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Integral Equations and Operator Theory

A Local Lifting Theorem for Jointly Subnormal Families of Unbounded Operators

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Abstract. A local lifting theorem for bounded operators that intertwine a pair of jointly subnormal families of unbounded operators is proved. Each family in question is assumed to be composed of operators defined on a common invariant domain consisting of "joint" analytic vectors. This result can be viewed as a generalization of the local lifting theorem proved by Sebestyén, Thomson and the present authors for pairs of bounded subnormal operators.

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1. Introduction

A local lifting theorem, originally formulated for pairs of bounded subnormal operators (cf. [6, Theorem 4.2]), states that an intertwining operator between two subnormal operators lifts to an intertwining operator between their minimal normal extensions if and only if (1) the restriction of the intertwining operator to each cyclic invariant subspace lifts, and (2) the supremum of the norms of the cyclic lifts is finite (recall that the first commutant lifting theorem for bounded subnormal operators was proved by Bram in [2]). Our aim in this paper is to generalize the local lifting theorem of [6] to the case of pairs of jointly subnormal families of unbounded operators (see [12–14, 17] for basic results on unbounded subnormal operators). The first problem to be overcome here concerns the question of which kind of minimality for normal extensions of unbounded operators should be chosen. A careful analysis

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of the results on lifting strong commutants contained in [11] reveals that the notion of minimality of cyclic type is appropriate for this purpose. The other difficulty is to guarantee the existence of minimal normal extensions of cyclic type for cyclic parts of a jointly subnormal family of operators. This can be achieved by assuming that each family in question have a dense set of "joint" analytic vectors (recall that the notion of an analytic vector was introduced by Nelson in [8]). The latter enables us to use a generalization of the Maserick theorem [7, Theorem 3.2], which was proved in [12], to establish the main result of the paper (cf. Theorem 5.3). In Sect. 6 we dwell upon a special kind of local intertwining referred to as local commutativity.

2. Preliminaries

As usual, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of complex numbers, real numbers and nonnegative integers, respectively. All linear spaces in this paper are assumed to be complex and all operators under consideration are assumed to be linear. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we denote by $\boldsymbol{B}(\mathcal{H},\mathcal{K})$ the set of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$. To simplify the writing, we put $\boldsymbol{B}(\mathcal{H}) := \boldsymbol{B}(\mathcal{H},\mathcal{H})$. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$. If X is a subset of $\mathcal{H}$, then $\mathrm{lin} X$ stands for the linear span of X. A family $\{f_\gamma\}_{\gamma\in\varLambda}$ of vectors in a linear space X is said to be *finite* if $f_\gamma = 0$ for all but a finite number of γ's.

Let A be an operator in $\mathcal{H}$. Denote by $\mathcal{D}(A)$, A^* and $\bar{A}$ the domain, the adjoint and the closure of A (in case they exist). We write $A \subseteq B$ if the operator B is an extension of A (A and B may act in distinct Hilbert spaces). We say that a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ *reduces* A if $PA \subseteq AP$, where P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$; if this is the case, then $A|_{\mathcal{M}}$ stands for the restriction of A to $\mathcal{M}$, i.e., $\mathcal{D}(A|_{\mathcal{M}}) = \mathcal{D}(A)\cap\mathcal{M}$ and $A|_{\mathcal{M}} f = Af$ for $f \in \mathcal{D}(A|_{\mathcal{M}})$. If A is closable and $\mathcal{E}$ is a linear subspace of $\mathcal{D}(A)$ such that $A \subseteq \overline{A|_{\mathcal{E}}}$ (or equivalently $\bar{A} = \overline{A|_{\mathcal{E}}}$), then $\mathcal{E}$ is called a *core* of A (here $A|_{\mathcal{E}}$ is the usual restriction of the mapping A to the set $\mathcal{E}$).

For an operator A in $\mathcal{H}$, we put $\mathcal{D}^\infty(A) = \bigcap_{n=1}^\infty \mathcal{D}(A^n)$. Elements of $\mathcal{D}^\infty(A)$ are referred to as $\mathcal{C}^\infty$-*vectors*. In the present paper we explore a special class of $\mathcal{C}^\infty$-vectors called analytic ones. Following [8], we say that $f \in \mathcal{D}^\infty(A)$ is an *analytic vector* for A if there exists a positive real number t such that

$$\sum_{n=0}^{\infty} \|A^n f\| \frac{t^n}{n!} < \infty.$$

The set of all analytic vectors for A, denoted by $\mathscr{A}(A)$, is a linear subspace of $\mathcal{H}$ which is invariant for A, i.e. $A(\mathscr{A}(A)) \subseteq \mathscr{A}(A)$.

Now, we recall some definitions from [12, Sect. 4]. Suppose that $(\varOmega, +, *)$ is a commutative $*$-semigroup with the zero element 0 and $\mathcal{E}$ is a linear space. By a form over $(\varOmega, \mathcal{E})$ we mean a mapping $\varphi : \varOmega \times \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ such that $\varphi(\upsilon; \cdot, \text{-}) : \mathcal{E}\times\mathcal{E} \to \mathbb{C}$ is a sesquilinear form for all $\upsilon \in \varOmega$. A form φ over

$(\Omega, \mathcal{E})$ is called *positive definite* if

$$\sum_{m,n=0}^{k} \varphi(v_n^* + v_m; f_m, f_n) \geq 0, \quad \{v_m\}_{m=0}^k \subseteq \Omega,\ \{f_m\}_{m=0}^k \subseteq \mathcal{E},\ k \in \mathbb{N}.$$

We say that a form φ over $(\Omega, \mathcal{E})$ is *weakly positive definite* if for every $f \in \mathcal{E}$, $\varphi(\cdot; f, f)$ is positive definite as a form over $(\Omega, \mathbb{C})$. Given a weakly positive definite form φ over $(\Omega, \mathcal{E})$ and $v \in \Omega$, we set

$$\mathscr{A}_\varphi(v) = \left\{ f \in \mathcal{E} \,\middle|\, \exists\, t > 0 : \sum_{n=0}^{\infty} \sqrt{\varphi(n \cdot (v^* + v); f, f)}\, \frac{t^n}{n!} < \infty \right\}.$$

Denote by $\widehat{\Omega}$ the set of all characters of Ω (by a *character* of Ω we mean a nonzero additive involution preserving complex mapping on Ω). Let G be a set of $*$-generators of a $*$-semigroup Ω. As in [12, p. 40], we write $\mathscr{C}_G$ for the Cartesian product $\prod_{v \in G} \mathscr{C}_v$, where $\mathscr{C}_v := \mathbb{C}$ for every $v \in G$. In what follows, we consider the mapping $j_G \colon \widehat{\Omega} \to \mathscr{C}_G$ given by the formula $j_G(\chi) = \{\chi(v)\}_{v \in G}$ for $\chi \in \widehat{\Omega}$. The following lemma is a version of [12, Corollary 3] (see also [12, Remarks 3 and 5]).

Lemma 2.1. *Let Ω, $\mathcal{E}$ and j_G be as above. Assume that $j_G(\widehat{\Omega}) = \mathscr{C}_G$. If φ is a weakly positive definite form over $(\Omega, \mathcal{E})$ such that $\mathscr{A}_\varphi(v) = \mathcal{E}$ for all $v \in G$, then the form φ is positive definite.*

Let Σ be a nonempty set. Denote by $\mathbb{N}_\Sigma$ the set of all mappings $\alpha \colon \Sigma \to \mathbb{N}$ such that $\alpha(\sigma) = 0$ for all but a finite number of elements $\sigma \in \Sigma$. The set $\Omega_\Sigma := \mathbb{N}_\Sigma \times \mathbb{N}_\Sigma$ becomes a commutative $*$-semigroup when equipped with the standard coordinate-wise addition and involution $(\alpha, \beta)^* = (\beta, \alpha)$ for $\alpha, \beta \in \mathbb{N}_\Sigma$. In this particular case, $G := \{(\delta_\sigma, 0) : \sigma \in \Sigma\}$ is a set of $*$-generators of Ω_Σ (δ_σ is the characteristic function of the singleton set $\{\sigma\}$). As observed in [12, p. 50],

$$j_G(\widehat{\Omega_\Sigma}) = \mathscr{C}_G. \tag{2.1}$$

3. Normal Extensions

Let $\mathcal{E}$ be an inner product space and let $\mathcal{H}$ be its Hilbert space completion. Denote by $\boldsymbol{L}(\mathcal{E})$ the algebra of all operators $A \colon \mathcal{E} \to \mathcal{E}$ (with composition as multiplication), and by $\boldsymbol{L}^\#(\mathcal{E})$ the subalgebra of $\boldsymbol{L}(\mathcal{E})$ consisting of all operators $A \in \boldsymbol{L}(\mathcal{E})$ for which there exists an operator $A^\# \in \boldsymbol{L}(\mathcal{E})$ such that $\langle Af, g\rangle = \langle f, A^\# g\rangle$ for all $f, g \in \mathcal{E}$. Such an operator $A^\#$ is uniquely determined. It is plain that $\boldsymbol{L}^\#(\mathcal{E})$ is a $*$-algebra with the involution $A \mapsto A^\#$. Note that if $\mathcal{E}$ is a linear subspace of a Hilbert space $\mathcal{K}$ and A is a densely defined operator in $\mathcal{K}$ such that $\mathcal{E} \subseteq \mathcal{D}(A)$, then $A|_{\mathcal{E}}$ belongs to $\boldsymbol{L}^\#(\mathcal{E})$ if and only if $\mathcal{E} \subseteq \mathcal{D}(A^*)$, $A(\mathcal{E}) \subseteq \mathcal{E}$ and $A^*(\mathcal{E}) \subseteq \mathcal{E}$; if this is the case, then $A^\# = A^*|_{\mathcal{E}}$.

Given a family $\boldsymbol{A} = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ of commuting operators (i.e. $A_\sigma A_\tau = A_\tau A_\sigma$ for all $\sigma, \tau \in \Sigma$) and $\alpha \in \mathbb{N}_\Sigma$, we set $\boldsymbol{A}^\alpha = \prod_{\sigma \in \Sigma} A_\sigma^{\alpha(\sigma)}$. A family $\boldsymbol{A} = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ is said to be *jointly subnormal* if there exist a Hilbert space $\mathcal{K}$ and a family $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ of spectrally commuting normal operators[1] acting in $\mathcal{K}$ such that $\mathcal{E} \subseteq \mathcal{K}$ and $A_\sigma \subseteq M_\sigma$ for all $\sigma \in \Sigma$; $\boldsymbol{M}$ is then called a *normal extension* of $\boldsymbol{A}$ (cf. [12, p. 49]).

We say that a family $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma}$ of operators acts in $\mathcal{H}$ if each operator A_σ acts in $\mathcal{H}$. Given a family $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma}$ of operators in $\mathcal{H}$, we set

$$\mathcal{D}^\infty(\boldsymbol{A}) = \bigcap\{\,\mathcal{D}(A_{\sigma_0}\cdots A_{\sigma_n})\colon \sigma_0,\ldots,\sigma_n \in \Sigma,\ n \in \mathbb{N}\}.$$

Clearly, $\mathcal{D}^\infty(\boldsymbol{A})$ is the greatest linear subspace of $\bigcap_{\sigma\in\Sigma}\mathcal{D}(A_\sigma)$ that is invariant (in the usual set-theoretical sense) for every A_σ. If each operator A_σ is densely defined, then we write $\boldsymbol{A}^* := \{A_\sigma^*\}_{\sigma\in\Sigma}$.

Now we move on to the concept of minimality of a normal extension. By [12, Remark 8] and the results contained in [18, p. 423], we may formulate the ensuing lemma.

Lemma 3.1. *Let $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ be a family of spectrally commuting normal operators in $\mathcal{K}$. Then*

(i) $\mathcal{D}^\infty(\boldsymbol{M}) = \mathcal{D}^\infty(\boldsymbol{M}^*) = \mathcal{D}^\infty(\boldsymbol{M}\cup\boldsymbol{M}^*)$,

(ii) $M_\sigma M_\tau f = M_\tau M_\sigma f$, $M_\sigma M_\tau^* f = M_\tau^* M_\sigma f$ *and* $M_\sigma^* M_\tau^* f = M_\tau^* M_\sigma^* f$ *for all* $f \in \mathcal{D}^\infty(\boldsymbol{M})$ *and* $\sigma,\tau \in \Sigma$,

(iii) $\{M_\sigma|_{\mathcal{D}^\infty(\boldsymbol{M})}\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}^\#(\mathcal{D}^\infty(\boldsymbol{M}))$,

(iv) $(M_\sigma|_{\mathcal{D}^\infty(\boldsymbol{M})})^\# = M_\sigma^*|_{\mathcal{D}^\infty(\boldsymbol{M})}$ *for all* $\sigma \in \Sigma$,

(v) *if* $\mathcal{E}$ *is a subset of* $\bigcap_{\sigma\in\Sigma}\mathcal{D}(M_\sigma)$ *such that* $M_\sigma(\mathcal{E}) \subseteq \mathcal{E}$ *for all* $\sigma \in \Sigma$, *then* $\mathcal{E} \subseteq \mathcal{D}^\infty(\boldsymbol{M}\cup\boldsymbol{M}^*)$.

Sketch of the proof. Note first that if $N_1,\ldots,N_n$ are spectrally commuting normal operators in $\mathcal{K}$ and E is the joint spectral measure[2] of $(N_1,\ldots,N_n)$, then

$$\begin{aligned}\mathcal{D}(N_n\cdots N_1) &= \mathcal{D}(N_n^*\cdots N_1^*)\\ &= \left\{f \in \mathcal{K}\colon \int_{\mathbb{C}^n}\sum_{k=1}^{n}\prod_{j=1}^{k}|z_j|^2\,\langle E(\mathrm{d}z)f,f\rangle < \infty\right\},\end{aligned} \tag{3.1}$$

$$\left.\begin{aligned} N_iN_j, N_jN_i &\subseteq \int_{\mathbb{C}^n} z_iz_j E(\mathrm{d}z),\\ N_iN_j^*, N_j^*N_i &\subseteq \int_{\mathbb{C}^n} z_i\bar{z}_j E(\mathrm{d}z),\\ N_i^*N_j^*, N_j^*N_i^* &\subseteq \int_{\mathbb{C}^n} \bar{z}_i\bar{z}_j E(\mathrm{d}z),\end{aligned}\right\} \tag{3.2}$$

for all $i,j \in \{1,\ldots,n\}$. It follows from (3.1) that $\mathcal{D}^\infty(\boldsymbol{M}) = \mathcal{D}^\infty(\boldsymbol{M}^*)$, which immediately implies that (i) holds. This combined with (3.2) gives (ii). The conditions (iii)–(v) follow from (i) and (ii). □

[1] We say that normal operators *spectrally commute* if their spectral measures commute.

[2] We refer the reader to [1] for more information on joint spectral measures.

It is now clear from Lemma 3.1 that $\boldsymbol{M}^{\alpha} f$, $\boldsymbol{M}^{*\alpha} f$ and $\boldsymbol{M}^{*\alpha}\boldsymbol{M}^{\beta} f$ make sense and that $\boldsymbol{M}^{*\alpha}\boldsymbol{M}^{\beta} f = \widetilde{\boldsymbol{M}}^{\#\alpha}\widetilde{\boldsymbol{M}}^{\beta} f$ for all $f \in \mathcal{D}^{\infty}(\boldsymbol{M})$ and $\alpha, \beta \in \mathbb{N}_{\Sigma}$, where $\widetilde{\boldsymbol{M}} := \{M_\sigma|_{\mathcal{D}^\infty(\boldsymbol{M})}\}_{\sigma\in\Sigma}$ and $\widetilde{\boldsymbol{M}}^{\#} := \{(M_\sigma|_{\mathcal{D}^\infty(\boldsymbol{M})})^{\#}\}_{\sigma\in\Sigma}$. In view of Lemma 3.1, any jointly subnormal family $\{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ is always commutative.

We say that a normal extension $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ of a jointly subnormal family $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ is *minimal of cyclic type* if

$$\mathcal{F}_{\boldsymbol{M}}(\mathcal{E}) := \operatorname{lin}\{\boldsymbol{M}^{*\alpha} f : f \in \mathcal{E}, \alpha \in \mathbb{N}_{\Sigma}\} \tag{3.3}$$

is a core of M_σ for every $\sigma \in \Sigma$ (according to Lemma 3.1, this definition is correct). We now prove that each jointly subnormal family possessing a rich set of analytic vectors has a minimal normal extension of cyclic type which can be built up from a normal extension.

Theorem 3.2. *Let $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ be a jointly subnormal family of operators such that $\mathcal{E} = \mathscr{A}(A_\sigma)$ for all $\sigma \in \Sigma$ and let $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ be a normal extension of $\boldsymbol{A}$. Then for every $\sigma \in \Sigma$, the closed linear space $\overline{\mathcal{F}_{\boldsymbol{M}}(\mathcal{E})}$ reduces M_σ to $N_\sigma := \overline{M_\sigma|_{\mathcal{F}_{\boldsymbol{M}}(\mathcal{E})}}$, i.e., $N_\sigma = M_\sigma|_{\overline{\mathcal{F}_{\boldsymbol{M}}(\mathcal{E})}}$. Moreover, $\boldsymbol{N} = \{N_\sigma\}_{\sigma\in\Sigma}$ is a minimal normal extension of $\boldsymbol{A}$ of cyclic type.*

Proof. Owing to Lemma 3.1, $\mathcal{F}_{\boldsymbol{M}}(\mathcal{E}) \subseteq \mathcal{D}^{\infty}(\boldsymbol{M} \cup \boldsymbol{M}^*)$ and $\{\widetilde{M}_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}^{\#}(\mathcal{F}_{\boldsymbol{M}}(\mathcal{E}))$, where $\widetilde{M}_\sigma := M_\sigma|_{\mathcal{F}_{\boldsymbol{M}}(\mathcal{E})}$. By our assumption, $\mathcal{E} \subseteq \mathscr{A}(\widetilde{M}_\sigma)$ for all $\sigma \in \Sigma$. Fix $\sigma \in \Sigma$. Since $\mathcal{F}_{\boldsymbol{M}}(\mathcal{E})$ is the linear span of $\bigcup_{\alpha\in\mathbb{N}_\Sigma} \widetilde{\boldsymbol{M}}^{\#\alpha}(\mathcal{E})$, and the operators $\widetilde{\boldsymbol{M}}^{\#\alpha}$ commute with $\widetilde{M}_\sigma$ and $\widetilde{M}_\sigma^{\#}$ (use again Lemma 3.1), we deduce from [12, Theorem 1] that N_σ is a normal operator. Hence, by [15, Corollary 1], the closed linear space $\overline{\mathcal{F}_{\boldsymbol{M}}(\mathcal{E})}$ reduces M_σ to N_σ. This implies that $N_\sigma^* \subseteq M_\sigma^*$. As σ is arbitrary, we deduce that $\mathcal{F}_{\boldsymbol{M}}(\mathcal{E}) = \mathcal{F}_{\boldsymbol{N}}(\mathcal{E})$. As a consequence, we see that $\mathcal{F}_{\boldsymbol{N}}(\mathcal{E})$ is a core of N_σ for every $\sigma \in \Sigma$. The normal operators N_σ, $\sigma \in \Sigma$, spectrally commute as restrictions of spectrally commuting normal operators M_σ, $\sigma \in \Sigma$. This completes the proof. □

In our paper we frequently consider unbounded subnormal operators A in a Hilbert space $\mathcal{H}$ such that $\mathcal{D}(A) = \mathscr{A}(A)$. The reader should be aware of the fact that such operators can never be closed (cf. [16, Theorem 7]).

Let $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ be a jointly subnormal family, and let $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ and $\boldsymbol{N} = \{N_\sigma\}_{\sigma\in\Sigma}$ be its normal extensions acting in Hilbert spaces $\mathcal{K}$ and $\mathcal{L}$ respectively. We say that $\boldsymbol{M}$ and $\boldsymbol{N}$ are $\mathcal{H}$-unitarily equivalent if there exists a unitary operator $U \in \boldsymbol{B}(\mathcal{K}, \mathcal{L})$ such that $U|_{\mathcal{H}} = I_{\mathcal{H}}$ and $UM_\sigma = N_\sigma U$ for all $\sigma \in \Sigma$. We show that minimal normal extensions of cyclic type of a jointly subnormal family are determined up to $\mathcal{H}$-unitary equivalence (this was suggested in [12, Remark 8]).

Proposition 3.3. *Let $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ be a jointly subnormal family of operators, and let $\boldsymbol{M}$ and $\boldsymbol{N}$ be minimal normal extensions of cyclic type of $\boldsymbol{A}$. Then $\boldsymbol{M}$ and $\boldsymbol{N}$ are $\mathcal{H}$-unitarily equivalent.*

Proof. Let $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces in which $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ and $\boldsymbol{N} = \{N_\sigma\}_{\sigma\in\Sigma}$ act. In view of Lemma 3.1, for all $\alpha, \beta \in \mathbb{N}_\Sigma$ and $f, g \in \mathcal{E}$ we have

$$\begin{aligned}\langle \boldsymbol{M}^{*\alpha} f, \boldsymbol{M}^{*\beta} g\rangle &= \langle \boldsymbol{M}^{\beta}\boldsymbol{M}^{*\alpha} f, g\rangle = \langle \boldsymbol{M}^{*\alpha}\boldsymbol{M}^{\beta} f, g\rangle = \langle \boldsymbol{M}^{\beta} f, \boldsymbol{M}^{\alpha} g\rangle \\ &= \langle \boldsymbol{A}^{\beta} f, \boldsymbol{A}^{\alpha} g\rangle = \langle \boldsymbol{N}^{\beta} f, \boldsymbol{N}^{\alpha} g\rangle = \cdots = \langle \boldsymbol{N}^{*\alpha} f, \boldsymbol{N}^{*\beta} g\rangle.\end{aligned}$$

Thus, by the cyclic minimality of $\boldsymbol{M}$ and $\boldsymbol{N}$, we deduce that there exists a unitary operator $U \in \boldsymbol{B}(\mathcal{K}, \mathcal{L})$ such that $U(\mathfrak{F}_{\boldsymbol{M}}(\mathcal{E})) = \mathfrak{F}_{\boldsymbol{N}}(\mathcal{E})$ and $U(\boldsymbol{M}^{*\alpha} f) = \boldsymbol{N}^{*\alpha} f$ for all $\alpha \in \mathbb{N}_\Sigma$ and $f \in \mathcal{E}$. Substituting $\alpha = 0$, we deduce that $U|_{\mathcal{H}} = I_{\mathcal{H}}$. By Lemma 3.1, we have

$$\begin{aligned}UM_\sigma(\boldsymbol{M}^{*\alpha} f) &= U(\boldsymbol{M}^{*\alpha} M_\sigma f) = U(\boldsymbol{M}^{*\alpha} A_\sigma f) = \boldsymbol{N}^{*\alpha} A_\sigma f \\ &= \boldsymbol{N}^{*\alpha} N_\sigma f = N_\sigma \boldsymbol{N}^{*\alpha} f = N_\sigma U(\boldsymbol{M}^{*\alpha} f), \quad \alpha\in\mathbb{N}_\Sigma, \sigma\in\Sigma, f\in\mathcal{E}.\end{aligned}$$

This, when combined with $U(\mathfrak{F}_{\boldsymbol{M}}(\mathcal{E})) = \mathfrak{F}_{\boldsymbol{N}}(\mathcal{E})$, implies that $U(M_\sigma|_{\mathfrak{F}_{\boldsymbol{M}}(\mathcal{E})}) = (N_\sigma|_{\mathfrak{F}_{\boldsymbol{N}}(\mathcal{E})})U$. Taking closures and using cyclic minimality, we deduce that $UM_\sigma = N_\sigma U$ for all $\sigma \in \Sigma$. This completes the proof. □

4. Lifting Criteria

Let $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma}$ and $\boldsymbol{B} = \{B_\sigma\}_{\sigma\in\Sigma}$ be families of operators acting in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ intertwines the families $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e. $TA_\sigma \subseteq B_\sigma T$ for all $\sigma \in \Sigma$, then we write $T\boldsymbol{A} \subseteq \boldsymbol{B}T$. The class of all operators $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ intertwining $\boldsymbol{A}$ and $\boldsymbol{B}$ will be denoted by $\mathcal{I}(\boldsymbol{A}, \boldsymbol{B})$.

Consider the following general situation.

$\mathcal{H}_1$ and $\mathcal{H}_2$ are closed linear subspaces of Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively; $\mathcal{E}_1$ and $\mathcal{E}_2$ are dense linear subspaces of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E}_1)$ and $\boldsymbol{B} = \{B_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E}_2)$ are jointly subnormal families of operators; $\boldsymbol{M} = \{M_\sigma\}_{\sigma\in\Sigma}$ and $\boldsymbol{N} = \{N_\sigma\}_{\sigma\in\Sigma}$ are their minimal normal extensions of cyclic type acting in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. (4.1)

Definition 4.1. We say that an operator $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ *lifts* to $\mathcal{I}(\boldsymbol{M}, \boldsymbol{N})$ if there exists an operator $R \in \mathcal{I}(\boldsymbol{M}, \boldsymbol{N})$ such that $T \subseteq R$; R is called a *lift* of T to $\mathcal{I}(\boldsymbol{M}, \boldsymbol{N})$.

We first state a Bram type criterion for the existence of a lift, which is a multioperator version of [11, Proposition 4.1] for intertwining operators.

Proposition 4.2. *Suppose that* (4.1) *holds and* $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ *is an operator such that* $T\mathcal{E}_1 \subseteq \mathcal{E}_2$. *Then the following conditions are equivalent*:

(i) *there exists a* (*unique*) *lift* $\widehat{T}$ *of* T *to* $\mathcal{I}(\boldsymbol{M}, \boldsymbol{N})$,

(ii) *there exists a real number* $c \geq 0$ *such that*

$$\sum_{\alpha,\beta}\langle \boldsymbol{B}^{\alpha} T f_\beta, \boldsymbol{B}^{\beta} T f_\alpha\rangle \leq c \sum_{\alpha,\beta}\langle \boldsymbol{A}^{\alpha} f_\beta, \boldsymbol{A}^{\beta} f_\alpha\rangle \qquad (4.2)$$

for all finite families $\{f_\alpha\}_{\alpha\in\mathbb{N}_\Sigma} \subseteq \mathcal{E}_1$.

Moreover, if (i) *holds, then*

(iii) $T\boldsymbol{A} \subseteq \boldsymbol{B}T$,

(iv) $\|\widehat{T}\|^2 = \min\{c \geq 0 : c$ *satisfies* (ii)$\}$.

Proposition 4.2 can be proved in a similar manner as [11, Proposition 4.1]: first, we may formulate a multioperator version of [11, Proposition 3.1] for intertwining operators, and then, with its help, adapt the proof of [11, Proposition 4.1] to the present context (use Lemma 3.1 to make the reasoning applicable).

Throughout what follows, a unique lift of T is denoted by $\widehat{T}$.

Now we are ready to formulate the main result of this section. Its proof resembles to some extent the proof of [6, Theorem 3.2]. To justify the implication (iv$_{\mathcal{M}}$)⇒(iii$_{\mathcal{M}}$) below, the pivotal part of the proof, we have to adapt the original idea from [6] to the context of unbounded operators, which seems to be the main difficulty in the whole procedure.

Theorem 4.3. *Suppose that* (4.1) *holds. Assume that* $\mathcal{E}_1 = \mathscr{A}(A_\sigma)$ *and* $\mathcal{E}_2 = \mathscr{A}(B_\sigma)$ *for all* $\sigma \in \varSigma$. *Let* $\mathcal{M}$ *be a linear subspace of* $\mathcal{E}_1$ *such that*

$$\mathcal{E}_1 = \operatorname{lin}\{\boldsymbol{A}^\alpha f : f \in \mathcal{M},\, \alpha \in \mathbb{N}_\varSigma\}, \tag{4.3}$$

and let $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ *be an operator such that* $T\mathcal{E}_1 \subseteq \mathcal{E}_2$. *Then the following conditions are equivalent:*

(i) T *lifts to* $\mathcal{I}(\boldsymbol{M}, \boldsymbol{N})$,

(ii) *there exists a real number* $c \geq 0$ *such that* (4.2) *holds for all finite families* $\{f_\alpha\}_{\alpha\in\mathbb{N}_\varSigma} \subseteq \mathcal{E}_1$,

(iii) *there exists a real number* $c \geq 0$ *such that*

$$\sum_{\alpha,\beta,\gamma,\delta} \langle \boldsymbol{B}^{\beta+\gamma} T f_{\alpha,\beta}, \boldsymbol{B}^{\alpha+\delta} T f_{\gamma,\delta}\rangle \leq c \sum_{\alpha,\beta,\gamma,\delta} \langle \boldsymbol{A}^{\beta+\gamma} f_{\alpha,\beta}, \boldsymbol{A}^{\alpha+\delta} f_{\gamma,\delta}\rangle \tag{4.4}$$

for all finite families $\{f_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_\varSigma} \subseteq \mathcal{E}_1$,

(iii$_{\mathcal{M}}$) $T\boldsymbol{A} \subseteq \boldsymbol{B}T$ *and there exists a real number* $c \geq 0$ *such that* (4.4) *holds for all finite families* $\{f_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_\varSigma} \subseteq \mathcal{M}$,

(iv) *there exists a real number* $c \geq 0$ *such that*

$$\sum_{\alpha,\beta,\gamma,\delta} \langle \boldsymbol{B}^{\beta+\gamma} T f, \boldsymbol{B}^{\alpha+\delta} T f\rangle \lambda_{\alpha,\beta}\overline{\lambda_{\gamma,\delta}} \leq c \sum_{\alpha,\beta,\gamma,\delta} \langle \boldsymbol{A}^{\beta+\gamma} f, \boldsymbol{A}^{\alpha+\delta} f\rangle \lambda_{\alpha,\beta}\overline{\lambda_{\gamma,\delta}} \tag{4.5}$$

for every $f \in \mathcal{E}_1$ *and for all finite families* $\{\lambda_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_\varSigma} \subseteq \mathbb{C}$,

(iv$_{\mathcal{M}}$) $T\boldsymbol{A} \subseteq \boldsymbol{B}T$ *and there exists a real number* $c \geq 0$ *such that* (4.5) *holds for every* $f \in \mathcal{M}$ *and for all finite families* $\{\lambda_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_\varSigma} \subseteq \mathbb{C}$.

Moreover, if (i) *holds, then*

(v) $T\boldsymbol{A} \subseteq \boldsymbol{B}T$,

(vi) *the smallest real number* $c \geq 0$ *satisfying the condition* (ii) (*resp.* (iii), (iii$_{\mathcal{M}}$), (iv), (iv$_{\mathcal{M}}$)), *is equal to* $\|\widehat{T}\|^2$.

Proof. The equivalence of conditions (i) and (ii) as well as the implication (i)⇒(v) follow from Proposition 4.2.

(ii)⇒(iii) From the implications (ii)⇒(i) and (i)⇒(v), we infer that $T\boldsymbol{A} \subseteq \boldsymbol{B}T$. Let $c \geq 0$ be as in (ii). Take a finite family $\{f_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_\Sigma} \subseteq \mathcal{E}_1$. We define a finite family $\{g_\alpha\}_{\alpha\in\mathbb{N}_\Sigma} \subseteq \mathcal{E}_1$ by

$$g_\alpha = \sum_\beta \boldsymbol{A}^\beta f_{\alpha,\beta}, \quad \alpha \in \mathbb{N}_\Sigma. \tag{4.6}$$

Then $T\boldsymbol{A} \subseteq \boldsymbol{B}T$ implies that

$$\begin{aligned}\sum_{\alpha,\beta}\langle \boldsymbol{B}^\alpha T g_\beta, \boldsymbol{B}^\beta T g_\alpha\rangle &= \sum_{\alpha,\beta}\Big\langle \boldsymbol{B}^\alpha T\Big(\sum_\gamma \boldsymbol{A}^\gamma f_{\beta,\gamma}\Big), \boldsymbol{B}^\beta T\Big(\sum_\delta \boldsymbol{A}^\delta f_{\alpha,\delta}\Big)\Big\rangle \\ &= \sum_{\alpha,\beta,\gamma,\delta}\langle \boldsymbol{B}^{\gamma+\alpha} T f_{\beta,\gamma}, \boldsymbol{B}^{\beta+\delta} T f_{\alpha,\delta}\rangle. \end{aligned} \tag{4.7}$$

Clearly, we have

$$\sum_{\alpha,\beta}\langle \boldsymbol{A}^\alpha g_\beta, \boldsymbol{A}^\beta g_\alpha\rangle = \sum_{\alpha,\beta,\gamma,\delta}\langle \boldsymbol{A}^{\gamma+\alpha} f_{\beta,\gamma}, \boldsymbol{A}^{\beta+\delta} f_{\alpha,\delta}\rangle. \tag{4.8}$$

That (4.4) is satisfied with the constant c follows from (4.2), (4.7) and (4.8).

(iii)⇒(ii) Let $c \geq 0$ be as in (iii). Take a finite family $\{f_\alpha\}_{\alpha\in\mathbb{N}_\Sigma} \subseteq \mathcal{E}_1$. For $\alpha, \beta \in \mathbb{N}_\Sigma$, set $g_{\alpha,\beta} = f_\alpha$ if $\beta = 0$ and $g_{\alpha,\beta} = 0$ otherwise. Then, we have

$$\sum_{\alpha,\beta,\gamma,\delta}\langle \boldsymbol{B}^{\beta+\gamma} T g_{\alpha,\beta}, \boldsymbol{B}^{\alpha+\delta} T g_{\gamma,\delta}\rangle = \sum_{\alpha,\gamma}\langle \boldsymbol{B}^\gamma T f_\alpha, \boldsymbol{B}^\alpha T f_\gamma\rangle,$$
$$\sum_{\alpha,\beta,\gamma,\delta}\langle \boldsymbol{A}^{\beta+\gamma} g_{\alpha,\beta}, \boldsymbol{A}^{\alpha+\delta} g_{\gamma,\delta}\rangle = \sum_{\alpha,\gamma}\langle \boldsymbol{A}^\gamma f_\alpha, \boldsymbol{A}^\alpha f_\gamma\rangle.$$

This and (4.4) imply (4.2) with the same constant c.

Consider the commutative $*$-semigroup $\Omega_\Sigma = \mathbb{N}_\Sigma \times \mathbb{N}_\Sigma$ (see Section 2). Given a real number $c \geq 0$, we define the form φ_c over $(\Omega_\Sigma, \mathcal{M})$ by

$$\varphi_c((\alpha,\beta); f, g) = c\langle \boldsymbol{A}^\alpha f, \boldsymbol{A}^\beta g\rangle - \langle \boldsymbol{B}^\alpha T f, \boldsymbol{B}^\beta T g\rangle, \quad f, g \in \mathcal{M}, (\alpha,\beta) \in \Omega_\Sigma.$$

(iv$_\mathcal{M}$)⇒(iii$_\mathcal{M}$) Let $c \geq 0$ be as in (iv$_\mathcal{M}$). It follows from (iv$_\mathcal{M}$) that the form φ_c is weakly positive definite. Consider the set $\mathcal{G} = \{(\delta_\sigma, 0) : \sigma \in \Sigma\}$ of $*$-generators of the $*$-semigroup Ω_Σ. We show that

$$\mathscr{A}_{\varphi_c}(\delta_\sigma, 0) = \mathcal{M} \quad \text{for every } \sigma \in \Sigma. \tag{4.9}$$

Indeed, if $\sigma \in \Sigma$ and $f \in \mathcal{M} \subseteq \mathscr{A}(A_\sigma)$, then there exists a real number $t = t(f) > 0$ such that

$$\sum_{n=0}^\infty \|A_\sigma^n f\| \frac{t^n}{n!} < \infty. \tag{4.10}$$

Since φ_c is weakly positive definite, we get

$$\begin{aligned} 0 &\leq \varphi_c(n\cdot((\delta_\sigma,0)^* + (\delta_\sigma,0)); f, f) = \varphi_c(n\cdot(\delta_\sigma,\delta_\sigma); f, f) \\ &= c\|A_\sigma^n f\|^2 - \|B_\sigma^n T f\|^2 \leq c\|A_\sigma^n f\|^2, \quad n \in \mathbb{N}, \sigma \in \Sigma. \end{aligned}$$

This together with (4.10) yields

$$\sum_{n=0}^\infty \sqrt{\varphi_c(n\cdot((\delta_\sigma,0)^* + (\delta_\sigma,0)); f, f)}\,\frac{t^n}{n!} < \infty.$$

Hence, $f \in \mathscr{A}_{\varphi_c}(\delta_\sigma, 0)$ for every $\sigma \in \Sigma$, which leads to (4.9). A combination of (2.1), (4.9) and Lemma 2.1 implies that the form φ_c is positive definite. As a consequence, (iii$_{\mathcal{M}}$) holds.

The implication (iii$_{\mathcal{M}}$)⇒(iv$_{\mathcal{M}}$) follows directly from the fact that every positive definite form is weakly positive definite.

Arguing as above with $\mathcal{M} = \mathcal{E}_1$, we see that (iii) and (iv) are equivalent (in the proofs of (iii$_{\mathcal{M}}$)⇔(iv$_{\mathcal{M}}$) and (iii)⇔(iv), we do not use the inclusion $T\boldsymbol{A} \subseteq \boldsymbol{B}T$).

(iii$_{\mathcal{M}}$)⇒(ii) Let $c \geq 0$ be as in (iii$_{\mathcal{M}}$). By (4.3), for a finite family $\{g_\alpha\}_{\alpha \in \mathbb{N}_\Sigma} \subseteq \mathcal{E}_1$, there exists a finite family $\{f_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}_\Sigma} \subseteq \mathcal{M}$ satisfying (4.6). Arguing as in the proof of the implication (ii)⇒(iii), we see that (4.2) holds with $\{g_\alpha\}_{\alpha \in \mathbb{N}_\Sigma}$ in place of $\{f_\alpha\}_{\alpha \in \mathbb{N}_\Sigma}$ (here we make of use of the assumption that $T\boldsymbol{A} \subseteq \boldsymbol{B}T$).

(iii$_{\mathcal{M}}$) follows from (ii), because, as shown above, (ii) implies (iii) and (v).

The reader can easily convince himself that all of conditions (ii), (iii), (iii$_{\mathcal{M}}$), (iv) and (iv$_{\mathcal{M}}$) are equivalent to each other with the same constants c. Thus, (vi) follows from the implication (ii)⇒(iv) of Proposition 4.2. This completes the proof. □

5. A Local Lifting Theorem for Unbounded Operators

Let $\mathcal{E}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$. Consider a family of commuting operators $\boldsymbol{A} = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$. For $f \in \mathcal{E}$ and $\sigma \in \Sigma$, we define

$$\mathcal{E}_{\boldsymbol{A},f} = \operatorname{lin}\{\boldsymbol{A}^\alpha f : \alpha \in \mathbb{N}_\Sigma\}, \quad \mathcal{Q}_{\boldsymbol{A},f} = \overline{\mathcal{E}_{\boldsymbol{A},f}} \quad \text{and} \quad A_{\sigma,f} = A_\sigma|_{\mathcal{E}_{\boldsymbol{A},f}}.$$

The family $\{A_{\sigma,f}\}_{\sigma \in \Sigma}$ will be denoted briefly by $\boldsymbol{A}_f$. Clearly, $\boldsymbol{A}_f \subseteq \boldsymbol{L}(\mathcal{E}_{\boldsymbol{A},f})$. Since $\boldsymbol{A}$ is a family of commuting operators, so also is $\boldsymbol{A}_f$. If $\boldsymbol{A}$ is jointly subnormal and $\boldsymbol{M} = \{M_\sigma\}_{\sigma \in \Sigma}$ is a normal extension of $\boldsymbol{A}$, then for every $f \in \mathcal{E}$, we have

$$A_{\sigma,f} \subseteq A_\sigma \subseteq M_\sigma, \quad \sigma \in \Sigma,$$

so the family $\boldsymbol{A}_f$ is jointly subnormal. For $g \in \mathcal{D}^\infty(\boldsymbol{M})$ and $\sigma \in \Sigma$, we put

$$\mathcal{F}^{\boldsymbol{M},g} = \operatorname{lin}\{\boldsymbol{M}^{*\alpha}\boldsymbol{M}^\beta g : \alpha, \beta \in \mathbb{N}_\Sigma\}, \quad \mathcal{Q}^{\boldsymbol{M},g} = \overline{\mathcal{F}^{\boldsymbol{M},g}}, \quad M_\sigma^g = \overline{M_\sigma|_{\mathcal{F}^{\boldsymbol{M},g}}}.$$

The definition of M_σ^g makes sense due to Lemma 3.1. Let us denote by $\boldsymbol{M}^g$ the family $\{M_\sigma^g\}_{\sigma \in \Sigma}$. If $f \in \mathcal{E}$, then $\mathcal{E}_{\boldsymbol{A},f} \subseteq \mathcal{F}^{\boldsymbol{M},f}$ and clearly $\mathcal{Q}_{\boldsymbol{A},f} \subseteq \mathcal{Q}^{\boldsymbol{M},f}$. Therefore,

$$A_{\sigma,f} \subseteq M_\sigma^f \subseteq M_\sigma, \quad \sigma \in \Sigma.$$

A relationship between families $\boldsymbol{A}_f$ and $\boldsymbol{M}^f$ is elucidated below.

Theorem 5.1. *Let $\boldsymbol{M} = \{M_\sigma\}_{\sigma \in \Sigma}$ be a normal extension of a jointly subnormal family $\boldsymbol{A} = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq \boldsymbol{L}(\mathcal{E})$ such that $\mathcal{E} = \mathscr{A}(A_\sigma)$ for all $\sigma \in \Sigma$. If $f \in \mathcal{E}$, then*

(i) *$\mathcal{Q}^{\boldsymbol{M},f}$ reduces M_σ to M_σ^f for every $\sigma \in \Sigma$,*
(ii) *$\boldsymbol{M}^f$ is a minimal normal extension of $\boldsymbol{A}_f$ of cyclic type.*

Proof. Since $\mathcal{E} = \mathscr{A}(A_\sigma)$ for every $\sigma \in \Sigma$, we see that $\mathcal{E}_{\boldsymbol{A},f} = \mathscr{A}(A_{\sigma,f})$ for all $\sigma \in \Sigma$. Noticing that $\mathcal{F}_{\boldsymbol{M}}(\mathcal{E}_{\boldsymbol{A},f}) = \mathcal{F}^{\boldsymbol{M},f}$ (see (3.3) for the notation), we may apply Theorem 3.2 to the jointly subnormal family $\boldsymbol{A}_f$ and its normal extension $\boldsymbol{M}$. This completes the proof. □

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be dense linear subspaces of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Consider families of operators $\boldsymbol{A} = \{A_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E}_1)$ and $\boldsymbol{B} = \{B_\sigma\}_{\sigma\in\Sigma} \subseteq \boldsymbol{L}(\mathcal{E}_2)$. Set $\boldsymbol{S} = (\boldsymbol{A}, \boldsymbol{B})$. Take an operator $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T\boldsymbol{A} \subseteq \boldsymbol{B}T$. Then $T\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $T(\mathcal{E}_{\boldsymbol{A},f}) \subseteq \mathcal{E}_{\boldsymbol{B},Tf}$ for all $f \in \mathcal{E}_1$. Owing to the boundedness of T, this yields $T(\mathcal{Q}_{\boldsymbol{A},f}) \subseteq \mathcal{Q}_{\boldsymbol{B},Tf}$ for all $f \in \mathcal{E}_1$. Define the operator $T_{\boldsymbol{S},f}$ via

$$T_{\boldsymbol{S},f} = T|_{\mathcal{Q}_{\boldsymbol{A},f}} \in \boldsymbol{B}(\mathcal{Q}_{\boldsymbol{A},f}, \mathcal{Q}_{\boldsymbol{B},Tf}).$$

It follows from $T\boldsymbol{A} \subseteq \boldsymbol{B}T$ that

$$T_{\boldsymbol{S},f}\, \boldsymbol{A}_f \subseteq \boldsymbol{B}_{Tf}\, T_{\boldsymbol{S},f}, \quad f \in \mathcal{E}_1. \tag{5.1}$$

Now we formulate a version of [6, Lemma 4.1] for families of operators.

Lemma 5.2. *Suppose that* (4.1) *holds. Assume that* $\mathcal{E}_1 = \mathscr{A}(A_\sigma)$ *and* $\mathcal{E}_2 = \mathscr{A}(B_\sigma)$ *for all* $\sigma \in \Sigma$. *Take an operator* $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $T\boldsymbol{A} \subseteq \boldsymbol{B}T$ *and fix a vector* $f \in \mathcal{E}_1$. *Then the following conditions are equivalent* (*with* $\boldsymbol{S} := (\boldsymbol{A}, \boldsymbol{B})$):

- (i) $T_{\boldsymbol{S},f}$ *lifts to* $\mathcal{I}(\boldsymbol{M}^f, \boldsymbol{N}^{Tf})$,
- (ii) *there exists a real number* $c \geq 0$ *such that* (4.5) *holds for all finite families* $\{\lambda_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_\Sigma} \subseteq \mathbb{C}$.

Moreover, if (i) *holds, then* $\|\widehat{T_{\boldsymbol{S},f}}\|^2 = \min\{c \geq 0 : c$ *satisfies* (ii)$\}$.

Proof. In view of (5.1) and Theorem 5.1, we can apply Theorem 4.3 to $\mathcal{M} = \mathbb{C}f$, the intertwining operator $T_{\boldsymbol{S},f}$, jointly subnormal families $\boldsymbol{A}_f$ and $\boldsymbol{B}_{Tf}$, and their minimal normal extensions of cyclic type $\boldsymbol{M}^f$ and $\boldsymbol{N}^{Tf}$, respectively. □

The following theorem, which is the main result of the paper, generalizes a local lifting theorem (cf. [6, Theorem 4.2]) to the case of pairs of jointly subnormal families of unbounded operators.

Theorem 5.3. *Suppose that* (4.1) *holds. Assume that* $\mathcal{E}_1 = \mathscr{A}(A_\sigma)$ *and* $\mathcal{E}_2 = \mathscr{A}(B_\sigma)$ *for all* $\sigma \in \Sigma$. *Let* $\mathcal{M}$ *be a linear subspace of* $\mathcal{E}_1$ *such that*

$$\mathcal{E}_1 = \operatorname{lin}\{\boldsymbol{A}^\alpha f : f \in \mathcal{M},\ \alpha \in \mathbb{N}_\Sigma\}$$

and let $T \in \boldsymbol{B}(\mathcal{H}_1, \mathcal{H}_2)$ *be an operator such that* $T\mathcal{E}_1 \subseteq \mathcal{E}_2$. *Then the following conditions are equivalent* (*with* $\boldsymbol{S} := (\boldsymbol{A}, \boldsymbol{B})$):

- (i) T *lifts to* $\mathcal{I}(\boldsymbol{M}, \boldsymbol{N})$,
- (ii) $T\boldsymbol{A} \subseteq \boldsymbol{B}T$, *the operator* $T_{\boldsymbol{S},f}$ *lifts to* $\mathcal{I}(\boldsymbol{M}^f, \boldsymbol{N}^{Tf})$ *for every* $f \in \mathcal{M}$, *and*

$$\sup_{f\in\mathcal{M}} \|\widehat{T_{\boldsymbol{S},f}}\| < \infty.$$

Moreover, if (i) *holds, then*

$$\|\widehat{T}\| = \sup_{f\in\mathcal{M}} \|\widehat{T_{\boldsymbol{S},f}}\|. \tag{5.2}$$

Proof. The proof of the implication (i)⇒(ii) requires more care than in the case of bounded operators. Let $\widehat{T}$ be the lift of T to $\mathcal{I}(\boldsymbol{M},\boldsymbol{N})$. Fix $f \in \mathcal{M}$. By the Putnam-Fuglede theorem (cf. [10]), $\widehat{T}\boldsymbol{M} \subseteq \boldsymbol{N}\widehat{T}$ implies $\widehat{T}\boldsymbol{M}^* \subseteq \boldsymbol{N}^*\widehat{T}$. These two inclusions and $T \subseteq \widehat{T}$ guarantee that $\widehat{T}|_{\mathcal{Q}^{\boldsymbol{M},f}} \in \boldsymbol{B}(\mathcal{Q}^{\boldsymbol{M},f}, \mathcal{Q}^{\boldsymbol{N},Tf})$ and $T_{\boldsymbol{S},f} \subseteq \widehat{T}|_{\mathcal{Q}^{\boldsymbol{M},f}}$. Next, we show that

$$\widehat{T}h \in \mathcal{D}(N_\sigma^{Tf}) \quad \text{for all } h \in \mathcal{D}(M_\sigma^f) \text{ and } \sigma \in \Sigma. \tag{5.3}$$

Indeed, since $h \in \mathcal{D}(M_\sigma^f) \subseteq \mathcal{D}(M_\sigma)$ and $\widehat{T}\boldsymbol{M} \subseteq \boldsymbol{N}\widehat{T}$, we deduce that $\widehat{T}h \in \mathcal{D}(N_\sigma)$. As $h \in \mathcal{Q}^{\boldsymbol{M},f}$, we have $\widehat{T}h = \widehat{T}|_{\mathcal{Q}^{\boldsymbol{M},f}}h \in \mathcal{Q}^{\boldsymbol{N},Tf}$. This, combined with the fact that $\mathcal{Q}^{\boldsymbol{N},Tf}$ reduces the operator N_σ to N_σ^{Tf} (cf. Theorem 5.1), leads to

$$\widehat{T}h \in \mathcal{D}(N_\sigma) \cap \mathcal{Q}^{\boldsymbol{N},Tf} = \mathcal{D}(N_\sigma^{Tf}),$$

which proves (5.3). The condition (5.3) together with $\widehat{T}\boldsymbol{M} \subseteq \boldsymbol{N}\widehat{T}$ implies that $\widehat{T}|_{\mathcal{Q}^{\boldsymbol{M},f}}\boldsymbol{M}^f \subseteq \boldsymbol{N}^{Tf}\,\widehat{T}|_{\mathcal{Q}^{\boldsymbol{M},f}}$. Hence, by the uniqueness of $\widehat{T_{\boldsymbol{S},f}}$, we have $\widehat{T_{\boldsymbol{S},f}} = \widehat{T}|_{\mathcal{Q}^{\boldsymbol{M},f}}$, which also shows that the right-hand side of (5.2) is less than or equal to its left-hand side.

The implication (ii)⇒(i) can be deduced from Lemma 5.2 and Theorem 4.3. On the way we also verify that the left-hand side of (5.2) is less than or equal to its right-hand side. This completes the proof. □

6. Local Commutativity

Let us concentrate on a single subnormal operator $A \in \boldsymbol{B}(\mathcal{H})$. For simplicity, we write "minimal normal extension" in place of "minimal normal extension of cyclic type". Suppose for a moment that $T \in \boldsymbol{B}(\mathcal{H})$ is an operator that commutes with A (this requirement is necessary for T to lift to the commutant of a minimal normal extension of A). Then $T\mathcal{Q}_{A,f} \subseteq \mathcal{Q}_{A,Tf}$ and consequently $T|_{\mathcal{Q}_{A,f}}A|_{\mathcal{Q}_{A,f}} = A|_{\mathcal{Q}_{A,Tf}}T|_{\mathcal{Q}_{A,f}}$ for all $f \in \mathcal{H}$. The question arises under what circumstances $T\mathcal{Q}_{A,f} \subseteq \mathcal{Q}_{A,f}$ for all $f \in \mathcal{H}$. Note that if this is the case, then $T|_{\mathcal{Q}_{A,f}}A|_{\mathcal{Q}_{A,f}} = A|_{\mathcal{Q}_{A,f}}T|_{\mathcal{Q}_{A,f}}$ for all $f \in \mathcal{H}$, which is referred to as the *local commutativity* of A and T. This question can be extended to the case in which A and T are not assumed to commute. In Proposition 6.1 we provide a complete answer to the extended question. It is worth pointing out that the operators A and T must commute if all cyclic invariant subspaces $\mathcal{Q}_{A,f}$ of A are invariant for T.

Given an operator $A \in \boldsymbol{B}(\mathcal{H})$ and a set $\mathcal{A}$ of closed linear subspaces of $\mathcal{H}$, we write $\mathrm{Lat}(A)$ for the set of all closed linear subspaces of $\mathcal{H}$ invariant for A and $\mathrm{Alg}(\mathcal{A})$ for the set of all operators $T \in \boldsymbol{B}(\mathcal{H})$ such that $\mathcal{A} \subseteq \mathrm{Lat}(T)$. Denote by $\mathcal{W}(A)$ the closure in the weak operator topology (equivalently: in the strong operator topology) of the algebra generated by $\{A, I_{\mathcal{H}}\}$. It is clear that $\mathcal{W}(A) \subseteq \mathrm{Alg}(\mathrm{Lat}(A))$. In general, the reverse implication is not true (cf. [5], [19, Example 7]; for more information on the subject see also [3,4]). The Olin-Thomson theorem states that $\mathcal{W}(A) = \mathrm{Alg}(\mathrm{Lat}(A))$ for any subnormal operator $A \in \boldsymbol{B}(\mathcal{H})$ (cf. [9, Theorem 3]).

Proposition 6.1. *Let $A \in \boldsymbol{B}(\mathcal{H})$ be a subnormal operator. If $T \in \boldsymbol{B}(\mathcal{H})$, then the following conditions are equivalent*:

(i) $T\mathcal{Q}_{A,f} \subseteq \mathcal{Q}_{A,f}$ *for all* $f \in \mathcal{H}$, *where* $\mathcal{Q}_{A,f}$ *is the smallest closed linear subspace of* $\mathcal{H}$ *containing* f *and invariant for* A,

(ii) $T \in \mathcal{W}(A)$.

Proof. (i)⇒(ii) In view of the Olin-Thomson theorem, it is enough to show that $T \in \mathrm{Alg}(\mathrm{Lat}(A))$. Take $\mathcal{M} \in \mathrm{Lat}(A)$. If $f \in \mathcal{M}$, then $f \in \mathcal{Q}_{A,f} \subseteq \mathcal{M}$. Hence, by (i), we have $Tf \in \mathcal{Q}_{A,f} \subseteq \mathcal{M}$. This means that $\mathcal{M} \in \mathrm{Lat}(T)$.

(ii)⇒(i) Take $f \in \mathcal{H}$. Since $\mathcal{Q}_{A,f} \in \mathrm{Lat}(A)$, we see that $\mathcal{Q}_{A,f} \in \mathrm{Lat}(p(A))$ for every complex polynomial p in one indeterminate, which implies that $\mathcal{Q}_{A,f} \in \mathrm{Lat}(R)$ for every $R \in \mathcal{W}(A)$. This, together with (ii), gives (i). □

We conclude the paper with a simple example of an operator T which lifts to the commutant of a minimal normal extension of a subnormal operator A, and which has the property that not all cyclic invariant subspaces of A are invariant for T. This shows that a (global) lift of a member T of the commutant of a subnormal operator A may exist though the local commutativity of A and T does not hold. In fact, it may even happen that for some vector f the linear spaces $\mathcal{Q}_{A,f}$ and $T\mathcal{Q}_{A,f}$ are orthogonal.

Example 6.2. Given a bounded Borel function ψ on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, we denote by M_ψ the operator of multiplication by ψ on $L^2(\mathbb{T})$. Let ξ be the identity function on $\mathbb{T}$. Set $T = M_\xi|_{H^2}$ and $A = T^2$, where H^2 stands for the Hardy space regarded as a closed linear subspace of $L^2(\mathbb{T})$. The operator A is subnormal (as an isometric operator) and $TA = AT$. Observe that if $f = \mathbb{1}$, then $A^n f = \xi^{2n}$ for all $n \in \mathbb{N}$, which implies that $\mathcal{Q}_{A,f} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}\cdot\xi^{2n}$. Since $Tf = \xi$, we deduce that $T\mathcal{Q}_{A,f} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}\cdot\xi^{2n+1}$. Therefore, the linear spaces $\mathcal{Q}_{A,f}$ and $T\mathcal{Q}_{A,f}$ are orthogonal. Note that the unitary operator M_{ξ^2} is a minimal normal extension of the isometry A (because $\bigvee_{n\in\mathbb{N}} M_{\xi^2}^{*n}(H^2) = L^2(\mathbb{T})$, which follows from the equalities $M_{\xi^2}^{*n} f = \bar{\xi}^{\,2n}$ and $M_{\xi^2}^{*n}\xi = \bar{\xi}^{\,2n-1}$, $n \geq 1$). Clearly, M_ξ is the lift of T to the commutant of M_{ξ^2}.

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