Abel's Functional Equation and Eigenvalues of Composition Operators on Spaces of Real Analytic Functions

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Abstract. We obtain full description of eigenvalues and eigenvectors of composition operators $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ for a real analytic self map $\varphi : \mathbb{R} \to \mathbb{R}$ as well as an isomorphic description of corresponding eigenspaces. We completely characterize those φ for which Abel's equation $f \circ \varphi = f + 1$ has a real analytic solution on the real line. We find cases when the operator C_φ has roots using a constructed embedding of φ into the so-called real analytic iteration semigroups.

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1. Introduction and Preliminaries

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-constant real analytic map and let $\mathscr{A}(\mathbb{R})$ be the space of real analytic functions defined on $\mathbb{R}$. Each symbol $\varphi : \mathbb{R} \to \mathbb{R}$ defines a composition operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ by $C_\varphi(f) := f \circ \varphi, f \in \mathscr{A}(\mathbb{R})$. When $\mathscr{A}(\mathbb{R})$ is endowed with its natural locally convex topology (see below), C_φ is a continuous linear operator on $\mathscr{A}(\mathbb{R})$. The first purpose of this article is to determine the eigenvalues and eigenvectors of composition operators $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

Of course, this is just to find a solution f of the equation

$$C_\varphi(f) = \lambda f \quad \text{for } \lambda \in \mathbb{C}. \tag{1}$$

This is a very classical topic. The equation appeared probably for the first time already in 1871 in a paper of Schröder [38] and was partially solved in 1884 in a paper of Königs [33] also for real analytic functions. That is why (1) is often called the Schröder equation. There is an impressive bibliography of the subject (see [34, Chapter 6], [35, Chapter 9] and citations therein or a

survey paper [8, Sec. 8], as well papers on the holomorphic case [18,40,41])
but, in spite of extensive literature inquires, we could not find any known
complete solution of the problem in our setting. The first main result of the
paper provides such a full solution describing also corresponding eigenspaces
(later on we also describe the eigenfunctions). In the rest of the article we
denote $\mathrm{id}\,(x) = x, x \in \mathbb{R}$, and, for a map $\varphi : \mathbb{R} \to \mathbb{R}$, we write $\varphi^{[0]} = \mathrm{id}$ and
$\varphi^{[n]}$ for the n-times composition of φ, $n \in \mathbb{N}$.

Theorem A. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map. Then the map
$C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ has the following set of eigenvalues:*

(a) $\mathbb{C} \setminus \{0\}$ *whenever either $\varphi > \mathrm{id}$ and the set of critical points of φ is
bounded from above or $\varphi < \mathrm{id}$ and the set of critical points is bounded
from below—in this case every eigenspace is topologically isomorphic to
the space $\mathscr{A}(\mathbb{T})$ of real analytic functions on the unit circle $\mathbb{T}$;*

(b) $\{(\varphi'(u))^n : n = 0, 1, 2, \dots\}$ *whenever $\varphi^{[2]}$ has exactly one fixed point u
and one of the following cases hold:*
   - $0 < |\varphi'(u)| < 1,$
   - $1 < |\varphi'(u)|$ *and φ has no critical points*

   *—in this case eigenspaces are one-dimensional of the form $\mathrm{span}\{f^n\}$
where f is an eigenvector of C_φ with the eigenvalue $\varphi'(u)$;*

(c) $\{1, -1\}$ *whenever $\varphi^{[2]} = \mathrm{id} \neq \varphi$—in this case the whole space is a direct
sum of eigenspaces and each of them is isomorphic to the space $\mathscr{A}_+(\mathbb{R})$
of even real analytic functions on $\mathbb{R}$;*

(d) $\{1\}$ *whenever $\varphi = \mathrm{id}$—in this case the whole space $\mathscr{A}(\mathbb{R})$ is an eigenspace;*

(e) $\{1\}$ *in all other cases—the eigenspace is one dimensional and consists
of constant functions.*

*In cases (a) and (b) the closed linear span of all eigenspaces is equal to $\mathscr{A}(\mathbb{R})$
if and only if φ has no critical points.*

The fact that for φ with a fixed point u the eigenvalues λ are powers
of $\varphi'(u)$ was already known to Königs (comp. [32, Satz 3]) who also proved
local existence in the case (b). Since under the assumption of (b) in that case
the attraction basin of the fixed point u is the whole $\mathbb{R}$, then this particular
case above is known (see for instance [34, Th. 6.4, 6.5], comp. [32, Satz 1]).
The case (c) is also known see [34, Th. 6.10]. Part (a) is closely related to the
so-called Abel equation

$$C_\varphi(f) = f + 1. \tag{2}$$

In fact, any solution f to the Abel equation produces a solution g, $g(x) :=
\exp(\log(\lambda)f(x))$ of the Schröder equation for any $\lambda \neq 0$ see [32, p. 57]. The
part (a) of Theorem A above is a consequence of our second main result
characterizing when the Abel equation is solvable in $\mathscr{A}(\mathbb{R})$ (in view of the
above remarks this case contains the main novelty in Theorem A). Note that
the isomorphic classification of eigenspaces as well as the last sentence of
Theorem A seems to be also new.

Theorem B. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map. Then the Abel equation*

$$f(\varphi(x)) = f(x) + 1 \quad \text{for every } x \in \mathbb{R}$$

has a real analytic solution $f : \mathbb{R} \to \mathbb{C}$ if and only if φ has no fixed points and the set of critical points of φ is bounded from above (in case $\varphi > \mathrm{id}$) or from below (in case $\varphi < \mathrm{id}$). Moreover, in that case there is a real analytic solution $f_0 : \mathbb{R} \to \mathbb{R}$ such that its set of critical points is bounded from above (in case $\varphi > \mathrm{id}$) or from below (in case $\varphi < \mathrm{id}$) and for every such f_0 each solution f of the Abel equation is of the form $f(x) = f_0(x) + g \circ f_0(x)$, where $g : \mathbb{R} \to \mathbb{R}$ is an arbitrary 1-periodic real analytic function.

The Abel equation is another classical subject. It was probably mentioned for the first time by Abel [1] in his note published posthumously (comp. also [45]) when a relation between f and f_0 as above was given in the case of strictly increasing solutions of the Abel equation with some additional assumptions on g. There is also a broad literature about the equation in various function classes [34, Chapter 7] or [8, Sec. 9]. There are also recent papers [45, 46], and papers on the holomorphic case, see for instance [15–18]. So far the Abel equation was solved in real analytic functions globally on $\mathbb{R}$ for $\varphi = \exp$ by Kneser (see [32, p. 64], comp. also a series of papers of Walker [48–51] for $\varphi(x) = \exp(x) - 1$ or $\varphi(x) = \exp(bx)$ inspired partially by numerical analysis) and there was a characterization of real analytic diffeomorphisms φ for which the Abel equation is solvable (iff φ has no fixed point, see [10, Main Theorem], comp. [11, Th. 3.6]). In [9, Th. 1.4, Cor. 4.2] it was proved that a necessary condition for real analytic solvability of the Abel equation is that all compact sets $K \subset \mathbb{R}$ are *wandering*, i.e., that there is $\nu \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$, $|n - m| > \nu$ holds $\varphi^{[n]}(K) \cap \varphi^{[m]}(K) = \emptyset$. This condition is strictly weaker than the condition we found.

The motivation of Kneser [32] for solving the Abel equation comes from his problem of finding an iteration root of exp, i.e., of a real analytic function r such that $r^{[2]} = \exp$. It is a folklore that if f is an invertible solution of the Abel or Schröder equations ($\lambda > 0$) then $G(t, x) = f^{-1}(f(x) + t)$ or $G(t, x) = f^{-1}(\lambda^t f(x))$, respectively, is a so-called real analytic iteration semigroup in which φ embeds, i.e., $G : (\mathbb{R}_+ \cup \{0\}) \times \mathbb{R} \to \mathbb{R}$ is real analytic satisfying the following conditions

$$G(t + s, x) = G(t, G(s, x)), \quad G(n, x) = \varphi^{[n]}(x), \quad \text{for } n = 0, 1, \ldots.$$

Clearly, $r(x) = G(1/2, x)$ is the required root. We prove the third main result of the paper:

Theorem C. *A real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$ embeds into a real analytic iteration semigroup whenever φ has no critical points and either φ has no fixed points or $\varphi^{[2]}$ has exactly one fixed point u and $0 < \varphi'(u) \neq 1$. In particular in that case there exist roots of the operator C_φ of arbitrary order.*

The notion of an iteration semigroup appears (and it is extensively studied) in [35, Ch. 9] and its group analogue in [34, Ch. 9] but the concept itself is much older, see for instance, [29, p. 194–195]. So far it was known that every real analytic diffeomorphism φ without fixed points embeds into a real analytic iteration semigroup [11, Th. 2.20], but if real analytic φ has no fixed and critical points, then there are real analytic iteration roots of arbitrary order [11, Th. 2.20]. On the other hand, there is no real analytic iteration

root for $\varphi(x) = \exp(x) - 1$ [34, Th. 15.13], for more information see a series of papers of Baker [2–7] and a papers of Szekeres [42–44]. For iteration roots we also have a broad literature: see [34, Chapter 15], [8, Sec. 2].

The composition operator is definitely one of the most natural linear operators of analysis and there is an extensive literature on that subject: see the monographs in case of spaces of holomorphic functions [19,39] and the papers on a real analytic case [13,14,20–24]. For a literature on the space of real analytic functions see a recent survey [20]. The paper is closely related to questions of real analytic (nonlinear) dynamics of one variable see [11] which in turn is connected with its holomorphic counterpart [37]. For the theory of one-dimensional manifolds we refer to [47] although the results are very classical and known much earlier. For functional analytic tools see [36].

Now, we present the organization of the paper in which we look at the considered problems from the functional analytic or operator theoretical point of view. We discuss in Sect. 2 the case when the self map φ has fixed points. Our main result is Theorem 2.9, that gives a complete picture of the eigenvalues, eigenvectors and eigenspaces of the composition operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ when φ has fixed points. Section 3 studies the case when φ has no fixed points and Theorem 3.11 gives several characterizations of those self maps $\varphi : \mathbb{R} \to \mathbb{R}$ such that the Abel equation (or a specific Schröder equation) has a real analytic solution. The method used is inspired by the method used in [10] (comp. [9]) via the orbit space $\mathbb{R}/\varphi$ of the real analytic map φ. Our approach allows us to give in Sect. 4. Theorem 4.2 a characterization of fixed point free φ which embeds into a real analytic iteration semigroup and give some sufficient conditions for in the case of φ with fixed points (Proposition 4.3). This implies an extension of Kneser's [32] and Belitskii and Tkachenko's [11, Th. 2.20] results about the existence of real analytic functions g such that $g \circ g = \varphi$.

In what follow $\mathbb{N}_0$ denotes the set of natural numbers $\mathbb{N}$ and 0. The open interval in $\mathbb{R}$ with extreme points $a < b$ is denoted by (a, b). Recall that we denote by id $: \mathbb{R} \to \mathbb{R}$ the identity map id $(x) = x$ and by $I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ the identity operator on $\mathscr{A}(\mathbb{R})$. The ball in the complex plane of center z and radius $r > 0$ is denoted by $B(z, r)$. If T is a continuous linear operator on a locally convex space E, its kernel and image are denoted respectively by $\ker T$ and $\operatorname{im} T$. The point spectrum $\sigma_p(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. Elements of $\sigma_p(T)$ are called eigenvalues of T. The spectrum $\sigma(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not a topological isomorphism from E onto E. By the open mapping theorem which works for surjective endomorphisms of $\mathscr{A}(\mathbb{R})$ (see [20]) $\lambda \in \sigma(C_\varphi)$ if and only if $C_\varphi - \lambda I : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is bijective.

A description of the natural topology on $\mathscr{A}(\mathbb{R})$, that goes back to Martineau, is given, for instance, in [25]. The space $\mathscr{A}(\mathbb{R})$ has very good properties: it is nuclear, separable, complete, satisfies the closed graph theorem and the uniform boundedness principle, but surprisingly it has no Schauder basis by [25]. To be precise, the space $\mathscr{A}(\mathbb{R})$ is equipped with the unique locally convex topology such that for any $U \subset \mathbb{C}$ open, $\mathbb{R} \subset U$, the restriction map $R : H(U) \longrightarrow \mathscr{A}(\mathbb{R})$ is continuous and for any compact set $K \subset \mathbb{R}$

the restriction map $r : \mathscr{A}(\mathbb{R}) \longrightarrow H(K)$ is continuous. We endow the space $H(U)$ of holomorphic functions on U with the compact-open topology and the space $H(K)$ of germs of holomorphic functions on K with its natural locally convex inductive limit topology:

$$H(K) = \operatorname*{ind}_{n \in \mathbb{N}} H^{\infty}(U_n),$$

where $(U_n)_{n \in \mathbb{N}}$ is a basis of $\mathbb{C}^d$-neighbourhoods of K. Martineau proved that there is exactly one topology on $\mathscr{A}(\mathbb{R})$ satisfying the condition above. For our purposes, it is important to recall that a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathscr{A}(\mathbb{R})$ tends to f if and only if there is a complex neighbourhood W of $\mathbb{R}$ such that each f_n and f extend to W holomorphically and $f_n \to f$ uniformly on compact subsets of W. The topology of $\mathcal{A}(J)$ for an open interval J in $\mathbb{R}$ is defined analogously. A long survey on spaces of real analytic functions with a very precise description of their topology is contained [20].

The following easy result is included for references in the rest of the paper.

Proposition 1.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-constant real analytic map.*
1. *0 is never an eigenvalue of C_{φ}. In particular, C_{φ} is injective.*
2. *1 is always an eigenvalue of C_{φ} and the constant functions are eigenvectors.*
3. *C_{φ} is surjective if and only if it is bijective if and only if $\varphi : \mathbb{R} \to \mathbb{R}$ is bijective and its inverse is real analytic, i.e. φ is a real analytic diffeomorphism.*
4. *$0 \in \sigma(C_{\varphi})$ if and only if φ is not a real analytic diffeomorphism.*

Proof. (1) If φ is not constant then $\varphi(\mathbb{R})$ is an interval. So if $C_{\varphi}(g) = 0$, then $g|_{\varphi(\mathbb{R})} \equiv 0$ and $g \equiv 0$. Statement (2) is trivial.

(3) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non-constant analytic map. If φ is a real analytic diffeomorphism, it is easy to see that C_{φ} is bijective. If C_{φ} is surjective, there is $f \in \mathscr{A}(\mathbb{R})$ such that $f(\varphi(x)) = x$ for all $x \in \mathbb{R}$. This implies right away that φ is injective in $\mathbb{R}$. As it is continuous, the image $\varphi(\mathbb{R})$ is an interval J and φ is strictly increasing or decreasing with a continuous inverse φ^{-1}. Clearly $f(y) = \varphi^{-1}(y)$ for each $y \in J$. This implies that φ^{-1} is real analytic in J and it is the restriction to J of the real analytic f defined on the whole $\mathbb{R}$. If J is not the whole real line, let a be an extreme of the interval. By continuity, $\varphi^{-1}(y)$ tends to $+\infty$ or $-\infty$, and φ^{-1} cannot be the restriction of $f \in \mathscr{A}(\mathbb{R})$ to J. Summarizing, φ is bijective from $\mathbb{R}$ onto itself, and the inverse is real analytic. Now (4) is a direct consequence of (3). $\square$

We conclude this section recalling when composition operators on $\mathscr{A}(\mathbb{R})$ are open, see [22].

Theorem 1.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map.*
(a) *im C_{φ} is dense in $\mathscr{A}(\mathbb{R})$ if and only if φ has no critical points.*
(b) *The following conditions are equivalent for any non-constant φ:*
- *φ is surjective;*
- *$C_{\varphi} : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is an isomorphism onto its image;*

- im C_φ *is closed.*

Proof. (a) If φ has no critical points, then it is a diffeomorphism onto im $\varphi = (a, b)$, where a could be $-\infty$ and b could be $+\infty$. Hence C_φ is an isomorphism of $\mathscr{A}(a, b)$ onto $\mathscr{A}(\mathbb{R})$. Since polynomials are dense in $\mathscr{A}(a, b)$ then $\mathscr{A}(\mathbb{R})$ is dense in $\mathscr{A}(a, b)$ thus $C_\varphi(\mathscr{A}(\mathbb{R}))$ is dense in $\mathscr{A}(\mathbb{R})$. On the other hand if $\varphi'(x) = 0$, then for any $f \in \mathrm{im}\, C_\varphi$ holds $f'(x) = 0$. Thus im $C_\varphi \subset \ker \delta'_x$, where $\delta'_x \in \mathscr{A}(\mathbb{R})'$, $\delta'_x(g) := g'(x)$.

(b) For every non-constant $\varphi : \mathbb{R} \to \mathbb{R}$ the map $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is injective. If φ is surjective, then it is semi-proper, i.e., for any compact set $K \subset \mathbb{R}$ there is a compact set $L \subset \mathbb{R}$ such that $\varphi(L) = K$. Then the result follows from [22, Th. 3.1]. $\qquad\square$

2. Self Map with Fixed Points

First, we describe precisely eigenvalues and eigenvectors in the case when φ has a fixed point. As it was explained in the introduction much about this case is known. Nevertheless we present the results and some proofs for the convenience of the reader, since the results are scattered in the literature. A description when eigenspaces span densely $\mathscr{A}(\mathbb{R})$ and the isomorphic classification of the eigenspaces seems to be new.

Lemma 2.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$. Then either*

(a) $\varphi^{[2]} = \mathrm{id}$ *or*
(b) *There is a convergent sequence $(x_n)_n$ in $\mathbb{R}$ such that for each $n \neq k$ we have $x_n \neq x_k$ and there is m such that $\varphi^{[m]}(x_n) = x_k$ or $\varphi^{[m]}(x_k) = x_n$.*

Proof. We distinguish several cases depending on the value of $\varphi'(u)$.

Case 1. If $|\varphi'(u)| < 1$, then the fixed point u is attractive and we can find a sequence $(x_n)_n$ of pairwise distinct points converging to u and such that $\varphi(x_n) = x_{n+1}$. Thus (b) in the statement is satisfied.

Case 2. If $|\varphi'(u)| > 1$, then the fixed point u is repelling and there is ε such that $|\varphi'(y)| > 1$ for each $y \in (u - \varepsilon, u + \varepsilon)$. This implies that either $\varphi'(y) > 1$ for each $y \in (u - \varepsilon, u + \varepsilon)$ or $\varphi'(y) < -1$ for each $y \in (u - \varepsilon, u + \varepsilon)$. Select a point $x_1 \in \varphi((u - \varepsilon, u + \varepsilon))$ and define $(x_n)_n$ the sequence of the iterates of x_1 by the inverse φ^{-1}. It is easy to see that the iteration can be accomplished and that (x_n) is a sequence of pairwise different points converging to u for which condition (b) in the statement holds.

Case 3. If $\varphi'(u) = 1$, there are two possible subcases. The first one is that $\varphi'(x) = 1$ for each $x \in \mathbb{R}$. In this case $\varphi(x) = x$ for each $x \in \mathbb{R}$. On the other hand, if we are not in this case, there is a neighbourhood of u in which φ' does not coincide with 1. Otherwise, since φ is real analytic, it would follow $\varphi(x) = x$ for each $x \in \mathbb{R}$. Now we have either $\varphi'(y) > 1$ for each $y \in (u, u + \varepsilon)$ or $0 < \varphi'(y) < 1$ for each $y \in (u, u + \varepsilon)$ for some $\varepsilon > 0$, and similarly at the other side $(u - \varepsilon, u)$. Proceeding as in cases 1 and 2 in the side where we have the inequality, we can find the desired sequence (x_n) for which statement (b) holds.

Case 4. In case $\varphi'(u) = -1$, then $(\varphi^{[2]})'(u) = 1$, and we can proceed as in Case 3. $\square$

Let us observe that the map $f \mapsto (x \mapsto x f(x))$ is an isomorphism of the space $\mathscr{A}_+(\mathbb{R})$ of even real analytic functions onto the space $\mathscr{A}_-(\mathbb{R})$ of odd real analytic functions.

Proposition 2.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$. The following holds:*

(i) *If $\varphi = \mathrm{id}$, then $C_\varphi = I$ on $\mathscr{A}(\mathbb{R})$ and 1 is the only eigenvalue of C_φ.*

(ii) *If $\varphi^{[2]} = \mathrm{id}$ but $\varphi \neq \mathrm{id}$, then C_φ has only two eigenvalues 1 and -1 and $\varphi'(u) = -1$. In this case each $f \in \mathscr{A}(\mathbb{R})$ can be decomposed as $f = f_1 + f_2$, where f_1 (resp. f_2) is an eigenvalue with eigenvector 1 (resp. -1). In this case both eigenspaces are isomorphic to the space of even real analytic functions $\mathscr{A}_+(\mathbb{R})$ on $\mathbb{R}$.*

(iii) *If $\varphi^{[2]} \neq \mathrm{id}$, then 1 is an eigenvalue of C_φ with one dimensional eigenspace consisting of constant functions.*

Proof. The proof of part (i) is trivial. In (ii), it is enough to set $f_1(x) := (f(x) + f(\varphi(x)))/2$ and $f_2(x) := (f(x) - f(\varphi(x)))/2$, $x \in \mathbb{R}$. It is clear that the decomposition is unique and using this decomposition one proves that no $\lambda \in \mathbb{C} \setminus \{1, -1\}$ is an eigenvalue.

In order to prove that $\varphi'(u) = -1$ in part (ii), we assume without loss of generality that $u = 0$. We observe that the graph of φ has to be symmetric with respect to the line $y = x$. Moreover, differentiating the equation $\varphi(\varphi(x)) = x$ we get $\varphi'(u) \cdot \varphi'(u) = 1$, so $\varphi'(u) = 1$ or $\varphi'(u) = -1$. In the first case, since φ is not the identity map but increasing around u, there is a neighbourhood of u where $\varphi(x)$ never takes value x except for u. For instance, for some $\varepsilon > 0$ and every $x \in (-\varepsilon, 0)$ we have $x < \varphi(x) < 0$. Such a function cannot be symmetric with respect to the line $y = x$. A similar proof works for the case $\varphi(x) < x$. We have proved that $\varphi'(u) = -1$.

Since $\varphi^{[2]} = \mathrm{id}$, φ is a real analytic diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ so it has no critical points. Thus for every $x \in \mathbb{R}$ we have $\varphi'(x) < 0$, in particular, φ has the only one fixed point u and $\lim_{x \to +\infty} \varphi(x) = -\infty$ and $\lim_{x \to -\infty} \varphi(x) = +\infty$. Define

$$\psi : \mathbb{R} \to \mathbb{R}, \quad \psi(x) := \frac{x - \varphi(x)}{2}.$$

The map ψ is strictly increasing, it has no critical points and it is surjective since

$$\psi'(x) = \frac{1}{2}(1 - \varphi'(x)) > 0, \quad \lim_{x \to +\infty} \psi(x) = +\infty, \quad \lim_{x \to -\infty} \psi(x) = -\infty.$$

Hence C_ψ is an isomorphism of $\mathscr{A}(\mathbb{R})$.

Let $f \in \mathscr{A}(\mathbb{R})$ be even then

$$C_\psi(f) \circ \varphi = f\left(\frac{\varphi(x) - x}{2}\right) = f\left(\frac{x - \varphi(x)}{2}\right) = C_\psi(f).$$

Analogously for odd f we have $C_\psi(f) \circ \varphi = -C_\psi(f)$. We have proved that C_ψ is an isomorphism of the space of even/odd real analytic functions

$\mathscr{A}_+(\mathbb{R})/\mathscr{A}_-(\mathbb{R})$ onto the eigenspace of C_φ corresponding to the eigenvalue $+1/-1$.

If the assumption of (iii) holds, we can apply Lemma 2.1 to find a convergent sequence $(x_n)_n$ in $\mathbb{R}$ of such that for each $n \neq k$ we have $x_n \neq x_k$ and there is m such that $\varphi^{[m]}(x_n) = x_k$ or $\varphi^{[m]}(x_k) = x_n$. Clearly $C_\varphi(f) = f$ holds if f is constant. Any other real analytic function g such that $C_\varphi(g) = g$ would be constant on the convergent sequence of pairwise different points $(x_n)_n$. This implies that g is constant.   □

Proposition 2.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$. If -1 is an eigenvalue of C_φ, then $\varphi^{[2]} = \mathrm{id}$.*

Proof. Let $f \in \mathscr{A}(\mathbb{R})$ be an eigenvector of C_φ for the eigenvalue -1. Then $f(\varphi(x)) = -f(x)$ for each $x \in \mathbb{R}$. Proceeding by contradiction, if the conclusion does not hold, we apply Lemma 2.1 to find a convergent sequence of pairwise different points $(x_n)_n$ such that $f(x_n) = f(x_1)$ or $f(x_n) = -f(x_1)$. Passing to a subsequence, it follows that the real analytic function f is constant. As $f(\varphi(x)) = -f(x)$, this constant value must be 0; a contradiction.   □

Proposition 2.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map with a fixed point $u \in \mathbb{R}$ such that $\varphi^{[2]} \neq \mathrm{id}$. Then the only possible eigenvalues λ of C_φ are of the form $\lambda = (\varphi'(u))^n$ for some $n \in \mathbb{N}$. All of them have at most a one dimensional eigenspace consisting of functions f with zero of order n at u.*

Proof. The argument is similar to the standard proof of the Königs' theorem in the holomorphic case [39, Ch. 6.1], see also [32, Satz 3] and [35, Th. 4.6.3]. The fact that eigenspaces are at most one dimensional follows from [34, Th. 6.1].   □

Proposition 2.5. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is a real analytic function with at least two fixed points, then 1 is the only eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.*

Proof. If $\varphi^{[2]} = \mathrm{id}$, then φ has more than one fixed point if and only if $\varphi = \mathrm{id}$.

Assume that $\varphi^{[2]} \neq \mathrm{id}$. By Propositions 2.3 and 1.1, -1 and 0 cannot be eigenvalues for C_φ. By Proposition 2.4, all eigenvalues are real.

Assume that $\lambda \neq 0$, $|\lambda| \neq 1$ is an eigenvalue of C_φ with a corresponding eigenvector f. Moreover, let $\varphi(u) = u$, $\varphi(w) = w$, $u < w$, and $\varphi(x) \neq x$ for all $x \in (u, w)$. By Proposition 2.4, $f(u) = f(w) = 0$.

There are two cases: either $\varphi(x) > x$ for all $x \in (u, w)$ or $\varphi(x) < x$ for all $x \in (u, w)$. We consider only the first case since the proof for the other is analogous. If $\varphi(x) < w$ for all $x \in (u, w)$ then $\varphi'(u) \geq 1$ and $0 \leq \varphi'(w) \leq 1$. By Proposition 2.4,

$$\lambda = (\varphi'(u))^n = (\varphi'(w))^m \quad \text{for some } n, m \geq 1.$$

Hence $\varphi'(u) = \varphi'(w) = 1$ and $\lambda = 1$; a contradiction. Thus $\varphi(x) > w$ for some $x \in (u, w)$. Define v_0 to be the smallest number $v \in (u, w)$ such that $\varphi(v) = w$. There is a sequence $(x_n)_{n \in \mathbb{N}} \subset (u, w)$ such that

$$\varphi^{[n]}(x_n) = v \quad \text{for every } n = 1, 2, \ldots.$$

Clearly, $u < x_{n+1} < x_n$ for $n = 1, 2, \ldots$ and

$$f(x_n) = (1/\lambda)f(\varphi(x_n)), \quad f(x_1) = (1/\lambda)f(v) = (1/\lambda^2)f(w) = 0.$$

We have proved that $f(x_n) = 0$ for every $n \in \mathbb{N}$ and so $f \equiv 0$; a contradiction.

□

Theorem 2.6. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function with a fixed point u such that $0 < |\varphi'(u)| < 1$. Then either*

(a) *$\varphi^{[2]}$ has at least two fixed points and then 1 is the only eigenvalue of C_φ;*
(b) *or $(\varphi'(u))^n$ is an eigenvalue for every $n \in \mathbb{N}$.*

In case (b) for every eigenfunction f of C_φ with the eigenvalue $\varphi'(u)$ the eigenspace corresponding to the eigenvalue $(\varphi'(u))^n$ is equal to $\mathrm{lin}\{f^n\}$ and f can be chosen real valued. Moreover, the closed linear span of all eigenspaces is equal to $\overline{\mathrm{im}\, C_f}$, i.e. it is equal to $\mathscr{A}(\mathbb{R})$ if and only if φ has no critical points (or, equivalently f has no critical points). If φ is surjective, then the closed linear span of all eigenspaces is isomorphic to $\mathscr{A}(\mathbb{R})$.

Proof. Define Ω to be the basin of attraction of u, i.e.,

$$\Omega := \left\{ x : \lim_{n \to \infty} \varphi^{[n]}(x) = u \right\}.$$

Clearly $\varphi^{-1}(\Omega) = \Omega$ and $\Omega \supset B(u, \varepsilon)$ for some $\varepsilon > 0$. This easily implies that Ω is open. Let $z \notin \Omega$ but $(u, z) \subset \Omega$. Clearly, $\varphi(z) \notin \Omega$ and the open interval joining $u = \varphi(u)$ and $\varphi(z)$ contains only points of Ω. We have proved that if Ω_0 is the connected component of Ω containing u then $\varphi(\partial\Omega_0) \subset \partial\Omega_0$. It is easily seen that at least one point in $\partial\Omega_0$ is fixed for either φ or $\varphi^{[2]}$.

Summarizing, we have shown that either $\varphi^{[2]}$ has at least two fixed points or the whole real line is the basin of attraction for u. Thus the statement (a) follows from Propositions 2.5. (comp. the proof of Proposition 2.7 (a) below).

Now, we consider the second case. It is easy to see that if $(\varphi'(u))^n$ is an eigenvalue of C_φ with an eigenvector f, then $(\varphi'(u))^{nk}$ is an eigenvalue with an eigenvector f^k. Accordingly, it suffices to prove the result for $n = 1$. The construction is now like [32, Satz 1] (comp. [34, Th. 6.4], [35, 4.6.1], [39, Ch. 6.1]) for a locally defined solution. The extension to the whole line goes by a standard argument (see [34, Th. 6.5]). The statement (b) is thus known (comp. Proposition 2.4). However, the part about the form of the closed linear span of eigenspaces seems to be new and we prove it below. Observe that in [12, Th. 4.5] it is proved that if φ is a diffeomorphism with several fixed points on which $|\varphi'| \neq 1$ then all eigenspaces are finite dimensional (even for more general operators).

Since $\varphi'(u)$ is real the real part of every eigenvector is also an eigenvector. It is clear that the closed linear span of all the eigenspaces is equal to $\overline{\mathrm{lin}}\{f^n : n \in \mathbb{N}\}$ for any eigenvector f of C_φ corresponding to the eigenvalue $\varphi'(u)$. Since polynomials are dense in $\mathscr{A}(\mathbb{R})$ we have for real valued f:

$$\overline{\mathrm{lin}}\{f^n : n \in \mathbb{N}\} = \overline{\mathrm{im}\, C_f}.$$

Now, we show that φ has a critical point if and only if f has a critical point. Indeed, if φ has a critical point x then differentiating

$$f(\varphi(x)) = \varphi'(u)f(x) \tag{3}$$

at x we get $f'(x) = 0$. On the other hand, $f'(u) \neq 0$ (Proposition 2.4) and if φ has no critical points, then again differentiating (3) at x we get that $f'(x) = 0$ if and only if $f'(\varphi(x)) = 0$. Thus if $f'(x) = 0$, then f' vanishes on the whole orbit of x which tends to u and it is infinite since φ is injective, so $f' \equiv 0$ and f is constant. This contradicts (3) since $\varphi'(u) \neq 1$. Hence, by Theorem 1.2, $\overline{\text{im } C_f} = \mathscr{A}(\mathbb{R})$ if and only if φ has no critical points.

If φ is surjective, then every point $x \in \mathbb{R}$ has an infinite backward orbit $\{\varphi^{[-n]}(x) : n \in \mathbb{N}\}$. Moreover, since $f(u) = 0$, $f'(u) \neq 0$, it follows that there exists x_+, x_- such that $f(x_+) > 0$, $f(x_-) < 0$. Additionally,

$$f\left(\varphi^{[-n]}(x_+)\right) = \frac{1}{(\varphi'(u))^n} f(x_+), \quad f\left(\varphi^{[-n]}(x_-)\right) = \frac{1}{(\varphi'(u))^n} f(x_-)$$

thus f is surjective. By Theorem 1.2,

$$C_f : \mathscr{A}(\mathbb{R}) \to \text{im } C_f = \overline{\text{im } C_f}$$

is a topological isomorphism. $\hfill\square$

Proposition 2.7. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function with a fixed point u such that $1 < |\varphi'(u)|$. The following holds*

(a) *If $\varphi^{[2]}$ has at least two fixed points, then 1 is the only eigenvalue of C_φ;*

(b) *$\varphi'(x) = 0$ for some $x \in \mathbb{R}$ and $\varphi^{[2]}$ has only one fixed point, then 1 is the only eigenvalue of C_φ*

(c) *If $\varphi'(x) \neq 0$ for each $x \in \mathbb{R}$ and $\varphi^{[2]}$ has only one fixed point, then $(\varphi'(u))^n$ is an eigenvalue for every $n \in \mathbb{N}$ with a real valued eigenfunction. Moreover, in that case, the closed linear span of all eigenspaces is equal to $\mathscr{A}(\mathbb{R})$ and every real valued eigenfunction f for an eigenvalue $\varphi'(u)$ is a diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}$.*

Proof. (a) If $\varphi^{[2]}$ has at least two fixed points, then we can apply Proposition 2.5 to conclude that 1 is the only eigenvalue of $C_{\varphi^{[2]}}$. This implies that the only possible eigenvalues of C_φ are 1 and -1. However, $\varphi^{[2]} \neq \text{id}$ since $1 < |\varphi'(u)|$, hence, by Proposition 2.3, -1 is not an eigenvalue of C_φ.

(b) The proof requires some preparation. We consider first the case when there is $x_0 \in \mathbb{R}$ such that $\varphi'(x_0) = 0$, $\varphi'(u) > 1$ and φ has exactly one fixed point. We have

$$\varphi(x) < x \quad \text{for all } x < u, \quad \varphi(x) > x \quad \text{for all } x > u.$$

It is easy to see that there is a sequence $(x_n)_n$ converging to u such that $\varphi(x_{n+1}) = x_n, n \in \mathbb{N}$, and $\varphi(x_1) = x_0$. Now, if $f \in \mathscr{A}(\mathbb{R})$ is an eigenvector of C_φ with an eigenvalue $\lambda \neq 0$, we have $f(\varphi(x)) = \lambda f(x), x \in \mathbb{R}$. Thus $f'(\varphi(x))\varphi'(x) = \lambda f'(x), x \in \mathbb{R}$. Since $\varphi'(x_0) = 0$, we have $f'(x_0) = 0$. Evaluating now at $x = x_1$, we get $f'(x_0)\varphi'(x_1) = f'(\varphi(x_1))\varphi'(x_1) = \lambda f'(x_1)$, hence $f'(x_1) = 0$. Proceeding by recurrence, $f'(x_n) = 0$ for each $n \in \mathbb{N}$. This implies that $f' \equiv 0$ so f is constant. Therefore 1 is the only eigenvalue of C_φ.

We are ready for the proof of (b). Suppose that φ satisfies $\varphi'(x_0) = 0$ for some $x_0 \in \mathbb{R}$ and that $\varphi^{[2]}$ has only one fixed point. Then $\varphi^{[2]}$ satisfies the assumptions of the proof just given, but $\varphi^{[2]} \neq \mathrm{id}$. So 1 is the only eigenvalue of $C_{\varphi^{[2]}}$ and, by Proposition 2.3, -1 is not the eigenvalue of C_φ. So the proof of part (b) is complete.

(c) Suppose now that $\varphi'(x) \neq 0$ for each $x \in \mathbb{R}$ and $\varphi^{[2]}$ has exactly one fixed point. Since $(\varphi^{[2]})'(u) > 1$, we conclude that $\varphi^{[2]}(x) > x$ if $x > u$ and $\varphi^{[2]}(x) < x$ if $x < u$. From this it follows easily that $\varphi^{[2]}$ is surjective, hence it is a diffeomorphism. Its inverse $\varphi^{[-2]}$ has exactly one fixed point and $0 < |(\varphi^{-1})'(u)| < 1$. Thus φ is a diffeomorphism of $\mathbb{R}$. We can apply Theorem 2.6 to φ^{-1} to conclude that $(\varphi'(u))^n$ is an eigenvalue of C_φ for every $n \in \mathbb{N}$.

Let f be an eigenvector of C_φ for the eigenvalue $\varphi'(u)$ and of $C_{\varphi^{-1}}$ for the eigenvalue $(\varphi'(u))^{-1}$. By Proposition 2.4, $f'(u) \neq 0$. Observe that by the proof of Theorem 2.6, u is an attracting fixed point for φ^{-1} with the attraction basin equal to $\mathbb{R}$. Moreover, $f'(\varphi(x)) = 0$ if and only if $f'(x) = 0$. Thus if f' has a critical point x, then it is zero on the full orbit $\{\varphi^{[k]}(x) : k \in \mathbb{Z}\}$ of φ (or φ^{-1}). The point u is a condensation point of such an orbit so $f' \equiv 0$; this contradicts $f(\varphi(y)) = \varphi'(u)f(y)$ for $\varphi'(u) \neq 1$. We have proved that f has no critical points.

We may take f real valued by taking the real part of any eigenfunction. Since $f(u) = 0$ and $f'(u) \neq 0$, f takes both positive and negative values. Since $|\varphi'(u)| > 1$ and

$$f(\varphi^{[k]}(x)) = (\varphi'(u))^k f(x) \quad \text{for } k \in \mathbb{N}$$

it follows that f is surjective on $\mathbb{R}$. We have proved that $f : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism.

Finally, the eigenspace for $(\varphi'(u))^n$ is equal to $\lim \{f^n\}$. The linear span of all eigenspaces is equal to the image of the subspace of polynomials in $\mathscr{A}(\mathbb{R})$ by the topological isomorphism $C_f : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ so dense in $\mathscr{A}(\mathbb{R})$ (see Theorem 1.2). $\qquad \square$

Let us note that from Theorem 2.6 and Proposition 2.7 the values $(\varphi'(u))^n$ are sometimes eigenvalues and sometimes they are not. Surprisingly, they are always elements of the spectrum by a result that is proved with a technique due to Hammond [28, Prop. 4.1].

Proposition 2.8. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function and let $C_\varphi :$ $\mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ be the associated composition operator. If u is a fixed point of φ such that $|\varphi'(u)| \neq 1, 0$, then $\varphi'(u)^n \in \sigma(C_\varphi)$ for each $n \in \mathbb{N}_0$.*

Proof. If $n = 0$, then $\varphi'(u)^n = 1 \in \sigma_p(C_\varphi)$ by Proposition 1.1. Fix $n \in \mathbb{N}$. Proceeding by contradiction, assume that there is $f \in \mathscr{A}(\mathbb{R})$ such that

$$f(\varphi(x)) - \varphi'(u)^n f(x) = (x - u)^n, \quad x \in \mathbb{R}.$$

Since $|\varphi'(u)| \neq 1$, $f(u) = 0$. Suppose by induction that $f^{(k)}(u) = 0, 0 \leq k \leq j - 1$. Taking the j-th derivative in the equality above, $j < n$, we get

$$\frac{d^j}{dx^j}(f(\varphi(x))\big|_{x=u} - \varphi'(u)^n f^{(j)}(u) = \frac{d^j}{dx^j}((x - u)^n)\big|_{x=u} = 0.$$

The first term consists of $\varphi'(u)^j f^{(j)}(u)$ plus some other summands involving lower order derivatives of f at u, that vanish by the induction hypothesis. Therefore

$$0 = \varphi'(u)^j (1 - \varphi'(u)^{n-j}) f^{(j)}(u),$$

hence $f^{(j)}(u) = 0$. Now taking the n-th derivative we reach a contradiction:

$$0 \neq \frac{d^n}{dx^n}((x-u)^n)\big|_{x=u} = \frac{d^n}{dx^n}(f(\varphi(x)))\big|_{x=u} - \varphi'(u)^n f^{(n)}(u)$$
$$= \varphi'(u)^n f^{(n)}(u) - \varphi'(u)^n f^{(n)}(u) = 0. \qquad \square$$

Summarizing, we have obtained complete description of eigenvalues and the dimension of the corresponding eigenspaces for all C_φ whenever φ has a fixed point as follows (so it completes the proof of Theorem A in the fixed point case).

Theorem 2.9. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function with a fixed point u and let us consider the map $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.*

(a) *If $\varphi'(u) = 1$, then 1 is the only eigenvalue and*
    (i) *either $\varphi = \mathrm{id}$ and in this case the eigenspace is equal to $\mathscr{A}(\mathbb{R})$*
    (ii) *or $\varphi \neq \mathrm{id}$ and the eigenspace is one dimensional.*
(b) *If $\varphi'(u) = -1$, then*
    (i) *either $\varphi^{[2]} = \mathrm{id}$ but $\varphi \neq \mathrm{id}$ and in this case there are two eigenvalues ± 1 and $\mathscr{A}(\mathbb{R})$ is a direct sum of two eigenspaces*
    (ii) *or $\varphi^{[2]} \neq \mathrm{id}$, 1 is the only eigenvalue and its eigenspace is one-dimensional.*
(c) *If $\varphi'(u) = 0$, then 1 is the only eigenvalue and its eigenspace is one-dimensional.*
(d) *If $0 < |\varphi'(u)| < 1$, then*
    (i) *either $\varphi^{[2]}$ has at least two fixed points and then 1 is the only eigenvalue and its eigenspace is one-dimensional*
    (ii) *or $((\varphi'(u))^n)_{n \in \mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces.*
(e) *If $1 < |\varphi'(u)|$, then*
    (i) *either $\varphi^{[2]}$ has at least two fixed points or φ has a critical point and then in both cases 1 is the only eigenvalue and its eigenspace is one-dimensional*
    (ii) *or $((\varphi'(u))^n)_{n \in \mathbb{N}}$ is the sequence of eigenvalues and all of them have one-dimensional eigenspaces.*

Our results in this section permit us to determine the eigenvalues and the eigenspaces of C_φ for $\varphi(x) = x^s, s \in \mathbb{N}$, $\varphi(x) = \sin(x)$, $\varphi(x) = e^x - 1$ and $\varphi(x) = \arctan(x)$ among other examples.

3. Self Map Without Fixed Points and the Abel Equation

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function without a fixed point. Recall that the Abel equation is the equation

$$f \circ \varphi = f + 1.$$

Clearly, if φ has a fixed point, there is no solution of the Abel equations.

Observe that if F_1 and F_2 are two real analytic solutions of the Abel equation $F(\varphi(x)) = F(x) + 1$, then $f := F_2 - F_1$ is a real analytic fixed point of the composition operator $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$.

First, we collect results about the relation between the solutions of the Abel type equations and eigenvalues of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$. They are known — see, for instance, [32, p. 57]:

Proposition 3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f_0. Then each $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ and this operator has an infinite dimensional eigenspace for the eigenvalue λ. Moreover, for every $\lambda \neq 0$ there is an eigenvector f which does not vanish at any point.*

Proof. Observe first that the function f_0 cannot be constant. Let p be a periodic function with period 1 and define $f := p \circ f_0$. We have

$$C_\varphi(f)(x) = f(\varphi(x)) = (p \circ f_0)(\varphi(x)) = p(f_0(x) + 1) = p(f_0(x)) = f(x).$$

Thus $C_\varphi(f) = f$. The infinite dimensionality follows varying p. This settles the case $\lambda = 1$. Taking a non-vanishing periodic function p we get the desired non vanishing eigenvector.

Take now $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Select a complex number μ such that $e^\mu = \lambda$. Set $G(x) := \exp(\mu f_0(x)), x \in \mathbb{R}$ (this is taken from [32, par. 1]). We have

$$C_\varphi(G)(x) = G(\varphi(x)) = \exp(\mu f_0(\varphi(x)))$$
$$= \exp(\mu(f_0(x) + 1)) = e^\mu \exp(\mu f_0(x)) = \lambda G(x).$$

Hence G is an eigenvector of C_φ with an eigenvalue λ.

If $F \in \mathscr{A}(\mathbb{R})$ is a fixed point of C_φ (and there is an infinite dimensional subspace of such functions), we get $C_\varphi(FG) = \lambda FG$. This implies that the eigenspace of the eigenvector λ is also infinite dimensional. □

The following proposition is in fact an observation due to Kneser [32, p. 57].

Proposition 3.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that some $\lambda \in \mathbb{C} \setminus \{0, 1\}$ is an eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ with a never vanishing eigenvector $f_0 \in \mathscr{A}(\mathbb{R})$. Then the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.*

Proof. Clearly f_0 extends to a non vanishing holomorphic function on some one-connected complex neighbourhood U of $\mathbb{R}$. Thus $f_0(x) = \exp(h(x))$ for some holomorphic function h on U (so the restriction of h to $\mathbb{R}$ is real analytic). Select a complex number μ such that $e^\mu = \lambda$. Since $f_0(\varphi(x)) = \lambda f_0(x), x \in \mathbb{R}$, we have:

$$\exp(h(\varphi(x))) = \exp(\mu + h(x)).$$

Since $\lambda \neq 1$ we have for some $k \in \mathbb{Z}$

$$h(\varphi(x)) = h(x) + \mu + 2k\pi i, \quad \text{where } \mu + 2k\pi i \neq 0.$$

Then f, $f(x) := \frac{1}{\mu + 2k\pi i} h(x)$, is the required solution of the Abel equation. □

The next three lemmas prepare the proof of a necessary condition for solvability of the Abel equation which is our crucial result. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous functions such that $\varphi > \mathrm{id}$. We can define the *lower hull* $\hat{\varphi}$ of φ as follows:

$$\hat{\varphi}(t) := \inf\{\varphi(s) : s \geq t\} = \inf\{\varphi(s) : t \leq s \leq \varphi(t)\},$$
$$m_\varphi := \inf\{\hat{\varphi}(t) : t \in \mathbb{R}\}.$$

Lemma 3.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\varphi > \mathrm{id}$. The function $\hat{\varphi} : \mathbb{R} \to \mathbb{R}$ is non-decreasing, continuous and $\hat{\varphi}(t) > t$ for every $t \in \mathbb{R}$. Thus m_φ exists (possibly $= -\infty$) and $\hat{\varphi}(\mathbb{R}) \supset (m_\varphi, +\infty)$.*

Proof. Only the continuity requires a proof. It is clearly enough to prove that $\hat{\varphi}$ is continuous on an arbitrary interval (a, b). Set $B := \max\{\varphi(x) | x \in [a, b]\}$. For each $t \in (a, b)$, $\hat{\varphi}(t) = \min\{\varphi(s) : t \leq s \leq B\}$. Fix $t_0 \in (a, b)$. Let $s_0 = \min\{s | t_0 \leq s \leq B, \varphi(s) = \hat{\varphi}(t_0)\}$. If $t_0 < s_0$, there is $r > 0$ such that $a < t_0 - r < t_0 + r < s_0$ and $\varphi(t) > \varphi(s_0)$ for $t \in (t_0 - r, s_0)$. In this case, it is easy to see that $\hat{\varphi}(t) = \varphi(s_0)$ for each $t \in (t_0 - r, s_0)$ and $\hat{\varphi}$ is continuous at t_0. Now, if $t_0 = s_0$, then $\hat{\varphi}(t_0) = \varphi(t_0)$. Given $\varepsilon > 0$, select $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset (a, b)$ and $|\varphi(t) - \varphi(t_0)| < \varepsilon$ when $t \in (t_0 - \delta, t_0 + \delta)$. It is now easy to see that $\hat{\varphi}(t_0) - \varepsilon \leq \hat{\varphi}(t) \leq \hat{\varphi}(t_0) + \varepsilon$ for each $t \in (t_0 - \delta, t_0 + \delta)$, and $\hat{\varphi}$ is also continuous at t_0 in this case. □

Lemma 3.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous functions such that $\varphi > \mathrm{id}$. For every $a > m_\varphi$ and every $y \geq a$ there is $\hat{y} \in [a, \hat{\varphi}(a)]$ such that $y = \varphi^{[k]}(\hat{y})$ for some $k \in \mathbb{N}$.*

Proof. We define (use Lemma 3.3)

$$\psi : (m_\varphi, +\infty) \to \mathbb{R}, \quad \psi(z) := \sup \hat{\varphi}^{-1}(z).$$

Since $\hat{\varphi}(z) > z$, the supremum above exists for any $z \in (m_\varphi, +\infty) \subset \hat{\varphi}(\mathbb{R})$. Moreover, $\hat{\varphi}(\psi(z)) = \varphi(\psi(z)) = z$, and so $\psi(z) < z$.

Let $y \geq a > m_\varphi$, then we construct a decreasing sequence $\psi^{[n]}(y)$ for all n such that $\psi^{[n-1]}(y) > m_\varphi$. If this sequence is infinite and bounded from below by a, it has a limit and

$$\varphi\left(\lim_n \psi^{[n]}(y)\right) = \lim_n \varphi\left(\psi^{[n]}(y)\right) = \lim_n \psi^{[n]}(y);$$

a contradiction since $\varphi > \mathrm{id}$. Thus there is $n \in \mathbb{N}$ such that

$$\psi^{[n+1]}(y) < a \leq \psi^{[n]}(y) \quad \text{and} \quad \hat{y} := \psi^{[n]}(y) = \hat{\varphi}\left(\psi^{[n+1]}(y)\right) \leq \hat{\varphi}(a)$$

by monotonicity of $\hat{\varphi}$. □

Lemma 3.5. *Let $\lambda \in \mathbb{C} \setminus \{0\}$, $r \in \mathbb{C}$, $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic function satisfying $\varphi > \mathrm{id}$. If there is a real analytic non-constant function $f : \mathbb{R} \to \mathbb{C}$ which solves the equation:*

$$f \circ \varphi = \lambda f + r,$$

then the set of critical points of φ is bounded from above; in particular, φ is strictly increasing from some point on.

Remark 3.6. Of course, an analogous result holds for $\varphi < \mathrm{id}$ if the set of critical points of φ is bounded from below.

Proof. We start with some claims.

Claim 1. If $\varphi^{[n]}$ has a critical point of order k at x_0 (i.e., the derivative has a zero of order k at x_0), then f has a critical point of order k at x_0.

Proof of the Claim 1. Since

$$f \circ \varphi^{[n]} = \lambda^n f + \text{constant}$$

we get

$$f^{(l)}(x_0) = \frac{1}{\lambda^n} \frac{d^l}{dx^l} \left(f \circ \varphi^{[n]} \right)(x_0).$$

It is easily seen that for $l = 1, \ldots, k$

$$\frac{d^l}{dx^l} \left(f \circ \varphi^{[n]} \right)(x_0) = 0.$$

This completes the proof of Claim 1.

Claim 2. If for some x_0 there is a sequence $0 \le n_1 < n_2 < \cdots < n_k$ of natural numbers such that $\varphi^{[n_j]}(x_0)$ for $j = 1, \ldots, k$ are critical points of φ, then $\varphi^{[n]}$ for $n > n_k$ has a critical point of order $\ge 2^k - 1$ at x_0.

Proof of Claim 2. If φ has a critical point of order l at x and of order m at $\varphi(x)$, then $\varphi^{[2]}$ has a critical point of order $ml + m + l$ at x. Indeed, let us take

$$\varphi(z) = \varphi(x) + \sum_{j=l+1}^{\infty} a_j (z-x)^j, \quad \varphi(w) = \varphi(\varphi(x)) + \sum_{j=m+1}^{\infty} b_j (w - \varphi(x))^j.$$

Hence

$$\varphi^{[2]}(z) = \varphi(\varphi(x)) + \sum_{j=m+1}^{\infty} b_j \left(\sum_{p=l+1}^{\infty} a_p (z-x)^p \right)^j$$

$$= \varphi(\varphi(x)) + \sum_{j=ml+m+l+1}^{\infty} c_j (z-x)^j.$$

Thus x is a critical point of order $(ml + m + l + 1) - 1$. Inductively we will get Claim 2.

Now, we prove our Lemma. Let $a > m_\varphi$. Then by Lemma 3.4, for every $y \ge a$ there is $\hat{y} \in [a, \hat{\varphi}(a)]$ such that

$$y = \varphi^{[n]}(\hat{y}) \quad \text{for some } n \in \mathbb{N}.$$

If φ has infinitely many distinct critical points $(x_n)_{n\in\mathbb{N}} \subset [a,\infty)$, then there is a sequence $(\hat{x}_n)_{n\in\mathbb{N}} \subset [a,\hat{\varphi}(a)]$ such that

$$\forall\, n \in \mathbb{N} \,\exists\, k_n \in \mathbb{N} \quad \varphi^{[k_n]}(\hat{x}_n) = x_n.$$

Case (a). There are infinitely many distinct points in the sequence $(\hat{x}_n)_{n\in\mathbb{N}}$. By Claim 1, there are infinitely many distinct critical points of f on $[a,\hat{\varphi}(a)]$. Thus the set of critical points has an accumulation point and thus $f' \equiv 0$; a contradiction.

Case (b). There is a point $\hat{x} \in [a,\hat{\varphi}(a)]$ and an infinite increasing sequence (k_n) such that for all $n \in \mathbb{N}$ the number $\varphi^{[k_n]}(\hat{x})$ is a critical point of φ. By Claim 2, for any $m > k_n$ the number $\hat{x}$ is a critical point of $\varphi^{[m]}$ of order $\geq 2^n - 1$. By Claim 1, f has a critical point $\hat{x}$ of order ∞ so $f' \equiv 0$; a contradiction. $\qquad\square$

Now, we are ready to formulate our necessary condition for real analytic solvability of the Abel equation (the condition turns out to be sufficient as well). So far the best necessary condition for solvability of the Abel equation is due to Belitskii and Lyubich [9, Th. 1.4]: all compact sets in $\mathbb{R}$ must be wandering, i.e., for every compact K there is an integer ν such that for any two $n, m \in \mathbb{N}$ such that $|n-m| > \nu$ the sets $\varphi^{[n]}(K)$ and $\varphi^{[m]}(K)$ are disjoint. This condition is strictly weaker than our condition. For diffeomorphisms φ the condition is equivalent with lack of fixed points but in general it only implies that φ is fixed point free. It is known that for solvability in continuous functions of all equations $f \circ \varphi = f + \gamma$ for every continuous function γ it is necessary and sufficient that φ has no fixed points and φ is strictly increasing on some set $[c,+\infty)$ (in case $\varphi > \mathrm{id}$) or on $(-\infty, c]$ (in case $\varphi < \mathrm{id}$), see [9, Th. 1.9] or [11, Th. 2.3], nevertheless this condition is not necessary for solvability in continuous functions of the Abel equation [11, Ex. 2.3].

Corollary 3.7. *If φ has no fixed point but the set of critical points is unbounded from above (if $\varphi > \mathrm{id}$) or from below (if $\varphi < \mathrm{id}$), then the only eigenvalue of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is 1 and the corresponding eigenspace consists of constant functions only. Moreover, the Abel equation $f \circ \varphi = f + 1$ has no real analytic solution $f \in \mathscr{A}(\mathbb{R})$.*

Proof. We consider only the case $\varphi > \mathrm{id}$—the other one is analogous. If $\lambda \neq 1$ then the eigenvectors cannot be constant. The result follows from Lemma 3.5. $\qquad\square$

Next part is devoted to the proof of sufficiency of our condition. The method of the proof is "topological" in nature and inspired by [10]. In that paper only the case of diffeomorphisms φ is considered. Transferring it to the general real analytic functions φ required some new ideas.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map. Then for any $x \in \mathbb{R}$ we denote by $O(x)$ the *full orbit* of x via φ, i.e.,

$$O(x) := \{y : \ \exists\, k, l \in \mathbb{N} : \varphi^{[k]}(x) = \varphi^{[l]}(y)\}.$$

The full orbits form a partition of $\mathbb{R}$. The quotient topological space with respect to that partition is denoted by $\mathbb{R}/\varphi$ and the corresponding (continuous) canonical quotient map we denote by $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$. It is worth noting that $\mathbb{R}/\varphi$ need not be Hausdorff. By Lemma 3.4 we get:

Corollary 3.8. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map without fixed points. Then $\mathbb{R}/\varphi$ is compact.*

Proof. We consider only the case $\varphi > \mathrm{id}$. Then for any $w \in \mathbb{R}$ its full orbit $O(w)$ is unbounded from above. By Lemma 3.4, there is $z \in O(w) \cap [a, \hat{\varphi}(a)]$ for any fixed $a > m_\varphi$. We have proved that π_φ maps continuously $[a, \hat{\varphi}(a)]$ onto $\mathbb{R}/\varphi$. This completes the proof by [26, 1.1.7.8]. $\qquad\square$

Now, we study the natural manifold structure on $\mathbb{R}/\varphi$. The next two results are generalizations to the case of non-diffeomorphic φ of the method presented by Belitskii and Lyubich in [10, Th. 3.1].

Lemma 3.9. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be real analytic and $\varphi > \mathrm{id}$. If the set of critical points of φ is bounded from above, then $\mathbb{R}/\varphi$ is homeomorphic to the circle $\mathbb{T}$ and there is a real analytic structure on $\mathbb{R}/\varphi$ which makes it diffeomorphic to $\mathbb{T}$ and makes the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic, such that its set of critical points coincides with the set of critical points of all the maps $\varphi^{[n]}$ for $n \in \mathbb{N}$.*

Proof. Assume that every critical point of φ is strictly smaller than a, in particular φ is strictly increasing on $[a, +\infty)$. Hence for $x \geq a$ we have $\varphi(x) = \hat{\varphi}(x)$. By Lemma 3.4, $[x, \varphi(x)]$ intersects every full orbit. If $a \leq x < y < z < \varphi(x)$ then $x < \varphi^{[n]}(x) < \varphi^{[n]}(y) < \varphi^{[n]}(z) < \varphi^{[n+1]}(x)$ for every $n \in \mathbb{N}$ and $O(y) \neq O(z)$. Moreover,

$$\bigcup_{w \in (y,z)} O(w) = \bigcup_{k \in \mathbb{N}} \left(\varphi^{[k]}\right)^{-1} \left(\bigcup_{n \in \mathbb{N}} \varphi^{[n]}(y, z)\right)$$

$$= \bigcup_{k \in \mathbb{N}} \left(\varphi^{[k]}\right)^{-1} \left(\bigcup_{n \in \mathbb{N}} \left(\varphi^{[n]}(y), \varphi^{[n]}(z)\right)\right)$$

is an open set in $\mathbb{R}$. Therefore $\pi_\varphi((y, z))$ is an open set.

We have proved that for every $x \geq a$ the map π_φ restricted to $(x, \varphi(x))$ is a homeomorphism onto an open set (clearly a Hausdorff one). Since every pair of distinct full orbits $O(y), O(z)$ is contained in one of these open sets it follows that the whole space $\mathbb{R}/\varphi$ is Hausdorff. Summarizing, by Corollary 3.8, the space $\mathbb{R}/\varphi$ is a connected compact Hausdorff one dimensional locally euclidean space, i.e., a connected compact manifold of dimension 1 without boundary. By [47, Th. 3.2], $\mathbb{R}/\varphi$ is homeomorphic to the circle $\mathbb{T}$.

Now, we define a real analytic structure on $\mathbb{R}/\varphi$. For any $x \geq a$ we define a chart

$$f_x : \pi_\varphi((x, \varphi(x))) \to (x, \varphi(x)), \quad f_x := \left(\pi_\varphi|_{(x,\varphi(x))}\right)^{-1}$$

Take $a \leq x < y$, $O(x) \neq O(y)$, then there is $n \in \mathbb{N}$ such that $\varphi^{[n]}(x) < y < \varphi^{[n+1]}(x)$. We consider $\pi_\varphi((x, \varphi(x)) \cap \pi_\varphi((y, \varphi(y))$ then

$$f_x\left[\pi_\varphi((x, \varphi(x))) \cap \pi_\varphi((y, \varphi(y)))\right] = \left(x, \left(\varphi^{[n]}\right)^{-1}(y)\right) \cup \left(\left(\varphi^{[n]}\right)^{-1}(y), \varphi(x)\right),$$

$$f_y\left[\pi_\varphi((x, \varphi(x))) \cap \pi_\varphi((y, \varphi(y)))\right] = \left(y, \varphi^{[n+1]}(x)\right) \cup \left(\varphi^{[n+1]}(x), \varphi(y)\right).$$

Therefore,

$$f_y \circ f_x^{-1}\Big|_{\left(\left(\varphi^{[n]}\right)^{-1}(y), \varphi(x)\right)} = \varphi^{[n]},$$

$$f_y \circ f_x^{-1}\Big|_{\left(x, \left(\varphi^{[n]}\right)^{-1}(y)\right)} = \varphi^{[n+1]},$$

$$f_x \circ f_y^{-1}\Big|_{\left(y, \varphi^{[n+1]}(x)\right)} = \left(\varphi^{[n]}\big|_{(x, \varphi(x))}\right)^{-1},$$

$$f_x \circ f_y^{-1}\Big|_{\left(\varphi^{[n+1]}(x), \varphi(y)\right)} = \left(\varphi^{[n+1]}\big|_{(x, \varphi(x))}\right)^{-1}.$$

Since for every $m \in \mathbb{N}$ the map $\varphi^{[m]}$ has no critical points on $[a, +\infty)$ the above maps are real analytic.

We have proved that the charts defined above form a real analytic atlas. Moreover, the real analytic structure on $\mathbb{R}/\varphi$ must be diffeomorphic to the standard one on $\mathbb{T}$ by [47, Th. 6.3].

Let us take any point $y \in \mathbb{R}$ then there is $n \in \mathbb{N}$ such that $\varphi^{[n]}(y) > a$ so some neighbourhood U of y is mapped via $\varphi^{[n]}$ into a neighbourhood of $\varphi^{[n]}(y)$, contained in $(x, \varphi(x))$ where $x > a$. Now, the map

$$f_x \circ \pi_\varphi : U \to (x, \varphi(x)), \quad f_x \circ \pi_\varphi|_U = \varphi^{[n]}|_U$$

is real analytic on U. The map π_φ has a critical point in y if and only if $\varphi^{[n]}$ has it. $\qquad \square$

We will make use of the notion of the *contractible map* by which we mean a map homotopic to a constant one.

Lemma 3.10. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map such that $\mathbb{R}/\varphi$ is a real analytic manifold with $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic. If $d : \mathbb{R}/\varphi \to \mathbb{T}$ is a non-contractible real analytic map, then the Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f on $\mathbb{R}$ with real values. The solution f has critical points exactly in the critical points of $d \circ \pi_\varphi$ (i.e., f has critical points if and only if d or π_φ has critical points).*

Proof. Let us denote by $q : \mathbb{R} \to \mathbb{T}$, $q(x) = \exp(2\pi i x)$, the standard quotient map. Clearly, q is a local diffeomorphism. Since $\mathbb{R}$ is contractible then, by [27, Th. 6.1] the map $d \circ \pi_\varphi : \mathbb{R} \to \mathbb{T}$ lifts to a continuous map $\Phi : \mathbb{R} \to \mathbb{R}$ with respect to q, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\ \Phi\ } & \mathbb{R} \\
\downarrow{\scriptstyle \pi_\varphi} & & \downarrow{\scriptstyle q} \\
\mathbb{R}/\varphi & \xrightarrow[\ d\ ]{} & \mathbb{T}.
\end{array}
$$

Since q is a real analytic local diffeomorphism then the lifting Φ has to be real analytic and its critical points are exactly critical points of $d \circ \pi_\varphi$.

Moreover, $\Phi(\varphi(x)) = \Phi(x) + n(x)$, where $n : \mathbb{R} \to \mathbb{Z}$ is a continuous (hence constant) function $n(x) := n$.

Now, we will use the well-known facts concerning liftings of homotopy but for the reader's convenience we provide details. If $n = 0$ then we can define a continuous map

$$F : \mathbb{R} \times [0,1] \to \mathbb{R}, \quad F(x,t) = t\Phi(x)$$

Therefore

$$F(\varphi(x), t) = t\Phi(\varphi(x)) = t\Phi(x) = F(x,t)$$

and $F(\cdot, t)$ is constant on full orbits of φ for every $t \in [0,1]$. We can define a continuous map

$$G : \mathbb{R}/\varphi \times [0,1] \to \mathbb{R}, \quad G(O(x), t) := F(x,t)$$

and $q \circ G$ is a homotopy joining d with the constant function; a contradiction.

We have proved that $n \neq 0$ so we can define $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{1}{n}\Phi(x)$, to be a real analytic solution of the Abel equation. As we have seen its critical points are exactly the critical points of $d \circ \pi_\varphi$. $\qquad\square$

Now, we summarize our knowledge about eigenvalues of $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ and their corresponding eigenspaces as well on solvability of the Abel equation in case of φ with no fixed point (so it completes the proof of Theorem B).

Theorem 3.11. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an analytic function. The following assertions are equivalent.*

(a) *Every complex $\lambda \neq 0$ is an eigenvalue of C_φ with at least one real analytic eigenvector non-vanishing at any point.*

(a') *Every complex $\lambda \neq 0$ is an eigenvalue of C_φ with an infinite dimensional eigenspace.*

(b) *There is a complex eigenvalue $\lambda \neq 1$ for C_φ with at least one real analytic eigenvector non-vanishing at any point.*

(b') *There is a complex eigenvalue $\lambda \neq 1$ for C_φ and φ has no fixed point.*

(c) *There is a non-constant eigenvector for the eigenvalue 1 and $\varphi^{[2]} \neq \mathrm{id}$.*

(d) *Either $\varphi > \mathrm{id}$ and the set of critical points of φ is bounded from above or $\varphi < \mathrm{id}$ and the set of critical points of φ is bounded from below.*

(e) *The space $\mathbb{R}/\varphi$ of full orbits of φ is a manifold homeomorphic to $\mathbb{T}$ which has a real analytic structure making the canonical map $\pi_\varphi : \mathbb{R} \to \mathbb{R}/\varphi$ real analytic (and, of course, then $\mathbb{R}/\varphi$ is real analytic diffeomorphic to $\mathbb{T}$).*

(f) *The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f.*

If these conditions hold, then for $\lambda > 0$ there is at least one strictly positive eigenvector. Moreover, there is a real analytic solution f_0 of the Abel equation with real values such that the set of critical points is bounded from above (in

case $\varphi > \mathrm{id}$) or bounded from below (in case $\varphi < \mathrm{id}$). In that case for every complex $\lambda \neq 0$, $e^\mu = \lambda$, the map

$$T_\lambda : \mathscr{A}(\mathbb{T}) \to \ker(C_\varphi - \lambda I), \quad T_\lambda(g) := [\exp \circ (\mu f_0)] \cdot [g \circ q \circ f_0],$$

is a topological isomorphism of $\mathscr{A}(\mathbb{T})$ onto the eigenspace of C_φ for λ (here $q : \mathbb{R} \to \mathbb{T}$, $q(x) := \exp(2\pi i x)$).

Remark 3.12. (1)   Theorem 3.11 generalizes simultaneously the result on the solvability of the Abel equation for $\varphi = \exp$ due to Kneser [32, p. 64] and the results for φ a fixed-point free diffeomorphism due to Belitskii and Lyubich [10, Main Theorem] that for a diffeomorphism φ the Abel equation is solvable in $\mathscr{A}(\mathbb{R})$ if and only if φ has no fixed point. In [12, Th. 4.1] (comp. [11, Th. 3.6]) it is proved that for a fixed point free diffeomorphism φ even a more general equations are real analytic solvable.

(2)   The last statement in Theorem 3.11 means that every fixed point of C_φ on $\mathscr{A}(\mathbb{R})$ is of the form $p \circ f_0$, where p is a periodic real analytic function with period 1. Clearly, for any real analytic solution f of the Abel equation all real analytic solutions of this equation are of the form $f + f_1$ where f_1 is a fixed point of C_φ, by the above theorem, $f_1 = g \circ q \circ f_0$ for some $g \in \mathscr{A}(\mathbb{T})$. This result improves the result of Abel true only for strictly increasing solutions; see [1] or [45].

(3)   If φ is a diffeomorphism without fixed points then in [12, Th. 4.2] some $1-1$ correspondence between periodic functions and eigenfunctions for C_φ with arbitrary fixed $\lambda \neq 0$ is established.

Proof. (a)⇒(b) is obvious. The implication (b)⇒(f) follows from Proposition 3.2 and (f)⇒(a), (a′) from Proposition 3.1. The implications (a′)⇒(b′), (c) are consequences of Theorem 2.9.

We prove now (b′), (c)⇒(d): first of all, by Theorem 2.9, (c) implies that φ has no fixed point, and (b′) implies that there is a non-constant eigenvector for some complex eigenvalue $\lambda \neq 1$ (comp. Proposition 1.1). Since both cases $\varphi > \mathrm{id}$ and $\varphi < \mathrm{id}$ are analogous, we assume $\varphi > \mathrm{id}$, and the conclusion (d) follows from Lemma 3.5.

(d)⇒(e) is Lemma 3.9 in case $\varphi > \mathrm{id}$, the other case is analogous.

(e)⇒(f): by [47, Th. 6.3], R/φ is real analytic diffeomorphic to $\mathbb{T}$ and, obviously, this diffeomorphism is a non-contractible real analytic map d from $\mathbb{R}/\varphi$ to $\mathbb{T}$. Apply Lemma 3.10 to conclude.

So far we have completed the proof of the equivalence of all the assertions.

For a real eigenvalue λ with a non-vanishing eigenvector f, the real part $\mathrm{Re}\, f$ and the imaginary part $\mathrm{Im}\, f$ of f are also such vectors. Clearly, $(\mathrm{Re}\, f)^2 + (\mathrm{Im}\, f)^2$ is a strictly positive eigenvector for the eigenvalue λ^2.

It remains to show the last part of the statement. We consider only the case $\varphi > \mathrm{id}$. By (e), there is a diffeomorphism $\tilde{d} : \mathbb{R}/\varphi \to \mathbb{T}$ and as in the proof of Lemma 3.10 we produce a solution of the Abel equation $f_0 : \mathbb{R} \to \mathbb{R}$. By Lemmas 3.10 and 3.9, f_0 has critical points exactly at critical points of

$\varphi^{[n]}$ for any $n \in \mathbb{N}$ so its set is bounded from above. Since $\exp \circ (\mu f_0)$ is a non-vanishing function and

$$C_\varphi(\exp \circ (\mu f_0)) = \lambda \exp \circ (\mu f_0),$$

we get that the map

$$G_\mu : \ker(C_\varphi - I) \to \ker(C_\varphi - \lambda I), \quad G_\mu(g) := (\exp \circ (\mu f_0)) \cdot g,$$

is a topological isomorphism. So it suffices to consider the case $\lambda = 1$, $\mu = 0$ and $T_1 = C_{q \circ f_0}$.

It is easy to observe that $T_1 : \mathscr{A}(\mathbb{T}) \to \ker(C_\varphi - I)$ continuously. Since $f_0(\mathbb{R})$ is a halfline unbounded from above (or the whole line) we get $q \circ f_0(\mathbb{R}) = \mathbb{T}$. Then by [21, Th. 3.2], $C_{q \circ f_0}$ is open onto its image. Since $C_{q \circ f_0}$ is injective it follows that this map is a topological isomorphism.

Now, we can prove that $C_{q \circ f_0}$ is surjective onto $\ker(C_\varphi - I)$ for the above choice of f_0. Let $C_\varphi(f) = f$, then f is constant on every full orbit $O(x)$, $x \in \mathbb{R}$. Thus $f = \hat{f} \circ \pi_\varphi$ for some continuous map $\hat{f} : \mathbb{R}/\varphi \to \mathbb{C}$. Since π_φ is a local diffeomorphism on a halfline of $\mathbb{R}$ mapped via π_φ onto the whole $\mathbb{R}/\varphi$ the map $\hat{f}$ is also real analytic.

Let us take the map $q \circ f_0 : \mathbb{R} \to \mathbb{T}$, clearly $q \circ f_0$ is constant on full orbits of φ so $q \circ f_0 = d \circ \pi_\varphi$ for some continuous map $d : \mathbb{R}/\varphi \to \mathbb{T}$. Since $q \circ f_0$ has no critical points on a halfline of $\mathbb{R}$ unbounded from above which is mapped via π_φ onto the whole $\mathbb{R}/\varphi$, π_φ is there a local diffeomorphism and $q \circ f_0(\mathbb{R}) = \mathbb{T}$ then $d : \mathbb{R}/\varphi \to \mathbb{T}$ is real analytic, surjective and has no critical points. We will show that d is injective. In order to show that we have to show that $q \circ f_0$ does not glue together two different orbits. Now, for x big enough φ is strictly increasing on $(x, \varphi(x))$ and also f_0 is injective on the same interval. Moreover, $f_0(\varphi(x)) = f_0(x) + 1$. This implies that for any $y \in (x, \varphi(x))$ the difference between $f_0(x)$ and $f_0(y)$ is non-integer. On the other hand by Lemma 3.4 every orbit different from $O(x)$ intersects $(x, \varphi(x))$. We have proved that $q \circ f_0$ cannot glue this orbit with $O(x)$. Finally, $d : \mathbb{R}/\varphi \to \mathbb{T}$ is a diffeomorphism.

Therefore

$$f = \hat{f} \circ d^{-1} \circ d \circ \pi_\varphi = C_{q \circ f_0}(\hat{f} \circ d^{-1}), \quad \text{where } \hat{f} \circ d^{-1} \in \mathscr{A}(\mathbb{T}). \qquad \square$$

Theorem 3.11 can be used to conclude that the Abel equation $f(\varphi(x)) = f(x) + 1$ has a real analytic solution $f \in \mathscr{A}(\mathbb{R})$ for $\varphi(x) = e^{\alpha x}$, $\alpha > 1/e$ (for $\alpha \le 1/e$ the map φ has a fixed point), and it does not have a real analytic solution for $\varphi(x) = x + 1 + \alpha \sin(\alpha^{-1} x), 0 < \alpha < 1$, since this function is real analytic, it is a (continuous) homeomorphism on $\mathbb{R}$, has no fixed points, but it has an unbounded sequence of critical points, namely $\varphi'(x) = 0$ if and only if $x = 2s\pi\alpha, s \in \mathbb{Z}$. The later example is mentioned in [9, Example 6.2] in connection with the smooth solvability of the cohomological equation. Observe that the point spectrum of C_φ for $\varphi(x) = e^{\alpha x}, \alpha > e^{-1}$ is equal to $\mathbb{C} \setminus \{0\}$ (the spectrum is equal to $\mathbb{C}$), and that for $\varphi(x) = x + 1 + \alpha \sin(\alpha^{-1} x), 0 < \alpha < 1$, the only eigenvalue of C_φ is 1.

Proposition 3.13. *If the conditions of Theorem 3.11 hold, then the closed linear span of all eigenspaces is equal to* $\overline{\operatorname{im} C_{f_0}} \subset \mathscr{A}(\mathbb{R})$ *for any solution*

f_0 *like in Theorem* 3.11*. This space is equal to the whole* $\mathscr{A}(\mathbb{R})$ *if and only if* φ *has no critical points. If* φ *is surjective, then this invariant space is isomorphic to* $\mathscr{A}(\mathbb{R})$.

Proof. The space of periodic real analytic functions is the closed linear span of

$$(\exp(2\pi i k \cdot))_{k\in\mathbb{Z}}.$$

Thus the closed linear span of all eigenspaces is equal to the closed linear span of

$$(\exp\circ((\mu+2\pi ik)f_0))_{k\in\mathbb{Z},\mu\in\mathbb{C}} = (C_{f_0}(\exp(\mu\cdot)))_{\mu\in\mathbb{C}}.$$

It is well-known that $(\exp(\mu\cdot))_{\mu\in\mathbb{C}}$ is linearly dense in $\mathscr{A}(\mathbb{R})$ (see e.g. [30, Prop. 3.2 and Cor. 3.3]), this yields the conclusion.

Observe that φ has critical points if and only if f_0 has critical points (see the proof of Theorem 3.11). Thus, by Theorem 1.2, im C_{f_0} is dense in $\mathscr{A}(\mathbb{R})$ if and only if φ has no critical points.

If φ is surjective then f_0 is also surjective and then, by Theorem 1.2, the map C_{f_0} is an isomorphism of $\mathscr{A}(\mathbb{R})$ onto a closed subspace of $\mathscr{A}(\mathbb{R})$. $\square$

We have completed the proof of Theorems A and B.

4. Iteration Semigroups and the Abel Equation

In the paper of Kneser [32] it is in fact constructed a solution of the Abel equation $f\circ\varphi = f+1$ for $\varphi = \exp$ which is additionally increasing without critical points. This allowed Kneser to show that there is a real analytic function $g:\mathbb{R}\to\mathbb{R}$, $g^{[2]} = \exp$. We find a generalization of this result nearly in its sharp form.

Let us say that the real analytic map $\varphi:\mathbb{R}\to\mathbb{R}$ *embeds into a real analytic iteration semigroup* whenever there is a real analytic map $\Phi:(\mathbb{R}_+\cup\{0\})\times\mathbb{R}\to\mathbb{R}$ (a real analytic iteration semigroup or flow)

$$\Phi(t+s,x) = \Phi(t,\Phi(s,x)) \quad \text{for every } t,s\in\mathbb{R}_+\cup\{0\},\ x\in\mathbb{R}$$

such that

$$\Phi(n,x) = \varphi^{[n]}(x) \quad \text{for every } n\in\mathbb{N}_0,\ x\in\mathbb{R}.$$

Clearly, if φ embeds into a real analytic iteration semigroup Φ, then there are real analytic functions g_n such that $g_n^{[n]} = \varphi$, one takes $g_n(x) := \Phi(1/n,x)$, i.e. there exists iteration roots of order n for φ. In the book of Kuczma [34, Chapter 9] it is considered the so-called *iteration group* (in case of dependence on $t\in\mathbb{R}$). Iteration roots are considered in [34, Chapter 15], [35, Chapter 11], [8, Section 2].

Lemma 4.1. *Let* $\varphi:\mathbb{R}\to\mathbb{R}$ *be a real analytic map embedded into a real analytic iteration semigroup* Φ.

(a) *The function* $\Phi(t,\cdot)$ *(in particular,* φ*) has no critical points for any* $t\in\mathbb{R}$ *and it is always strictly increasing.*

(b) *If* φ *has no fixed point, then*

$$\Phi(t, x) \neq \Phi(s, x), \quad \frac{\partial \Phi}{\partial t}(t, x) \neq 0 \quad \text{for every } s, t \in \mathbb{R}_+ \cup \{0\}, \ s \neq t, \ x \in \mathbb{R}.$$

Proof. (a) Denote by ∂_2 to be a partial derivative with respect to the second variable. Observe that if $\partial_2 \Phi(s, y) = 0$ then

$$\partial_2 \Phi(t + s, y) = \partial_2 \Phi(t, \Phi(s, y)) \cdot \partial_2 \Phi(s, y) = 0, \quad \text{for every } t > 0.$$

Thus the real analytic function $\frac{\partial}{\partial y}\Phi(\cdot, y) \equiv 0$. This is a contradiction, since $\Phi(0, x) \equiv x$ and so $\partial_2 \Phi(0, y) = 1$.

Now, the derivative of φ must be real and cannot change the sign. If φ has no fixed point it must be increasing. If φ has a fixed point u, then $(\varphi^{[2]})'(u) > 0$. If $\varphi'(u) < 0$, then the real function $\partial_2 \Phi(t, y)$ must change the sign somewhere between $t = 1$ and $t = 2$ and this contradicts the first statement. We have proved that $\varphi'(u) > 0$ and so $\varphi' > 0$ everywhere.

(b) Assume that $\Phi(t, x) = \Phi(s, x)$ for some $t < s$. Hence

$$\Phi(t, x) = \Phi(s - t, \Phi(t, x)),$$

i.e., there is $y \in \mathbb{R}$ and $t > 0$ such that $\Phi(t, y) = y$, and, obviously, $\Phi(nt, y) = y$ for any $n \in \mathbb{N}$. We consider only the case $\varphi > \text{id}$ since the other case is analogous.

In that case $\Phi(n, y) = \varphi^{[n]}(y)$ tends to $+\infty$ for $n \to +\infty$. This means that t is not rational but then for any $\varepsilon > 0$, $N > 0$ there are $m, n \in \mathbb{N}$, $m > N$, such that

$$0 < m - nt < \varepsilon.$$

Since $\Phi(\cdot, y)$ is continuous this is a contradiction with:

$$\Phi(m, y) = \Phi(m - nt, \Phi(nt, y)) = \Phi(m - nt, y).$$

Now, observe that $\Phi(\cdot, x)$ is the only real analytic solution of the Cauchy problem

$$F'(t) = G(F(t)), \quad F(0) = x,$$

where $G(z) := \frac{\partial \Phi}{\partial t}(0, z)$. If $\frac{\partial \Phi}{\partial t}(w, u) = 0$ then $G(\Phi(w, u)) = 0$ and the constant function $F \equiv \Phi(w, u)$ is the solution of the above Cauchy problem for $x = \Phi(w, u)$. Hence $\Phi(\cdot, \Phi(w, u))$ is constant; a contradiction. $\square$

Theorem 4.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map. The following assertions are equivalent:*

(a) *The map φ has no fixed point and it embeds into a real analytic iteration semigroup.*

(b) *The Abel equation $f \circ \varphi = f + 1$ has a real analytic solution f without critical points and with only real values.*

(c) *The map φ has no fixed and no critical points.*

If these equivalent conditions are satisfied, then there is a real analytic function g such that $g^{[n]} = \varphi$, i.e., the map $C_\varphi : \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ has a n-th root C_g in the algebra of operators on $\mathscr{A}(\mathbb{R})$.

Kneser [32] considered the case $\varphi = \exp$, Belitskii and Tkachenko [11, Th. 2.20]proved that (c) implies existence of arbitrary iteration roots without using iteration semigroup. They also showed in the same place that if φ is a diffeomorphism, then (a) holds.

Proof. (a)⇒(c) Lemma 4.1 (a).

(b)⇒(a) Solvability of the Abel equation implies that φ has no fixed point. Since f has no critical points it is either strictly increasing or strictly decreasing. The image of f is always unbounded from above so it must be of the form $(a, +\infty)$ where a could be $-\infty$. It is easily seen (in fact this is an observation of Abel [1], comp. [34, p. 198] and [32, p. 57]) that

$$\Phi(t, x) := f^{-1}(f(x) + t), \quad t \geq 0,$$

is a real analytic flow in which φ embeds.

(c)⇒(b): Follows from Lemmas 3.9 and 3.10, see the proof of (e)⇒(f) in Theorem 3.11. □

The observation that the existence of invertible solution of a Schröder equation implies embedding into an iteration semigroup is a folklore.

Proposition 4.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real analytic map without critical points. If $\varphi^{[2]}$ has exactly one fixed point u and $0 < \varphi'(u) \neq 1$, then φ embeds into a real analytic iteration semigroup. In particular, in that case there exist roots of the operator C_φ of arbitrary order.*

Proof. By Proposition 2.7, if $\varphi'(u) > 1$, then the solution $f : \mathbb{R} \to \mathbb{R}$ of the equation $C_\varphi(f) = \varphi'(u)f$ is a diffeomorphism. We get the iteration semigroup:

$$\Phi(t, x) := f^{-1}\left((\varphi'(u))^t \cdot f(x) \right). \tag{4}$$

Assume that $0 < \varphi'(u) < 1$. By Theorem 2.6, there is a solution $f : \mathbb{R} \to \mathbb{R}$ of the equation $C_\varphi(f) = \varphi'(u)f$ without critical points, $f(u) = 0$. Since $\varphi'(u) < 1$ then for any $x \in \mathbb{R}$ the value $(\varphi'(u))^t \cdot f(x)$ belongs to the image of f. So we can define the iteration semigroup be the formula (4). □

We have completed the proof of Theorem C. Unfortunately, in the case of self maps φ with fixed points we cannot characterize when φ embeds into a real analytic iteration semigroup. Nevertheless, we suspect that the following conjecture is true.

Conjecture. *A real analytic map $\varphi : \mathbb{R} \to \mathbb{R}$ embeds into a real analytic iteration semigroup if and only if it has no critical point, has at most one fixed point u and in that case $0 < \varphi'(u) \neq 1$.*

In [8, Section 2] examples of polynomials φ without embedding into an iteration semigroup are mentioned. By [34, Th. 15.13], the function $\varphi(x) = \exp(x) - 1$ without critical points which has exactly one fixed point $u = 0$ with $\varphi'(u) = 1$ cannot be embedded into an iteration semigroup. Let us observe that if φ embeds into an iteration semigroup Φ then φ commutes with every map $g_t := \Phi(t, \cdot)$, i.e., $g_t \circ \varphi = \varphi \circ g_t$. Then the necessary condition for existence of Φ is the existence of many functions commuting with φ. This is

the method used by Baker to show that such Φ does not exist for a wide class of φ defined locally around a fixed point or for some meromorphic φ; see [34, Th. 10.11] and the papers [2–7,44], where a description of analytic functions defined locally around the fixed point and commuting with fixed φ is given as well as some criteria of existence of entire functions commuting with some φ are given.

Especially interesting are papers [5,6] where the case of φ with a fixed point u, $\varphi'(u) = 1$, is considered — i.e., the case we cannot decide here. In particular, from [5, Th. 2] it follows that if a real analytic function $\varphi : \mathbb{R} \to \mathbb{R}$ extends to a function meromorphic on $\mathbb{C}$ or to an entire function with a fixed point u, $\varphi'(u) = 1$, then the function φ cannot be embedded into an iteration semigroup. In [6] more examples related to embeddability into an iteration semigroup are given but mostly they are not real analytic on the whole line.

The case of functions φ with two fixed points is considered in [31], again it implies that many functions φ cannot be embed into an iteration semigroup. In order to solve the problem of characterization of φ which embeds into an iteration semigroup we need to describe global real analytic functions which commute with φ. We believe that the question of characterizing φ with real analytic iteration roots is even more difficult although also in that case iteration roots of φ commute with φ so the same necessary condition works.

Acknowledgments

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