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Integral Equations and Operator Theory

The Pair of Operators $T^{[*]}T$ and $TT^{[*]}$: J-Dilations and Canonical Forms

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Abstract. We describe a procedure of dilating an operator T in an infinite dimensional Krein space, such that many of the spectral and algebraic properties of the operators $T^{[*]}T$ and $TT^{[*]}$ are preserved. We use the procedure to study canonical forms of those two operators in a finite dimensional Krein space.

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0. Introduction

The problem of comparing the operators $T^{[*]}T$ and $TT^{[*]}$ in indefinite inner product spaces has already attracted some attention. One of the motivations was a result in [13] stating that a matrix T admits polar decomposition if and only if the canonical forms of $T^{[*]}T$ and $TT^{[*]}$ are the same. In the finite dimensional situation canonical forms of the matrices in question were considered in [9] for some special cases. Later on those results were generalized in Theorem 3.2 in [12] to provide a full description. A related result concerning an analogue of the singular value decomposition can be found in [3]. On the other hand, the infinite dimensional case is far from being fully understood. For example, zero can be a singular critical point of one of the operators, while it is in the positive spectrum of the other operator. Further examples can be found in [15], where the notions of regular and singular critical point were studied for the pair $T^{[*]}T$ and $TT^{[*]}$. In [14] the same pair of operators was studied in the context of local definitizability. The present paper treats both the infinite and the finite dimensional case, since in its course we shall present an alternative proof of one of the main results of [12]. This will

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follow from more general considerations which hold in the infinite dimensional situation.

The main tool in this paper is a method of dilation (reduction) for the operator T, which is quite natural for the study of the properties of $T^{[*]}T$ and $TT^{[*]}$. This construction, which we called J-dilation, has its origins in [8], while being also similar to a construction implicitly used in [12]. Despite its usefulness, so far this kind of dilation has not been studied systematically. Therefore, in the present paper we consider which properties of $T^{[*]}T$ are preserved under the J-dilation procedure.

The first four sections are devoted to the general situation in infinite dimensional Krein spaces. We prove that spectral properties at nonzero points in the complex plane, as well as definitizability and nilpotency are preserved. In Sect. 5 we present material that is valid in the general setting, but is on the other hand tailored to the study of the finite dimensional case. Our result here is a complete description of how Jordan chains corresponding to the zero eigenvalue of $T^{[*]}T$ behave under the J-dilation. This leads to the proof of the quoted result in [12], which is presented in Sect. 6. In this light, in the subsequent section, we see the result on polar decomposition of [13] from a different angle. We conclude the paper with a concrete example.

1. *J*-Dilations and *J*-Restrictions

We assume background knowledge on Krein spaces, see [1,6] for wide treatments of the subject. The indefinite inner product on a Krein space is always denoted by $[\,\cdot\,,\,\cdot\,]$, even if there is more than one space in question. We use a Hilbert space structure on a Krein space only in a few examples. The theorems are formulated entirely in the Krein space language.

By a *subspace* of a Krein space $\mathcal{H}$ we mean a closed linear space $\mathcal{H}_0 \subseteq \mathcal{H}$ with the indefinite inner product inherited from $\mathcal{H}$. The space $\mathcal{H}_0$ is not necessarily a Krein space itself. If $\mathcal{H}$ is a Pontriagin space then the necessary and sufficient condition for $\mathcal{H}_0$ being a Pontriagin space is its nondegeneracy. By $\mathcal{E} \dotplus \mathcal{F}$ we mean a direct sum of two subspaces, we will write $\mathcal{E} \,[\dotplus]\, \mathcal{F}$ if the spaces are additionally $[\,\cdot\,,\,\cdot\,]$-orthogonal.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Krein spaces and let T belong to the space $\mathbf{B}(\mathcal{H},\mathcal{K})$ of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Then by $T^{[*]}$ we mean the Krein space adjoint of the operator T. We define now the main object of the paper.

Definition 1.1. Let $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{H}$, $\mathcal{K}$ be Krein spaces. We say that an operator $T \in \mathbf{B}(\mathcal{H},\mathcal{K})$ is a *J-dilation of* $T_0 \in \mathbf{B}(\mathcal{H}_0,\mathcal{K}_0)$ (or conversely T_0 is an *J-restriction of* T) if the following three conditions are satisfied:

(i) $\mathcal{H}_0$ is a subspace of $\mathcal{H}$, $\mathcal{K}_0$ is a subspace of $\mathcal{K}$.

(ii) There exist subspaces $\mathcal{H}_i$ of $\mathcal{H}$ and $\mathcal{K}_i$ of $\mathcal{K}$ $(i = 1, 2, 3)$ such that

$$\mathcal{H} = \mathcal{H}_0 \,[\dotplus]\, \mathcal{H}_1 \,[\dotplus]\, (\mathcal{H}_2 \dotplus \mathcal{H}_3), \qquad \mathcal{K} = \mathcal{K}_0 \,[\dotplus]\, \mathcal{K}_1 \,[\dotplus]\, (\mathcal{K}_2 \dotplus \mathcal{K}_3),$$

where $\mathcal{H}_1$ and $\mathcal{K}_1$ are Krein spaces, $\mathcal{H}_2$ and $\mathcal{H}_3$ ($\mathcal{K}_2$ and $\mathcal{K}_3$) are skewly linked neutral spaces such that $\mathcal{H}_2 \dotplus \mathcal{H}_3$ ($\mathcal{K}_2 \dotplus \mathcal{K}_3$) is a Krein space.

(iii) The operator T has a following representation with respect to the above decomposition

$$T = \begin{pmatrix} T_0 & 0 & T_{02} & 0 \\ 0 & 0 & 0 & 0 \\ T_{20} & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

Note that (ii) implies that with $x_i, y_i \in \mathcal{H}_i$ $(0 = 1, \ldots, 3)$ we have

$$[x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3] = [x_0, y_0] + [x_1, y_1] + [x_2, y_3] + [x_3, y_2]. \tag{1.2}$$

A similar formula holds for $\mathcal{K}$ as well. The J-dilation (J-restriction) will be called *rigid* if

$$\ker T = \mathcal{H}_1 \dot{+} \mathcal{H}_3, \qquad \overline{\operatorname{im} T} = \mathcal{K}_0 \dot{+} \mathcal{K}_2. \tag{1.3}$$

Note that in such case

$$\mathcal{H}_3 = \ker T \cap \ker T^{[\perp]}, \qquad \mathcal{K}_2 = \overline{\operatorname{im} T} \cap \overline{\operatorname{im} T}^{[\perp]}. \tag{1.4}$$

Example 1. Let us analyze the following classical example (see e.g. [8,15] for extensions and modifications). Let $\mathcal{H} = \mathcal{K}$ be the space $L^2[0,1] \times \mathbb{C}^2$ with the Π_1-inner product defined by the fundamental symmetry $J(f, x, y) = (f, y, x)$ for all $f \in L^2[0,1]$, $x, y \in \mathbb{C}$. Consider the selfadjoint operator

$$T := \begin{pmatrix} M_{\sqrt{t}} & 0 & \pi(\mathbf{1}) \\ \langle \cdot, \mathbf{1} \rangle & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $M_\phi \in \mathbf{B}(L^2[0,1])$ denotes the multiplication operator by a bounded measurable function ϕ, $\pi(g)$ (where $g \in L^2[0,1]$) maps $x \in \mathbb{C}$ to xg and $\mathbf{1} \in L^2[0,1]$ is a function constantly equal one. Note that T, after interchanging the last two columns, is already in the form (1.1) with $T_0 = M_{\sqrt{t}}$. Zero is a singular critical point of the operator

$$T^{[*]}T = T^2 = \begin{pmatrix} M_t & 0 & \pi(\sqrt{t}) \\ \langle \cdot, \sqrt{t} \rangle & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

since $\ker T^2 = \ker T$ is a degenerate space. On the other hand T_0^2 does not have any critical points. And so we have discovered the first property that is not being preserved by J-dilations. The next, a kind of obvious one, is the number of negative squares of the underlying space, see also Example 3.

Although we have put the definitons of J-dilation and J-restriction in together, they are actually two different notions. The first one requires finding an outer space, while the latter one says something about the inner structure of the operator. This splitting reflects also in the following three results.

Proposition 1.2. *Given T_0 there always exists a rigid J-dilation T of T_0 with $\operatorname{im} T$ closed.*

Proof. If both spaces $\mathcal{H}_0$ and $\mathcal{K}_0$ are trivial then the construction of a rigid J-dilation is obvious. Suppose now that at least one of the spaces $\mathcal{H}_0$ and $\mathcal{K}_0$ is nontrivial. We show first that there exist operators $A \in \mathbf{B}(\mathcal{H}_0)$, $B \in \mathbf{B}(\mathcal{K}_0, \mathcal{H}_0)$, $C \in \mathbf{B}(\mathcal{K}_0)$ such that the block operator matrix

$$\begin{pmatrix} A & B \\ T_0 & C \end{pmatrix} \in \mathbf{B}(\mathcal{H}_0 \times \mathcal{K}_0)$$

is boundedly invertible. Here we understand $\mathcal{H}_0 \times \mathcal{K}_0$ as being equipped with the (unique) Banach space topology it has as a product of two Krein spaces (each of which has a unique Banach space topology induced by the Krein space structure). If the space $\mathcal{H}_0$ ($\mathcal{K}_0$) is trivial, then it is enough to set C (or A) as a boundedly invertible operator. If both spaces $\mathcal{K}_0$ and $\mathcal{H}_0$ are nontrivial, it is enough to choose A and C boundedly invertible and set $B = 0$. Then, by the Schur's complement reasoning, zero is in the resolvent of the above block operator matrix. We set $\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_1 \times \mathcal{K}_0 \times \mathcal{K}_0$ and $\mathcal{K} = \mathcal{K}_0 \times \mathcal{K}_1 \times \mathcal{H}_0 \times \mathcal{H}_0$, where the spaces $\mathcal{H}_1$ and $\mathcal{K}_1$ can be chosen arbitrary. The indefinite inner product on $\mathcal{H}$ is given by

$$\left[\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}\right] := [x_0, y_0]_{\mathcal{H}_0} + [x_1, y_1]_{\mathcal{H}_1} + [x_2, y_3]_{\mathcal{K}_0} + [x_3, y_2]_{\mathcal{K}_0},$$

an analogous formula defines the inner product on $\mathcal{K}$. We identify $\mathcal{H}_0$ and $\mathcal{H}_1$ ($\mathcal{K}_0$ and $\mathcal{K}_1$) with the first and the second component of $\mathcal{H}$ ($\mathcal{K}$) respectively. Moreover, we set $\mathcal{H}_2 := \{0\} \times \{0\} \times \mathcal{K}_0 \times \{0\}$, $\mathcal{H}_3 =: \{0\} \times \{0\} \times \{0\} \times \mathcal{K}_0$, $\mathcal{K}_2 := \{0\} \times \{0\} \times \mathcal{H}_0 \times \{0\}$, $\mathcal{K}_3 =: \{0\} \times \{0\} \times \{0\} \times \mathcal{H}_0$. Finally, we define

$$T = \begin{pmatrix} T_0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \\ A & 0 & B & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

□

Example 2. We present[1] an operator that does not have a rigid J-restriction. Let $\mathcal{L}$ be a closed, strictly positive but not uniformly positive subspace of a Krein space $\mathcal{K}$ and let $G \in \mathbf{B}(\mathcal{K})$ be a fundamental symmetry. The space $\mathcal{K}$ can be written as an $\langle \cdot, \cdot \rangle$-orthogonal sum of $\mathcal{L}$ and its $\langle \cdot, \cdot \rangle$-orthogonal complement $\mathcal{L}^{\langle \perp \rangle}$, where $\langle \cdot, \cdot \rangle$ stands for the Hilbert space inner product $[G \cdot, \cdot]$. The operator T defined as zero on $\mathcal{L}$ and identity on $\mathcal{L}^{\langle \perp \rangle}$ is continuous. Suppose it has a rigid J-restriction T_0. Since $\ker T$ is non-degenerate, $\mathcal{H}_3$ equals $\{0\}$. Consequently, $\mathcal{H}_1 = \ker T$. But $\ker T$ endowed with the original inner product inherited from $\mathcal{K}$ is not a Krein space, contradiction.

On the other hand T has a nontrivial J-restriction. Indeed, let $e \in \mathcal{L} \setminus \{0\}$. Then $\mathcal{H}_1 = \operatorname{lin}\{e\}$ and $\mathcal{K}_1 = \operatorname{lin}\{Ge\}$ are a Krein spaces (with the original inner product $[\cdot, \cdot]$). We set $\mathcal{H}_0 = \mathcal{H}_1^{[\perp]}$, $\mathcal{K}_0 = \mathcal{K}_1^{[\perp]}$, $\mathcal{H}_2 = \mathcal{H}_3 =$

[1]We thank the referee for his suggestions on this example.

$\mathcal{K}_2 = \mathcal{K}_3 = \{0\}$. Note that

$$[Ge, \mathcal{L}^{\langle\perp\rangle}] = \left\langle e, \mathcal{L}^{\langle\perp\rangle}\right\rangle = 0,$$

which leads to $\operatorname{im} T = \mathcal{L}^{\langle\perp\rangle} \subseteq \mathcal{K}_1^{[\perp]} = \mathcal{K}_0$. Since $Te = 0$, the operator

$$T_0 := T|_{\mathcal{H}_0} \colon \mathcal{H}_0 \to \mathcal{L}^{\langle\perp\rangle} \subseteq \mathcal{K}_0$$

is a J-restriction of T.

Proposition 1.3. *Let $\mathcal{H}$, $\mathcal{K}$ be nonzero Pontryagin spaces. Given T there exists a rigid J-restriction T_0 of T.*

Proof. We apply Theorem IX.2.5 of [1] to the subspace $\ker T$ of $\mathcal{H}$ and to the subspace $\overline{\operatorname{im} T}$ of $\mathcal{K}$. As a consequence we get the decompositions

$$\mathcal{H} = \mathcal{H}_0 \,[\dot+]\, \mathcal{H}_1 \,[\dot+]\, (\mathcal{H}_2 \dot+ \mathcal{H}_3), \quad \mathcal{K} = \mathcal{K}_0 \,[\dot+]\, \mathcal{K}_1 \,[\dot+]\, (\mathcal{K}_2 \dot+ \mathcal{K}_3),$$

satisfying (1.3), (1.4) and points (i) and (ii) of the definition of J-dilation. It is also apparent that T, with respect to the above decompositions, has the form (1.1). Hence, T_0 appearing in (1.1) is a rigid J-restriction of T. □

We refer the reader to Thm. IX.2.5 of [1] for questions connected with uniqueness of this construction.

2. Further Properties of *J*-Dilations, The Adjoint Operator

Treating $\mathcal{H} = \mathcal{H}_0 \,[\dot+]\, \mathcal{H}_1 \,[\dot+]\, (\mathcal{H}_2 \dot+ \mathcal{H}_3)$ as a direct and orthogonal sum of three Krein spaces and likewise for $\mathcal{K} = \mathcal{K}_0 \,[\dot+]\, \mathcal{K}_1 \,[\dot+]\, (\mathcal{K}_2 \dot+ \mathcal{K}_3)$ we see that the operator $T^{[*]}$ has the following block operator form

$$T^{[*]} = \left(\begin{array}{c|c|c} T_0^{[*]} & 0 & \begin{pmatrix} T_{20} \\ 0 \end{pmatrix}^{[*]} \\ \hline 0 & 0 & 0 \\ \hline (T_{02}\ 0)^{[*]} & 0 & \begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}^{[*]} \end{array}\right). \tag{2.1}$$

A simple indefinite inner product argument shows that

$$\begin{pmatrix} T_{20} \\ 0 \end{pmatrix}^{[*]} = \begin{pmatrix} 0 & T_{20}^+ \end{pmatrix}, \quad \begin{pmatrix} T_{02} & 0 \end{pmatrix}^{[*]} = \begin{pmatrix} 0 \\ T_{02}^+ \end{pmatrix}, \quad \begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}^{[*]} = \begin{pmatrix} 0 & 0 \\ 0 & T_2^+ \end{pmatrix}$$

with some $T_{20}^+ \in \mathbf{B}(\mathcal{K}_3, \mathcal{H}_0)$, $T_{02}^+ \in \mathbf{B}(\mathcal{K}_0, \mathcal{H}_3)$ and $T_2^+ \in \mathbf{B}(\mathcal{K}_3, \mathcal{H}_3)$, respectively. Substituting this into (2.1) we obtain

$$T^{[*]} = \begin{pmatrix} T_0^{[*]} & 0 & 0 & T_{20}^+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T_{02}^+ & 0 & 0 & T_2^+ \end{pmatrix}. \tag{2.2}$$

Hence, if we interchange the roles of $\mathcal{H}_2$ and $\mathcal{H}_3$ and interchange the roles of $\mathcal{K}_2$ and $\mathcal{K}_3$, we can see $T^{[*]}$ as a J-dilation of $T_0^{[*]}$. This fact and the lemma below will allow us to interchange the roles of $T^{[*]}$ and T in further reasonings.

We define the operators $T_{i\bullet} \in \mathbf{B}(\mathcal{H}, \mathcal{K}_i)$ and $T_{\bullet j} \in \mathbf{B}(\mathcal{H}_j, \mathcal{K})$ as respectively the ith row and jth column of the matrix T. Our main hero will be definitely the operator

$$T_{0\bullet} = \begin{pmatrix} T_0 & 0 & T_{02} & 0 \end{pmatrix}, \tag{2.3}$$

though some of the others will appear as well. Note that $(T_{0\bullet})^{[*]} = (T^{[*]})_{\bullet 0}$ and $(T_{\bullet 0})^{[*]} = (T^{[*]})_{0\bullet}$.

Lemma 2.1. *T is a rigid J-dilation of T_0 if and only if $T^{[*]}$ is a rigid J-dilation of $T_0^{[*]}$.*

Proof. Suppose that T is a rigid J-dilation of T_0. The inclusion $\mathcal{K}_1 [\dotplus] \mathcal{K}_2 \subseteq \ker T^{[*]}$ is obvious. To see the opposite one takes $y = y_0 + y_1 + y_2 + y_3 \in \ker T^{[*]}$, $y_i \in \mathcal{K}_i$ $(i = 0, 1, 2, 3)$. Then for every $x \in \mathcal{H}$ we have $[Tx, y] = 0$. By (1.3) T maps $\mathcal{H}$ onto a dense subspace of $\mathcal{K}_0 [\dotplus] \mathcal{K}_2$. Consequently, by (1.2), $y_0 = 0$, $y_3 = 0$. Hence, $y \in \mathcal{K}_1 [\dotplus] \mathcal{K}_2$.

Suppose now that $\overline{\mathrm{im}} T^{[*]} \subsetneq \mathcal{K}_0 [\dotplus] \mathcal{K}_3$. Then either $\overline{\mathrm{im}}(T^{[*]})_{0\bullet} \subsetneq \mathcal{K}_0$ or $\overline{\mathrm{im}}(T^{[*]})_{3\bullet} \subsetneq \mathcal{K}_3$. In the first case there exists a nonzero $x_0 \in \mathcal{K}_0$ which satisfies $[T^{[*]}y, x_0] = 0$ for all $y \in \mathcal{K}$. Consequently $x_0 \in \ker T$, which contradicts the rigidity of T. In the latter case there exists a nonzero $x_2 \in \mathcal{K}_2$ such that $[T^{[*]}y, x_2] = 0$ for all $y \in \mathcal{K}$, which is again in contradiction with the rigidity of T. □

At this point we can derive formulas for the operators $T^{[*]}T$ and $TT^{[*]}$:

$$T^{[*]}T = \begin{pmatrix} T_0^{[*]}T_0 & 0 & T_0^{[*]}T_{02} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ T_{02}^{+}T_0 & 0 & T_{02}^{+}T_{02} & 0 \end{pmatrix}, \qquad TT^{[*]} = \begin{pmatrix} T_0T_0^{[*]} & 0 & 0 & T_0T_{20}^{+} \\ 0 & 0 & 0 & 0 \\ T_{20}T_0^{[*]} & 0 & 0 & T_{20}T_{20}^{+} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

Using our row–column notation we can rewrite (2.4) as

$$T^{[*]}T = (T_{0\bullet})^{[*]}T_{0\bullet}, \qquad TT^{[*]} = T_{\bullet 0}(T_{\bullet 0})^{[*]}. \tag{2.5}$$

Lemma 2.2. *If T is a rigid J-dilation of T_0 then the operators $T_{\bullet 0}$ and $(T_{0\bullet})^{[*]}$ are injective and* $\ker T^{[*]}T = \ker T_{0\bullet}$.

Proof. The operator $T_{\bullet 0}$ is clearly injective by (1.3). By Lemma 2.1 we have $\ker T^{[*]} = \mathcal{K}_1 [\dotplus] \mathcal{K}_2$, hence $(T^{[*]})_{\bullet 0} = (T_{0\bullet})^{[*]}$ is injective as well. The equality $\ker T^{[*]}T = \ker T_{0\bullet}$ follows now from (2.5). □

3. Powers of Operators $T^{[*]}T$, Annihilating Polynomials

Starting from (2.5) one can easily prove by induction the following formulas

$$(T^{[*]}T)^j = (T_{0\bullet})^{[*]}(T_0T_0^{[*]})^{j-1}T_{0\bullet}, \quad j \geq 1, \tag{3.1}$$

$$(TT^{[*]})^j = T_{\bullet 0}(T_0^{[*]}T_0)^{j-1}(T_{\bullet 0})^{[*]}, \quad j \geq 1. \tag{3.2}$$

Observe that, denoting by P_0 the projection $P_0(x_0+\cdots+x_3)=x_0$, ($x_i \in \mathcal{H}_i$, $i=1,\ldots,3$), we get

$$P_0(T^{[*]}T)^j|_{\mathcal{H}_0} = (T_0^{[*]}T_0)^j, \quad j \geq 0. \tag{3.3}$$

A similar formula can be derived for $(TT^{[*]})^j$. Note also the following intertwining relations

$$(T_0T_0^{[*]})^j T_{0\bullet} = T_{0\bullet}(T^{[*]}T)^j, \quad j=0,1,\ldots, \tag{3.4}$$

$$(T_0^{[*]}T_0)^j (T_{\bullet 0})^{[*]} = (T_{\bullet 0})^{[*]}(TT^{[*]})^j, \quad j=0,1,\ldots, \tag{3.5}$$

$$(T_{0\bullet})^{[*]}(T_0T_0^{[*]})^j = (T^{[*]}T)^j(T_{0\bullet})^{[*]}, \quad j=0,1,\ldots. \tag{3.6}$$

If an operator A is nilpotent then we put $\nu(A) := \min\{n \in \mathbb{N} : A^n = 0\}$. It is an easy fact that $T^{[*]}T$ is nilpotent if and only if $TT^{[*]}$ is nilpotent, in such case $|\nu(T^{[*]}T) - \nu(TT^{[*]})| \leq 1$ (see also [4]). Equation (3.1) implies the following proposition.

Proposition 3.1. *Let T be a J-dilation of T_0. If $p(t)$ is an annihilating polynomial for $T_0T_0^{[*]}$ (for $T^{[*]}T$) then $tp(t)$ is an annihilating polynomial for $T^{[*]}T$ (for $T_0T_0^{[*]}$, respectively). Consequently the operator $T_0T_0^{[*]}$ is nilpotent if and only if $T^{[*]}T$ is nilpotent and*

$$|\nu(T_0T_0^{[*]}) - \nu(T^{[*]}T)| \leq 1.$$

Proof. Let $p(T_0T_0^{[*]}) = 0$. Then by (3.1)

$$T^{[*]}Tp(T^{[*]}T) = (T_{0\bullet})^{[*]}p(T_0T_0^{[*]})T_{0\bullet} = 0.$$

On the other hand if $p(T^{[*]}T) = 0$ then by (3.3) we have $p(T_0^{[*]}T_0) = 0$ and consequently,

$$T_0T_0^{[*]}p(T_0T_0^{[*]}) = T_0p(T_0^{[*]}T_0)T_0^{[*]} = 0.$$

□

Proposition 3.2. *Let T be a closed range, rigid J-dilation of an operator T_0. Then the spaces*

$$\ker(T^{[*]}T)^{j+1}/\ker(T^{[*]}T)^j, \qquad \ker(T_0T_0^{[*]})^j/\ker(T_0T_0^{[*]})^{j-1} \tag{3.7}$$

are linearly isomorphic for $j \geq 1$. Consequently, if, in addition, $T^{[*]}T$ is nilpotent then $\nu(T^{[*]}T) = \nu(T_0T_0^{[*]}) + 1$.

Proof. Since T is a closed range rigid dilation of T_0, we get $T_{0\bullet}$ surjective and $(T_{0\bullet})^{[*]}$ injective (Lemma 2.2). Consequently, by (3.1) we have

$$T_{0\bullet}(\ker(T^{[*]}T)^j) = \ker(T_0T_0^{[*]})^{j-1}, \quad j \geq 1.$$

Therefore, for $j \geq 1$ the mapping

$$\Phi_j(x + \ker(T^{[*]}T)^j) := T_{0\bullet}x + \ker(T_0T_0)^{j-1}, \qquad x \in \ker(T^{[*]}T)^{j+1}$$

is a well defined linear isomorphism between the spaces listed in (3.7). □

As usually we define the *Segre characteristic* for the nilpotent operator A in a finite dimensional space as a decreasing sequence of sizes of Jordan blocks in the Jordan canonical form of A, extended with an infinite number of zeros. Since, for $j = 1, 2, \ldots$, the dimension of $\ker A^j / \ker A^{j-1}$ equals the number of Jordan chains in the Jordan canonical form of A of length larger or equal then j, we get the following.

Corollary 3.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional. If T is a rigid dilation of T_0 and $T^{[*]}T$ is nilpotent with the Segre characteristic $(n_k)_{k=1}^{\infty}$, then the Segre characteristic of $T_0 T_0^{[*]}$ is*

$$(\max\{n_k - 1, 0\})_{k=1}^{\infty}.$$

Example 3. Let $\mathcal{H}_0 = \mathcal{K}_0$ be an infinite dimensional Hilbert space and let T_0 be the zero operator on $\mathcal{H}_0$. Consider now an arbitrary closed range, rigid J-dilation T of T_0. According to Proposition 3.2 the space $\ker(T^{[*]}T)^2/\ker(T^{[*]}T)$ is infinite dimensional. Hence, there are infinitely many Jordan chains of length two corresponding to the zero eigenvalue. Therefore, neither $\mathcal{H}$ nor $\mathcal{K}$ is a Pontryagin space.

4. Spectral Properties and Definitizabilty

Proposition 4.1. *Let T be a J-dilation of T_0 with $\mathcal{H}_0 \neq \mathcal{H}$. Then $\sigma(T^{[*]}T) = \sigma(T_0 T_0^{[*]}) \cup \{0\}$.*

Proof. A simple argument involving the Schur complement applied to the block operator matrix (2.4) shows that $\sigma(T^{[*]}T) = \sigma(T_0^{[*]}T_0) \cup \{0\}$. By a well known result the latter set is equal to $\sigma(T_0 T_0^{[*]}) \cup \{0\}$. □

Note also the following proposition, which shows, besides other things, that the nonzero point spectra of the operators $T^{[*]}T$ and $T_0 T_0^{[*]}$ coincide. By *the algebraic root space of an operator* A we mean the space

$$\mathcal{S}_\lambda(A) := \{f \in \mathcal{D}(A) : \exists n \in \mathbb{N} : (A - \lambda)^n f = 0\}.$$

Proposition 4.2. *Let T be a J-dilation of T_0 and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then*

(i) *$T_{0\bullet}$ maps $\mathcal{S}_\lambda(T^{[*]}T)$ bijectively to $\mathcal{S}_\lambda(T_0 T_0^{[*]})$;*

(ii) *$(T_{0\bullet})^{[*]}$ maps $\mathcal{S}_\lambda(T_0 T_0^{[*]})$ bijectively to $\mathcal{S}_\lambda(T^{[*]}T)$;*

(iii) *$\mathcal{S}_\lambda(T^{[*]}T)$ is finite dimensional if and only if $\mathcal{S}_\lambda(T_0 T_0^{[*]})$ is finite dimensional;*

(iv) *$\mathcal{S}_\lambda(T^{[*]}T)$ is non-degenerate if and only if $\mathcal{S}_\lambda(T_0 T_0^{[*]})$ is non-degenerate.*

Proof. (i)&(ii) First note that by the intertwining relation (3.4) $T_{0\bullet}$ maps $\mathcal{S}_\lambda(T^{[*]}T)$ into $\mathcal{S}_\lambda(T_0 T_0^{[*]})$. Similarly, by (3.6), $(T_{0\bullet})^{[*]}$ maps $\mathcal{S}_\lambda(T_0 T_0^{[*]})$ into $\mathcal{S}_\lambda(T^{[*]}T)$. Since $T_{0\bullet}(T_{0\bullet})^{[*]} = T_0 T_0^{[*]}$ and the latter operator is clearly injective on $\mathcal{S}_\lambda(T_0 T_0^{[*]})$, the operator $(T_{0\bullet})^{[*]}|_{\mathcal{S}_\lambda(T_0 T_0^{[*]})}$ is injective as well. The mapping $T^{[*]}T = (T_{0\bullet})^{[*]} T_{0\bullet}$ maps $\mathcal{S}_\lambda(T^{[*]}T)$ bijectively onto itself. By injectivity of

$(T_{0\bullet})^{[*]}|_{\mathcal{S}_\lambda(T_0T_0^{[*]})}$, each of the mappings $T_{0\bullet}|_{\mathcal{S}_\lambda(T^{[*]}T)}$ and $(T_{0\bullet})^{[*]}|_{\mathcal{S}_\lambda(T_0T_0^{[*]})}$ is bijective.

Point (iii) is now obvious. Point (iv) follows directly from bijectivity of $T_{0\bullet}|_{\mathcal{S}_\lambda(T^{[*]}T)}$, $(T_{0\bullet})^{[*]}|_{\mathcal{S}_\lambda(T_0T_0^{[*]})}$, $(T^{[*]}T)|_{\mathcal{S}_\lambda(T^{[*]}T)}$ and $(T_0T_0^{[*]})|_{\mathcal{S}_\lambda(T_0T_0^{[*]})}$. □

The reader might have already guessed that the case $\lambda = 0$ is much more difficult, we will deal with it in the next section. For the notions of definitizability, definitizing polynomial and spectral function see e.g. [7,11]. Note, that in our setting all operators are bounded. We also take the usual definitions of the set of critical points $c(A)$ and the positive and negative spectrum $\sigma_{\pm\pm}(A)$. We set $\mathbb{R}_\pm := \{x \in \mathbb{R} : \pm x > 0\}$.

Proposition 4.3. *The operator $T^{[*]}T$ is definitizable if and only if $T_0T_0^{[*]}$ is definitizable. If $p(t)$ is a definitizing polynomial for $T_0T_0^{[*]}$ (for $T^{[*]}T$) then $tp(t)$ is a definitizing polynomial for $T^{[*]}T$ (for $T_0T_0^{[*]}$, respectively). Consequently, if $T^{[*]}T$ is definitizable then*

$$\sigma_{\pm\pm}(T^{[*]}T) \cap \mathbb{R}_+ = \sigma_{\pm\pm}(T_0T_0^{[*]}) \cap \mathbb{R}_+, \quad \sigma_{\pm\pm}(T^{[*]}T) \cap \mathbb{R}_- = \sigma_{\mp\mp}(T_0T_0^{[*]}) \cap \mathbb{R}_-$$

and

$$c(T^{[*]}T) \cup \{0\} = c(T_0T_0^{[*]}) \cup \{0\}.$$

Proof. Let $T_0T_0^{[*]}$ be definitizable with the definitizing polynomial $p(t)$. By (3.1) we have

$$[(T^{[*]}T)^j x, y] = [(T_0T_0^{[*]})^{j-1}T_{0\bullet}x, T_{0\bullet}y], \quad j \geq 1,\ x, y \in \mathcal{H}.$$

In consequence, $tp(t)$ is a definitizing polynomial for $T^{[*]}T$. On the other hand if $p(t)$ is a definitizing polynomial for $T^{[*]}T$ then formula (3.3) shows that $p(t)$ is a definitizing polynomial for $T_0^{[*]}T_0$ as well. By [15, Theorem 3.1] $tp(t)$ is definitizing for $T_0T_0^{[*]}$. The 'consequently' part is now obvious. □

By $\mathfrak{R}_0$ we denote the semiring generated by finite intervals and their complements with endpoints not in $c(T^{[*]}T) \cup \{0\}$ and by E and E_0 we denote the spectral function of $T^{[*]}T$ and $T_0T_0^{[*]}$ respectively.

Theorem 4.4. *Let T be a J-dilation of T_0 and let $T^{[*]}T$ be definitizable. Then*

$$T_{0\bullet}E(\sigma) = E_0(\sigma)T_{0\bullet}, \qquad \lambda \in \rho(T^{[*]}T) \setminus \{0\}. \tag{4.1}$$

Consequently, a spectral point $\lambda \in c(T^{[*]}T) \setminus \{0\}$ is a singular critical point for $T^{[*]}T$ if and only if it is a singular critical point for $T_0T_0^{[*]}$.

Proof. The intertwining relation (3.4) implies

$$T_{0\bullet}(T^{[*]}T - \lambda)^{-1} = (T_0T_0^{[*]} - \lambda)^{-1}T_{0\bullet}, \qquad \lambda \in \rho(T^{[*]}T) \setminus \{0\},$$

which after integration over a suitable contour becomes (4.1) (cf. [15], proof of Theorem 4.1).

Suppose that λ is a regular critical point of $T^{[*]}T$ and let us take a bounded closed neighborhood τ of λ such that $\tau \cap (c(T^{[*]}T) \cup \{0\}) = \emptyset$.

By the properties of the spectral function ([11, p.30], see also [15, Theorem 4.2] for usage of similar arguments as those below) there exists $X \in \mathbf{B}(\mathcal{K}_0)$ such that $T_0 T_0^{[*]} X \tilde{E}(\tau) = E_0(\tau)$. Since λ is a regular critical point of $T^{[*]}T$, there exists a constant $c \geq 0$ such that

$$\|E(\sigma)\| \leq c, \qquad \sigma \subseteq \tau, \quad \sigma \in \mathfrak{R}_0. \tag{4.2}$$

Now for $\sigma \subseteq \tau$ such that $\lambda \in \sigma \in \mathfrak{R}_0$ we get

$$\begin{aligned} \|E_0(\sigma)\| &= \|E_0(\sigma)E_0(\tau)\| = \left\| E_0(\sigma) T_0 T_0^{[*]} X E_0(\tau) \right\| \\ &\leq \left\| E_0(\sigma) T_{0\bullet} \left(T_{0\bullet}\right)^{[*]} \right\| \|X\| \cdot c \leq \left\| T_{0\bullet} E(\sigma) (T_{0\bullet})^{[*]} \right\| \|X\| \cdot c \\ &\leq \|T_{0\bullet}\| \cdot c \cdot \left\| (T_{0\bullet})^{[*]} \right\| \|X\| \cdot c. \end{aligned}$$

Hence, λ is a regular critical point for $T_0 T_0^{[*]}$. A similar argument shows the opposite implication. □

5. Decomposing Spaces

We say that a pair of subspaces $\mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{K}'$ of $\mathcal{K}$ *decomposes* T if $\mathcal{H}'$ and $\mathcal{K}'$ are Krein spaces, $T\mathcal{H}' \subseteq \mathcal{K}'$ and $T^{[*]}\mathcal{K}' \subseteq \mathcal{H}'$. Note that in such case the pair $\mathcal{H}'^{[\perp]}$ and $\mathcal{K}'^{[\perp]}$ decomposes T as well and T can be written in the form $T = T' \,[\dotplus]\, T''$, while $T^{[*]} = T'^{[*]} \,[\dotplus]\, T''^{[*]}$.

For the definition of a canonical form of an H-symmetric matrix we refer to [5]. Since our paper contains both finite and infinite dimensional situations we view matrices as operators. Let A be a selfadjoint operator in a finite dimensional Krein space $\mathcal{E}$. A linear basis $(e_j)_j$ of $\mathcal{E}$ will be called a *canonical basis for the operator* A if and only if the matrix representation of A in $(e_j)_j$ is in a Jordan canonical form and the Gramm matrix $([e_i, e_j])_{ij}$ is of a special type as outlined in [5]. By the *sign characteristic of* A we understand the sign characteristic of the pair consisting of the matrix representation of A in a canonical bases $(e_j)_j$ and the Gramm matrix $([e_i, e_j])_{ij}$. Obviously this notion does not depend on the choice of a canonical bases. If $e_1, \ldots, e_k$ belong to some canonical basis of A and form a full Jordan chain, then by the *sign* of the chain we mean as usually the number $[e_1, e_k]$, which is either plus or minus one.

From now on we concentrate on the zero eigenvalue. Note that the nonzero eigenvalues were analyzed in the previous section. One can easily apply the methods used in the proof of Proposition 4.2 to analyze the sign characteristic for nonzero eigenvalues in the finite dimensional case (see also Proposition 3 in [9]).

Motivated by the finite dimensional situation we introduce the following definition. However, note that neither the assumption of the finite dimensionality of the space nor nilpotency of the operator $T^{[*]}T$ is needed in this section.

Definition 5.1. Let $\mathcal{E}$, $\mathcal{F}$ be a pair of finite dimensional spaces that decomposes T. We say that it is of type (i) if $\dim\mathcal{E} = \dim\mathcal{F} = 2k$ (with some $k \geq 1$) and there exist linear bases $g_1, \dots, g_{2k}$ of $\mathcal{E}$ and $h_1, \dots, h_{2k}$ of $\mathcal{F}$ such that

$$\begin{aligned} Tg_j &= \begin{cases} h_{j-1} & : \ j = 2, \dots, 2k \\ 0 & : \ j = 1 \end{cases} \\ T^{[*]}h_j &= \begin{cases} g_{j-1} & : \ j = 2, \dots, 2k \\ 0 & : \ j = 1 \end{cases} \end{aligned} \tag{5.1}$$

and

$$[g_i, g_j] = [h_i, h_j] = \begin{cases} 0 & : \ i + j \neq 2k+1, \\ 1 & : \ i + j = 2k+1 \end{cases};$$

type (ii) if $\dim\mathcal{E} = \dim\mathcal{F} - 1 \geq 1$, there exist canonical bases $e_1, \dots, e_k$ for $T^{[*]}T$ and $f_1, \dots, f_{k+1}$ for $TT^{[*]}$ such that

$$\begin{aligned} T^{[*]}f_j &= \begin{cases} e_{j-1} & : \ j = 2, \dots, k+1 \\ 0 & : \ j = 1 \end{cases} \\ Te_j &= f_j, \quad j = 1, \dots, k \end{aligned}$$

and

$$\begin{aligned} [e_i, e_j] &= \varepsilon\delta_{i+j,k+1} \quad i, j = 1, \dots, k, \\ [f_i, f_j] &= \varepsilon\delta_{i+j,k+2} \quad i, j = 1, \dots, k+1 \end{aligned} \tag{5.2}$$

with some $\varepsilon \in \{-1, 1\}$;

type (iii) if $\dim\mathcal{F} = \dim\mathcal{E} - 1 \geq 1$, there exist canonical bases $e_1, \dots, e_{k+1}$ for $T^{[*]}T$ and $f_1, \dots, f_k$ ($k = \dim\mathcal{F}$) for $TT^{[*]}$ such that

$$\begin{aligned} Te_j &= \begin{cases} f_{j-1} & : \ j = 2, \dots, k+1 \\ 0 & : \ j = 1 \end{cases} \\ T^{[*]}f_j &= e_j, \quad j = 1, \dots, k \end{aligned}$$

and

$$\begin{aligned} [e_i, e_j] &= \varepsilon\delta_{i+j,k+2}, \quad i, j = 1, \dots, k+1, \\ [f_i, f_j] &= \varepsilon\delta_{i+j,k+1}, \quad i, j = 1, \dots, k \end{aligned} \tag{5.3}$$

with some $\varepsilon \in \{-1, 1\}$;

type (iv) if $\dim\mathcal{F} = 1$, $\dim\mathcal{E} = 0$;

type (v) if $\dim\mathcal{E} = 1$, $\dim\mathcal{F} = 0$.

We will refer to the bases appearing in each point above as a *corresponding (to a type) basis.*

Remark 5.2. While in the definitions of types (ii)–(iii) the operators $T^{[*]}T|_{\mathcal{E}}$ and $TT^{[*]}|_{\mathcal{F}}$ are presented in their canonical bases, in the definition of type (i) this is not the case. Although the bases $g_1, \dots, g_{2k}$ and $h_1, \dots, h_{2k}$ are not canonical it is easy to transform them into canonical ones. In fact, the canonical basis of $T^{[*]}T|_{\mathcal{E}}$ consists in this case of two Jordan chains

$$\frac{1}{2}(g_{2k} + g_{2k-1}), \frac{1}{2}(g_{2k-2} + g_{2k-3}), \dots, \frac{1}{2}(g_2 + g_1)$$

and

$$\frac{1}{2}(g_{2k}-g_{2k-1}), \frac{1}{2}(g_{2k-2}-g_{2k-3}), \dots, \frac{1}{2}(g_2-g_1)$$

of opposite signs, the same concerns the canonical basis of $TT^{[*]}|_{\mathcal{F}}$ with g everywhere replaced by h.

Note that in each of the types zero is the only eigenvalue of the operators $T^{[*]}T|_{\mathcal{E}}$ and $TT^{[*]}|_{\mathcal{F}}$, or one of the operators is trivial and the second one is zero. Hence, a necessary condition for an operator T to have a decomposing pair of spaces of one of the types (i)–(v) is that zero is an eigenvalue of at least one of the operators $T^{[*]}T$ and $TT^{[*]}$. In the next section we will see that this condition is also sufficient in the finite dimensional case. On the other hand Example 1 shows that in the infinite dimensional Π_1-space zero can be an eigenvalue of both operators $T^{[*]}T$ and $TT^{[*]}$, but there is no decomposing pair for T.

Proposition 5.3. *Let T be a rigid J-dilation of T_0 and let $\tilde{\mathcal{E}}$, $\tilde{\mathcal{F}}$ be a pair of decomposing spaces for T_0 of type* (i) *with the corresponding bases $\tilde g_1,\dots,\tilde g_{2k}$ and $\tilde h_1,\dots,\tilde h_{2k}$. Then there exists a pair of spaces $\mathcal{E}$, $\mathcal{F}$ decomposing T and of type* (i) *with a corresponding bases $g_{-1},g_0,\dots,g_{2k}$ and $h_{-1},h_0,\dots,h_{2k}$, such that*

$$T_{0\bullet}\, g_j=\tilde h_j, \quad (T_{\bullet\, 0})^{[*]}h_j=\tilde g_j, \quad j=1,\dots,2k, \tag{5.4}$$

Proof. First note that by surjectivity of $T_{0\bullet}$ there exist $g_{2k}\in\mathcal{H}$ and $h_{2k}\in\mathcal{K}$ such that

$$T_{0\bullet}\, g_{2k}=\tilde h_{2k}, \qquad (T_{\bullet\, 0})^{[*]}h_{2k}=\tilde g_{2k}. \tag{5.5}$$

We define now g_j and h_j by a recursive relation

$$g_{j-1}:=T^{[*]}h_j \qquad h_{j-1}:=Tg_j, \quad j=0,\dots,2k. \tag{5.6}$$

By Lemma 2.2 the operator $(T_{0\bullet})^{[*]}$ is injective and hence

$$g_0=T^{[*]}Tg_2=(T_{0\bullet})^{[*]}T_{0\bullet}\, g_2=(T_{0\bullet})^{[*]}\tilde h_2\neq 0.$$

For the same reasons g_{-1}, h_0, h_{-1} are nonzero. On the other hand

$$Tg_{-1}=TT^{[*]}Tg_1=T(T_{0\bullet})^{[*]}T_{0\bullet}\, g_1=T(T_{0\bullet})^{[*]}\tilde h_1=T_0T_0^{[*]}\tilde h_1=0. \tag{5.7}$$

Similarly

$$T^{[*]}h_{-1}=0. \tag{5.8}$$

Hence, $g_{-1},g_0,\dots,g_{2k}$ and $h_{-1},h_0,\dots,h_{2k}$ are nonzero vectors satisfying

$$\begin{aligned} Tg_j&=\begin{cases} h_{j-1} & : \ j=0,\dots,2k\\ 0 & : \ j=-1\end{cases}\\ T^{[*]}h_j&=\begin{cases} g_{j-1} & : \ j=0,\dots,2k\\ 0 & : \ j=-1\end{cases}, \end{aligned} \tag{5.9}$$

cf. (5.1). Equations (5.4) are also satisfied by the exploited intertwining relations (3.4), (3.5). Define now the spaces $\mathcal{E}$ and $\mathcal{F}$ by

$$\mathcal{E} := \operatorname{lin}\{g_{-1},\dots,g_{2k}\}, \quad \mathcal{F} := \operatorname{lin}\{h_{-1},\dots,h_{2k}\}.$$

Clearly $T\mathcal{E} \subseteq \mathcal{F}$ and $T^{[*]}\mathcal{F} \subseteq \mathcal{E}$, by (5.6),(5.7) and (5.8).

We show now that

$$[g_i, g_j] = [h_i, h_j] = \delta_{i+j,2k-1}, \quad i,j = -1,\dots,2k,\ i+j < 4k-1. \tag{5.10}$$

Without loss of generality we can assume that $i \le j$. Note that

$$[g_i, g_j] = [T^{[*]}Tg_i, g_{j+2}] = 0, \quad i,j = -1,0.$$

For $j > 0$ and $i \le 2k-2$ we have

$$\begin{aligned}[g_i, g_j] &= [T^{[*]}Tg_{i+2}, g_j] = [T_{0\bullet}\, g_{i+2}, T_{0\bullet}\, g_j] \\ &= [\tilde{h}_{i+2}, \tilde{h}_j] = \delta_{i+j+2,2k+1} = \delta_{i+j,2k-1}.\end{aligned}$$

Next,

$$[g_{2k-1}, g_{2k-1}] = [Th_{2k}, Th_{2k}] = [\tilde{h}_{2k}, \tilde{h}_{2k}] = 0.$$

The same calculations hold for g and h interchanged, which finishes the proof of (5.10).

The cases $i,j = 2k$ and $i = 2k-1$, $j = 2k$ of (5.10) are more difficult. Suppose we replace g_{2k} and h_{2k} respectively by

$$\hat{g}_{2k} := g_{2k} + \alpha_0 g_0 + \alpha_{-1} g_{-1}, \qquad \hat{h}_{2k} := h_{2k} + \beta_0 h_0 + \beta_{-1} h_{-1}$$

with some $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 0,-1)$ and we replace g_{2k-1} and h_{2k-1} respectively by

$$\hat{g}_{2k-1} := T^{[*]}\hat{h}_{2k}, \qquad \hat{h}_{2k-1} := T\hat{g}_{2k}.$$

In such case the modified systems

$$g_{-1},\dots,g_{2k-2},\hat{g}_{2k-1},\hat{g}_{2k}, \quad h_{-1},\dots,h_{2k-2},\hat{h}_{2k-1},\hat{h}_{2k}$$

still satisfy (5.9) since $g_i \in \ker T^{[*]}T = \ker T_{0\bullet}$ and $h_i \in \ker TT^{[*]} = \ker(T_{\bullet 0})^{[*]}$ for $i = 0,-1$. Note that (5.4) holds for any choice of α_i, β_i $(i = 0,-1)$ as well. Observe that by (5.10) we have

$$[\hat{g}_{2k}, \hat{g}_{2k}] = [g_{2k}, g_{2k}] + \alpha_{-1}, \tag{5.11}$$
$$[\hat{h}_{2k}, \hat{h}_{2k}] = [h_{2k}, h_{2k}] + \beta_{-1} \tag{5.12}$$

and

$$[\hat{g}_{2k}, \hat{g}_{2k-1}] = [\hat{h}_{2k-1}, \hat{h}_{2k}] = [g_{2k}, g_{2k-1}] + \alpha_0 + \beta_0. \tag{5.13}$$

Hence, it is easy to choose α_i, β_i $(i = 0,-1)$ such that the inner products in (5.11), (5.12) and (5.13) are all zero. Consequently, $\mathcal{E}$ and $\mathcal{F}$ are non-degenerate spaces and hence they decompose T. From the construction it is obvious that the spaces $\mathcal{E}$ and $\mathcal{F}$ are of type (i) with

$$g_{-1},\dots,g_{2k-2},\hat{g}_{2k-1},\hat{g}_{2k},$$

and

$$h_{-1},\dots,h_{2k-2},\hat{h}_{2k-1},\hat{h}_{2k}$$

as corresponding bases. □

Proposition 5.4. *Let T be a rigid J-dilation of T_0 and let $\tilde{\mathcal{E}}$, $\tilde{\mathcal{F}}$ be a pair of decomposing spaces for T_0 of type* (ii) *with the corresponding bases $\tilde{e}_1, \ldots, \tilde{e}_k$ and $\tilde{f}_1, \ldots, \tilde{f}_{k+1}$. Then there exists a pair of spaces $\mathcal{E}$, $\mathcal{F}$ decomposing T and of type* (iii) *with a corresponding bases $e_0, \ldots, e_{k+1}$ and $f_0, \ldots, f_k$, such that*

$$\begin{aligned} T_{0\bullet}\, e_j &= \tilde{f}_j, \quad j = 1, \ldots, k+1, \\ (T_{\bullet 0})^{[*]} f_j &= \tilde{e}_j, \quad j = 1, \ldots, k. \end{aligned} \tag{5.14}$$

Moreover,

$$[e_{k+1}, e_0] = [\tilde{e}_{k+1}, \tilde{e}_1], \qquad [f_k, f_0] = [\tilde{f}_k, \tilde{f}_1]. \tag{5.15}$$

Proof. Since the mapping $T_{0\bullet}$ is onto, there exist $e_{k+1} \in \mathcal{H}$ such that

$$T_{0\bullet}\, e_{k+1} = \tilde{f}_{k+1}.$$

Now let e_j and f_j be defined by

$$f_j := T e_{j+1}, \quad e_j := T^{[*]} f_j, \quad j = 0, \ldots, k. \tag{5.16}$$

By definition, $e_0, \ldots, e_{k+1}$ and $f_0, \ldots, f_k$ are Jordan chains for $T^{[*]}T$ and $TT^{[*]}$ respectively. By the intertwining relations (3.4) and (3.5) we have that (5.14) is satisfied. Furthermore, note that

$$e_0 = T^{[*]} T e_1 = (T_{0\bullet})^{[*]} T_{0\bullet}\, e_1 = (T_{0\bullet})^{[*]} \tilde{f}_1. \tag{5.17}$$

By Lemma 2.2 the operator $(T_{0\bullet})^{[*]}$ is injective, thus $e_0 \neq 0$. Since $T^{[*]} f_0 = T^{[*]} T e_1 = e_0$, we get $f_0 \neq 0$ as well. On the other hand,

$$T e_0 = T (T_{0\bullet})^{[*]} \tilde{f}_1 = T_0 T_0^{[*]} \tilde{f}_1 = 0. \tag{5.18}$$

Consequently, $T^{[*]} T e_0 = 0$ and $TT^{[*]} f_0 = TT^{[*]} T e_1 = T e_0 = 0$.

Define now the spaces $\mathcal{E}$ and $\mathcal{F}$ by

$$\mathcal{E} := \operatorname{lin}\{e_0, \ldots, e_{k+1}\}, \quad \mathcal{F} := \operatorname{lin}\{f_0, \ldots, f_k\}.$$

Clearly $T\mathcal{E} \subseteq \mathcal{F}$ and $T^{[*]}\mathcal{F} \subseteq \mathcal{E}$, by (5.16) and (5.18). What remains to prove is that

$$[e_i, e_j] = \delta_{i+j,k+1} [\tilde{f}_{k+1}, \tilde{f}_1], \quad i, j = 0, \ldots, k+1 \tag{5.19}$$

and

$$[f_i, f_j] = \delta_{i+j,k} [\tilde{f}_{k+1}, \tilde{f}_1] = \delta_{i+j,k+1} [\tilde{e}_k, \tilde{e}_1], \quad i, j = 0, \ldots, k. \tag{5.20}$$

Namely, if the two above formulas are satisfied then $\mathcal{E}$ and $\mathcal{F}$ are non-degenerate spaces and (5.15) holds as well. We proceed now like in the proof of the last proposition, the details are left to the reader. The cases $i+j < 2k+2$ of (5.19) can be proved directly. We replace e_{k+1}, if necessary, by $\hat{e}_{k+1} := e_{k+1} + \alpha e_0$, where $\alpha \in \mathbb{R}$ is chosen in such way that $[\hat{e}_{k+1}, \hat{e}_{k+1}] = 0$. Since

$$[f_i, f_j] = [T e_{i+1}, T e_{j+1}] = [e_{i+1}, e_j], \quad i, j = 0, \ldots, k$$

we have (5.20). □

Proposition 5.5. *Let T be a rigid J-dilation of T_0 and let $\tilde{\mathcal{E}} = \{0\}$, $\tilde{\mathcal{F}}$ be a pair of decomposing spaces for T_0 of type* (iv) *and let $\tilde{f}_1 \in \tilde{\mathcal{F}}$ be such that $[\tilde{f}_1, \tilde{f}_1] = \pm 1$. Then there exists a pair of spaces $\mathcal{E}$, $\mathcal{F}$ decomposing T and of type* (iii) *with a corresponding canonical bases e_0, e_1 and f_0, such that*

$$T_{0\bullet}\, e_1 = \tilde{f}_1.$$

Moreover,

$$[e_1, e_0] = [f_0, f_0] = [\tilde{f}_1, \tilde{f}_1]. \tag{5.21}$$

The proof is similar to the proof of Proposition 5.4. Substituting $T^{[*]}$ for T we get the analogues of Propositions 5.4 and 5.5 for types (iii) and (v).

6. Canonical Forms for the Finite Dimensional Case

In this section we give an alternative proof of [12, Theorem 3.2], cf. also the main result in [3]. It is worth mentioning that the paper [12] contains a broad and interesting study of the problem of comparing the operators $T^{\star}T$ and $TT^{\star}$ also in the presence of different types of involutions. Nevertheless, we find it important to present the proof below for two reasons. Firstly, our proof highlights the induction argument, which is only implicitly present in the proof of [12]. Secondly, our proof is shorter than the one in [12] and has a more geometrical nature thanks to the J-dilation procedure. We restrict ourselves to the nilpotent case, since the essential difficulty lies in the zero eigenvalue. From now on we assume that every space is finite dimensional.

Theorem 6.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional and let the operators $TT^{[*]}$ and $T^{[*]}T$ be nilpotent. Then there exists subspaces $\mathcal{E}_i$ of $\mathcal{H}$ and $\mathcal{F}_i$ of $\mathcal{K}$ $(i = 1, \ldots, n)$ such that*

$$\mathcal{H} = \mathcal{E}_1 \,[\dot{+}]\, \cdots \,[\dot{+}]\, \mathcal{E}_n, \qquad \mathcal{K} = \mathcal{F}_1 \,[\dot{+}]\, \cdots \,[\dot{+}]\, \mathcal{F}_n$$

and each pair $\mathcal{E}_i$, $\mathcal{F}_i$ decomposes T and is of one of the types (i)–(v)*.*

Note that our decomposing pairs of type (i) are the same as blocks of type (2) in [12], decomposing pairs of type (ii) correspond to blocks of type (3) in [12], decomposing pairs of type (iii) correspond to blocks of type (4) in [12] and the sum of decomposing pairs of type (iv) and (v) constitutes the block of type (1) in [12].

Proof. We will prove the claim by induction with respect to

$$N := \max\left\{\nu(T^{[*]}T), \nu(TT^{[*]})\right\}.$$

Let us suppose first that $N = 1$. Then $T^{[*]}T = 0$ and $TT^{[*]} = 0$. Hence, $\operatorname{im} T$ as well as $\operatorname{im} T^{[*]}$ are neutral spaces. We fix a skewly linked companion $\mathcal{K}'$ of $\operatorname{im} T$ and let $\mathcal{K}'' := (\operatorname{im} T \dot{+} \mathcal{K}')^{[\perp]}$, so that

$$\mathcal{K} = (\operatorname{im} T \dot{+} \mathcal{K}') \,[\dot{+}]\, \mathcal{K}''.$$

Since $\mathcal{K}''[\perp] \operatorname{im} T$, it is contained in the kernel of $T^{[*]}$. Also $\operatorname{im} T \subseteq \ker T^{[*]}$ by $T^{[*]}T = 0$. On the other hand if $x \in \mathcal{K}'$ then, by definition of $\mathcal{K}'$, $[T^{[*]}x, z] =$

$[x, Tz] = 1$ for some $z \in \mathcal{H}$. Hence, $T^{[*]}x \neq 0$ and consequently $\ker T^{[*]} = \operatorname{im} T \dotplus \mathcal{K}''$ Now consider a similar decomposition

$$\mathcal{H} = (\operatorname{im} T^{[*]} \dotplus \mathcal{H}') \boxplus \mathcal{H}''.$$

Again, we have $\ker T = \operatorname{im} T^{[*]} \dotplus \mathcal{H}''$. Moreover,

$$\operatorname{im} T = T\mathcal{H}', \quad \operatorname{im} T^{[*]} = T^{[*]}\mathcal{K}'. \tag{6.1}$$

The pairs $\mathcal{H}''$, $\{0\}$ and $\mathcal{K}''$, $\{0\}$ decompose T, since $T|_{\mathcal{H}''} = 0$, $T^{[*]}|_{\mathcal{K}''} = 0$. Furthermore, $T|_{\mathcal{H}''}\colon \mathcal{H}'' \to \{0\}$ and $T|_{\{0\}}\colon \{0\} \to \mathcal{K}''$ satisfy the theorem with respectively blocks of type (v) and (iv) only. To finish the proof of case $N = 1$ we need to show that T restricted to $(\operatorname{im} T^{[*]} \dotplus \mathcal{H}')$ decomposes into blocks of types (i),(ii) and (iii). Obviously, type (ii) as well as (iii) are not possible, since $T^{[*]}T$ and $TT^{[*]}$ do not have Jordan chains longer then one. Fix a linear basis $g_2^1, \dots, g_2^l$ of $\mathcal{H}'$ and choose vectors $g_1^1, \dots, g_1^l \in \operatorname{im} T^{[*]}$ such that

$$[g_1^i, g_2^j] = \delta_{ij} \quad i, j = 1, \dots, l.$$

Set $h_1^i := Tg_2^i$, $i = 1, \dots, l$ and let $h_2^i \in \mathcal{K}'$ for $i = 1, \dots, l$ be such that $T^{[*]}h_2^i = g_1^i$. Note that

$$[h_1^i, h_2^j] = [Tg_2^i, h_2^j] = [g_2^i, T^{[*]}h_2^j] = [g_2^i, g_1^j] = \delta_{ij}, \quad i, j = 1, \dots, l.$$

Hence the theorem holds for $T|_{\operatorname{im} T^{[*]} \dotplus \mathcal{H}'}$ with $\mathcal{E}_i := \operatorname{lin}\{g_1^i, g_2^i\}$, $\mathcal{F}_i := \operatorname{lin}\{h_1^i, h_2^i\}$ $(i = 1, \dots, l)$ being of type (i).

Now let us assume that the claim is true for N and consider the $N+1$ case. Let T_0 be a rigid J-restriction of T. By Proposition 3.2 we can apply the induction hypothesis to the operator $T_0^{[*]}$. Hence, $\mathcal{H}_0 = \tilde{\mathcal{E}}_1 \boxplus \cdots \boxplus \tilde{\mathcal{E}}_n$, $\mathcal{K}_0 = \tilde{\mathcal{F}}_1 \boxplus \cdots \boxplus \tilde{\mathcal{F}}_n$ with each pair $\tilde{\mathcal{E}}_i$, $\tilde{\mathcal{F}}_i$ $(i = 1, \dots, n)$ decomposing T_0 and being of one of the types (i)–(v). By multiple use of Propositions 5.3, 5.4 and 5.5 we get a system of spaces $\mathcal{E}_1, \dots, \mathcal{E}_n$, $\mathcal{F}_1, \dots, \mathcal{F}_n$. Since $\tilde{\mathcal{E}}_i \cap \tilde{\mathcal{E}}_j = \{0\}$ for $i \neq j$, and $T_{0\bullet} E_i = \tilde{E}_i$ $(i = 1, \dots, n)$, we get $\mathcal{E}_i \cap \mathcal{E}_j = \{0\}$ for $i \neq j$. Similarly, $\mathcal{F}_i \cap \mathcal{F}_j = \{0\}$. Since each pair $\mathcal{E}_i$, $\mathcal{F}_i$ decomposes T, the spaces $\mathcal{E} := \mathcal{E}_1 \dotplus \cdots \dotplus \mathcal{E}_n$ and $\mathcal{F} := \mathcal{F}_1 \dotplus \cdots \dotplus \mathcal{F}_n$ are non-degenerate and decompose T as well.

Set $S := T|_{\mathcal{E}^{[\perp]}}$ and note that

$$T^{[*]}T = (T^{[*]}T)|_{\mathcal{E}} \boxplus S^{[*]}S.$$

The Segre characteristic of $(T^{[*]}T)|_{\mathcal{E}}$ is given by Propositions 5.3–5.5. On the other hand the Segre characteristics of $T^{[*]}T$ is determined up to Jordan chains of length one by the Segre characteristic of $T_0T_0^{[*]}$ (see Corollary 3.3). Combining these two facts we conclude, that the operator $S^{[*]}S$ has Jordan chains of length one only. The same property concerns $SS^{[*]}$. Hence, we can apply the first induction step to S. What remains to show is that $T|_{\mathcal{E}}$ satisfies the theorem. First note that

$$\text{if } i \neq j \text{ and } x = T^{[*]}Ty \in \mathcal{E}_i,\ z \in \mathcal{E}_j \text{ then } [x, z] = 0. \tag{6.2}$$

Indeed, $[x, z] = [T_{0\bullet}\, y, T_{0\bullet}\, z]$. The latter equals zero, since $T_{0\bullet}\, y \in \tilde{\mathcal{E}}_i$, $T_{0\bullet}\, z \in \tilde{\mathcal{E}}_j$.

Now let us consider the following construction. For each $i, j = 1, \dots, n$, $i > j$ there exists a pair of spaces $\hat{\mathcal{E}}_i$, $\hat{\mathcal{F}}_i$, decomposing T, of the same type as $\mathcal{E}_i$, $\mathcal{F}_i$ and such that

$$\mathcal{E} = \mathcal{E}_1 \dotplus \cdots \dotplus \mathcal{E}_{i-1} \dotplus \hat{\mathcal{E}}_i \dotplus \mathcal{E}_{i+1} \dotplus \cdots \dotplus \mathcal{E}_n,$$
$$\mathcal{F} = \mathcal{F}_1 \dotplus \cdots \dotplus \mathcal{F}_{i-1} \dotplus \hat{\mathcal{F}}_i \dotplus \mathcal{F}_{i+1} \dotplus \cdots \dotplus \mathcal{F}_n,$$

and such that

$$[\hat{\mathcal{E}}_i, \mathcal{E}_j] = \{0\}, \tag{6.3}$$

and

$$[\hat{\mathcal{E}}_i, \mathcal{E}_p] = [\mathcal{E}_i, \mathcal{E}_p], \quad p \neq j. \tag{6.4}$$

By repeating recursively this procedure we get the desired decomposition of $\mathcal{H}$ and $\mathcal{K}$.

We will consider now several cases, corresponding to the types of pairs $\mathcal{E}_i$, $\mathcal{F}_i$ and $\mathcal{E}_j$, $\mathcal{F}_j$. Without loss of generality we can assume that the type of the pair $\mathcal{E}_i$, $\mathcal{F}_i$ increases with i. Moreover, note that there are neither pairs of type (iv) nor (v).

1. *The ith and the jth pair are of type* (iii). Let $e_0^i, \dots, e_{k+1}^i$, $f_0^i, \dots, f_k^i$ ($e_0^j, \dots, e_{l+1}^j$, $f_0^j, \dots, f_l^j$) be bases corresponding to the type for $\mathcal{E}_i$ and $\mathcal{F}_i$ ($\mathcal{E}_j$ and $\mathcal{F}_j$) respectively. We modify the e_{k+1}^i vector only. Namely, we define

$$\hat{e}_{k+1}^i = e_{k+1}^i + \alpha e_0^j,$$

where $\alpha \in \mathbb{C}$ is such that

$$[\hat{e}_{k+1}^i, e_{l+1}^i] = 0.$$

We set $\hat{\mathcal{E}}_i := \operatorname{lin}\{e_0^i, \dots, e_k^i, \hat{e}_{k+1}^i\}$. Note that by (6.2) and the above we have $[\hat{\mathcal{E}}_i, \mathcal{E}_j] = \{0\}$. We also set $\hat{\mathcal{F}}_i := \mathcal{F}_i$. In this case we already have $[\mathcal{F}_i, \mathcal{F}_j] = \{0\}$, since

$$[f_k^i, f_l^j] = [Te_{k+1}^i, Te_{l+1}^j] = [T^{[*]}Te_{k+1}^i, e_{l+1}^j],$$

which is zero by (6.2). Thanks to (6.2) we get (6.4) as well. Note that the systems $e_0^i, \dots, e_k^i, \hat{e}_{k+1}^i$ and $f_0^i, \dots, f_k^i$ are bases of $\hat{\mathcal{E}}_i$, $\mathcal{F}_i$ corresponding to type (i).

2. *The ith pair is of type* (iii) *and the jth pair is of type* (ii). Let $e_0^i, \dots, e_{k+1}^i$, $f_0^i, \dots, f_k^i$ ($e_0^j, \dots, e_l^j$, $f_0^j, \dots, f_{l+1}^j$) be bases corresponding to the type for $\mathcal{E}_i$ and $\mathcal{F}_i$ ($\mathcal{E}_j$ and $\mathcal{F}_j$) respectively. We define

$$\hat{e}_{k+1}^i = e_{k+1}^i + \alpha e_0^j,$$

where $\alpha \in \mathbb{C}$ is such that

$$[\hat{e}_{k+1}^i, e_l^i] = 0.$$

Moreover, we define $\hat{f}_k^i := T(\hat{e}_{k+1}^i) = f_k^i + \alpha f_k^0$. Hence,

$$[\hat{f}_k^i, f_{l+1}^j] = [\hat{e}_{k+1}^i, T^{[*]} f_{l+1}^j] = [\hat{e}_{k+1}^i, e_l^j] = 0.$$

The spaces $\hat{\mathcal{E}}_i := \operatorname{lin}\{e_0^i, \ldots, e_k^i, \hat{e}_{k+1}^i\}$, $\hat{\mathcal{F}}_i := \operatorname{lin}\left\{f_0^i, \ldots, f_{k-1}^i, \hat{f}_k^i\right\}$ satisfy now the requirements.

3. *The ith pair is of type* (iii) *and the jth pair is of type* (i). Let $e_0^i, \ldots, e_{k+1}^i$, $f_0^i, \ldots, f_k^i$ ($g_{-1}^j, \ldots, g_{2l}^j$, $h_{-1}^j, \ldots, h_{2l}^j$) be bases corresponding to the type for $\mathcal{E}_i$ and $\mathcal{F}_i$ ($\mathcal{E}_j$ and $\mathcal{F}_j$) respectively. We set

$$\hat{e}_{k+1}^i = e_{k+1}^i + \alpha_{-1} g_{-1}^j + \alpha_0 g_0^j,$$

where $\alpha_{-1}, \alpha_0 \in \mathbb{C}$ are such that

$$[\hat{e}_{k+1}^i, g_{2l}^j] = [\hat{e}_{k+1}^i, g_{2l-1}^j] = 0.$$

Furthermore, we set $\hat{f}_k^i := T\hat{e}_{k+1}^i$ and proceed as before.

4. *The ith and the jth pair are of type* (ii). Similarly to 1., interchanging the roles of the spaces.

5. *The ith pair is of type* (ii) *and the jth pair is of type* (i). Similarly to 3., interchanging the roles of the spaces.

6. *The ith and the jth pair are of type* (i). We set

$$\hat{g}_{2k}^i := g_{2k}^i + \alpha_{-1} g_{-1}^j + \alpha_0 g_0^j, \quad \hat{h}_{2k}^i := h_{2k}^i + \beta_{-1} h_{-1}^j + \beta_0 h_0^j$$
$$\hat{g}_{2k-1}^i = T^{[*]} h_{2k}^i, \quad \hat{h}_{2k-1}^i = T g_{2k}^i$$

with $\alpha_p, \beta_q \in \mathbb{C}$ $(p, q = -1, 0)$ such that

$$[\hat{g}_r^i, g_s^j] = [\hat{h}_r^i, h_s^j] = 0, \qquad r = 2k-1, 2k,\ s = 2l-1, 2l.$$

The pair of spaces

$$\hat{\mathcal{E}}_i := \operatorname{lin}\{g_{-1}^i, \ldots, g_{2k-2}^i, \hat{g}_{2k-1}^i, \hat{g}_{2k}^i\},\ \hat{\mathcal{F}}_i := \operatorname{lin}\left\{f_{-1}^i, \ldots, f_{2k-2}^i, \hat{f}_{2k-1}^i, \hat{f}_{2k}^i\right\}$$

now satisfies the requirements. This completes the proof. □

7. Polar Decomposition Revisited

Let $\mathcal{H}$ and $\mathcal{K}$ be finite dimensional Krein spaces. We call an operator $U \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ *unitary* if $UU^{[*]} = I_{\mathcal{K}}$ and $U^{[*]}U = I_{\mathcal{H}}$. We say that $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ *admits a polar decomposition* if there exists a unitary $U \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ and a selfadjoint $A \in \mathbf{B}(\mathcal{H})$ such that $T = UA$. Corollary 6 of [13] says the following.

Theorem 7.1. *The operator $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, where the Krein spaces $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional, admits a polar decomposition if and only if the sign characteristics of $T^{[*]}T$ and $TT^{[*]}$ are the same.*

Note that the 'only if' part of the theorem is obvious. In the light of Theorem 6.1 we are now able to give a new proof of the 'if' part by showing an explicit construction of the unitary transformation U. We again restrict ourselves to the nilpotent case, the nonzero eigenvalues can be handled for example as in [13].

Part of the proof. Suppose that $T^{[*]}T$ is nilpotent and the sign characteristics of $T^{[*]}T$ and $TT^{[*]}$ are the same. We apply Theorem 6.1 and obtain a system of decomposing pairs $\mathcal{E}_i$, $\mathcal{F}_i$, $i = 1, \ldots, n$. Since the sign characteristics of

$T^{[*]}T$ and $TT^{[*]}$ are the same, we can renumerate and group the decomposing spaces in the following way:

$$\mathcal{H} = \mathcal{E}_1 [\dotplus] \cdots [\dotplus] \mathcal{E}_r [\dotplus] \left(\mathcal{E}_{r+1} [\dotplus] \mathcal{E}_{\phi(r+1)}\right) [\dotplus] \cdots [\dotplus] \left(\mathcal{E}_{r+q} [\dotplus] \mathcal{E}_{\phi(r+q)}\right)$$
$$\mathcal{K} = \mathcal{F}_1 [\dotplus] \cdots [\dotplus] \mathcal{F}_r [\dotplus] \left(\mathcal{F}_{r+1} [\dotplus] \mathcal{F}_{\phi(r+1)}\right) [\dotplus] \cdots [\dotplus] \left(\mathcal{F}_{r+q} [\dotplus] \mathcal{F}_{\phi(r+q)}\right)$$

where

- $r, q \geq 0$ and ϕ is a bijection from the set $\{r+1, \ldots, r+q\}$ to the set $\{r+q+1, \ldots, r+2q\}$,
- each decomposing pair $\mathcal{E}_i$, $\mathcal{F}_i$ is of type (i) for $i = 1, \ldots, r$,
- either $\mathcal{E}_{r+i}$, $\mathcal{F}_{r+i}$ is of type (ii) and $\mathcal{E}_{\phi(r+i)}$, $\mathcal{F}_{\phi(r+i)}$ is of type (iii) or $\mathcal{E}_{r+i}$, $\mathcal{F}_{r+i}$ is of type (iv) and $\mathcal{E}_{\phi(r+i)}$, $\mathcal{F}_{\phi(r+i)}$ is of type (v),
- $\dim \mathcal{E}_{r+i} = \dim \mathcal{F}_{\phi(r+i)}$,
- if the four systems

  $$e_1^{r+i}, \ldots, e_k^{r+i}, f_1^{r+i}, \ldots, f_{k+1}^{r+i} \text{ and } e_1^{\phi(r+i)}, \ldots, e_{k+1}^{\phi(r+i)}, f_1^{\phi(r+i)}, \ldots, f_k^{\phi(r+i)}$$

  ($k \geq 0$) are the corresponding bases for the pairs $\mathcal{E}_{r+i}$, $\mathcal{F}_{r+i}$ and $\mathcal{E}_{\phi(r+i)}$ $\mathcal{F}_{\phi(r+i)}$ respectively then

  $$[f_1^{r+i}, f_{k+1}^{r+i}] = [e_1^{\phi(r+i)}, e_{k+1}^{\phi(r+i)}]. \tag{7.1}$$

For $i = 1, \ldots, r$ the operator $T|_{\mathcal{E}_i}$ is already in the polar decomposition form. Indeed, if $g_1, \ldots, g_{2k}$ and $h_1, \ldots, h_{2k}$ are corresponding bases for $\mathcal{E}_i$ and $\mathcal{F}_i$ respectively, then the mapping $U : \mathcal{E}_i \to \mathcal{F}_i$ defined by $Ug_j := h_j$ ($j = 1, \ldots, k$) is unitary and $A := U^{-1}T$ is selfadjoint. We show now that each restriction $T|_{\mathcal{E}_{r+i} [\dotplus] \mathcal{E}_{\phi(r+i)}}$ has a polar decomposition as well. Suppose that the pair $\mathcal{E}_{r+i}$, $\mathcal{F}_{r+i}$ is of type (ii) and $\mathcal{E}_{\phi(r+i)}$, $\mathcal{F}_{\phi(r+i)}$ is of type (iii), the proof for the (iv)–(v)-case goes the same way. We define a mapping

$$U : \mathcal{E}_{r+i} [\dotplus] \mathcal{E}_{\phi(r+i)} \to \mathcal{F}_{r+i} [\dotplus] \mathcal{F}_{\phi(r+i)}$$

by

$$Ue_j^{r+i} = f_j^{\phi(r+i)}, \quad j = 1, \ldots, k, \qquad Ue_j^{\phi(r+i)} = f_j^{r+i}, \quad j = 1, \ldots, k+1.$$

It is unitary by (5.2), (5.3) and (7.1). The reader can now easily check that the operator $A := U^{-1}T$ is selfadjoint. □

We refer the reader to [2,3,10,16] for topics related to the polar decomposition.

8. Explicit Example

Suppose that we want to generate a matrix T such that $T^{[*]}T$ and $TT^{[*]}$ are nilpotent and have given sign characteristics. Constructing the canonical forms and then using a random basis transformation seems to be the simplest method. However, the Jordan chains are unstable and while performing this numerically the Jordan structure may be destroyed. We describe now how to apply our dilation procedure to get desired examples.

Our method consists of several steps. In step $(j+1)$ we construct spaces $\mathcal{H}^{(j+1)} = \mathbb{C}^{n_j}$, $\mathcal{K}^{(j+1)} = \mathbb{C}^{m_j}$ and a matrix $T^{(j)} \in \mathbb{C}^{n_j \times m_j}$. The indefinite inner products on $\mathcal{H}^{(j+1)}$ and $\mathcal{K}^{(j+1)}$ will be given by invertible, selfadjoint matrices $H^{(j)}$ and $K^{(j)}$ respectively, which means

$$[x,y] = \left\langle H^{(j)}x, y \right\rangle, \quad x,y \in \mathcal{H}^j, \qquad [z,w] = \left\langle K^{(j)}z, w \right\rangle, \quad z,w \in \mathcal{K}^j,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. Moreover, $T^{(j+1)}$ will always be a rigid J-dilation of $T^{(j)}$, i.e. $T_0^{(j+1)} = T^{(j)}$. For simplicity we restrict ourselves to generating matrices with given Segre characteristic, since fixing the signs is a minor problem here.

Example 4. We construct a matrix T such that the Segre characteristics of $T^{[*]}T$ and $TT^{[*]}$ are $(4,3,3,2)$ and $(3,3,3,3,1)$ respectively. Note that such a situation is possible according to Theorem 6.1. Namely, there is one pair of decomposing spaces of type (iii) with the lengths of Jordan blocks 4 and 3, one pair of decomposing spaces of type (i) with two Jordan blocks of length 3 for $T^{[*]}T$ and two Jordan blocks of length 3 of $TT^{[*]}$, one pair of decomposing spaces of type (ii) with lengths of Jordan blocks 2 and 3 and finally one pair of decomposing spaces of type (iv).

Step 0. We start with

$$\mathcal{H}^0 = \mathbb{C}, \quad \mathcal{K}^0 = \{0\},$$

and $T^{(0)}$ being the zero operator from $\mathcal{H}^0$ to $\mathcal{K}^0$. The Segre characteristics for $T^{(0)[*]}T^{(0)}$ $T^{(0)}T^{(0)[*]}$ are (1) and (0) respectively.

Step 1. In this step we want to find a rigid J-dilation $T^{(1)}$ of $T^{(0)}$ such that $T^{(1)[*]}T^{(1)}$ and $T^{(1)}T^{(1)[*]}$ have the Segre characteristics $(1,1,1,1)$ and $(2,1,1)$ respectively. Moreover, two blocks of length one for $T^{(1)[*]}T^{(1)}$ and two blocks of length one for $T^{(1)}T^{(1)[*]}$ has to form a decomposing pair of spaces of type (i), one decomposing pair of type (ii) and one decomposing pair of type (v). This means that the spaces $\mathcal{H}^1$ and $\mathcal{K}^1$ are both of dimension 4 and

$$\dim\ker T = 2. \tag{8.1}$$

A priori we have two possible choices for dimensions of the spaces $\mathcal{H}_1^1$, $\mathcal{H}_2^1$, $\mathcal{H}_3^1$ and two possible choices for dimensions of $\mathcal{K}_1^1$, $\mathcal{K}_2^1$, $\mathcal{K}_3^1$. Namely, we can have

$$\dim\mathcal{H}_0^1 = \dim\mathcal{H}^0 = 1, \quad \dim\mathcal{H}_1^1 := 3, \quad \dim\mathcal{H}_2^1 = \dim\mathcal{H}_3^1 := 0 \tag{8.2}$$

or

$$\dim\mathcal{H}_0^1 = \dim\mathcal{H}^0 = 1, \quad \dim\mathcal{H}_1^1 := 1, \quad \dim\mathcal{H}_2^1 = \dim\mathcal{H}_3^1 := 1, \tag{8.3}$$

analogously

$$\dim\mathcal{K}_0^1 = \dim\mathcal{K}^0 = 0, \quad \dim\mathcal{K}_1^1 := 2, \quad \dim\mathcal{K}_2^1 = \dim\mathcal{K}_3^1 := 1 \tag{8.4}$$

or

$$\dim\mathcal{K}_0^1 = \dim\mathcal{K}^0 = 0, \quad \dim\mathcal{K}_1^1 := 0, \quad \dim\mathcal{K}_2^1 = \dim\mathcal{K}_3^1 := 2. \tag{8.5}$$

However, if we set the dimensions according to (8.2) and (8.5) then the J-dilation will not be rigid, since the necessary condition that $\dim(\mathcal{H}_0^1 \dot{+} \mathcal{H}_2^1) = \dim(\mathcal{K}_0^1 \dot{+} \mathcal{K}_2^1)$ is in such case violated. The same happens if we take (8.3) together with (8.4). What is left is (8.2) with (8.4) and (8.3) with (8.5). In the first case we get the matrix

$$T^{(1)} = \begin{pmatrix} 0 & 0 \\ T_{20}^{(1)} & 0 \\ 0 & 0 \end{pmatrix}.$$

The J-dilation is rigid if and only if the 1×1 matrix $T_{20}^{(1)}$ is invertible. Observe that $\dim\ker T^{(1)} = 3$. Hence, by (8.1), we have to reject this case as well.

Let us analyze the case (8.3) with (8.5). The matrix $T^{(1)}$ has then the block form

$$T^{(1)} = \begin{pmatrix} T_{20}^{(1)} & 0 & T_{2}^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the J-dilation is rigid if and only if the 2×2 matrix $\begin{pmatrix} T_{20}^{(1)} & T_{2}^{(1)} \end{pmatrix}$ is invertible. Keeping this constraint we can even pick the matrices $T_{20}^{(1)}$ and $T_{2}^{(1)}$ at random. Note that in this case we have $\dim\ker T^{(1)} = 2$, which agrees with (8.1).

The inner products on $\mathcal{H}_1$ and $\mathcal{K}_1$ are given by the matrices

$$H_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad K_1 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

respectively.[2]

Step 2. Now we want to obtain Segre characteristics $(3, 2, 2, 1)$ and $(2, 2, 2, 2)$, hence $\dim\mathcal{H}^2 = \dim\mathcal{K}^2 = 8$. There are two possibilities of setting the dimensions, namely

$$\begin{aligned} &\dim\mathcal{H}_1^2 := \dim\mathcal{K}_1^2 := 0, \qquad \dim\mathcal{H}_2^2 = \dim\mathcal{H}_3^2 := \dim\mathcal{K}_2^2 = \dim\mathcal{K}_3^2 := 2. \\ &\dim\mathcal{H}_1^2 := \dim\mathcal{K}_1^2 := 2, \qquad \dim\mathcal{H}_2^2 = \dim\mathcal{H}_3^2 := \dim\mathcal{K}_2^2 = \dim\mathcal{K}_3^2 := 1, \end{aligned}$$

otherwise the necessary condition $\dim(\mathcal{H}_0^2 \dotplus \mathcal{H}_2^2) = \dim(\mathcal{K}_0^2 \dotplus \mathcal{K}_2^2)$ for rigidity of the J-dilation is violated. However, we need to reject the latter case. Indeed, since $\dim\ker T^{(1)} = 2$ we are not able to extend $T^{(1)}$ to an invertible matrix $\begin{pmatrix} T^{(1)} & T_{20}^{(2)} \\ T_{02}^{(2)} & T_{02}^{(2)} \end{pmatrix}$ in a five-dimensional space. We set

$$H_2 := \begin{pmatrix} H_1 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix} \qquad K_2 := \begin{pmatrix} K_1 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}.$$

[2] Here is a chance to fix the signs as well.

Again, the numbers for above blocks $T^{(2)}_{20}$, $T^{(2)}_{2}$, $T^{(2)}_{02}$ of the $T^{(2)}$ matrices could be chosen arbitrary, with the only requirement that the matrix $\begin{pmatrix} T^{(1)} & T^{(2)}_{20} \\ T^{(2)}_{02} & T^{(2)}_{02} \end{pmatrix}$ is invertible.

Step 3. Finally, we construct operator $T^{(4)}$ such that the Segre characteristics of $T^{(4)[*]}T^{(4)}$ and $T^{(4)}T^{(4)[*]}$ are respectively $(3,3,3,3,1)$ and $(4,3,3,2)$. As in the previous steps we determine the dimensions:

$$\dim \mathcal{H}_3 = 13 = 8+1+2+2, \quad \dim \mathcal{K}_3 = 12 = 8+0+2+2,$$

we set

$$H_3 = \begin{pmatrix} H_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} K_2 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}$$

and we choose $T^{(3)}_{20}$, $T^{(3)}_{2}$ and $T^{(3)}_{02}$ in such way, that the matrix $\begin{pmatrix} T^{(2)} & T^{(3)}_{20} \\ T^{(3)}_{02} & T^{(3)}_{02} \end{pmatrix}$ is invertible.

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