Integr. Equ. Oper. Theory 76 (2013), 145–149
DOI 10.1007/s00020-013-2044-6
Published online March 30, 2013

Integral Equations and Operator Theory

Erratum

Erratum to: Closed-Range Composition Operators on Weighted Bergman Spaces

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Dedicated to Bartholomew

**Erratum to: Integr Equ Oper Theory 72 (2012), 103–114
DOI 10.1007/s00020-011-1912-1**

1. Introduction

The proof of [1, Lemma 2.1] is based on work in [3] that has been discovered to contain an error; see [4]. This lemma holds under an additional hypothesis, but not in general; cf. [4, Corollary 2.2 and Example 2.4]. This has implications for [1, Theorem 2.3] since part of its proof makes use of [1, Lemma 2.1]. Statements (1) and (2) of this theorem are equivalent in general; see Theorem 1.3 and its proof, below. And statement (3) (of [1, Theorem 2.3]) implies statements (1) and (2); but the converse of this does not hold in general. We provide the details in our work here, starting with a review of the terminology involved. Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and let $\mathbb{T}$ denote its boundary $\{z : |z| = 1\}$. Let A denote normalized two-dimensional Lebesgue measure on $\mathbb{D}$ and let m denote normalized Lebesgue measure on $\mathbb{T}$; "normalized" so that these are probability measures. For $\alpha > -1$, let A_α denote the probability measure on $\mathbb{D}$ given by: $dA_\alpha = c_\alpha(1-|z|^2)^\alpha dA$, where $c_\alpha = \alpha + 1$. For any such α and for $1 \le p < \infty$, let $\mathbb{A}^p_\alpha$ denote the Banach space of functions f that are analytic in $\mathbb{D}$ such that

$$||f||^p_{p,\alpha} := \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty.$$

The online version of the original article can be found under doi:10.1007/s00020-011-1912-1.

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If φ is an *analytic self-map* of the unit disk—that is, φ is analytic in $\mathbb{D}$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$—then $C_\varphi(f) := f \circ \varphi$ defines a bounded composition operator on $\mathbb{A}^p_\alpha$; cf. [8, Theorem 11.6]. For any such function φ, there is a corresponding Nevanlinna counting function that is defined on $\mathbb{D}$ by

$$N_\varphi(w) := \sum_{z \in \varphi^{-1}(\{w\})} \log(1/|z|);$$

where the sum is rendered according the multiplicity of any zero of $\varphi - w$, and is zero if $\varphi^{-1}(\{w\}) = \emptyset$. Moreover, for $\varepsilon > 0, \Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ and $G_\varepsilon := \varphi(\Omega_\varepsilon)$; supressing any reference to φ in our notation. A Borel subset E of $\mathbb{D}$ is said to satisfy condition (L) provided:

(L) There exist constants s and $c, 0 < s, c < 1$, such that $A(E \cap D(z, s)) \geq c \cdot A(D(z, s))$ for all z in $\mathbb{D}$;

where $D(z, s) := \{w \in \mathbb{D} : |\frac{z-w}{1-\overline{w}z}| < s\}$ – the *psuedohyperbolic* disk of radius s about z. An *outer annulus* is a set of the form: $\{z : s < |z| < 1\}$, for some $s, 0 < s < 1$. Among the results in Section 2 of [1] are Lemma 2.1 and Theorem 2.3, whose statements are as follows.

Lemma 1.1. *For any analytic self-map φ of $\mathbb{D}$, the following are equivalent.*

(1) *There exists $\varepsilon > 0$ such that $G_\varepsilon := \varphi(\Omega_\varepsilon)$ satisfies condition (L).*
(2) *There exist $\varepsilon > 0, M > 1$ and a subset U of Ω_ε such that $\varphi(U)$ contains an outer annulus and $\frac{1}{M} \leq |\varphi'(z)| \leq M$ for all z in U.*

Theorem 1.2. *If $1 \leq p < \infty, \alpha > -1$ and φ is an analytic self-map of $\mathbb{D}$, then the following are equivalent.*

(1) *C_φ is closed-range on $\mathbb{A}^p_\alpha$.*
(2) *There exists $\varepsilon > 0$ such that $G_\varepsilon := \varphi(\Omega_\varepsilon)$ satisfies condition (L).*
(3) *There exist $\varepsilon > 0, M > 1$ and a subset U of Ω_ε such that $\varphi(U)$ contains an outer annulus and $\frac{1}{M} \leq |\varphi'(z)| \leq M$ for all z in U.*

The proof of the lemma above is based on [3, Theorem 3.4] which we now know does not hold in general (cf. [4]). Indeed, this lemma does not hold in general (cf. [4, Example 2.4]). In this lemma, statement (2) implies (1), always. If (1) holds, then, at every point of $K := \mathbb{T} \cap \overline{\Omega}_\varepsilon, \varphi$ has an angular derivative that is bounded above, in modulus, by $1/\varepsilon$, and $\varphi^*(K) = \mathbb{T}$; see the proof of Theorem 2.5 and Remark 2.6, in [2]. If one can choose $\varepsilon > 0$ sufficiently small so that K contains finitely many closed subarcs $\{J_\nu\}_{\nu=1}^N$ of $\mathbb{T}$ such that $\varphi^*(\cup_{\nu=1}^N J_\nu) = \mathbb{T}$, then (2) holds; see [4, Corollary 2.2] and the proof of [1, Lemma 2.1]. Statements (1) and (2) in the theorem above are equivalent in general; as is shown below. In fact, part of the proof of Theorem 2.3 in [1] is quite independent of Lemma 2.1 and needs no revision; namely, the implication: (1) $\implies$ (2). And statement (3) always implies (2), and hence (1). By the discussion above, if (2) holds and, additionally, K contains finitely many closed subarcs $\{J_\nu\}_{\nu=1}^N$ of $\mathbb{T}$ such that $\varphi^*(\cup_{\nu=1}^N J_\nu) = \mathbb{T}$, then we have (3). We now state and prove our revised version of [1, Theorem 2.3]. This result holds without any restrictions and extends [2, Theorem 2.4] to our weighted Bergman spaces.

Theorem 1.3. *If $1 \le p < \infty, \alpha > -1$ and φ is an analytic self-map of $\mathbb{D}$, then the following are equivalent.*

(1) *C_φ is closed-range on $\mathbb{A}^p_\alpha$.*
(2) *There exists $\varepsilon > 0$ such that $G_\varepsilon := \varphi(\Omega_\varepsilon)$ satisfies condition (L).*

Proof. In the proof of [1, Theorem 2.3] the implication (1) $\Longrightarrow$ (2) was established without any reference to [1, Lemma 2.1] and thus still stands. What remains to be shown here is that (2) $\Longrightarrow$ (1). As in the proof of [2, Theorem 2.4] we can use versions of Lemmas 2.1, 2.2 and 2.3 in [2] to reduce to the case that φ is nonconstant and fixes zero. And we may restrict our attention to C_φ on $\mathbb{A}^p_{\alpha,0} := \{f \in \mathbb{A}^p_\alpha : f(0) = 0\}$. By the proof of [8, Theorem 4.28] there is a constant $C > 1$ such that, for all f in $\mathbb{A}^p_{\alpha,0}$,

$$\frac{1}{C}||f||_{p,\alpha} \le \left\{ \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^p \, dA_\alpha(w) \right\}^{\frac{1}{p}} \le C||f||_{p,\alpha};$$

which we indicate by: $||f||_{p,\alpha} \approx \left\{\int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^p \, dA_\alpha(w)\right\}^{\frac{1}{p}}$. And, by the work of Luecking in [6], if G_ε satisfies condition (L), then there is a positive constant η such that

$$\int_{G_\varepsilon} |f'(w)|^p(1-|w|^2)^p \, dA_\alpha(w) \ge \eta \int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^p \, dA_\alpha(w),$$

whenever $f \in \mathbb{A}^p_{\alpha,0}$. We first consider the case: $1 \le p < 2$, and proceed as in the proof of [2, Theorem 2.4] by decomposing $\{z \in \mathbb{D} : \varphi'(z) \ne 0\}$ into at most countably many semi-closed polar rectangles R_n, on each of which φ is univalent; cf. [7, page 186]. By the Schwarz–Pick lemma (cf. [5, page 2]), $(1-|z|^2)|\varphi'(z)|/(1-|\varphi(z)|^2) \le 1$ for all z in $\mathbb{D}$. So, if $z \in \Omega_\varepsilon$, then $\varepsilon|\varphi'(z)| < 1$ and hence: $\varepsilon^2|\varphi'(z)|^2 \le \varepsilon^p|\varphi'(z)|^p$. Letting $S_n = \varphi(\Omega_\varepsilon \cap R_n)$, we find that

$$\begin{aligned}
||f\circ\varphi||^p_{p,\alpha} &\approx \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^p(1-|z|^2)^{\alpha+p}\, dA(z) \\
&\ge \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{\alpha+p}\, dA(z) \\
&\ge \varepsilon^{2-p} \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^2(1-|z|^2)^{\alpha+p}\, dA(z) \\
&\ge \varepsilon^{\alpha+2} \int_{\Omega_\varepsilon} |f'(\varphi(z))|^p|\varphi'(z)|^2(1-|\varphi(z)|^2)^{\alpha+p}\, dA(z) \\
&= \varepsilon^{\alpha+2}\cdot\sum_n \int_{\Omega_\varepsilon\cap R_n} |f'(\varphi(z))|^p|\varphi'(z)|^2(1-|\varphi(z)|^2)^{\alpha+p}\, dA(z)
\end{aligned}$$

$$= \varepsilon^{\alpha+2} \cdot \sum_n \int_{G_\varepsilon} |f'(w)|^p (1-|w|^2)^{\alpha+p} \chi_{S_n}(w)\, dA(w)$$

$$= \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1-|w|^2)^{\alpha+p} \left(\sum_n \chi_{S_n}(w) \right) dA(w)$$

$$\geq \varepsilon^{\alpha+2} \int_{G_\varepsilon} |f'(w)|^p (1-|w|^2)^{\alpha+p}\, dA(w)$$

$$\geq \eta\varepsilon^{\alpha+2} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{\alpha+p}\, dA(w)$$

$$\approx \int_{\mathbb{D}} |f(w)|^p\, dA_\alpha(w).$$

From this it follows that C_φ is closed-range on $\mathbb{A}^p_\alpha$; when $1 \leq p < 2$. We now apply [1, Lemma 2.2] to get that C_φ is closed-range on $\mathbb{A}^p_\alpha$, for all p, $1 \leq p < \infty$. Thus, (1) holds and our proof is complete. □

Remark 1.4. The comments and results in [1] that are subsequent to Theorem 2.3 still hold, with the substitution of Theorem 1.3 above. The only reference made in this latter portion of [1] to Lemma 2.1 occurs at the bottom of page 111 in this reference. Using the notation of the discussion there, if $\cup_{\nu=1}^N B(I_\nu) = \mathbb{T}$, then, there exists $\varepsilon > 0$ and, for each $\nu, 1 \leq \nu \leq N$, there is a closed subarc J_ν of I_ν, such that $\cup_{\nu=1}^N J_\nu \subseteq \overline{\Omega}_\varepsilon$ and $\cup_{\nu=1}^N B(J_\nu) = \mathbb{T}$. Thus, by [4, Corollary 2.2] C_φ is closed-range on $\mathbb{A}^2$ and hence, by Theorem 1.3 above, is closed-range on all of the $\mathbb{A}^p_\alpha$-spaces. One may consult [4, Remark 1.5] for similar observations.

Acknowledgments

We thank the referee for suggestions that have led to an improvement of the exposition of this erratum, and the editor for kind help along the way.

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