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Matrices for finite group representations that respect Galois automorphisms

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Abstract. We are given a finite group H, an automorphism τ of H of order r, a Galois extension L/K of fields of characteristic zero with cyclic Galois group $\langle\sigma\rangle$ of order r, and an absolutely irreducible representation $\rho\colon H \to \mathsf{GL}(n, L)$ such that the action of τ on the character of ρ is the same as the action of σ. Then the following are equivalent.

- ρ is equivalent to a representation $\rho'\colon H \to \mathsf{GL}(n, L)$ such that the action of σ on the entries of the matrices corresponds to the action of τ on H, and
- the induced representation $\mathsf{ind}_{H,H\rtimes\langle\tau\rangle}(\rho)$ has Schur index one; that is, it is similar to a representation over K.

As examples, we discuss a three dimensional irreducible representation of A_5 over $\mathbb{Q}[\sqrt{5}]$ and a four dimensional irreducible representation of the double cover of A_7 over $\mathbb{Q}[\sqrt{-7}]$.

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1. Introduction. This paper begins with the following question, suggested to the author by Richard Parker. The alternating group A_5 has a three dimensional representation over the field $\mathbb{Q}[\sqrt{5}]$ which induces up to the symmetric group S_5 to give a six dimensional irreducible that can be written over $\mathbb{Q}$. Given an involution in S_5 that is not in A_5, is it possible to write down a 3×3 matrix representation of A_5 such that the Galois automorphism of $\mathbb{Q}[\sqrt{5}]$ acts on matrices in the same way as the involution acts on A_5 by conjugation?

More generally, we are given a finite group H, an automorphism τ of order r, a Galois extension L/K of fields of characteristic zero with cyclic Galois group $\mathsf{Gal}(L/K) = \langle\sigma\rangle$ of order $[L : K] = r$, and an absolutely irreducible representation $\rho\colon H \to \mathsf{GL}(n, L)$. We assume that the action of τ on the character of the representation ρ is the same as the action of σ. Then the question is whether it is possible to conjugate to a representation $\rho' \colon H \to \mathsf{GL}(n, L)$ with the property that the Galois automorphism σ acts on matrices in the same way as τ acts on H. In other words, we are asking whether the following diagram can be made to commute.

$$\begin{tikzcd} H \arrow[r, "\rho'"] \arrow[d, "\tau"'] & \mathsf{GL}(n, L) \arrow[d, "\sigma"] \\ H \arrow[r, "\rho'"] & \mathsf{GL}(n, L). \end{tikzcd}$$

We answer this using the invariant $\lambda(\rho)$ in the relative Brauer group

$$\mathsf{Br}(L/K) = H^2(\langle\sigma\rangle, L^\times) \cong K^\times/N_{L/K}(L^\times)$$

that defines the division algebra associated to the representation obtained by inducing to the semidirect product.

Theorem 1.1. *Let $\rho \colon H \to \mathsf{GL}(n, L)$ be as above. Then there is an invariant $\lambda(\rho) \in K^\times/N_{L/K}(L^\times)$ such that the following are equivalent.*

(1) $\lambda(\rho) = 1$.
(2) *There is a conjugate ρ' of ρ making the diagram above commute.*
(3) *If G is the semidirect product $H \rtimes \langle\tau\rangle$, then the induced representation $\mathsf{ind}_{H,G}(\rho)$ has Schur index equal to one; in other words, it can be written over K.*

More generally, the order of $\lambda(\rho)$ in $K^\times/N_{L/K}(L^\times)$ is equal to the Schur index of the induced representation, and the associated division algebra is the one determined by $\lambda(\rho)$.

The equivalence of (1) and (2) is proved in Section 3. The equivalence of (1) and (3) is more standard, see for example Turull [8], and is proved in Section 4. Combining these gives the more interesting statement of the equivalence of (2) and (3). We end with some examples. In the case of the three dimensional representations of A_5, we have $\lambda(\rho) = 1$, and we write down explicit matrices for ρ', though they are not very pleasant. In the case of the four dimensional irreducible representations of $2A_7$, we have $\lambda(\rho) = -2$, which is not a norm from $\mathbb{Q}[\sqrt{-7}]$, and the division ring associated to the induced representation is the quaternion algebra with symbol $(-2, -7)_{\mathbb{Q}}$.

2. The matrix X. Consider the composite $\sigma \circ \rho \circ \tau^{-1}$:

$$H \xrightarrow{\tau^{-1}} H \xrightarrow{\rho} \mathsf{GL}(n, L) \xrightarrow{\sigma} \mathsf{GL}(n, L).$$

This representation is equivalent to ρ, and so there exists a matrix X, well defined up to scalars in $L^\times$, such that conjugation by X takes ρ to $\sigma \circ \rho \circ \tau^{-1}$. Write c_X for conjugation by X, so that $c_X(A) = XAX^{-1}$. Then we have

$$\sigma \circ \rho \circ \tau^{-1} = c_X \circ \rho. \tag{2.1}$$

By abuse of notation, we shall also write σ for the automorphism of $\mathsf{GL}(n, L)$ given by applying σ to each of its entries. Then

$$c_{\sigma(X)}(\sigma(A)) = \sigma(X)\sigma(A)\sigma(X)^{-1} = \sigma(XAX^{-1}),$$

so we have

$$c_{\sigma(X)} \circ \sigma = \sigma \circ c_X.$$

So equation (2.1) gives

$$\begin{aligned}\sigma^2 \circ \rho \circ \tau^{-2} &= \sigma \circ c_X \circ \rho \circ \tau^{-1} \\ &= c_{\sigma(X)} \circ \sigma \circ \rho \circ \tau^{-1} \\ &= c_{\sigma(X)} \circ c_X \circ \rho \\ &= c_{\sigma(X).X} \circ \rho.\end{aligned}$$

Continuing this way, for any $i > 0$, we have

$$\sigma^i \circ \rho \circ \tau^{-i} = c_{\sigma^{i-1}(X)\cdots\sigma(X).X} \circ \rho.$$

Taking $i = r$, we have $\sigma^r = 1$ and $\tau^r = 1$, so

$$\rho = c_{\sigma^{r-1}(X)\cdots\sigma(X).X} \circ \rho. \tag{2.2}$$

Definition 2.3. If A is an $n \times n$ matrix over L, we define the norm of A to be

$$N_{L/K}(A) = \sigma^{r-1}(A)\cdots\sigma(A).A$$

as an $n \times n$ matrix over K.

Equation (2.2) now reads

$$\rho = c_{N_{L/K}(X)} \circ \rho.$$

By Schur's lemma, it follows that the matrix $N_{L/K}(X)$ is a scalar multiple of the identity,

$$N_{L/K}(X) = \lambda I.$$

Applying σ and rotating the terms on the left, we see that $\lambda = \sigma(\lambda)$, so that $\lambda \in K^\times$. If we replace X by a scalar multiple μX, then the scalar λ gets multiplied by $\sigma^{r-1}(\mu)\cdots\sigma(\mu)\mu$, which is the norm $N_{L/K}(\mu)$. Thus the scalar λ is well defined only up to norms of elements in $L^\times$. We define it to be the λ-invariant of ρ:

$$\lambda(\rho) \in K^\times/N_{L/K}(L^\times).$$

Thus $\lambda(\rho) = 1$ if and only if X can be replaced by a multiple of X to make $N_{L/K}(X) = I$.

3. The matrix $\boldsymbol{Y}$. The goal is to find a matrix Y conjugating ρ to a representation ρ' such that $\sigma \circ \rho' \circ \tau^{-1} = \rho'$. Thus we wish Y to satisfy

$$\sigma \circ c_Y \circ \rho \circ \tau^{-1} = c_Y \circ \rho.$$

We rewrite this in stages:

$$\begin{aligned}c_{\sigma(Y)} \circ \sigma \circ \rho \circ \tau^{-1} &= c_Y \circ \rho \\ \sigma \circ \rho \circ \tau^{-1} &= c_{\sigma(Y)^{-1}} \circ c_Y \circ \rho \\ c_X \circ \rho &= c_{\sigma(Y)^{-1}Y} \circ \rho.\end{aligned}$$

Again applying Schur's lemma, $\sigma(Y)^{-1}Y$ is then forced to be a multiple of X. Since $N_{L/K}(\sigma(Y)^{-1}Y) = I$, it follows that if there is such a Y, then $\lambda(\rho)$ is the identity element of $K^\times/N_{L/K}(L^\times)$. This proves one direction of Theorem 1.1. The other direction is now an immediate consequence of the version of Hilbert's Theorem 90 given in Serre [6, Chapter X, Proposition 3]:

Theorem 3.1. *Let L/K be a finite Galois extension of fields with Galois group $\mathsf{Gal}(L/K)$. Then $H^1(\mathsf{Gal}(L/K), \mathsf{GL}(n, L)) = 0$.*

Corollary 3.2. *Let L/K be a Galois extension with cyclic Galois group $\mathsf{Gal}(L/K) = \langle\sigma\rangle$ of order r. If a matrix $X \in \mathsf{GL}(n, L)$ satisfies $N_{L/K}(X) = I$, then there is a matrix Y such that $\sigma(Y)^{-1}Y = X$.*

Proof. This is the case of a cyclic Galois group of Theorem 3.1. □

This completes the proof of the equivalence of (1) and (2) in Theorem 1.1.

Remark 3.3. In order to make effective use of the implication (1) ⇒ (2) of Theorem 1.1, given X with $N_{L/K}(X) = I$, it is necessary to be able to construct a matrix Y such that $\sigma(Y)^{-1}Y = X$. The practical implementation of this is discussed in closely related contexts in Fieker [1], Glasby et al. [2–4].

4. The induced representation. Let $G = H \rtimes \langle\tau\rangle$, so that for $h \in H$, we have $\tau(h) = \tau h \tau^{-1}$ in G. Then the induced representation $\mathsf{ind}_H^G(\rho)$ is an LG-module with character values in K, but cannot necessarily be written as an extension to L of a KG-module. So we restrict the coefficients to K and examine the endomorphism ring.

Lemma 4.1. $\mathsf{End}_{KG}(\mathsf{ind}_{H,G}(\rho|_K))$ *has dimension r^2 over K.*

Proof. The representation $\rho|_K$ is an irreducible KH-module, whose extension to L decomposes as the sum of the Galois conjugates of ρ, so $\mathsf{End}_{KH}(\rho|_K)$ is r dimensional over K. For the induced representation $\mathsf{ind}_{H,G}(\rho|_K) = \mathsf{ind}_{H,G}(\rho)|_K$, as vector spaces we then have

$$\begin{aligned}&\mathsf{End}_{KG}(\mathsf{ind}_{H,G}(\rho|_K))\\ &\quad\cong \mathsf{Hom}_{KH}(\rho|_K, \mathsf{res}_{G,H}\mathsf{ind}_{H,G}(\rho|_K)) \cong r.\mathsf{End}_{KH}(\rho|_K). \quad \square\end{aligned}$$

Proposition 4.2. *The algebra $\mathsf{End}_{KG}(\mathsf{ind}_{H,G}(\rho|_K))$ is a crossed product algebra, central simple over K, with generators m_λ for $\lambda \in L$ and an element ξ, satisfying*

$$m_\lambda + m_{\lambda'} = m_{\lambda+\lambda'}, \quad m_\lambda m_{\lambda'} = m_{\lambda\lambda'}, \quad m_\lambda \circ \xi = \xi \circ m_{\sigma(\lambda)}, \quad \xi^r = m_{\lambda(\rho)}.$$

Proof. We can write the representation $\mathsf{ind}_{H,G}(\rho|_K)$ in terms of matrices as follows.

$$g \mapsto \begin{pmatrix} \rho(g)|_K &&& \\ & \sigma\rho\tau^{-1}(g)|_K && \\ && \ddots & \\ &&& \sigma^{-1}\rho\tau(g)|_K \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} &&& I \\ I &&& \\ & \ddots && \\ && I & \end{pmatrix} \circ \sigma.$$

It is easy to check that the following are endomorphisms of this representation.

$$m_\lambda = \begin{pmatrix} \lambda I & & & \\ & \sigma(\lambda)I & & \\ & & \ddots & \\ & & & \sigma^{-1}(\lambda)I \end{pmatrix}, \qquad \xi = \begin{pmatrix} & & & \sigma^{-1}(X) \\ X & & & \\ & \sigma(X) & & \\ & & \ddots & \\ & & & \sigma^{-2}(X) & \end{pmatrix}$$

with $\lambda \in L$ and X as in Section 2. Since these generate an algebra of dimension r^2 over K, by Lemma 4.1, they generate the algebra $\mathsf{End}_{KG}(\mathsf{ind}_{H,G}(\rho|_K))$. The given relations are easy to check, and present an algebra which is easy to see has dimension at most r^2, and therefore no further relations are necessary. □

Corollary 4.3. *The Schur index of the induced representation* $\mathsf{ind}_{H,G}(\rho)$ *is equal to the order of* $\lambda(\rho)$ *as an element of* $K^\times/N_{L/K}(L^\times)$. *In particular, the Schur index is one if and only if* $\lambda(\rho) = 1$ *as an element of* $K^\times/N_{L/K}(L^\times)$.

Proof. This follows from the structure of the central simple algebra $\mathsf{End}_{KG}(\rho|_K)$ given in Proposition 4.2, using the theory of cyclic crossed product algebras, as developed for example in Pierce [5, Section 15.1], particularly Proposition b of that section. □

This completes the proof of the equivalence of (1) and (3) in Theorem 1.1. In particular, it shows that $\lambda(\rho)$ can only involve primes dividing the order of G.

5. Examples. Our first example is a three dimensional representation of A_5. There are two algebraically conjugate three dimensional irreducible representations of A_5 over $\mathbb{Q}[\sqrt{5}]$ swapped by an outer automorphism of A_5, and giving a six dimensional representation of the symmetric group S_5 over $\mathbb{Q}$.

Setting $\alpha = \frac{1+\sqrt{5}}{2}$, $\bar{\alpha} = \frac{1-\sqrt{5}}{2}$, we can write the action of the generators on one of these three dimensional representations as follows.

$$(12)(34) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (153) \mapsto \begin{pmatrix} -1 & 1 & \alpha \\ \alpha & 0 & -\alpha \\ -\alpha & 0 & 1 \end{pmatrix}.$$

Taking this for ρ, we find a matrix X conjugating this to $\sigma \circ \rho \circ \tau^{-1}$ where σ is the field automorphism and τ is conjugation by (12). Using the fact that if $a = (12)(34)$ and $b = (153)$, then $ab^2abab^2 = (253)$, we find that

$$X = \begin{pmatrix} 1 & -\bar{\alpha} & \bar{\alpha} \\ -\bar{\alpha} & 1 & -\bar{\alpha} \\ \bar{\alpha} & -\bar{\alpha} & 1 \end{pmatrix}.$$

We compute that $\sigma(X).X$ is minus the identity. Now -1 is in the image of $N_{\mathbb{Q}[\sqrt{5}],\mathbb{Q}}$, namely we have $(2-\sqrt{5})(2+\sqrt{5}) = -1$. So we replace X by $(2-\sqrt{5})X$ to achieve $\sigma(X).X = I$. Having done this, by Hilbert 90, there exists Y with $\sigma(Y)^{-1}.Y = X$. Such a Y conjugates ρ to the desired form. For example, we can take

$$Y = \begin{pmatrix} 1-2\sqrt{5} & 3-2\sqrt{5} & -3+2\sqrt{5} \\ 3-2\sqrt{5} & 1-2\sqrt{5} & 3-2\sqrt{5} \\ -3+2\sqrt{5} & 3-2\sqrt{5} & 1-2\sqrt{5} \end{pmatrix}.$$

Thus we end up with the representation

$$(12)(34) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$(153) \mapsto \frac{1}{40}\begin{pmatrix} 10-4\sqrt{5} & -5+19\sqrt{5} & 25-9\sqrt{5} \\ -10-4\sqrt{5} & 25+9\sqrt{5} & -5-19\sqrt{5} \\ -50 & 35-5\sqrt{5} & -35-5\sqrt{5} \end{pmatrix},$$

$$(253) \mapsto \frac{1}{40}\begin{pmatrix} 10+4\sqrt{5} & -5-19\sqrt{5} & 25+9\sqrt{5} \\ -10+4\sqrt{5} & 25-9\sqrt{5} & -5+19\sqrt{5} \\ -50 & 35+5\sqrt{5} & -35+5\sqrt{5} \end{pmatrix}.$$

Denoting these matrices by a, b, and c, it is routine to check that $a^2 = b^3 = (ab)^5 = 1$, $a^2 = c^3 = (ac)^5 = 1$, and $c = \sigma(b) = ab^2abab^2$.

More generally, if H is an alternating group A_n and G is the corresponding symmetric group S_n, then all irreducible representations of G are rational and so the invariant $\lambda(\rho)$ is equal to one for any irreducible character of H that is not rational. So an appropriate matrix Y may always be found in this case.

Our second example is one with $\lambda(\rho) \neq 1$. Let H be the group $2A_7$, namely a non-trivial central extension of A_7 by a cyclic group of order two. Let τ be an automorphism of H of order two, lifting the action of a transposition in S_7 on H, and let G be the semidirect product $H \rtimes \langle\tau\rangle$. Then H has two Galois conjugate irreducible representations of dimension four over $\mathbb{Q}[\sqrt{-7}]$. Let ρ be one of them. The induced representation is eight dimensional over $\mathbb{Q}[\sqrt{-7}]$. Restricting coefficients to $\mathbb{Q}$ produces a 16 dimensional rational representation whose endomorphism algebra E is a quaternion algebra. Thus the induced representation can be written as a four dimensional representation over $E^{\mathsf{op}} \cong E$. This endomorphism algebra was computed by Turull [7] in general for the double covers of symmetric groups. In this case, by Corollary 5.7 of that paper, the algebra E is generated over $\mathbb{Q}$ by elements u and v satisfying $u^2 = -2$, $v^2 = -7$, and $uv = -vu$. Thus the invariant $\lambda(\rho)$ is equal to -2 as an element of $\mathbb{Q}^\times/N_{\mathbb{Q}[\sqrt{-7}],\mathbb{Q}}(\mathbb{Q}[\sqrt{-7}]^\times)$ in this case.

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