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Rational numbers with small denominators in short intervals

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Abstract. We use bounds on bilinear forms with Kloosterman fractions and improve the error term in the asymptotic formula of Balazard and Martin (Bull Sci Math 187:Art. 103305, 2023) on the average value of the smallest denominators of rational numbers in short intervals.

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1. Introduction. Given an integer $N \geqslant 1$, and $j = 1, \ldots, N$, we denote by $q_j(N)$ the smallest integer q such that, for some a, we have

$$\frac{a}{q} \in \left(\frac{j-1}{N}, \frac{j}{N}\right].$$

Next, we consider the average value

$$S(N) = \frac{1}{N} \sum_{j=1}^{N} q_j(N).$$

Recently, Balazard and Martin [2] have confirmed the conjecture of Kruyswijk and Meijer [10] that

$$S(N) \sim \frac{16}{\pi^2} N^{3/2}$$

and in fact established the following much more precise asymptotic formula

$$S(N) = \frac{16}{\pi^2} N^{3/2} + O\left(N^{4/3} (\log N)^2\right), \tag{1.1}$$

see [2, Equation (1)]. Note that the asymptotic formula (1.1) improves on previous upper and lower bounds of Kruyswijk and Meijer [10] and Stewart [13], for example on the previous inequalities

$$1.35 N^{3/2} < S(N) < 2.04 N^{3/2}$$

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in [13] (note that $16/\pi^2 = 1.6211\ldots$). For other related results, see [1,4,5,7, 11,12] and references therein.

The bound on the error term in (1.1) is based on the classical bound of Kloosterman sums, see, for example, [9, Corollary 11.12].

Here, we use bounds on bilinear forms with Kloosterman fractions due to Duke et al. [6], and improve the error term in the asymptotic formula (1.1) as follows.

Theorem 1.1. *We have*

$$S(N) = \frac{16}{\pi^2} N^{3/2} + O\left(N^{29/22+o(1)}\right),$$

as $N \to \infty$.

2. Preliminary reductions. As usual, we use the expressions $U \ll V$ and $U = O(V)$ to mean $|U| \le cV$ for some constant $c > 0$ which throughout this paper is absolute.

We have

$$S(N) = \frac{16}{\pi^2} N^{3/2} + R(N), \tag{2.1}$$

where by [2, Equations (19), (20), and (21)], we can write

$$R(N) \ll T_{11}(N) + T_{12}(N) + T_2(N) \tag{2.2}$$

for some quantities $T_{11}(N)$, $T_{12}(N)$, and $T_2(N)$ which are estimated in [2] separately. In particular, by [2, Equations (23) and (26)], we have

$$T_{12}(N) \ll N^{5/4}(\log N)^2 \quad \text{and} \quad T_2(N) \ll N^{5/4}(\log N)^2. \tag{2.3}$$

Therefore, the error term in (1.1) comes from the bound

$$T_{11}(N) \ll N^{4/3}(\log N)^2 \tag{2.4}$$

given by [2, Equation (22)].

We now see from (2.1), (2.2), and (2.3) that in order to establish Theorem 1.1 we only need to improve (2.4) as

$$T_{11}(N) \ll N^{29/22+o(1)}. \tag{2.5}$$

We first recall the following expression for $T_{11}(N)$ given in [2, Section 5.3]:

$$T_{11}(N) = \sum_{s \geqslant \sqrt{N}} \sum_{\substack{1 \leqslant r \leqslant R_s \\ \gcd(r,s)=1}} r B_1\left(\frac{Nr^{-1}}{s}\right) \tag{2.6}$$

with the Bernoulli function

$$B_1(u) = \begin{cases} 0 & \text{if } u \in \mathbb{Z}, \\ \{u\} - 1/2 & \text{if } u \in \mathbb{Z}, \end{cases}$$

where $\{u\}$ is the fractional part of a real u, the inversion r^{-1} in the fractional part $\{Nr^{-1}/s\}$ is computed modulo s, and R_s is a certain sequence of positive integers, satisfying

$$R_s \ll N/s \tag{2.7}$$

(we refer to [2] for an exact definition, which is not important for our argument).

It is more convenient for us to work with the function

$$\psi(u) = \{u\} - 1/2,$$

which coincides with $B_1(u)$ for all $u \notin \mathbb{Z}$.

In particular,

$$B_1\left(\frac{Nr^{-1}}{s}\right) = \psi\left(\frac{Nr^{-1}}{s}\right)$$

unless $s \mid N$.

Using the classical bound on the divisor function

$$\tau(k) = k^{o(1)}, \tag{2.8}$$

for a positive integer $k \to \infty$ (see, for example, [9, Equation (1.81)]), we infer from (2.6) that

$$T_{11}(N) = U(N) + E(N), \tag{2.9}$$

where

$$U(N) = \sum_{s \geqslant \sqrt{N}} \sum_{\substack{1 \leqslant r \leqslant R_s \\ \gcd(r,s)=1}} r\psi\left(\frac{Nr^{-1}}{s}\right), \tag{2.10}$$

and, using (2.7),

$$E(N) \ll \sum_{\substack{s \geqslant \sqrt{N} \\ s \mid N}} R_s^2 \ll N^2 \sum_{\substack{s \geqslant \sqrt{N} \\ s \mid N}} s^{-2} \leqslant N^{1+o(1)}. \tag{2.11}$$

3. Vaaler polynomials. For a real z, let $\mathbf{e}(z) = \exp(2\pi z)$. By a result of Vaaler [14], see also [8, Theorem A.6], we have the following approximation of $\psi(u)$.

Lemma 3.1. *For any integer $H \geqslant 1$, there is a trigonometric polynomial*

$$\psi_H(u) = \sum_{1 \le |h| \le H} \frac{a_h}{-2i\pi h}\,\mathbf{e}(hu)$$

for coefficients $a_h \in [0,1]$ and such that

$$|\psi(u) - \psi_H(u)| \leqslant \frac{1}{2H+2} \sum_{|h| \le H} \left(1 - \frac{|h|}{H+1}\right)\mathbf{e}(hu). \tag{3.1}$$

4. Bilinear forms with Kloosterman fractions. Here we collect some estimates on bilinear forms with exponentials $\mathbf{e}\left(hr^{-1}/s\right)$ where, as before, r^{-1} in the argument is computed modulo s.

For $U \geqslant 1$, we also use $u \sim U$ to indicate $U \leqslant u < 2U$.

We start with recalling the following bound of Duke et al. [6, Theorem 1].

Lemma 4.1. *For sequences* $\boldsymbol{\alpha} = \{\alpha_r\}_{r=1}^{\infty}$, $\boldsymbol{\beta} = \{\beta_s\}_{s=1}^{\infty}$ *of complex numbers, a nonzero integer* K, *and real positive* R *and* S, *we have*

$$\left| \sum_{s\sim S} \sum_{\substack{r\sim R\\ \gcd(r,s)=1}} \alpha_r\beta_s\,\mathbf{e}\left(Kr^{-1}/s\right) \right| \leqslant \|\boldsymbol{\alpha}\|\|\boldsymbol{\beta}\| \left((R+S)^{1/2} + \left(1+\frac{K}{RS}\right)^{1/2} \min\{R,S\} \right) (RS)^{o(1)},$$

where

$$\|\boldsymbol{\alpha}\| = \left(\sum_{r\sim R} |\alpha_r|^2\right)^{1/2} \quad \text{and} \quad \|\boldsymbol{\beta}\| = \left(\sum_{s\sim S} |\beta_s|^2\right)^{1/2}.$$

Next, given two sequences of complex numbers

$$\boldsymbol{\alpha} = \{\alpha_r\}_{r=1}^{\infty} \quad \text{and} \quad \boldsymbol{\beta} = \{\beta_s\}_{s=1}^{\infty},$$

a sequence of positive integers

$$\mathcal{R} = \{\beta_s\}_{s=1}^{\infty},$$

and an integer h, for $S \geqslant 1$, we define the bilinear form

$$\mathscr{B}_K(S;\mathcal{R},\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{s\sim S} \sum_{\substack{r=1\\ \gcd(r,s)=1}}^{R_s} \alpha_r\beta_s\,\mathbf{e}\left(Kr^{-1}/s\right).$$

Note that in the sums $\mathscr{B}_K(S;\mathcal{R},\boldsymbol{\alpha},\boldsymbol{\beta})$ the range of summation over r depends on s and hence Lemma 4.1 does not directly apply.

We observe that for

$$\alpha_r = r, \quad \beta_s \ll 1, \quad R_s \ll \min\{N/s, s\}, \qquad r,s = 1,2,\ldots, \tag{4.1}$$

the argument in [2, Section 3] (in which we also inject the bound (2.8)) immediately implies that for

$$0 < |K| = N^{O(1)} \quad \text{and} \quad 0 < S \ll N,$$

we have

$$\begin{aligned} \mathscr{B}_K(S;\mathcal{R},\boldsymbol{\alpha},\boldsymbol{\beta}) &\ll \sum_{s\sim S} \gcd(K,s)^{1/2} R_s s^{1/2}\log s\\ &\leqslant N^{1+o(1)} \sum_{s\sim S} \gcd(K,s)^{1/2} s^{-1/2}\\ &\leqslant N^{1+o(1)} S^{-1/2} \sum_{d\mid K} d^{1/2} \sum_{\substack{s\leqslant 2S\\ d\mid s}} 1\\ &\leqslant N^{1+o(1)} S^{-1/2} \sum_{d\mid K} d^{1/2} \lfloor 2S/d\rfloor\\ &\leqslant N^{1+o(1)} S^{1/2} \sum_{d\mid K} d^{-1/2} \end{aligned}$$

$$\leqslant N^{1+o(1)} S^{1/2}. \tag{4.2}$$

Note that one can also derive (4.2) via [6, Lemma 8] and partial summation.

In fact, using the bound (4.2) for $S \leqslant N^{2/3}$ and the trivial bound

$$\mathscr{B}_K(S; \mathcal{R}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \ll \sum_{s \sim S} R_s^2 \ll N^2 S^{-1}$$

in our argument below, one recovers the asymptotic formula (1.1). However, using some other bounds, we achieve a stronger result.

We also remark that for us only the choice of $\boldsymbol{\alpha} = \{\alpha_r\}_{r=1}^{\infty}$ satisfying (4.1) matters. However we present the below results for a more general $\boldsymbol{\alpha}$ (but still they admit even more general forms).

Using Lemma 4.1 together with the standard completing technique, see, for example, [9, Section 12.2], we derive our main technical tool.

Lemma 4.2. *For sequences $\boldsymbol{\alpha} = \{\alpha_r\}_{r=1}^{\infty}$, $\boldsymbol{\beta} = \{\beta_s\}_{s=1}^{\infty}$, and $\mathcal{R} = \{R_s\}_{s=1}^{\infty}$, a nonzero integer K and real S with*

$$\alpha_r \ll A, \quad \beta_s \ll B, \quad R_s \ll \min\{N/s, s\}, \qquad r, s = 1, 2, \ldots,$$

and

$$N^{1/2} \ll S \ll N,$$

we have

$$|\mathscr{B}_K(S; \mathcal{R}, \boldsymbol{\alpha}, \boldsymbol{\beta})| \leqslant AB(RS)^{1/2} \left(S^{1/2} + R + K^{1/2} S^{-1/2} R^{1/2} \right) N^{o(1)},$$

where

$$R = \max\{R_s : \ s \sim S\}.$$

Proof. Note that

$$R \ll N/S \ll S. \tag{4.3}$$

Using the orthogonality of exponential functions, we write

$$\begin{aligned}
&\mathscr{B}_K(S; \mathcal{R}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
&\quad = \sum_{s \sim S} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R_s} \alpha_r \beta_s \, \mathbf{e}\left(K r^{-1}/s\right) \\
&\quad = \sum_{s \sim S} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R} \alpha_r \beta_s \, \mathbf{e}\left(K r^{-1}/s\right) \frac{1}{R} \sum_{u=0}^{R-1} \sum_{t=1}^{R_s} \mathbf{e}(u(t-r)/R) \\
&\quad = \frac{1}{R} \sum_{u=0}^{R-1} \sum_{s \sim S} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R} \alpha_r \, \mathbf{e}(-ur/R) \beta_s \, \mathbf{e}\left(K r^{-1}/s\right) \sum_{t=1}^{R_s} \mathbf{e}(ut/R).
\end{aligned}$$

Using that

$$\sum_{t=1}^{R_s} \mathbf{e}(ut/R) \ll \frac{R}{\min\{u, R-u\} + 1},$$

see [9, Equation (8.6)], we derive

$$\mathscr{B}_K(S;\mathcal{R},\boldsymbol{\alpha},\boldsymbol{\beta}) \ll \frac{1}{R}\sum_{u=0}^{R-1}\frac{R}{\min\{u,R-u\}+1} \times \left|\sum_{s\sim S}\sum_{\substack{r=1\\ \gcd(r,s)=1}}^{R}\alpha_r\,\mathbf{e}(-ur/R)\beta_s\,\mathbf{e}\left(Kr^{-1}/s\right)\right|.$$

It remains to observe that, for each $u = 0,\ldots,R-1$, the bound of Lemma 4.1 applies to the inner sum and implies

$$|\mathscr{B}_K(S;\mathcal{R},\boldsymbol{\alpha},\boldsymbol{\beta})| \leqslant AB(RS)^{1/2}\left((R+S)^{1/2}+\left(1+\frac{K}{RS}\right)^{1/2}\min\{R,S\}\right)N^{o(1)}.$$

Recalling (4.3), this now simplifies as

$$\begin{aligned}|\mathscr{B}_K(S;\mathcal{R},\boldsymbol{\alpha},\boldsymbol{\beta})| &\leqslant AB(RS)^{1/2}\left(S^{1/2}+\left(1+\frac{K}{RS}\right)^{1/2}R\right)N^{o(1)}\\ &= AB(RS)^{1/2}\left(S^{1/2}+R+K^{1/2}S^{-1/2}R^{1/2}\right)N^{o(1)},\end{aligned}$$

which concludes the proof. □

Remark 4.3. Instead of using Lemma 4.1, that is, essentially [6, Theorem 1], one can also derive a version of Lemma 4.2 from [6, Theorem 2], or from a stronger result due to Bettin and Chandee [3, Theorem 1]. However these bounds do not seem to improve our main result.

5. Proof of Theorem 1.1. As we have noticed in Section 2, it is enough to only estimate $T_{11}(N)$, as we borrow the bounds on $T_{12}(N)$ and $T_2(N)$ from [2]. Furthermore, we see from (2.9) and (2.11) that it is enough to estimate $U(N)$ given by (2.10).

We note that it is important to observe that the sum defining $\psi_H(u)$ in Lemma 3.1 does not contain the term with $h=0$, while the sum on the right hand side of (3.1) does. Hence, for any integer $H \geqslant 1$, by Lemma 3.1, we have

$$\begin{aligned}U(N) \ll H^{-1}&\sum_{s\geqslant\sqrt{N}}\sum_{\substack{r=1\\ \gcd(r,s)=1}}^{R_s} r\\ &+\sum_{1\le|h|\le H}\frac{1}{h}\left|\sum_{s\geqslant\sqrt{N}}\sum_{\substack{r=1\\ \gcd(r,s)=1}}^{R_s} r\,\mathbf{e}\left(hNr^{-1}/s\right)\right|,\\ &+\frac{1}{H}\sum_{1\le|h|\le H}\left|\sum_{s\geqslant\sqrt{N}}\sum_{\substack{r=1\\ \gcd(r,s)=1}}^{R_s} r\,\mathbf{e}\left(hNr^{-1}/s\right)\right|\end{aligned}$$

$$\ll H^{-1} \sum_{s \geqslant \sqrt{N}} R_s^2 + \sum_{1 \le |h| \le H} \frac{1}{h} \left| \sum_{s \geqslant \sqrt{N}} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R_s} r\, \mathbf{e}\left(hNr^{-1}/s\right) \right|$$

$$\ll H^{-1} N^{3/2} + \sum_{1 \le |h| \le H} \frac{1}{h} \left| \sum_{s \geqslant \sqrt{N}} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R_s} r\, \mathbf{e}\left(hNr^{-1}/s\right) \right|.$$

Note that $R_s \geqslant 1$ implies $s \ll N$. Therefore, partitioning the corresponding summation over s into dyadic intervals, we see that there is some integer S with

$$N^{1/2} \ll S \ll N$$

and such that

$$U(N) \ll H^{-1} N^{3/2} + V(N,S) \log N, \tag{5.1}$$

where

$$V(N,S) = \sum_{1 \le |h| \le H} \frac{1}{h} \left| \sum_{s \sim S} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R_s} r\, \mathbf{e}\left(hNr^{-1}/s\right) \right|.$$

Now, if $S \leqslant H^{1/5} N^{3/5}$, then we use the bound (4.2) and easily derive

$$V(N,S) \leqslant N^{1+o(1)} S^{1/2} \leqslant H^{1/10} N^{13/10+o(1)}. \tag{5.2}$$

On the other hand, for $S > H^{1/5} N^{3/5}$, Lemma 4.2 (used with $A \ll N/S$ and $B \ll 1$), after recalling that $R \ll N/S$, implies the same bound:

$$\begin{aligned} \sum_{s \sim S} \sum_{\substack{r=1 \\ \gcd(r,s)=1}}^{R_s} & r\, \mathbf{e}\left(hNr^{-1}/s\right) \\ & \leqslant (N/S) N^{1/2+o(1)} \left(S^{1/2} + NS^{-1} + h^{1/2} NS^{-1}\right) \\ & \leqslant (N/S) N^{1/2+o(1)} \left(S^{1/2} + h^{1/2} NS^{-1}\right). \end{aligned}$$

Therefore, recalling that $S > H^{1/5} N^{3/5}$, we obtain

$$\begin{aligned} V(N,S) & \leqslant (N/S) N^{1/2+o(1)} \left(S^{1/2} + H^{1/2} NS^{-1}\right) \\ & = N^{3/2+o(1)} S^{-1/2} + H^{1/2} N^{5/2+o(1)} S^{-2} \\ & \leqslant H^{-1/10} N^{6/5+o(1)} + H^{1/10} N^{13/10+o(1)} \\ & \leqslant H^{1/10} N^{13/10+o(1)}. \end{aligned}$$

Therefore, the bound (5.2) holds for any S. Substituting (5.2) in (5.1) yields

$$U(N) \ll H^{-1} N^{3/2} + H^{1/10} N^{13/10+o(1)}$$

and choosing

$$H = \left\lceil N^{2/11} \right\rceil$$

to optimise the bound, we obtain

$$U(N) \ll N^{29/22+o(1)}.$$

Finally, recalling (2.9) and (2.11), we derive (2.5) and conclude the proof.

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