



A unified observability result for non-autonomous observation problems

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Abstract. A final-state observability result in the Banach space setting for non-autonomous observation problems is obtained that covers and extends all previously known results in this context, while providing a streamlined proof that follows the established Lebeau-Robbiano strategy.

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1. Introduction. Observability and null-controllability results for (non-) autonomous Cauchy problems are relevant especially in the field of control theory of partial differential equations and have recently attracted a lot of attention in the literature. Here, the most common approach towards *final-state observability* is a so-called *Lebeau-Robbiano strategy*, which combines a suitable *uncertainty principle* with a corresponding *dissipation estimate* for the evolution family describing the evolution of the system, see ([essUCP](#)) and ([DE](#)) below, respectively. Certain null-controllability results can then be inferred from final-state observability via a standard duality argument, see, e.g., [[4](#)] for more information and also [[9](#)] for an holistic overview of duality theory for control systems.

Such a Lebeau-Robbiano strategy has been considered, for instance, in [[1, 2, 4, 8, 10, 11, 13, 14](#)], see also [[5](#)] for a review of other related results in this context. The two most general results in this direction so far are [[4](#), Theorem 3.3] and [[1](#), Theorem 13], each highlighting different aspects and exhibiting certain advantages and disadvantages over the other, both with regard to hypotheses and the asserted conclusion, see the discussion below. The aim of the present

work is to present a unified extension of both mentioned results, taking the best of each, thus allowing to apply the Lebeau-Robbiano strategy to a broader range of observation problems and, at the same time, providing a streamlined proof.

2. Lebeau-Robbiano strategy for non-autonomous observation problems. For the reader’s convenience, let us fix the following notational setup.

Hypothesis 2.1. *Let X and Y be Banach spaces, $T > 0$, $E \subseteq [0, T]$ be measurable with positive Lebesgue measure, and $(U(t, s))_{0 \leq s \leq t \leq T}$ be an exponentially bounded evolution family on X . Let $C : [0, T] \rightarrow \mathcal{L}(X, Y)$ be essentially bounded on E such that $[0, T] \ni t \mapsto \|C(t)U(t, 0)x_0\|_Y$ is measurable for all $x_0 \in X$.*

Here, we denote by $\mathcal{L}(X, Y)$ the space of bounded operators from X to Y . Also recall that $(U(t, s))_{0 \leq s \leq t \leq T} \subseteq \mathcal{L}(X) := \mathcal{L}(X, X)$ is called an *evolution family* of bounded operators on X if

$$U(s, s) = \text{Id} \quad \text{and} \quad U(t, s)U(s, r) = U(t, r) \quad \text{for } 0 \leq r \leq s \leq t \leq T. \tag{2.1}$$

It is called *exponentially bounded* if there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $0 \leq s \leq t \leq T$, we have the bound $\|U(t, s)\|_{\mathcal{L}(X)} \leq Me^{\omega(t-s)}$.

Evolution families are oftentimes used to describe the evolution of non-autonomous Cauchy problems, see, e.g., [4, Section 2] and the references cited therein. The family $(C(t))_{t \in [0, T]}$ in the mapping $t \mapsto \|C(t)U(t, 0)x_0\|_Y$ can be understood as *observation operators* through which the state of the system is observed at each time $t \geq 0$. In the context of L^p -spaces, these are often chosen as multiplication operators by characteristic functions for some (time-dependent) sensor sets, see, e.g., Example 2.5 below.

The following theorem now covers and extends all known previous results in this context, see the discussion below.

Theorem 2.2. *Assume Hypothesis 2.1. Let $(P_\lambda)_{\lambda > 0}$ be a family in $\mathcal{L}(X)$ such that for some constants $d_0, d_1, \gamma_1 > 0$, we have*

$$\forall \lambda > 0, \forall x \in X: \|P_\lambda x\|_X \leq d_0 e^{d_1 \lambda \gamma_1} \text{ess inf}\{E \ni \tau \mapsto \|C(\tau)P_\lambda x\|_Y\}. \quad (\text{essUCP})$$

Suppose also that for some constants $d_2 \geq 1, d_3, \gamma_2, \gamma_3 > 0$ with $\gamma_2 > \gamma_1$ and $\gamma_4 \geq 0$, we have

$$\forall \lambda > 0, \forall 0 \leq s < t \leq T, \forall x \in X: \|(\text{Id} - P_\lambda)U(t, s)x\|_X \leq d_2 \max\{1, (t - s)^{-\gamma_4}\} e^{-d_3 \lambda \gamma_2 (t-s)^{\gamma_3}} \|x\|_X. \tag{DE}$$

Then, there exists a constant $C_{\text{obs}} > 0$ such that for each $r \in [1, \infty]$ and all $x_0 \in X$, we have the final-state observability estimate

$$\|U(T, 0)x_0\|_X \leq C_{\text{obs}} \begin{cases} \left(\int_E \|C(t)U(t, 0)x_0\|_Y^r dt\right)^{1/r}, & r < \infty, \\ \text{ess sup}_{t \in E} \|C(t)U(t, 0)x_0\|_Y, & r = \infty. \end{cases} \tag{OBS}$$

Moreover, if for some interval $(\tau_1, \tau_2) \subseteq [0, T]$ with $\tau_1 < \tau_2$ we have $|(\tau_1, \tau_2) \cap E| = \tau_2 - \tau_1$, then, depending on the value of r , the constant C_{obs} can be bounded as

$$C_{\text{obs}} \leq \frac{C_1}{(\tau_2 - \tau_1)^{1/r}} \exp\left(\frac{C_2}{(\tau_2 - \tau_1)^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}} + C_3 T\right), \quad (2.2)$$

with the usual convention $1/r = 0$ if $r = \infty$, where $C_1, C_2, C_3 > 0$ are constants not depending on r, T, E, τ_1 , or τ_2 .

The above theorem represents the established Lebeau–Robbiano strategy, in which an uncertainty principle (**essUCP**), here with respect to the given family $(P_\lambda)_{\lambda>0}$ and uniform only on the subset $E \subseteq [0, T]$, and a corresponding dissipation estimate in the form **(DE)** are used as an input; it should be emphasized that the requirement $\gamma_1 < \gamma_2$ is essential here. The output in the form of **(OBS)** then constitutes a so-called *final-state observability estimate* for the evolution family $(U(t, s))_{0 \leq s \leq t \leq T} \subseteq \mathcal{L}(X)$ with respect to the family $(C(t))_{t \in [0, T]}$ of observation operators. The corresponding constant C_{obs} in **(OBS)** is called *observability constant*. An explicit form of the constants C_1, C_2, C_3 in **(2.2)** is given in Remark 3.1 below for easier reference.

Discussion and extensions. We first comment on two minor extensions of Theorem 2.2.

Remark 2.3. (1) It becomes clear from the proof, see **(3.7)** below, that instead of the polynomial blow-up in the dissipation estimate **(DE)** for small differences $t - s$ one can also allow a certain (*sub-*)*exponential* blow-up. More precisely, one may replace the term $\max\{1, (t - s)^{-\gamma_4}\}$ in estimate **(DE)** by a factor of the form $\exp(c(t - s)^{-\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}})$ with some constant $c > 0$.

(2) If $|(\tau_1, \tau_2) \cap E| = \tau_2 - \tau_1$ for some interval $(\tau_1, \tau_2) \subseteq [0, T]$ with $\tau_1 < \tau_2$, then the dissipation estimate **(DE)** is actually needed only for $\tau_1 < s < t \leq \tau_2$, cf. part (1) of Remark 3.2 below.

Let us now compare Theorem 2.2 to earlier results in the literature.

Remark 2.4. (1) In the particular case where $[\tau_1, \tau_2] = [0, T]$, the bound on C_{obs} in **(2.2)** is completely consistent, except perhaps for some minor differences in the explicit form of the constants C_1, C_2, C_3 in Remark 3.1 below, with all bounds obtained earlier for $E = [0, T]$ in **[3, 8, 14]** in the autonomous case (i.e. $C(t) \equiv C$ and the evolution family being actually a semigroup) and in **[4]** in the non-autonomous case.

(2) Theorem 2.2 covers **[4, Theorem 3.3]**, while allowing a polynomial blow-up for small differences $t - s$ in the dissipation estimate **(DE)**. Such a blow-up has first been considered in **[1, Theorem 13]**, but under much more restrictive assumptions, see item (4) below. Moreover, in contrast to **[4, Theorem 3.3]**, Theorem 2.2 requires the uncertainty relation (**essUCP**) only on a subset of $[0, T]$ of positive measure, instead of the whole interval $[0, T]$, and thus allows far more general families of observation operators. These families also need to be uniformly bounded only on this measurable subset and not on the whole interval $[0, T]$.

(3) The results from [3, Theorem A.1] and [8, Theorem 2.1] formulate a variant of our Theorem 2.2 in the autonomous case with $E = [0, T]$, but assume (essUCP) and (DE) only for $\lambda > \lambda^*$ with some $\lambda^* \geq 0$. Our current formulation with the whole range $\lambda > 0$, just as in [4, Theorem 3.3], is not really a restriction to that. Indeed, by a change of variable, one may then simply consider the family $(P_{\lambda+\lambda^*})_{\lambda>0}$ instead, with a straightforward adaptation of the parameters d_0 and d_1 in (essUCP). In this sense, Theorem 2.2 completely covers [3, Theorem A.1] and [8, Theorem 2.1].

(4) Let $\tau_1 \in [0, T)$ be such that $|(\tau_1, T) \cap E| > 0$. By a change of variable, namely via considering $C(\cdot + \tau_1)$ on $[0, T - \tau_1]$ and the evolution family $U(t + \tau_1, s + \tau_1)_{0 \leq s < t \leq T - \tau_1}$, one may replace $U(T, 0)$, $U(t, 0)$, and E in (OBS) by $U(T, \tau_1)$, $U(t, \tau_1)$, and $(\tau_1, T) \cap E$ respectively; note that (essUCP) and (DE) then remain valid with the same constants. In this sense, Theorem 2.2 entirely covers [1, Theorem 13], while leaving the Hilbert space setting and not requiring strong continuity or contractivity of the evolution family. At the same time, our bound on C_{obs} in (2.2) contains an additional prefactor $1/(\tau_2 - \tau_1)^{1/r}$ in front of the exponential term, which significantly changes the asymptotics of the estimate as $\tau_2 - \tau_1$ (and thus also T) gets large. Such improved asymptotics has proved extremely useful in the past, for instance, when considering homogenization limits as in [14].

In order to support the above comparison, we briefly revisit [4, Theorem 4.8] in the following example.

Example 2.5. Let \mathbf{a} be a uniformly strongly elliptic polynomial of degree $m \geq 2$ in \mathbb{R}^d with coefficients $a_\alpha \in L^\infty([0, T])$, that is, $\mathbf{a}: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{C}$ with

$$\mathbf{a}(t, \xi) = \sum_{|\alpha| \leq m} a_\alpha(t)(i\xi)^\alpha, \quad t \in [0, T], \xi \in \mathbb{R}^d,$$

such that for some $c > 0$, we have

$$\operatorname{Re} \sum_{|\alpha|=m} a_\alpha(t, \xi) \geq c|\xi|^m, \quad t \in [0, T], \xi \in \mathbb{R}^d.$$

Let $p \in [1, \infty]$. It was shown in [4, Theorem 4.4] that there is an exponentially bounded evolution family $(U_p(t, s))_{0 \leq s \leq t \leq T}$ in $L^p(\mathbb{R}^d)$ associated to \mathbf{a} . Let $(\Omega(t))_{t \in [0, T]}$ be a family of measurable subsets of \mathbb{R}^d such that the mapping $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \mathbf{1}_{\Omega(t)}(x)$ is measurable. Then, $\|\mathbf{1}_{\Omega(\cdot)} U_p(\cdot, 0) u_0\|_{L^p(\mathbb{R}^d)}$ is for all $u_0 \in L^p(\mathbb{R}^d)$ measurable on $[0, T]$ by [4, Lemma 4.7], so that Hypothesis 2.1 with $C(t) = \mathbf{1}_{\Omega(t)}$ is satisfied for every choice of measurable $E \subseteq [0, T]$ with positive measure. Moreover, a dissipation estimate as in (DE) with $\gamma_2 = m$ (but without blow-up, i.e., $\gamma_4 = 0$) was established in the proof of [4, Theorem 4.8] with P_λ being some smooth frequency cutoffs. It remains to consider a corresponding essential uncertainty principle (essUCP).

Suppose that the family $(\Omega(t))_{t \in [0, T]}$ of subsets is *uniformly thick* on E in the sense that there are $L, \rho > 0$ such that for all $x \in \mathbb{R}^d$ and all $t \in E$, we have

$$|\Omega(t) \cap ((0, L)^d + x)| \geq \rho L^d. \tag{2.3}$$

Following the proof of [4, Theorem 4.8], we see that an essential uncertainty principle as in (essUCP) holds with $\gamma_1 = 1 < \gamma_2$, so that Theorem 2.2 can be applied. Here, the set E for which (2.3) needs to hold could be any measurable subset of $[0, T]$ with positive measure, for instance, $E = [0, T] \setminus \mathbb{Q}$ (satisfying $|E| = T$) or even some fractal set, say of Cantor-Smith-Volterra type. In particular, this allows for completely arbitrary choices of measurable $\Omega(t)$ for $t \notin E$, even $\Omega(t) = \emptyset$. By contrast, these choices for $\Omega(t)$ would not be allowed in [4, Theorem 4.8], where (2.3) is required to hold for all $x \in \mathbb{R}^d$ and all $t \in [0, T]$ and is thus much more restrictive on the choice of $(\Omega(t))_{t \in [0, T]}$.

Remark 2.6. In the situation of Example 2.5 with $p < \infty$, it was shown in [4, Theorem 4.10] that an observability estimate as in (OBS) can hold with $r < \infty$ only if the family $(\Omega(t))_{t \in [0, T]}$ is *mean thick* in the sense that for some $L, \rho > 0$, we have

$$\frac{1}{T} \int_0^T |\Omega(t) \cap ((0, L)^d + x)| \, dt \geq \rho L^d \quad \text{for all } x \in \mathbb{R}^d.$$

It is easy to see that families which are uniformly thick on a subset of $[0, T]$ of positive measure as in (2.3) are also mean thick in the above sense (with possibly different parameters), but the converse need not be true. A corresponding example in \mathbb{R} is the family $(\Omega(t))_{t \in [0, T]}$ with $\Omega(t) = (0, \infty)$ for $t \leq T/2$ and $\Omega(t) = (-\infty, 0)$ for $T/2 < t \leq T$. It is yet unclear whether such choices also lead to an observability estimate as in (OBS) or anything similar. In this sense, Example 2.5 and Theorem 2.2 still leave a gap between necessary and sufficient conditions on the family $(\Omega(t))_{t \in [0, T]}$ towards final-state observability.

3. Proof of Theorem 2.2. Our proof of Theorem 2.2 is a streamlined adaptation of earlier approaches, especially of that from [4] and its predecessors [14], [8], and [1]. It avoids the interpolation argument in [4] and is thus much more direct and, at the same time, requires an uncertainty relation only on a measurable subset of $[0, T]$ of positive measure.

Proof of Theorem 2.2. Let us fix $x_0 \in X$. For $0 \leq t \leq T$, we abbreviate

$$F(t) := \|U(t, 0)x_0\|_X, \quad G(t) := \|C(t)U(t, 0)x_0\|_Y.$$

By Hölder’s inequality, we clearly have

$$\|G(\cdot)\|_{L^1(E)} \leq |E|^{1-\frac{1}{r}} \|G(\cdot)\|_{L^r(E)} \tag{3.1}$$

with the usual convention $1/\infty = 0$. Hence, estimate (OBS) for $r > 1$ follows from the one for $r = 1$ by multiplying the corresponding constant C_{obs} by $|E|^{1-\frac{1}{r}} \leq \max\{1, |E|\}$. It therefore suffices to show (OBS) for $r = 1$, which in the new notation reads

$$F(T) \leq C_{\text{obs}} \int_E G(t) \, dt. \tag{3.2}$$

Upon possibly removing from E a set of measure zero, we may assume without loss of generality that (essUCP) holds with ess inf replaced by inf

and that $C(\cdot)$ is uniformly bounded on E . Let us then show that there exist constants $c_1, c_2 > 0$ such that for all $0 \leq s < t \leq T$ with $t \in E$ and all $\varepsilon \in (0, 1)$, we have

$$F(t) \leq c_1 \exp\left(c_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}\right) (\varepsilon^{-1} G(t) + \varepsilon F(s)). \tag{3.3}$$

To this end, let $\varepsilon \in (0, 1)$ and fix $0 \leq s < t \leq T$ with $t \in E$. For $\lambda > 0$, we introduce

$$F_\lambda := \|P_\lambda U(t, 0)x_0\|_X, \quad F_\lambda^\perp := \|(\text{Id} - P_\lambda)U(t, 0)x_0\|_X,$$

as well as

$$G_\lambda := \|C(t)P_\lambda U(t, 0)x_0\|_Y, \quad G_\lambda^\perp := \|C(t)(\text{Id} - P_\lambda)U(t, 0)x_0\|_Y.$$

The uncertainty relation (**essUCP**) and the uniform boundedness of $C(\cdot)$ on E then give

$$F_\lambda \leq d_0 e^{d_1 \lambda^{\gamma_1}} G_\lambda \quad \text{and} \quad G_\lambda^\perp \leq \|C(\cdot)\|_{E, \infty} F_\lambda^\perp \quad \text{for all } \lambda > 0,$$

where $\|C(\cdot)\|_{E, \infty} := \sup_{\tau \in E} \|C(\tau)\|_{\mathcal{L}(X, Y)} < \infty$. Since by the triangle inequality $F(t) \leq F_\lambda + F_\lambda^\perp$ and $G_\lambda \leq G(t) + G_\lambda^\perp$, the latter implies that for all $\lambda > 0$, we have

$$\begin{aligned} F(t) &\leq d_0 e^{d_1 \lambda^{\gamma_1}} (G(t) + \|C(\cdot)\|_{E, \infty} F_\lambda^\perp) + F_\lambda^\perp \\ &\leq e^{d_1 \lambda^{\gamma_1}} (d_0 G(t) + (d_0 \|C(\cdot)\|_{E, \infty} + 1) F_\lambda^\perp). \end{aligned} \tag{3.4}$$

Now, writing $U(t, 0)x_0 = U(t, s)U(s, 0)x_0$ by (2.1), we obtain from (DE) with $x = U(s, 0)x_0$ that

$$F_\lambda^\perp \leq d_2 \max\{1, (t-s)^{-\gamma_4}\} e^{-d_3 \lambda^{\gamma_2} (t-s)^{\gamma_3}} F(s),$$

and inserting this into the preceding estimate (3.4) yields for all $\lambda > 0$ that

$$\begin{aligned} F(t) &\leq c_1 e^{d_1 \lambda^{\gamma_1}} \max\{1, (t-s)^{-\gamma_4}\} (G(t) + e^{-d_3 \lambda^{\gamma_2} (t-s)^{\gamma_3}} F(s)) \\ &= c_1 e^{f(\lambda)} \max\{1, (t-s)^{-\gamma_4}\} (e^{\frac{d_3}{2} \lambda^{\gamma_2} (t-s)^{\gamma_3}} G(t) + e^{-\frac{d_3}{2} \lambda^{\gamma_2} (t-s)^{\gamma_3}} F(s)) \end{aligned} \tag{3.5}$$

with

$$c_1 := \max\{d_0, (d_0 \|C(\cdot)\|_{E, \infty} + 1)d_2\} \geq 1 \quad \text{and} \quad f(\lambda) := d_1 \lambda^{\gamma_1} - \frac{d_3}{2} \lambda^{\gamma_2} (t-s)^{\gamma_3}. \tag{3.6}$$

Let us maximize $f(\lambda)$ with respect to λ . In light of $\gamma_2 > \gamma_1$ by hypothesis, a straightforward calculation reveals that f takes its maximal value on $(0, \infty)$ at the point

$$\lambda^* := \left(\frac{2d_1 \gamma_1}{d_3 \gamma_2}\right)^{\frac{1}{\gamma_2 - \gamma_1}} \left(\frac{1}{t-s}\right)^{\frac{\gamma_3}{\gamma_2 - \gamma_1}} > 0.$$

Taking into account the relation $\frac{\gamma_1}{\gamma_2 - \gamma_1} + 1 = \frac{\gamma_2}{\gamma_2 - \gamma_1}$, we observe that

$$\begin{aligned} \frac{d_3}{2}(\lambda^*)^{\gamma_2}(t-s)^{\gamma_3} &= \frac{d_3}{2} \left(\frac{2d_1\gamma_1}{d_3\gamma_2} \right)^{\frac{\gamma_2}{\gamma_2-\gamma_1}} \left(\frac{1}{t-s} \right)^{\frac{\gamma_2\gamma_3}{\gamma_2-\gamma_1}-\gamma_3} \\ &= \frac{d_1\gamma_1}{\gamma_2} \left(\frac{2d_1\gamma_1}{d_3\gamma_2} \right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} \left(\frac{1}{t-s} \right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}} = \frac{d_1\gamma_1}{\gamma_2}(\lambda^*)^{\gamma_1}. \end{aligned}$$

We may therefore estimate $f(\lambda)$ as

$$f(\lambda) \leq f(\lambda^*) = d_1 \left(1 - \frac{\gamma_1}{\gamma_2} \right) (\lambda^*)^{\gamma_1} = d_1 \left(1 - \frac{\gamma_1}{\gamma_2} \right) \left(\frac{2d_1\gamma_1}{d_3\gamma_2} \right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} \left(\frac{1}{t-s} \right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}.$$

Moreover, using the elementary bound $\xi^\alpha \leq e^{\alpha\xi}$ for $\alpha, \xi > 0$, we have

$$\max\{1, (t-s)^{-\gamma_4}\} \leq \exp\left(\frac{\gamma_4(\gamma_2-\gamma_1)}{\gamma_1\gamma_3} \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}\right). \tag{3.7}$$

Inserting this and the preceding bound on $f(\lambda)$ into (3.5), we arrive for all $\lambda > 0$ at

$$F(t) \leq c_1 \exp\left(c_2 \left(\frac{1}{t-s}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}\right) \left(e^{\frac{d_3}{2}\lambda^{\gamma_2}(t-s)^{\gamma_3}} G(t) + e^{-\frac{d_3}{2}\lambda^{\gamma_2}(t-s)^{\gamma_3}} F(s) \right)$$

with

$$c_2 := d_1 \left(1 - \frac{\gamma_1}{\gamma_2} \right) \left(\frac{2d_1\gamma_1}{d_3\gamma_2} \right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} + \frac{\gamma_4(\gamma_2-\gamma_1)}{\gamma_1\gamma_3}. \tag{3.8}$$

We finally choose $\lambda > 0$ such that $\varepsilon = e^{-\frac{d_3}{2}\lambda^{\gamma_2}(t-s)^{\gamma_3}}$, which shows that (3.3) is valid; note that indeed neither c_1 nor c_2 depend on s or t .

Let $q := \left(\frac{3}{4}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1\gamma_3}} < 1$, and choose by Lebesgue’s differentiation theorem a (right) density point $\ell \in [0, T] \cap E$ of E in the sense of Appendix A. Proposition A.1 then guarantees that there is a strictly decreasing sequence $(\ell_m)_{m \in \mathbb{N}}$ in $(\ell, T]$ of the form $\ell_m = \ell + q^{m-1}(\ell_1 - \ell)$, $m \in \mathbb{N}$, satisfying

$$|(\ell_{m+1}, \ell_m) \cap E| \geq \frac{\delta_m}{3}, \quad m \in \mathbb{N}, \tag{3.9}$$

where $\delta_m := \ell_m - \ell_{m+1}$, $m \in \mathbb{N}$. It is also easy to see that

$$\delta_{m+1} = q\delta_m, \quad m \in \mathbb{N}. \tag{3.10}$$

Since the evolution family $(U(t, s))_{0 \leq s \leq t \leq T}$ is exponentially bounded by hypothesis, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$F(t) = \|U(t, s)U(s, 0)x_0\|_X \leq Me^{\omega(t-s)}F(s) \quad \text{for all } 0 \leq s \leq t \leq T.$$

Setting $\omega_+ := \max\{\omega, 0\}$, this in particular implies for each $m \in \mathbb{N}$ and all $t \in (\ell_{m+1}, \ell_m)$ that

$$F(\ell_m) \leq Me^{\omega(\ell_m-t)}F(t) \leq Me^{\omega_+\delta_1}F(t). \tag{3.11}$$

Define

$$\xi_m := \ell_{m+1} + \frac{\delta_m}{6} \in (\ell_{m+1}, \ell_m), \quad m \in \mathbb{N},$$

which in light of (3.9) satisfies

$$\begin{aligned}
 |(\xi_m, \ell_m) \cap E| &= |(\ell_{m+1}, \ell_m) \cap E| - |(\ell_{m+1}, \xi_m) \cap E| \\
 &\geq \frac{\delta_m}{3} - (\xi_m - \ell_{m+1}) = \frac{\delta_m}{6} > 0.
 \end{aligned}
 \tag{3.12}$$

Combining (3.11) and (3.3) with $s = \ell_{m+1}$, we obtain for all $m \in \mathbb{N}$, $t \in (\xi_m, \ell_m) \cap E$, and $\varepsilon \in (0, 1)$ that

$$F(\ell_m) \leq c_3 \exp\left(\frac{c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) (\varepsilon^{-1} G(t) + \varepsilon F(\ell_{m+1}))$$

with

$$c_3 := M c_1 e^{\omega + \delta_1} \geq 1 \quad \text{and} \quad c_4 := c_2 6^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}} > 0,$$

where we have taken into account that $t - \ell_{m+1} \geq \xi_m - \ell_{m+1} = \delta_m/6$. With the particular choice

$$\varepsilon = c_3^{-1} q \exp\left(-\frac{2c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) \in (0, 1),$$

the latter turns into

$$F(\ell_m) \leq c_3^2 q^{-1} \exp\left(\frac{3c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) G(t) + q \exp\left(-\frac{c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) F(\ell_{m+1}). \tag{3.13}$$

Observing that

$$\exp\left(\frac{4c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) = \exp\left(\frac{3c_4}{\delta_{m+1}^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right)$$

by (3.10) and the choice of q , multiplying (3.13) by δ_m and rearranging terms yields

$$\delta_m \exp\left(-\frac{3c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) F(\ell_m) - \delta_{m+1} \exp\left(-\frac{3c_4}{\delta_{m+1}^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) F(\ell_{m+1}) \leq c_3^2 q^{-1} \delta_m G(t)$$

for all $t \in (\xi_m, \ell_m) \cap E$, $m \in \mathbb{N}$. Taking into account (3.12), integrating the latter with respect to $t \in (\xi_m, \ell_m) \cap E$ leads to

$$\begin{aligned}
 \delta_m \exp\left(-\frac{3c_4}{\delta_m^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) F(\ell_m) - \delta_{m+1} \exp\left(-\frac{3c_4}{\delta_{m+1}^{\frac{\gamma_1 \gamma_3}{\gamma_2 - \gamma_1}}}\right) F(\ell_{m+1}) \\
 \leq 6c_3^2 q^{-1} \int_{\ell_{m+1}}^{\ell_m} \mathbf{1}_E(t) G(t) dt
 \end{aligned}$$

for all $m \in \mathbb{N}$. Note here that the exponential boundedness of the evolution family guarantees that the sequence $(F(\ell_m))_{m \in \mathbb{N}}$ is bounded. Since also $\delta_m \rightarrow 0$

and $\ell_m \rightarrow \ell$ as $m \rightarrow \infty$, summing the last inequality over all $m \in \mathbb{N}$ implies by a telescoping sum argument that

$$\delta_1 \exp\left(-\frac{3c_4}{\delta_1^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) F(\ell_1) \leq 6c_3^2 q^{-1} \int_{\ell}^{\ell_1} \mathbf{1}_E(t) G(t) dt \leq 6c_3^2 q^{-1} \int_E G(t) dt,$$

which can be rewritten as

$$F(\ell_1) \leq 6c_3^2 q^{-1} \delta_1^{-1} \exp\left(\frac{3c_4}{\delta_1^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) \int_E G(t) dt.$$

Now, we have $F(T) \leq M e^{\omega(T-\ell_1)} F(\ell_1)$ by using once more the exponential boundedness of the evolution family, which shows that (3.2) holds with

$$C_{\text{obs}} = 6c_3^2 q^{-1} \delta_1^{-1} \exp\left(\frac{3c_4}{\delta_1^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}}\right) M e^{\omega(T-\ell_1)}. \tag{3.14}$$

Finally, suppose that $|(\tau_1, \tau_2) \cap E| = \tau_2 - \tau_1$ for some interval $(\tau_1, \tau_2) \subseteq [0, T]$ with $\tau_1 < \tau_2$. We may then simply choose $\ell = \tau_1$ and $\ell_1 = \tau_2$ in the above reasoning, leading to $\delta_1 = (1 - q)(\tau_2 - \tau_1)$. For $r \geq 1$, in light of (3.1) with E replaced by $(\tau_1, \tau_2) \cap E$, we conclude that C_{obs} in (OBS) can be bounded as in (2.2), which completes the proof. \square

For organizational purposes, we extract from the above proof the following more explicit bound on the observability constant.

Remark 3.1. In the case where $|(\tau_1, \tau_2) \cap E| = \tau_2 - \tau_1$ for some interval $(\tau_1, \tau_2) \subseteq [0, T]$ with $\tau_1 < \tau_2$ in (the proof of) Theorem 2.2, we actually have $|(\ell_{m+1}, \ell_m) \cap E| = \delta_m$ instead of the weaker (3.9). Consequently, one may choose $\xi_m := \ell_{m+1} + \delta_m/2$, resulting in $|(\xi_m, \ell_m) \cap E| = \delta_m/2$ instead of (3.12). We may therefore replace the numerical factor 6 in (3.14) and the constant c_4 by 2, so that

$$C_{\text{obs}} \leq \frac{C_1}{(\tau_2 - \tau_1)^{1/r}} \exp\left(\frac{C_2}{(\tau_2 - \tau_1)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}} + C_3 T\right)$$

with

$$C_1 := \frac{2M^3 c_1^2}{q(1-q)}, \quad C_2 := 3c_2 \left(\frac{2}{1-q}\right)^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}, \quad C_3 := 3\omega_+,$$

where

$$c_1 = \max\{d_0, (d_0 \|C(\cdot)\|_{E,\infty} + 1)d_2\} \geq 1, \\ c_2 = d_1 \left(1 - \frac{\gamma_1}{\gamma_2}\right) \left(\frac{2d_1\gamma_1}{d_3\gamma_2}\right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} + \frac{\gamma_4(\gamma_2 - \gamma_1)}{\gamma_1\gamma_3}$$

are as in (3.6) and (3.8), respectively, and where $M \geq 1$ and $\omega_+ = \max\{\omega, 0\}$, $\omega \in \mathbb{R}$, are such that $\|U(t, s)\|_{\mathcal{L}(X)} \leq M e^{\omega(t-s)}$ for all $0 \leq s \leq t \leq T$.

Remark 3.2. (1) The proof of Theorem 2.2 actually requires the dissipation estimate (DE) only for $\ell \leq s < t \leq \ell_1$.

(2) It is worth to emphasize that in contrast to [1, Theorem 13], our proof of Theorem 2.2 does *not* require the sequence $(\ell_m)_{m \in \mathbb{N}}$ to belong to the set E , but only to $(\ell, T]$. This relaxed requirement is much easier to satisfy than the stricter one in [1, Proposition 14] and is reviewed in Proposition A.1 in the appendix.

Appendix A. Approximating density points of measurable subsets of the real line. Recall that a point $\ell \in \mathbb{R}$ is called a *right density point* (resp. *density point*) of a measurable set $E \subseteq \mathbb{R}$ with positive Lebesgue measure if

$$\lim_{\theta \rightarrow 0} \frac{|(\ell, \ell + \theta) \cap E|}{\theta} = 1 \quad (\text{resp. } \lim_{\theta \rightarrow 0} \frac{|(\ell - \theta, \ell + \theta) \cap E|}{2\theta} = 1). \quad (\text{A.1})$$

It is easy to see that every density point is also a right density point, cf. [6, p. 32], and it follows from Lebesgue’s differentiation theorem that almost every point $\ell \in E$ is a density point of E , and thus a right density point, see, e.g., [12, Corollary 2.14].

The following result is an adapted version of [15, Proposition 2.1]; see also [6, Lemma 2.1.5] for a similar statement. We give a proof here in order to be self-contained.

Proposition A.1. *Let $E \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure, and let $\ell \in \mathbb{R}$ be a right density point of E . Then, given $q \in (0, 1)$, for every $\ell_1 > \ell$ sufficiently close to ℓ , the strictly decreasing sequence $(\ell_m)_{m \in \mathbb{N}}$ with*

$$\ell_m = \ell + q^{m-1}(\ell_1 - \ell), \quad m \in \mathbb{N}, \quad (\text{A.2})$$

satisfies

$$\ell_m - \ell_{m+1} \leq 3|(\ell_{m+1}, \ell_m) \cap E|, \quad m \in \mathbb{N}. \quad (\text{A.3})$$

Proof. Since ℓ is a right density point of E by hypothesis, there is by (A.1) some $\theta_0 > 0$, depending on E and q , such that

$$|(\ell, \ell + \theta) \setminus E| < \frac{1 - q}{2(1 + q)} |(\ell, \ell + \theta) \cap E| \quad \text{for all } \theta < \theta_0. \quad (\text{A.4})$$

We fix an arbitrary $\ell_1 \in (\ell, \ell + \theta_0)$ and define the sequence $(\ell_m)_{m \in \mathbb{N}}$ as in (A.2). It remains to show that (A.3) holds. To this end, let $m \in \mathbb{N}$, and set $\theta := \ell_m - \ell$, so that $\ell + \theta = \ell_m$ and $\theta < \ell_1 - \ell < \theta_0$. Inserting this into (A.4) gives

$$|(\ell_{m+1}, \ell_m) \setminus E| \leq |(\ell, \ell_m) \setminus E| < \frac{1 - q}{2(1 + q)} |(\ell, \ell_m) \cap E|. \quad (\text{A.5})$$

In order to bound the right-hand side further, we observe that

$$\begin{aligned} \ell_{m+1} - \ell &= q^m(\ell_1 - \ell) = \frac{q}{1 - q} (1 - q)q^{m-1}(\ell_1 - \ell) \\ &= \frac{q}{1 - q} (q^{m-1} - q^m)(\ell_1 - \ell) = \frac{q}{1 - q} (\ell_m - \ell_{m+1}). \end{aligned}$$

This leads to the estimate

$$\begin{aligned} |(\ell, \ell_m) \cap E| &\leq |(\ell, \ell_{m+1})| + |(\ell_{m+1}, \ell_m) \cap E| \\ &\leq \frac{q}{1-q}(\ell_m - \ell_{m+1}) + |(\ell_{m+1}, \ell_m) \cap E|, \end{aligned}$$

which together with (A.5) and the bounds $q/(1+q) < 1$ and $(1-q)/(1+q) < 1$ yields

$$|(\ell_{m+1}, \ell_m) \setminus E| < \frac{1}{2} \left(|(\ell_{m+1}, \ell_m) \cap E| + (\ell_m - \ell_{m+1}) \right).$$

From this, we conclude that

$$\begin{aligned} \ell_m - \ell_{m+1} &= |(\ell_{m+1}, \ell_m) \cap E| + |(\ell_{m+1}, \ell_m) \setminus E| \\ &\leq \frac{3}{2} |(\ell_{m+1}, \ell_m) \cap E| + \frac{1}{2} (\ell_m - \ell_{m+1}), \end{aligned}$$

and rearranging terms shows the desired estimate (A.3). This completes the proof. \square

Remark A.2. The proof of Proposition A.1 shows that (A.4) can actually be relaxed to

$$|(\ell, \ell + \theta) \setminus E| < \min \left\{ \frac{1-q}{2q}, \frac{1}{2} \right\} |(\ell, \ell + \theta) \cap E| \quad \text{for all } \theta < \theta_0$$

with the corresponding modification to (A.5). The rest of the proof then carries over.

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