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Terms of recurrence sequences in the solution sets of norm form equations

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Abstract. The structure as well as several arithmetic properties of the solution sets of norm form equations are of classical and recent interest. In this paper, we give a finiteness result for terms of linear recurrence sequences appearing in the coordinates of solutions of norm form equations. Our main theorem yields a common generalization of certain recent results from the literature.

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1. Introduction. Arithmetic properties of solutions of norm form equations have a considerable literature. Here we mention only one result (which is the most important from our viewpoint), and for generalities on norm form equations or other related result, we refer the interested reader to the papers [1] or [2]. Fuchs and Heintze considered terms of so-called multi-recurrences (being natural generalizations of linear recurrence sequences) among the coordinates of solutions of norm form equations. They set the problem in the general case, however, their results concern the case of multi-recurrences which are simple. Simplifying their result to the case of linear recurrence sequences, this means that the characteristic polynomials of these sequences have simple zeroes. They could prove that a simple multi-recurrence, under certain necessary assumptions, can have only finitely many terms among the coordinates of solutions of a norm form equation.

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In the special situation when the underlying field is quadratic, we in fact investigate terms of recurrence sequences among the solutions of generalized Pell equations. For an account on these results, see [4]. In particular, the main theorem of [4] states that under certain assumptions, a recurrence sequence has only finitely many terms among the coordinates of solutions of generalized Pell equations.

In this paper, we provide a common extension of the main results of [2] (more precisely, its corollary to the case of linear recurrence sequences) and of [4]. That is, we prove that, under certain assumptions, an arbitrary linear recurrence sequence can have only finitely many terms in the coordinates of solutions of norm form equations. We show that the imposed assumptions are necessary. In our proof, we shall combine the finiteness of solutions of polynomial-exponential equations proved by Schlickewei and Schmidt [5] with some other tools and ideas.

2. The main result. To formulate our main result, we need to introduce some notation. Let K be an algebraic number field of degree k, and write $\mathcal{N}(\alpha)$ for the norm of $\alpha \in K$ (over $\mathbb{Q}$). Let $\alpha_1, \dots, \alpha_k \in K$ be linearly independent over $\mathbb{Q}$, and let m be a non-zero integer. Consider the norm form equation

$$\mathcal{N}(x_1\alpha_1 + \cdots + x_k\alpha_k) = m \tag{1}$$

in integers $x_1, \dots, x_k$. Write X_i $(i = 1, \dots, k)$ for the coordinate sets of solutions of (1).

Let r be a positive integer, $a_1, \dots, a_r \in \mathbb{Z}$ such that $a_r \neq 0$, and $U_0, \dots, U_{r-1} \in \mathbb{Z}$ not all zero. If

$$U_n = a_1U_{n-1} + \cdots + a_rU_{n-r} \quad (n \geq r) \tag{2}$$

and r is minimal such that (U_n) satisfies a relation above, then $U = (U_n) = (U_n)_{n\geq 0}$ is called a linear recurrence sequence (of integers) of order r. Throughout this paper, we always assume that a recurrence sequence is given by its minimal length relation (2). We shall also use the notation

$$U = U(a_1, \dots, a_r, U_0, \dots, U_{r-1}).$$

The characteristic polynomial of (U_n) is defined by

$$f(x) := x^r - a_1x^{r-1} - \cdots - a_r = \prod_{i=1}^{d} (x - \beta_i)^{m_i} \tag{3}$$

where $\beta_1, \dots, \beta_d$ are distinct algebraic numbers and $m_1, \dots, m_t$ are positive integers. Then as it is well-known (see e.g. [9, Theorem C.1 in part C]), we have

$$U_n = \sum_{i=1}^{d} g_i(n)\beta_i^n \quad (n \geq 0). \tag{4}$$

Here $g_i(x)$ is a not identically zero polynomial of degree at most $m_i - 1$ $(i = 1, \dots, s)$ with coefficients in the number field $\mathbb{Q}(\beta_1, \dots, \beta_d)$. (The fact that the polynomials g_i are not identically zero is not formulated in [9, Theorem C.1], however it is clear from its proof.) We say that the sequence (U_n) is degenerate if there are integers i, j with $1 \leq i < j \leq s$ such that α_i/α_j is a root of unity; otherwise it is non-degenerate.

Now we can formulate our main result.

Theorem 2.1. *Let K be an algebraic number field of degree k, and (U_n) be a non-degenerate linear recurrence sequence of integers of order $r \geq 2$ given by (2) and (4). If $\beta_i \neq \pm 1$ for some $i = 1, \ldots, d$ and one of the conditions*

(i) $a_r \neq \pm 1$,
(ii) $K \cap \mathbb{Q}(\beta_i) = \mathbb{Q}$ $(i = 1, \ldots, d)$

is satisfied, then

$$U_n \in X_1 \cup \cdots \cup X_k \tag{5}$$

holds only for finitely many indices n, where X_i $(i = 1, \ldots, k)$ are the sets of the coordinates of the solutions of any norm form equation, as defined in (1). Further, the number of such indices is bounded by c_1, where $c_1 = c_1(m, k, r)$ is an effectively computable constant depending only on m, k, r.

Remark. It is important to mention that (1) can have infinitely many solutions (with m chosen appropriately), unless K is $\mathbb{Q}$ or an imaginary quadratic field. This follows from results of Schmidt (see [6, Satz 2] or [7, Chapter VII]). In certain cases, (1) can have infinitely many solutions even if we take only k' linearly independent algebraic integers $\alpha_i \in K$ $(i = 1, \ldots, k')$ with $k' < k$. This happens e.g. if we take $\alpha_1 = 1$ and $\alpha_2 = \sqrt{2}$ in $K = \mathbb{Q}(\sqrt[4]{2})$, together with $m = 1$. (For a precise description of these cases, see [6,7] again). Clearly, Theorem 2.1 remains valid also in these cases. Indeed, choose algebraic integers $\alpha_{k'+1}, \ldots, \alpha_k$ from K such that $\alpha_1, \ldots, \alpha_k$ are linearly independent over $\mathbb{Q}$. As $X_1 \cup \cdots \cup X_{k'} \subseteq X_1 \cup \cdots \cup X_k$, the finiteness of the set of indices n with

$$U_n \in X_1 \cup \cdots \cup X_{k'}$$

immediately follows from Theorem 2.1.

We also note that the conditions in the theorem are all necessary. To show this, we exhibit some examples. However, we do so after the proof of Theorem 2.1 since then we shall have the required machinery and notation.

3. Lemmas and proofs. To prove our theorem, we need two lemmas. The first one is due to Schlickewei and Schmidt [5]. It concerns the finiteness of the solutions of polynomial-exponential equations. For its formulation, we need to introduce some new notation.

Let L be an algebraic number field, and let $P_1, \ldots, P_s$ be not identically zero polynomials in t variables over L. Further, let $\kappa_{i1}, \ldots, \kappa_{it}$ $(i = 1, \ldots, s)$ be non-zero elements of L. Consider the equation

$$\sum_{i=1}^{s} P_i(\boldsymbol{x})\boldsymbol{\kappa}_i^{\boldsymbol{x}} = 0 \tag{6}$$

in tuples $\boldsymbol{x} = (x_1, \ldots, x_t) \in \mathbb{Z}^t$, where $\boldsymbol{\kappa}_i = (\kappa_{i1}, \ldots, \kappa_{it})$ $(i = 1, \ldots, s)$ and

$$\boldsymbol{\kappa}_i^{\boldsymbol{x}} = \kappa_{i1}^{x_1} \cdots \kappa_{it}^{x_t} \quad (i = 1, \ldots, s). \tag{7}$$

Let $\mathcal{P}$ be a partition of the set $\Lambda = \{1, \ldots, s\}$. Then the system of equations

$$\sum_{i \in \lambda} P_i(\boldsymbol{x}) \boldsymbol{\kappa}_i^{\boldsymbol{x}} = 0 \quad (\lambda \in \mathcal{P}) \tag{8}$$

is called a refinement of (6). Let $\mathcal{S}(\mathcal{P})$ be the set of solutions of (8) which are not solutions of

$$\sum_{i \in \lambda} P_i(\boldsymbol{x}) \boldsymbol{\kappa}_i^{\boldsymbol{x}} = 0 \quad (\lambda \in \mathcal{P}')$$

with any proper refinement $\mathcal{P}'$ of $\mathcal{P}$. Set $i_1 \overset{\mathcal{P}}{\sim} i_2$ if i_1 and i_2 lie in the same subset λ of $\mathcal{P}$. Let $G(\mathcal{P})$ be the subgroup of $\mathbb{Z}^t$ consisting of tuples $\boldsymbol{z} = (z_1, \ldots, z_t)$ with

$$\boldsymbol{\kappa}_{i_1}^{\boldsymbol{z}} = \boldsymbol{\kappa}_{i_2}^{\boldsymbol{z}} \quad \text{for any } i_1, i_2 \text{ with } i_1 \overset{\mathcal{P}}{\sim} i_2.$$

Lemma 3.1. *Using the above notation, if $G(\mathcal{P}) = \{\mathbf{0}\}$, then we have*

$$|\mathcal{S}(\mathcal{P})| < 2^{35A^3} D^{6A^2}$$

with $D = \deg(L)$ and

$$A = \max\left(t, \sum_{i \in \Lambda} \binom{t + \delta_i}{t}\right),$$

where δ_i is the total degree of the polynomial P_i.

Proof. The statement is [5, Theorem 1]. □

Our second lemma is an application of Lemma 3.1 with $t = 1$ to linear recurrence sequences. Note that it is closely related to the zero multiplicity of linear recurrence sequences, known to be finite and bounded due to a deep result of Schmidt [8].

Lemma 3.2. *Let (U_n) be a non-degenerate linear recurrence sequence of integers of order $r \geq 2$. Using (4), let $I \subseteq \{1, \ldots, d\}$. Then there are only finitely many indices n for which*

$$\sum_{i \in I} g_i(n) \beta_i^n = 0 \tag{9}$$

holds. Further, the number of such indices n can be bounded by c_2, where $c_2 = c_2(r)$ is an effectively computable constant depending only on r.

Proof. We apply Lemma 3.1 with $t = 1$. Let n be a solution of (9). First observe that if $g_i(n) = 0$ for some $i \in I$, then n comes from a finite set of cardinality bounded in terms of r. So we may assume that $g_i(n) \neq 0$ $(i \in I)$. Clearly, there is a partition $\mathcal{P}$ of I such that $n \in \mathcal{S}(\mathcal{P})$. If I has a class λ with $|\lambda| = 1$, then n is a root of one of the polynomials g_i $(i \in I)$, which is excluded. So we can suppose that $|\lambda| \geq 2$ for all $\lambda \in \mathcal{P}$. Clearly, since (U_n) is non-degenerate, we have $G(\mathcal{P}) = \{0\}$. Thus, by Lemma 3.1, n comes again from a finite set of cardinality bounded in terms of r, and our claim follows. □

Now we can prove our main result.

Proof of Theorem 2.1. By Lemma 5 of Győry [3], we know that there are only finitely many pairwise non-associate algebraic integers μ in K of norm m, and their number can be bounded in terms of k, m. That is, if $(x_1, \dots, x_k) \in \mathbb{Z}^k$ is any solution of (1), then we have

$$x_1\alpha_1 + \cdots + x_k\alpha_k = \varepsilon\mu \tag{10}$$

with such a μ, where ε is a unit of K. As we need to bound the number of indices n for which $U_n = x_i$ for some solution $(x_1, \dots, x_k)$ and i with $1 \leq k$, we may assume that μ is fixed. Let $\varepsilon_1, \dots, \varepsilon_\ell$ be a system of fundamental units in K. Then (10) yields

$$x_1\alpha_1 + \cdots + x_k\alpha_k = \delta\varepsilon_1^{u_1}\cdots\varepsilon_\ell^{u_\ell}\mu, \tag{11}$$

where δ is a root of unity in K. Since the number of roots of unity in K is bounded in terms of k, we may assume that here δ is fixed. Taking the conjugates in (11) (following arguments from [1] and [2]), we get

$$\begin{pmatrix} \sigma_1(\alpha_1) & \cdots & \sigma_1(\alpha_k) \\ \vdots & \ddots & \vdots \\ \sigma_k(\alpha_1) & \cdots & \sigma_k(\alpha_k) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} \sigma_1(\delta\mu)\nu_{11}^{u_1}\cdots\nu_{1\ell}^{u_\ell} \\ \vdots \\ \sigma_k(\delta\mu)\nu_{k1}^{u_1}\cdots\nu_{k\ell}^{u_\ell} \end{pmatrix}, \tag{12}$$

where $\sigma_1, \dots, \sigma_k$ are the isomorphisms of K into $\mathbb{C}$ (in any order), and $\nu_{ij} = \sigma_i(\varepsilon_j)$ $(1 \leq i \leq k,\ 1 \leq j \leq \ell)$. As the determinant of the matrix on the left hand side of (12) is known to be non-zero, we can write

$$x_i = b_{1i}\nu_{11}^{u_1}\cdots\nu_{1\ell}^{u_\ell} + \cdots + b_{ki}\nu_{k1}^{u_1}\cdots\nu_{k\ell}^{u_\ell} \quad (i = 1, \dots, k) \tag{13}$$

with some algebraic numbers b_{ij} (belonging to the normal closure of K). Then, using the notation (7), by (4) and (13), relation (5) gives

$$b_{1i}\boldsymbol{\nu}_1^{\boldsymbol{u}} + \cdots + b_{ni}\boldsymbol{\nu}_n^{\boldsymbol{u}} = P_1(n)\beta_1^n + \cdots + P_d(n)\beta_d^n \tag{14}$$

with $\boldsymbol{u} = (u_1, \dots, u_\ell)$ for some i with $1 \leq i \leq k$. We may clearly assume that i is fixed. We shall apply Lemma 3.1 to handle the solutions $\boldsymbol{u}, n$ of (14). For this, we need to introduce some new notation. Define the $(k+1)$-tuples $\boldsymbol{\vartheta}_j$ $(1 \leq j \leq k+d)$ by

$$\boldsymbol{\vartheta}_j = \begin{cases} (\nu_{j1}, \dots, \nu_{j\ell}, 1) & \text{for } 1 \leq j \leq k, \\ (1, \dots, 1, \beta_{j-k}) & \text{for } k+1 \leq j \leq k+d, \end{cases}$$

and the polynomials Q_j $(1 \leq j \leq k+d)$ in $\ell+1$ variables $\boldsymbol{z} = (z_1, \dots, z_{\ell+1})$ by

$$Q_j(\boldsymbol{z}) = \begin{cases} b_{ji} & \text{for } 1 \leq j \leq k, \\ -P_j(z_{k+1}) & \text{for } k+1 \leq i \leq k+d. \end{cases}$$

Then we can rewrite (14) as

$$\sum_{j=1}^{k+d} Q_j(\boldsymbol{z})\boldsymbol{\vartheta}_j^{\boldsymbol{z}} = 0. \tag{15}$$

Let $\mathcal{P}$ be any partition of the set $J=\{1,\dots,k+d\}$. Observe that the number of such partitions is bounded in terms of k and r. Consider the refinement

$$\sum_{j\in\lambda} Q_j(\boldsymbol{z})\boldsymbol{\vartheta}_j^{\boldsymbol{z}}=0 \quad (\lambda\in\mathcal{P}) \tag{16}$$

of (15). We shall be concerned with the solutions of (16) which are not solutions of proper refinements of it; that is, with $\mathcal{S}(\mathcal{P})$. If there is a $\lambda\in\mathcal{P}$ such that $\lambda\subseteq\{k+1,\dots,k+d\}$, then any solution of (16) comes from a case where we have a vanishing subsum in the right hand side of (14). However, since by Lemma 3.2 the number of indices n which allow this is bounded in terms of r, we may assume that it is not the case. We study the set $G(\mathcal{P})$ - in fact, we show that $G(\mathcal{P})=\{\mathbf{0}\}$ in any case. Take an index j_1 with $k+1\le j_1\le k+d$ such that $\beta_{j_1}\neq\pm1$. (By our assumptions, such an index exists.) Then $j_1\in\lambda$ for some $\lambda\in\mathcal{P}$. By the above argument, we see that there is a $j_2\in\lambda$ with $1\le j_2\le k$ such that Q_{j_2} is not identically zero. If $\boldsymbol{z}\in G(\mathcal{P})$, then as $j_1\overset{\mathcal{P}}{\sim} j_2$, we have

$$\nu_{j_21}^{z_1}\cdots\nu_{j_2\ell}^{z_\ell}=\beta_{j_1}^{z_{\ell+1}}. \tag{17}$$

If (i) holds, then since β_{j_1} is not a unit in $\sigma_{j_2}(K)$, we get $z_{\ell+1}=0$. Then, as $\nu_{j_21},\dots,\nu_{j_2\ell}$ is a system of fundamental units in $\sigma_{j_2}(K)$, we obtain $z_v=0$ $(1\le v\le\ell)$. Hence $\boldsymbol{z}=\mathbf{0}$, so $G(\mathcal{P})=\{\mathbf{0}\}$, and our statement follows from Lemma 3.1 in this case. Assume that (ii) holds. Taking the inverse $\sigma_{j_2}^{-1}$ of σ_{j_2}, (17) yields

$$\varepsilon_1^{z_1}\cdots\varepsilon_\ell^{z_\ell}=\beta_h^{z_{\ell+1}}, \tag{18}$$

where $\beta_h=\sigma_{j_2}^{-1}(\beta_{j_1})$. Clearly, β_h is an algebraic conjugate of β_{j_2}, so it is a root of $f(x)$ in (3). Thus (ii) gives that the unit $\beta_h^{z_{\ell+1}}$ is rational - that is, it is ±1. Since $\beta_{j_1}\neq\pm1$ and by the non-degenerate property of (U_n), β_h cannot be a root of unity, this gives $z_{\ell+1}=0$. Then as $\varepsilon_1,\dots,\varepsilon_\ell$ form a system of fundamental units in K, we also get $z_1=\dots=z_\ell=0$. So $\boldsymbol{z}=\mathbf{0}$, and $G(\mathcal{P})=\{\mathbf{0}\}$. Thus our statement follows from Lemma 3.1 also in this case. The proof of the theorem is complete. □

As we mentioned earlier, the assumptions made in Theorem 2.1 are necessary. We conclude our paper by some examples showing that this is the case indeed.

Example. Let (U_n) be given by $U_0=0, U_1=1$, and $U_{n+2}=U_{n+1}-U_n$ $(n\ge0)$. Then as one can easily check, (U_n) is given by $0,1,1,0,-1,-1,0,1,\dots$. Taking $K=\mathbb{Q}(\sqrt{2})$ (so $k=2$) and $\alpha_1=1$, $\alpha_2=\sqrt{2}$, $m=1$, (1) reads as

$$\mathcal{N}(x_1+x_2\sqrt{2})=1 \tag{19}$$

in integers x_1,x_2. Since $1+\sqrt{2}$ is a fundamental unit in K of norm -1, all solutions x_1,x_2 of (19) come from the coefficients of 1 and $\sqrt{2}$ in $\pm(1+\sqrt{2})^\ell$ with $\ell\in\mathbb{Z}$, ℓ even. From this, we easily obtain that

$$X_1=\{\pm P_\ell:\ell\ge0\},\quad X_2=\{\pm Q_\ell:\ell\ge0\},$$

where $P_0 = 1$, $P_1 = 3$, $Q_0 = 0$, $Q_1 = 2$, and

$$P_{\ell+2} = 6P_{\ell+1} - P_\ell, \quad Q_{\ell+2} = 6Q_{\ell+1} - Q_\ell.$$

In particular, this shows that $U_n \in X_1 \cup X_2$ for every $n \geq 0$. However, in this case, the characteristic polynomial of (U_n) is $f(x) = x^2 - x + 1$, so the sequence is degenerate. Hence it is necessary to exclude this property.

Next, consider the sequence (U_n) defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+2} = 2U_{n+1} - U_n$ $(n \geq 0)$. Then one readily gets that $U_n = n$ $(n \geq 0)$. So considering again (19), we obtain that $U_n \in X_1 \cup X_2$ for infinitely many n, namely, whenever n belongs to either (P_ℓ) or (Q_ℓ). Now we have $f(x) = x^2 - 2x + 1 = (x-1)^2$, so the condition that one of the roots of f is different from ± 1 is violated.

Finally, assume that none of the conditions (i) and (ii) in Theorem 2.1 is satisfied. Let $\alpha = \sqrt[3]{2}$ and $K = \mathbb{Q}(\alpha)$. Then $1, \alpha, \alpha^2$ form an integral basis of K. Further, $\alpha - 1$ is a fundamental unit of K, and the only roots of unity in K are ± 1. Putting $(\alpha_1, \alpha_2, \alpha_3) = (1, \alpha, \alpha^2)$ and $m = 1$, the norm form equation (1) is given by

$$\mathcal{N}(x_1 + x_2\alpha + x_3\alpha^2) = 1.$$

Since the norm of $\alpha - 1$ is 1, this equation is equivalent to

$$x_1 + x_2\alpha + x_3\alpha^2 = (\alpha - 1)^u \quad (u \in \mathbb{Z}).$$

Thus, in this case, (12) reads as

$$\begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \xi\alpha & \xi^2\alpha^2 \\ 1 & \xi^2\alpha & \xi\alpha^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} (\alpha - 1)^u \\ (\xi\alpha - 1)^u \\ (\xi^2\alpha - 1)^u \end{pmatrix},$$

where $\xi = (-1 + \sqrt{-3})/2$ (which is a primitive third root of unity). From this, by a simple calculation, we obtain that

$$x_1 = \frac{1}{3}(\alpha - 1)^u + \frac{1}{3}(\xi\alpha - 1)^u + \frac{1}{3}(\xi^2\alpha - 1)^u \quad (u \in \mathbb{Z}). \tag{20}$$

Set $U_0 = 1$, $U_1 = -1$, $U_2 = 1$, and $U_{n+3} = -3U_{n+2} - 3U_{n+1} + U_n$ $(n \geq 0)$. So (U_n) is a linear recurrence sequence of order $r = 3$, with characteristic polynomial $f(x) = x^3 + 3x^2 + 3x - 1$. As one can easily check, the roots of $f(x)$ are given by $\alpha - 1$, $\xi\alpha - 1$, $\xi^2\alpha - 1$, and also that in this case (4) is given by

$$U_n = \frac{1}{3}(\alpha - 1)^n + \frac{1}{3}(\xi\alpha - 1)^n + \frac{1}{3}(\xi^2\alpha - 1)^n \quad (n \geq 0). \tag{21}$$

Comparing (20) and (21), we see that $U_n \in X_1$ for every $n \geq 0$. Observe that now (i) does not hold as $a_3 = -1$, and (ii) does not hold since $K \cap \mathbb{Q}(\alpha - 1) = K$. So we need to require the validity of (i) or (ii), indeed.

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