if and only if S is a linear automorphism of $\mathbb{R}^{2n}$ such that $S^*\sigma = \sigma$. Equivalently, $S^T J S = S J S^T = J$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. The symplectic group has a unique (connected) covering group of order two; the latter has a true representation as a group $\operatorname{Mp}(n)$ of unitary operators on $L^2(\mathbb{R}^n)$; this group is called the metaplectic group. The covering projection $\Pi\colon \operatorname{Mp}(n) \longrightarrow \operatorname{Sp}(n)$ is uniquely defined up to inner automorphisms; we calibrate this projection so that we have $\Pi(\widehat{J}) = J$ where $\widehat{J} \in \operatorname{Mp}(n)$ is the modified Fourier transform defined by

$$\widehat{J}\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^{n/2} e^{-in\pi/4} \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}xx'} \psi(x')dx'.$$

(we refer to [14,15,23] for detailed studies of the metaplectic representation).

The modulation spaces $M_s^q(\mathbb{R}^n)$ have remarkable invariance properties:

Proposition 15. (i) *Each space $M_s^q(\mathbb{R}^n)$ is invariant under the action of the Heisenberg–Weyl operators $\widehat{T}(z)$; in fact there exists a constant $C > 0$ such that*

$$||\widehat{T}(z)\psi||_{M_s^q}^\phi \leq C\langle z\rangle^s ||\psi||_{M_s^q}^\phi. \tag{65}$$

(ii) *For* $1 \leq q < \infty$ *the space* $M_s^q(\mathbb{R}^n)$ *is invariant under the action of the metaplectic group* $\mathrm{Mp}(n)$: *if* $\widehat{S} \in \mathrm{Mp}(n)$ *then* $\widehat{S}\psi \in M_s^q(\mathbb{R}^n)$ *if and only if* $\psi \in M_s^q(\mathbb{R}^n)$. *In particular* $M_s^q(\mathbb{R}^n)$ *is invariant under the Fourier transform.*

(See de Gosson and Luef [18], Gröchenig [24]).

A remarkable property of the Feichtinger algebra is that it is the smallest Banach space invariant under the action of the Heisenberg–Weyl operators (16) and of the metaplectic group.

5.3. The Sjöstrand Symbol Classes

In [35,36] Sjöstrand introduced a class $M^{\infty,1}(\mathbb{R}^{2n})$ of general pseudodifferential symbols; Gröchenig [25] showed that this class is identical to the weighted modulation space $M_s^{\infty,1}(\mathbb{R}^{2n})$ when $s = 0$.

5.4. Definition and Main Properties

Let us set, for $s \geq 0$,

$$\langle\langle z, \zeta \rangle\rangle^s = (1 + |z|^2 + |\zeta|^2)^{s/2}. \tag{66}$$

By definition, $M_s^{\infty,1}(\mathbb{R}^{2n})$ consists of all $a \in \mathcal{S}'(\mathbb{R}^{2n})$ such that there exists a function $\Phi \in \mathcal{S}(\mathbb{R}^{2n})$ for which

$$\int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} |\widetilde{\mathrm{Wig}}(a, \Phi)(z, \zeta)| \langle\langle z, \zeta \rangle\rangle^s d\zeta < \infty \tag{67}$$

where $\widetilde{\mathrm{Wig}}(a, \Phi)$ is the cross-Wigner transform on $\mathbb{R}^{2n}$:

$$\widetilde{\mathrm{Wig}}(a, \Phi)(z, \zeta) = \left(\tfrac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta z'} a(z + \tfrac{1}{2}z') \overline{\Phi(z - \tfrac{1}{2}z')} dz'. \tag{68}$$

When $s = 0$ one obtains the Sjöstrand class: $M^{\infty,1}(\mathbb{R}^{2n}) = M_0^{\infty,1}(\mathbb{R}^{2n})$. It is easy to check that for every window $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ the formula

$$\|a\|_{M_s^{\infty,1}}^{\Phi} = \int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} [|\widetilde{\mathrm{Wig}}(a, \Phi)(z, \zeta)| \langle\langle z, \zeta \rangle\rangle^s] d\zeta < \infty \tag{69}$$

defines a norm on $M_s^{\infty,1}(\mathbb{R}^{2n})$. As for the modulation spaces $M_s^q(\mathbb{R}^n)$ condition (69) is independent of the choice of window Φ, and when Φ runs through $\mathcal{S}(\mathbb{R}^{2n})$ the functions $\|\cdot\|_{M_s^{\infty,1}}^{\Phi}$ form a family of equivalent norms on $M_s^{\infty,1}(\mathbb{R}^{2n})$. It turns out that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach space for the topology defined by any of these norms; moreover the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ is dense in $M_s^{\infty,1}(\mathbb{R}^{2n})$.

The Sjöstrand classes $M_s^{\infty,1}(\mathbb{R}^{2n})$ contain many of the usual pseudodifferential symbol classes and we have the inclusion

$$C_b^{2k+1}(\mathbb{R}^{2n}) \subset M_0^{\infty,1}(\mathbb{R}^{2n}) \tag{70}$$

where $C_b^{2k+1}(\mathbb{R}^{2n})$ is the vector space of all functions which are differentiable up to order $2n + 1$ with bounded derivatives. In fact, for every window Φ there exists a constant $C_\Phi > 0$ such that

$$\|a\|_{M_s^{\infty,1}}^{\Phi} \leq C_\Phi \|a\|_{C^{2k+1}} = C_\Phi \sum_{|\alpha| \leq 2k+1} \|\partial_z^\alpha a\|_\infty. \tag{71}$$

We first recall the following result, which says that these space are invariant under linear changes of variables:

Proposition 16. *Let M be a real invertible $2n \times 2n$ matrix. If $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ then $a \circ M \in M_s^{\infty,1}(\mathbb{R}^{2n})$, and there exists a constant $C_M > 0$ such that for every window Φ and every $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ we have*

$$||a \circ M||_{M_s^{\infty,1}}^{\Phi} \leq C_M ||a||_{M_s^{\infty,1}}^{\psi} \tag{72}$$

where $\Psi = \Phi \circ M^{-1}$

For a proof of this result, see Proposition 7 in de Gosson and Luef [18].

We are next going to show that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is invariant under the action of the metaplectic group $\mathrm{Mp}(2n)$. Denoting by $\widetilde{S}$ the generic element of $\mathrm{Mp}(2n)$ we have:

Proposition 17. *Let $\widetilde{S} \in \mathrm{Mp}(2n)$ and $a \in \mathcal{S}'(\mathbb{R}^{2n})$. We have $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ if and only if $\widetilde{S}a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ and we have*

$$||\widetilde{S}a||_{M_s^{\infty,1}}^{\widetilde{S}\Phi} \leq \lambda_{\max}^s ||Sa||_{M_s^{\infty,1}}^{\Phi} \tag{73}$$

where $\lambda_{\max}^s$ is the largest eigenvalue of $S^T S \in \mathrm{Sp}(2n)$, $S = \Pi(\widetilde{S})$.

Proof. Let $S \in \mathrm{Sp}(2n)$ be the projection of $\widetilde{S}$. We have

$$||\widetilde{S}a||_{M_s^{\infty,1}}^{\widetilde{S}\Phi} = \int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} [|\widetilde{\mathrm{Wig}}(\widetilde{S}a, \widetilde{S}\Phi)(z,\zeta)| \langle\langle(z,\zeta)\rangle\rangle^s] d\zeta$$

$$= \int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} [|\widetilde{\mathrm{Wig}}(a, \Phi)(S^{-1}(z,\zeta))| \langle\langle(z,\zeta)\rangle\rangle^s] d\zeta$$

$$= \int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} [|\widetilde{\mathrm{Wig}}(a, \Phi)(z,\zeta))| \langle\langle S(z,\zeta)\rangle\rangle^s] d\zeta.$$

Now $\langle\langle S(z,\zeta)\rangle\rangle \leq \lambda_{\max}\langle\langle z,\zeta\rangle\rangle$ hence

$$||\widetilde{S}a||_{M_s^{\infty,1}}^{\widetilde{S}\Phi} \leq \lambda_{\max}^s \int_{\mathbb{R}^{2n}} \sup_{z \in \mathbb{R}^{2n}} [|\widetilde{\mathrm{Wig}}(a, \Phi)(z,\zeta))| \langle\langle z,\zeta\rangle\rangle^s] d\zeta$$

which is the inequality (73). $\square$

5.5. Regularity Properties

The following result is well-known (see e.g. Gröchenig [25]); it shows that the Weyl correspondence $a \overset{\mathrm{Weyl}}{\longleftrightarrow} A$ is a continuous mapping $M_s^{\infty,1}(\mathbb{R}^{2n}) \longrightarrow M_s^q(\mathbb{R}^n)$:

Proposition 18. *Let $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$. The Weyl operator $A \overset{\mathrm{Weyl}}{\longleftrightarrow} a$ is bounded on $M_s^q(\mathbb{R}^n)$ for every $q \in [1, \infty]$, and there exists a constant $C > 0$ independent of q such that following uniform estimate holds*

$$||A||_{B(M_s^q)} \leq C ||a||_{M_s^{\infty,1}}$$

for all $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ ($|| \cdot ||_{M_s^q}$ is the operator norm on the Banach space $M_s^q(\mathbb{R}^n)$).

The Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2n})$ contains the Hörmander symbol class $S_{0,0}^0(\mathbb{R}^{2n})$ consisting of all $a \in C^\infty(\mathbb{R}^{2n})$ such that for every pair of multi-indices $\alpha, \beta \in N^n$ there exists $C_{\alpha\beta} \geq 0$ such that $|\partial_x^\alpha \partial_p^\beta a(x,p)| \leq C_{\alpha\beta}$. The result above implies as a particular case a Calderón and Vaillancourt [9] type result: if $a \in S_{0,0}^0(\mathbb{R}^{2n})$ then $A \overset{\text{Weyl}}{\longleftrightarrow} a$ is bounded on $L^2(\mathbb{R}^n)$.

For our purposes the following property is very important:

Proposition 19. *Let* $a, b \in M_s^{\infty,1}(\mathbb{R}^{2n})$. *Then* $a \star_\hbar b \in M_s^{\infty,1}(\mathbb{R}^{2n})$. *In particular, for every window of the type* $\Phi = \text{Wig}\,\varphi$ *where* $\varphi \in \mathcal{S}(\mathbb{R}^n)$, *there exists a constant* $C_\Phi > 0$ *such that*

$$||a \star_\hbar b||_{M_s^{\infty,1}}^\Phi \leq C_\Phi ||a||_{M_s^{\infty,1}}^\Phi ||b||_{M_s^{\infty,1}}^\Phi.$$

Since obviously $\overline{a} \in M_s^{\infty,1}(\mathbb{R}^{2n})$ if and only if $a \in M_s^{\infty,1}(\mathbb{R}^{2n})$ the property above can be restated by saying that $M_s^{\infty,1}(\mathbb{R}^{2n})$ is a Banach $*$-algebra with respect to the Moyal product $\star_\hbar$ if and only if $C_\Phi \leq 1$ and the involution $a \longmapsto \overline{a}$.

6. Spectral Properties of the BBJ Operators

Recall that intertwining properties (54) hold more generally:

$$\widetilde{A}_{\text{BJ}} U_\phi = U_\phi A_{\text{BJ}}, \quad U_\phi^* \widetilde{A}_{\text{BJ}} = A_{\text{BJ}} U_\phi^*$$

hold for all ϕ in Feichtinger's algebra $S_0(\mathbb{R}^n)$. Feichtinger has shown that a kernel theorem holds for $S_0(\mathbb{R}^n)$, see [21]. Suppose A_{BJ} is a mapping from $S_0(\mathbb{R}^n)$ to its dual space $S_0'(\mathbb{R}^n)$. Then there exists a $K \in S_0(\mathbb{R}^{2n})$ such that

$$A_{\text{BJ}}\phi(x) = \int K(x,y)\phi(y)dy.$$

In this section we want to discuss generalized eigenvectors and generalized eigenvalues for Bopp Born Jordan operators that map $S_0(\mathbb{R}^n)$ to $S_0'(\mathbb{R}^n)$ based on the Gelfand triple $(S_0(\mathbb{R}^n), L^2(\mathbb{R}^n), S_0'(\mathbb{R}^n))$ as advocated by Feichtinger with various collaborators in a series of papers [10,22] and provide an example of a Banach Gelfand triple. More concretely, we have that $S_0(\mathbb{R}^n)$ is continuously and densely embedded into $L^2(\mathbb{R}^n)$ and the Hilbert space $L^2(\mathbb{R}^n)$ is w^*-continuously and densely embedded into $S_0'(\mathbb{R}^n)$.

Note that the scalar product $(.,.)_{L^2}$ on $L^2(\mathbb{R}^n)$ extends in a natural way to a duality between $S_0(\mathbb{R}^n)$ and $S_0'(\mathbb{R}^n)$.

The usefulness of the Gelfand triple $(S_0(\mathbb{R}^n), L^2(\mathbb{R}^n), S_0'(\mathbb{R}^n))$ lies in the treatment of generalized eigenvectors for operators mapping Feichtinger's algebra into its dual space. As motivation for the notion of generalized eigenvectors we consider the translation operator $T_x f(t) = f(t-x)$ on $S_0(\mathbb{R}^n)$. Then the eigenvectors of T_x are given by the exponentials $\chi_\omega(t) = e^{2\pi i \omega t}$ for the eigenvalues $e^{-2\pi i \omega x}$ for every $\omega \in \mathbb{R}^n$, but the eigenvectors are not in $S_0(\mathbb{R}^n)$. One way to cope with this problem, is to interpret the eigenvalue problem in a weak sense, see Maurin [32]. In our situation we have the following result, see [19]:

Lemma 20. *Suppose A is a self-adjoint operator on $S_0(\mathbb{R}^n)$. Then there exists a complete family of distributions $(\psi_\alpha)_{\alpha \in \mathbb{A}}$ in $S_0'(\mathbb{R}^n)$ (the so-called generalized eigenvectors of A) such that*

$$(\psi_\alpha, A\phi) = \lambda_\alpha(\psi_\alpha, \phi) \quad \text{for each } \phi \in S_0(\mathbb{R}^n)$$

and that there exists a least one ψ_α such that $(\psi_\alpha, \phi) \neq 0$ for each $\phi \in S_0(\mathbb{R}^n)$, and the generalized eigenvalues λ_α of A.

Based on this useful fact we are going to treat spectral properties for Bopp Born Jordan operators. The Banach-Gelfand triple $(S_0(\mathbb{R}^n), L^2(\mathbb{R}^n), S_0'(\mathbb{R}^n))$ provides a convenient setting for extending eigenvalue problems from the Hilbert space setting to a distributional framework.

Proposition 21. *Suppose A_{BJ} is an essentially self-adjoint operator from $S_0(\mathbb{R}^n)$ to $S_0'(\mathbb{R}^n)$, i.e. the symbol of A_{BJ} is real-valued.* (i) *There exists a complete family of generalized eigenvectors $\{\psi_\alpha\}_{\alpha \in \mathbb{A}}$ and generalized eigenvectors $\{\lambda_\alpha\}_{\alpha \in \mathbb{A}}$ for A_{BJ} with ψ_α in $S_0'(\mathbb{R}^n)$ for each $\alpha \in \mathbb{A}$.* (ii) *Furthermore, $\widetilde{A}_{\mathrm{BJ}} \colon S_0(\mathbb{R}^{2n}) \longrightarrow S_0'(\mathbb{R}^{2n})$ has a complete set $\{\psi\}_{\alpha \in \mathbb{A}}$ of generalized eigenvectors with respect to generalized eigenvalues $\{\lambda_\alpha\}_{\alpha \in \mathbb{A}}$.*

Proof. By assumption the operators A_{BJ} has a complete family of eigenvectors ψ_α by the preceding lemma if one considers the operator A_{BJ} on $S_0(\mathbb{R}^n)$ and one extends it to $S_0'(\mathbb{R}^n)$. The correspondence between the eigenvectors of A_{BJ} and $\widetilde{A}_{\mathrm{BJ}}$ follows from the intertwining relations (54) that extend naturally to this setting. If ψ_α is a generalized eigenvector of A_{BJ}, then $\psi_\alpha = U_\phi \psi_\alpha$ is a generalized eigenvector of $\widehat{A}_{\mathrm{BJ}}$ for the same eigenvalue. Suppose on the other hand, that ψ_α is a generalized eigenvector of $\widetilde{A}_{\mathrm{BJ}}$, then $U_\phi^* \psi_\alpha$ is an eigenvector of A_{BJ} corresponding to the same eigenvalue. $\qquad\square$

Acknowledgments

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