Arch. Math. 122 (2024), 213–225

0003-889X/24/020213-13
published online December 19, 2023
https://doi.org/10.1007/s00013-023-01928-0

Archiv der Mathematik

On the optimal effective stability bounds for quasi-periodic tori of finitely differentiable and Gevrey Hamiltonians

Gerard Farré

Abstract. It is known that a Diophantine quasi-periodic torus with frequency $\omega \in \Omega_\tau^d$ of a C^l Hamiltonian is effectively stable for a time $T(r)$ that is polynomial on the inverse of the distance to the torus, that we denote by r, with exponent $1 + (l-2)/(\tau+1)$. It is also known that a Diophantine quasi-periodic torus of a Gevrey Hamiltonian $H \in G^{\alpha,L}$ is effectively stable for an exponentially long time on the inverse of the distance to the torus with exponent $1/(\alpha(1+\tau))$. In this note, we see that following the methods in [11] one can show the almost optimality of these exponents. We also show that, for a dense subset of non-resonant vectors, for quasi-periodic tori of finitely differentiable and Gevrey Hamiltonians, the naive lower bound $T(r) \geq Cr^{-1}$ is optimal in terms of the exponent.

Mathematics Subject Classification. Primary 37J40; Secondary 37J25.

Keywords. Hamiltonian systems, Quasi-periodic tori, Effective stability.

1. Introduction. The theory of effective stability in dynamical systems tries to answer questions regarding the bounds for the time that a vector field or a map on a differentiable manifold exhibits a certain type of dynamics. A particularly interesting question is to measure how much time do nearby orbits to invariant sets stay nearby forwards in time. This idea leads to an effective version of Lyapunov stability for invariant sets, in which we are no longer interested in knowing if nearby orbits will remain close for all positive time but only if they will do so for a finite interval of time. The question becomes then how long these intervals of time can be. In this note, we are interested in the theory of effective stability in the context of Hamiltonian dynamics, more specifically for Lagrangian invariant tori of Hamiltonians which are close to integrable.

In particular, we study Hamiltonian systems $H \in C^3(\mathcal{D}_R)$, $\mathcal{D}_R = \mathbb{T}^d \times B_R$ where $B_R \subset \mathbb{R}^d$ is the closed ball of radius $R > 0$ centered at the origin, $d \geq 3$, and H is of the form

$$H(\theta, I) = \omega \cdot I + \mathcal{O}(|I|^2). \tag{1.1}$$

In this particular case, the integrable part is considered to be linear and the perturbation to be a function of order two in the action variable I. If ω is non-resonant, it follows from (1.1) that $\mathcal{T}_0 = \mathbb{T}^d \times \{0\}$ is indeed an invariant and Lagrangian quasi-periodic torus.

In Theorems A and B, we will assume an arithmetic strong non-resonance condition on the frequency ω, namely that it is Diophantine, i.e., that there exist $\gamma > 0$, $\tau \geq d-1$ such that

$$|\omega \cdot k| \geq \frac{\gamma}{\|k\|^\tau}, \quad \forall\, k \in \mathbb{Z}^d \setminus \{0\}.$$

We will denote the set of vectors satisfying this condition for a fixed τ and γ by $\Omega^d_{\tau,\gamma}$ and $\Omega^d_\tau = \cup_{\gamma>0}\Omega^d_{\tau,\gamma}$. In case $\omega \in \Omega^d_\tau$ in (1.1), we call $\mathcal{T}_0$ a Diophantine quasi-periodic torus, or DQP-torus. In the case where ω is non-resonant and $\omega \notin \Omega^d_\tau$ for any $\tau \geq d-1$, we say that ω is a Liouville vector. In other words, the set of Liouville vectors is the complement of the set of Diophantine vectors inside the set of non-resonant vectors.

It is known that if H as in (1.1) is real-analytic and $\omega \in \Omega^d_\tau$, then there exist $C > 0$, $r^* > 0$ such that for all $0 < r < r^*$, for $|I| \leq r$, we have that

$$\|\Pi_I \Phi^t_H(\theta, I)\| \leq 2r, \quad 0 < t < \exp(Cr^{-\frac{1}{\tau+1}}).$$

Thus DQP-tori of real-analytic Hamiltonians are exponentially stable. This fact, up to small differences in the exponent, extensions to lower dimensional tori, and extensions in terms of regularity, has been proved in the following references [18,19,22,23]. In this work, we consider the case in which H is less regular, either $H \in C^l$ for some $l \geq 3$ or $H \in G^{\alpha,L}$ for some $\alpha \geq 1, L > 0$. It is well known that, the more regular the Hamiltonian, the larger become the effective stability lower bounds known for $\mathcal{T}_0$. It follows from [4,5] that in the case of H being finitely differentiable we have effective stability for a polynomial time, and for the Gevrey case we have exponential stability with exponent $1/(\alpha(\tau+1))$, where α is the Gevrey exponent. One of our goals in this note is to show that these stability times are close to optimal and optimal in the finitely differentiable and Gevrey cases respectively.

In [11], the optimality of the exponents in the effective stability bounds for DQP-tori of real-analytic Hamiltonians was proved by providing, for any $C, \varepsilon > 0$, Hamiltonians in the form of explicit convergent series with a sequence of initial conditions $z_n = (\theta_n, I_n)$ with $\|I_n\|$ tending to zero and such that the flow of H satisfies, for all n,

$$\|\Pi_I \Phi^{t_n}_H(\theta_n, I_n)\| > 2\|I_n\|, \quad 0 < t_n \leq \exp(C\|I_n\|^{-\frac{1}{\tau+1}-\varepsilon}).$$

In this work, we provide analogous results for lower regularities. We also provide results on the optimality of the naive bound $T(r) \geq Cr^{-1}$ for a dense subset of Liouville frequencies in both the finite differentiable and Gevrey cases. We will see in the following section what are the exact statements. Let us also mention that under certain non-degeneracy assumptions, Diophantine quasi-periodic tori of Gevrey Hamiltoninas are known to be doubly exponentially stable ([7,20]). Clearly our examples cannot satisfy these non-degeneracy conditions. The question of the optimality for the effective stability bounds in these non-degenerate cases has been adressed in [15] for Gevrey Hamiltonians.

Finally, let us mention that for the case of elliptic fixed points there are also several results on the effective stability and examples of Lyapunov unstable elliptic fixed points, for which we refer the reader to the following (incomplete list of) references [8,9,12,16,18,25].

1.1. Statement of the results. Let us introduce some preliminary definitions. We will be considering the Hamiltonians lying in the class of real-valued C^l functions, $l \geq 3$, on $\mathcal{D}_R = \mathbb{T}^d \times B_R$ with the norm

$$\|f\|_l = \max_{i \in \mathbb{N}^{2d}, |i| \leq l} |\partial^i f|_{C^0(\mathcal{D}_R)}, \quad f \in C^l(\mathcal{D}_R), \tag{1.2}$$

which form a Banach space.

We will also consider, given $\alpha \geq 1$ and $L > 0$, Hamiltonians in the space of real-valued functions $f \in C^\infty(\mathcal{D}_R)$ which are (α, L)-Gevrey, meaning that we have

$$\|f\|_{G^{\alpha,L}(\mathcal{D}_R)} = \sup_{i \in \mathbb{N}^{2d}} |f|_{\alpha,L,i,R} < \infty, \quad |f|_{\alpha,L,i,R} = L^{|i|\alpha}(i!)^{-\alpha} |\partial^i f|_{C^0(\mathcal{D}_R)}. \tag{1.3}$$

Here we use the standard multi-index notation $|i| = |i_1| + \cdots + |i_d|, \quad i! = i_1! \cdots i_d!$. For $\alpha = 1$, one recovers the space of real-analytic functions with a certain analyticity band width. We will denote this space by $G^{\alpha,L}(\mathcal{D}_R)$, which is also a Banach space. When no subindex is used, for $v \in \mathbb{R}^d$, we denote $\|v\| = \max\{|v_i|,\ i = 1, \ldots, d\}$.

For a Hamiltonian as in (1.1), let us define the time of diffusion away from $\mathcal{T}_0$ from a ball of radius $r > 0$ as

$$T(r) = \inf_{\theta_0 \in \mathbb{T}^d, |I_0| \leq r} \left\{ t > 0,\ \mathrm{dist}(\Phi_H^t(\theta_0, I_0), \mathcal{T}_0) = 2r \right\}.$$

It follows from [5, Corollary 2.2] that for a DQP-torus of a C^l Hamiltonian as in (1.1), there exist $r^* > 0$ and $C > 0$ (depending on γ, τ, l, d, R) such that for all $0 < r < r^*$,

$$T(r) \geq C \frac{1}{r} \left(\frac{1}{r} \right)^{\frac{l-2}{\tau+1}}. \tag{1.4}$$

In order to see this, consider [5, Corollary 2.2] for $\delta = r$ and $F = \mathbb{R}^d$. Our first result is that the exponent $1 + (l-2)/(\tau+1)$ in equation (1.4) is almost optimal.

Theorem A. *For any $\tau > d - 1$, $l \geq 3$, $C > 0$, and $\varepsilon > 0$, there exist $H \in C^l(\mathcal{D}_R)$ as in (1.1) with $\omega \in \Omega_\tau^d$, $c > 0$, and a sequence $r_n \to 0$ such that for every $n \geq 0$, we have $z_n = (\theta_n, I_n)$ with $\|I_n\| = |r_n|$ such that*

$$\sup_{[0,t_n]} \|\Pi_I \Phi_H^t(z)\| \geq 2|r_n|, \quad z \in B_{\frac{|r_n|}{4} e^{-ct_n}}(z_n),$$

where

$$t_n = C\frac{1}{|r_n|}\left(\frac{1}{|r_n|}\right)^{\frac{l-1}{\tau+1}+\varepsilon}. \tag{1.5}$$

Remark 1. Notice that there is a small discrepancy between the stability exponent and the exponent for the rate of escape that we obtain in Theorem A, namely there is a gap of size $1/(\tau+1)$. It has been pointed out by the referee that it may be possible to prove that the stability exponent can be improved to $1+(l-1)/(\tau+1)$ by using the methods in [2]. The result in Theorem A would imply, in case this is possible, that this new stability exponent is the optimal one.

Similarly, it follows from [4,19] that for a Hamiltonian as in (1.1) belonging to $G^{\alpha,L}(\mathcal{D}_R)$, we have that there exist $C>0$ and $r^*>0$ (again depending on γ,τ,R,d,α,L) such that for any $0<r<r^*$, we have

$$T(r) \geq \exp(Cr^{-\frac{1}{\alpha(\tau+1)}}). \tag{1.6}$$

The result follows from [4, Corollary 2.5], again considering $\delta=r$ and $F=\mathbb{R}^d$.

Theorem B. *For any $\tau>d-1$, $\alpha\geq 1$, $C>0$, and $\varepsilon>0$, there exist $L>0$, $H\in G^{\alpha,L}(\mathcal{D}_R)$ as in (1.1) with $\omega\in\Omega^d_\tau$, $c>0$, and a sequence $r_n\to 0$ such that for every $n\geq 0$, we have $z_n=(\theta_n,I_n)$ with $\|I_n\|=|r_n|$ and*

$$\sup_{[0,t_n]}\|\Pi_I\Phi^t_H(z)\|\geq 2|r_n|, \quad z\in B_{\frac{|r_n|}{4}e^{-ct_n}}(z_n),$$

where $t_n=\exp(C|r_n|^{-\frac{1}{\alpha(\tau+1)}-\varepsilon})$.

Notice that none of the results above follow from the real-analytic case shown in [11] since the times of diffusion for lower regularities need to be smaller. Nevertheless, it seems that the existence of bump functions in the strictly Gevrey and finitely differentiable case should allow us to obtain stronger results. In particular, we would expect that following the methods in [10,14] one could build examples with diffusion in a larger positive measure subset of the phase space, and eventually allow us to show that in general an invariant Diophantine torus is not accumulated by a set of initial conditions with large relative measure and which have larger diffusion times than the strictly optimal ones. This would contrast with the results for the analytic case in [6]. The existence of such Hamiltonians would be in line with the examples of smooth Hamiltonians with Diophantine tori which are not accumulated by a positive measure set of invariant tori (see again [10,14]).

Let us now state a result regarding frequencies which are not Diophantine. For a Hamiltonian of type (1.1), it is easy to derive, independently of the frequency ω, a naive bound for $T(r)$. There exists a constant $C>0$ and $r^*>0$ such that for all $0<r<r^*$, we have that

$$T(r)\geq \frac{C}{r}. \tag{1.7}$$

In the case where $H\in C^l(\mathcal{D}_R)$, comparing the latter bound with (1.4), we see that they are the same up to an improvement of the exponent (which increases with the regularity) by $(l-2)/(\tau+1)$ when $\omega \in \Omega_\tau^d$. Therefore it is natural to ask what happens when the frequency ω does not belong to Ω_τ^d. Is then the bound in (1.7) optimal? The statement of Theorem C below is that there exists a dense subset of Liouville vectors $\mathcal{L}^d \subset \mathbb{R}^d$ such that the bound (1.7) is optimal. Let us define $\omega = (\omega_1, \ldots, \omega_d) \in \mathcal{L}^d$ if ω is non-resonant and $\tilde{\omega} := (\omega_1, \ldots, \omega_{d-1}) \notin \Omega_\tau^{d-1}$ for any $\tau \geq d-2$ (meaning that $\tilde{\omega}$ is a Liouville vector in $\mathbb{R}^{d-1}$). Our statement is the following.

Theorem C. *For $\omega \in \mathcal{L}^d$, $l \geq 3$, $C > 0$, there exist $H \in C^l(\mathcal{D}_R)$ as in (1.1), $c > 0$, and sequences $r_n \to 0$, $\varepsilon_n \to 0$ such that for every $n \geq 0$, we have $z_n = (\theta_n, I_n)$ with $\|I_n\| = |r_n|$ and*

$$\sup_{[0,t_n]} \|\Pi_I \Phi_H^t(z)\| \geq 2|r_n|, \quad z \in B_{\frac{|r_n|}{4} e^{-ct_n}}(z_n),$$

where $t_n = C\left(\frac{1}{|r_n|}\right)^{1+\varepsilon_n}$.

Finally, let us define an analogous arithmetic condition to define a set of frequencies for which an analogous statement as the one in Theorem C holds as well for the case of Gevrey Hamiltonians. We will say that $\omega \in \mathcal{L}_\alpha^d$ if it is a non-resonant vector and such that $\tilde{\omega}$ satisfies that there is a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that

$$\lim_{j\to\infty} \frac{\ln|\tilde{\omega}\cdot k_j|}{\|k_j\|^{\frac{1}{\alpha}}} = -\infty.$$

Theorem D. *For $\omega \in \mathcal{L}_\alpha^d$, $\alpha \geq 1$, $C > 0$, there exist $L > 0$, $H \in G^{\alpha,L}(\mathcal{D}_R)$ as in (1.1), $c > 0$, and sequences $r_n \to 0$, $\varepsilon_n \to 0$ such that for every $n \geq 0$, we have $z_n = (\theta_n, I_n)$ with $\|I_n\| = |r_n|$ and*

$$\sup_{[0,t_n]} \|\Pi_I \Phi_H^t(z)\| \geq 2|r_n|, \quad z \in B_{\frac{|r_n|}{4} e^{-ct_n}}(z_n),$$

where $t_n = C\left(\frac{1}{|r_n|}\right)^{1+\varepsilon_n}$.

For the general perturbative case, meaning for Hamiltonians of type $H = \omega \cdot I + \varepsilon f(\theta, I)$, the almost optimality and optimality (understood in the same sense as in this work) of the exponents for the lower time bounds analogous to (1.4) and (1.6) were proved in [4,5] by constructing examples that diffuse for a time arbitrarily close to the lower bounds in terms of the exponent. Similar examples had already been used by Nekhoroshev in order to motivate the need for steepness assumptions in Nekhoroshev theory (see [21]) and by Sevryuk in the context of KAM theory in order to prove that in the absence of Russmann non-degeneracy conditions we can find perturbations that destroy all tori ([24]). Although the stability results proved in [4,5] for the general perturbative setting can be translated, after a suitable rescaling, to Hamiltonians of the type (1.1), this is no longer the case when we seek examples of instability since the examples used in [4,5] are not of the particular type (1.1). The type of examples found in [11] for the analytic case allow us to overcome this problem. We refer the reader to [3, Section 3] for a discussion on the comparison between these two settings.

Remark 2. All the examples that we present in this paper could have been built in such a way that the resulting quasi-periodic tori are Lyapunov unstable. We have not shown this explicitly because in this particular case the existence of Lyapunov unstable quasi-periodic tori has already been proved for real-analytic Hamiltonians in [11].

Remark 3. All the constructions in this note are given as explicit convergent series in the appropriate topologies that are considered. Alternatively, they can be seen as the limit of an infinite conjugacy scheme of symplectic changes of variables à la Anosov Katok (see [1,13]).

2. Proof of the results. From now on, we will consider, for the sake of clarity, the case $d = 3$. The proofs work in an analogous manner in the case where $d > 3$. For any $\omega \in \mathbb{R}^3$, let us define the function

$$\omega(I_3) = \omega + (I_3, 0, 0).$$

We will need the following arithmetic lemmas. The proof of Lemma 1 can be found in [11], and Lemma 2 below is a generalization of [11, Lemma 2]. Let us denote by $\tilde{\omega} \in \mathbb{R}^2$ the vector obtained by omitting the last component in $\omega \in \mathbb{R}^3$, and similarly we denote $\tilde{\omega}(I_3) := (\omega_1 + I_3, \omega_2)$.

Lemma 1. *For any $\tau > 2$ and $\tilde{\omega} \in \Omega^2_\tau$, a.e. $\omega_3 \in \mathbb{R}$ satisfies $\omega := (\tilde{\omega}, \omega_3) \in \Omega^3_\tau$.*

The main interest of the Lemma 1 is to show that the class of vectors for which Lemma 2 yields inequality (2.1) is non-empty.

Lemma 2. *Consider $\omega \in \mathbb{R}^3$ non-resonant. There exists a sequence $\{r_j\} \subset \mathbb{R}$ and an increasing sequence in norm $\{k_j\} \subset \mathbb{Z}^2$ such that*

(a) $\lim_{j\to\infty} |r_j| = 0$,
(b) $\lim_{j\to\infty} \|k_j\| = \infty$,
(c) $\tilde{\omega}(r_j) \cdot k_j = 0, \ \forall j \geq 1$.

Furthermore:

(d) *If $\omega \in \mathbb{R}^3$ is such that $\tilde{\omega} \notin \Omega^2_\tau$, then we may assume without loss of generality that*

$$\|k_j\| < |r_j|^{-(\tau+1)^{-1}}. \tag{2.1}$$

(e) *If $\omega \in \mathcal{L}^3$, then we may assume without loss of generality that*

$$\|k_j\| < |r_j|^{-(j+1)^{-1}}. \tag{2.2}$$

(f) *If $\omega \in \mathcal{L}^3_\alpha$, then we may assume without loss of generality that there exists an increasing sequence $u_j \to \infty$ such that*

$$\|k_j\| < \left(\ln|r_j|^{-\frac{1}{u_j}}\right)^\alpha. \tag{2.3}$$

Proof. Consider an increasing sequence $\{k_j\} \subset \mathbb{Z}^2$. Then if we impose

$$0 = \tilde{\omega}(r_j) \cdot k_j = \omega_1 k_{j,1} + \omega_2 k_{j,2} + r_j k_{j,1},$$

we obtain $r_j = (\tilde{\omega}\cdot k_j)/k_{j,1}$. Up to a permutation (consider simply $\tilde{\omega}(r_j) = \tilde{\omega}+(0, r_j)$ instead of $\tilde{\omega}(r_j) = \tilde{\omega}+(r_j, 0)$ if necessary) and considering a subsequence of $\{k_j\}$, we can assume without loss of generality that $\|k_j\| = |k_{j,1}|$. Thus

$$|r_j| \leq |\tilde{\omega}\cdot k_j|/\|k_j\|.$$

Additionally, by Dirichlet's approximation theorem, we can assume that the increasing sequence $\{k_j\}$ has been chosen in such a way that $|\tilde{\omega}\cdot k_j| \to 0$, and this gives us a) to c). If we assume that ω satisfies the assumptions in d), then we can additionally assume without loss of generality that the sequence k_j is such that $|\tilde{\omega}\cdot k_j| < \|k_j\|^{-\tau}$, which leads to (2.1). Analogously, if we assume that ω satisfies the assumptions in e), then we can assume without loss of generality that the sequence k_j is such that $|\tilde{\omega}\cdot k_j| < \|k_j\|^{-j}$, leading to (2.2). The proof of f) follows in the same manner. □

Let us also state a consequence of the Gronwall inequalities [17, Lemma 4.1.2] adapted to our context in the form of the following lemma.

Lemma 3. *Assume that $\{H_n\} \subset C^3(\mathcal{D}_R)$ is a convergent sequence of Hamiltonians as in (1.1) and denote the limit by H. There exist constants $K, c > 0$ such that, for any given $T > 0$ and $\xi > 0$, if $\|H_n - H\|_3 < \xi$, then for all $z, z_0 \in \mathcal{D}_{R/4}$, we have that, if the solutions remain in $\mathcal{D}_R$,*

$$\|\Phi^t_H(z) - \Phi^t_{H_n}(z_0)\| < (\|z - z_0\| + K\xi)e^{cT}, \quad t \in [0, T].$$

Proof. From the differential equation corresponding to the difference between the Hamiltonian vector fields H_n and H, we have

$$\dot{y} - \dot{x} = J\nabla H_n(y) - J\nabla H_n(x) + f(y),$$

where $\|f\|_{C^0(\mathcal{D}_R)} \leq \xi$. Thus we obtain, by the mean value theorem, that

$$\|y(t) - x(t)\| \leq \|y(0) - x(0)\| + \int_0^t \|f\|_{C^0(\mathcal{D}_R)}\, ds$$
$$+ \int_0^t 3\|J\nabla^2 H_n\|_{C^0(\mathcal{D}_R)}\|y - x\| ds.$$

This leads to, by the Gronwall inequalities and denoting $\|J\nabla^2 H_n\|_{C^0(\mathcal{D}_R)} \leq L$,

$$\|y(t) - x(t)\| \leq \|y(0) - x(0)\| \exp\left(\int_0^t 3L ds\right) + \int_0^t \xi \left(\exp \int_s^t 3L d\tau\right) ds.$$

Therefore

$$\|y(t) - x(t)\| \leq \left(\|y(0) - x(0)\| + \frac{\xi}{3L}\right)\exp(3Lt).$$

Notice finally that because of the convergence of the sequence H_n in $C^3(\mathcal{D}_R)$, L can be taken sufficiently large so that the above argument works uniformly on n, and thus the statement follows by considering $K = 1/(3L), c = 3L$. □

Proof of Theorem A. Consider the Hamiltonian

$$H = \lim_{n\to\infty} H_n,$$
$$H_n(\theta, I) = \omega(I_3)\cdot I - \sum_{j=2}^{n} I_3^2 \frac{1}{C\|k_j\|^{l+\delta}} \sin(2\pi(k_{j,1}\theta_1 + k_{j,2}\theta_2)) \tag{2.4}$$

for a sequence $\{k_j\}$ as in Lemma 2 with $\omega \in \Omega^3_\tau$ such that $\tilde{\omega} \notin \Omega^2_{\tau-\rho}$ and where the choices of $\rho, \delta > 0$ will be made precise later. It is an elementary consequence of Lemma 1 that such frequencies exist for $\tau > 2$. First of all let us consider the convergence. It is clear from (2.4) that, for any $m > n \geq N$,

$$\max_{i\in\mathbb{N}^6, |i|\leq l} |\partial^i (H_n - H_m)|_{C^0(D_R)} \lesssim \sum_{j=n}^{m} \frac{1}{C\|k_j\|^{\delta}} \to 0$$

as $N \to \infty$ if we assume $\|k_j\|$ to increase sufficiently fast, and so the limit exists and $H \in C^l(\mathcal{D}_R)$.

When considering the corresponding Hamiltonian vector field to (2.4), notice first of all that $\dot{I}_3 = 0$. In particular, we obtain that, for an initial condition $z_{0,n} = (\theta(0), I(0)) = (0, \ldots, 0, r_n)$, $I_3 \equiv r_n$ and

$$(\dot{\theta}_1, \dot{\theta}_2) = \tilde{\omega}(r_n),$$
$$(\dot{I}_1, \dot{I}_2) = 2\pi \sum_{j=2}^{n} (k_{j,1}, k_{j,2}) r_n^2 \frac{1}{C\|k_j\|^{l+\delta}} \cos(2\pi k_j \cdot \tilde{\omega}(r_n)t)$$
$$= 2\pi k_n r_n^2 \frac{1}{C\|k_n\|^{l+\delta}} + 2\pi \sum_{j=2}^{n-1} k_j r_n^2 \frac{1}{C\|k_j\|^{l+\delta}} \cos(2\pi k_j \cdot \tilde{\omega}(r_n)t).$$

This leads, by integration, to the following expression for the action variables

$$(I_1(t), I_2(t)) = 2\pi k_n r_n^2 \frac{1}{C\|k_n\|^{l+\delta}} t$$
$$+ 2\pi \sum_{j=2}^{n-1} k_j \frac{r_n^2}{C(2\pi k_j \cdot \omega(r_n))\|k_j\|^{l+\delta}} \sin(2\pi k_j \cdot \tilde{\omega}(r_n)t).$$

Since we can assume without loss of generality that $|k_{j,1}| = \|k_j\|$, we have that if $\|k_n\|$ is increasing sufficiently fast, for $j = 2, \ldots, n-1$,

$$|k_j \cdot \tilde{\omega}(r_n)| = |k_{j,1}\omega_1 + k_{j,2}\omega_2 + k_{j,1} r_j + k_{j,1}(r_n - r_j)| \geq \frac{\|k_j\| r_n}{2}$$

and also that for all $n \geq 3$,

$$\left\| 2\pi \sum_{j=2}^{n-1} k_j \frac{r_n^2}{C(2\pi k_j \cdot \omega(r_n))\|k_j\|^{l+\delta}} \sin(2\pi k_j \cdot \tilde{\omega}(r_n)t) \right\| \leq 3r_n.$$

Thus it follows that

$$\|I(t)\| \geq 2\pi r_n^2 \frac{1}{C\|k_n\|^{l-1+\delta}} t - 3r_n. \tag{2.5}$$

Using now Lemma 2, we have $\|k_n\| < |r_n|^{-(\tau+1-\rho)^{-1}}$ and we obtain that, assuming $\delta, \rho > 0$ to be sufficiently small,

$$t_n = C\frac{1}{|r_n|}\left(\frac{1}{|r_n|}\right)^{\frac{l-1+\delta}{\tau+1-\rho}} \leq C\frac{1}{|r_n|}\left(\frac{1}{|r_n|}\right)^{\frac{l-1}{\tau+1}+\varepsilon} \tag{2.6}$$

and we have $\|I(t_n)\| > 3r_n$. The following step is to see that the limit Hamiltonian H inherits these dynamics by Gronwall's lemma. Assume that $\|k_{n+1}\|$ is sufficiently large so that

$$\|H_n - H\|_{C^3(\mathcal{D}_R)} < \xi := \frac{r_n e^{-ct_n}}{4K}$$

where K, c are given by Lemma 3. Then by this very same lemma, we have that for $z \in B_{\frac{r_n}{4}e^{-ct_n}}(z_{0,n})$,

$$\|\Phi^{t_n}_{H_n}(z_{0,n}) - \Phi^{t_n}_{H}(z)\| \leq (\|z_{0,n} - z\| + K\xi)e^{ct_n} \leq \frac{r_n}{2}.$$

Therefore we have that there exists $c > 0$ such that for all initial conditions $z \in B_{\frac{r_n}{4}e^{-ct_n}}(z_{0,n})$, we have that $\|\Pi_I \Phi^{t_n}_H(z)\| > 2\|z\|$. □

Proof of Theorem B. Consider the Hamiltonian

$$H = \lim_{n\to\infty} H_n,$$

$$H_n(\theta, I) = \omega(I_3)\cdot I - \sum_{j=2}^{n} I_3^2 e^{-3\alpha L\|k_j\|^{\frac{1}{\alpha}}}\sin(2\pi(k_{j,1}\theta_1 + k_{j,2}\theta_2)) \tag{2.7}$$

with $L = \frac{C}{4\alpha}$, and a sequence $\{k_j\}$ as in Lemma 2 for an $\omega \in \Omega^3_\tau$ such that $\tilde{\omega} \notin \Omega^2_{\tau-\rho}$ (where the choice of $\rho > 0$ will be made precise later). Let us show first the convergence of the sequence. We have that, for a given $N \in \mathbb{N}$, for all $m > n \geq N$, $i \in \mathbb{N}^6$,

$$\begin{aligned}
&|H_m - H_n|_{\alpha,L,i,R}\\
&\quad \lesssim \frac{L^{|i|\alpha}}{i!^\alpha}\sum_{j=n}^{m}\|k_j\|^{|i|}e^{-3\alpha L\|k_j\|^{\frac{1}{\alpha}}} = \sum_{j=n}^{m}\frac{L^{|i|\alpha}}{i!^\alpha}\|k_j\|^{|i|}e^{-3\alpha L\|k_j\|^{\frac{1}{\alpha}}}\\
&\quad = \sum_{j=n}^{m}\left(\frac{L^{|i|}}{i!}\|k_j\|^{\frac{|i|}{\alpha}}\right)^{\alpha} e^{-3\alpha L\|k_j\|^{\frac{1}{\alpha}}} \leq \sum_{j=n}^{m} e^{2\alpha L\|k_j\|^{\frac{1}{\alpha}}}e^{-3\alpha L\|k_j\|^{\frac{1}{\alpha}}}\\
&\quad = \sum_{j=n}^{m} e^{-\alpha L\|k_j\|^{\frac{1}{\alpha}}}.
\end{aligned}$$

Thus

$$\|H_m - H_n\|_{G^{\alpha,L}(\mathcal{D}_R)} = \sup_{i\in\mathbb{N}^6}|H_m - H_n|_{\alpha,L,i,R} \lesssim \sum_{j=n}^{m} e^{-\alpha L\|k_j\|^{\frac{1}{\alpha}}} \to 0$$

as $N \to \infty$. Therefore the sequence H_n is a Cauchy sequence and the limit $H \in G^{\alpha,L}(\mathcal{D}_R)$. Analogously as in the proof of Theorem A, we obtain that for

$z_{0,n} = (\theta(0), I(0)) = (0, \dots, 0, r_n)$, we have that the two first action variables of the flow of H_n satisfy

$$(I_1(t), I_2(t)) = 2\pi(k_{n,1}, k_{n,2}) r_n^2 e^{-3\alpha L \|k_n\|^{\frac{1}{\alpha}}} t$$
$$+ 2\pi \sum_{j=2}^{n-1} r_n^2 \frac{(k_{j,1}, k_{j,2})}{(2\pi k_j \cdot \omega(r_n))} e^{-3\alpha L \|k_j\|^{\frac{1}{\alpha}}} \sin(2\pi k_j \cdot \tilde{\omega}(r_n) t).$$

Assuming again $\|k_n\|$ to increase sufficiently fast, we have

$$\|I(t)\| \geq 2\pi r_n^2 e^{-3\alpha L \|k_n\|^{\frac{1}{\alpha}}} t - 3r_n. \tag{2.8}$$

Using that $\|k_n\| \leq |r_n|^{-(\tau+1-\rho)^{-1}}$, for $C = 4\alpha L > 0$, we have that for all n sufficiently large, for

$$t_n = \exp(C|r_n|^{-\frac{1}{\alpha(\tau+1)} - \varepsilon}), \tag{2.9}$$

we get $\|I(t_n)\| > 3r_n$ (again by considering $\rho > 0$ sufficiently small with respect to ε). We conclude as before that, due to Lemma 3, there exists $c > 0$ such that for all $n \in \mathbb{N}$,

$$\|\Pi_I \Phi_H^{t_n}(z)\| \geq 2\|z\|, \quad z \in B_{\frac{|r_n|}{4} e^{-ct_n}}(z_{0,n}).$$

□

Proof of Theorem C. Consider the Hamiltonian

$$H = \lim_{n \to \infty} H_n, \quad H_n(\theta, I) = \omega(I_3) \cdot I - \sum_{j=2}^{n} I_3^2 \frac{1}{C\|k_j\|^{l+1}} \sin(2\pi(k_{j,1}\theta_1 + k_{j,2}\theta_2))$$

for a sequence $\{k_j\}$ as in Lemma 2 with $\omega \in \mathcal{L}^3$. The result follows exactly in the same lines as the proof of Theorem A, the only difference being that in (2.6) we will obtain

$$t_n = C \frac{1}{|r_n|} \left(\frac{1}{|r_n|} \right)^{l/(n+1)} = C \left(\frac{1}{|r_n|} \right)^{1+\varepsilon_n}$$

for $\varepsilon_n := l/(n+1)$. This finishes the proof. □

Proof of Theorem D. Consider the Hamiltonian

$$H = \lim_{n \to \infty} H_n,$$
$$H_n(\theta, I) = \omega(I_3) \cdot I - \sum_{j=2}^{n} C^{-1} I_3^2 e^{-3\alpha L \|k_j\|^{\frac{1}{\alpha}}} \sin(2\pi(k_{j,1}\theta_1 + k_{j,2}\theta_2)) \tag{2.10}$$

with $L > 0$, and a sequence $\{k_j\}$ as in Lemma 2 for $\omega \in \mathcal{L}_\alpha^3$. The result follows exactly in the same lines as the proof of Theorem B, the only difference being that in (2.8) we will obtain that, due to (2.3),

$$\|I(t)\| \geq 2\pi C^{-1} r_n^2 e^{-3\alpha L \|k_n\|^{\frac{1}{\alpha}}} t - 3r_n \geq 2\pi C^{-1} r_n^2 e^{-\frac{3\alpha L}{u_n} \ln \frac{1}{|r_n|}} t - 3r_n$$
$$= 2\pi C^{-1} r_n^2 |r_n|^{3\alpha L/u_n} t - 3r_n.$$

By choosing now $\varepsilon_n = 3\alpha L/u_n \to 0$, we obtain the desired result. □

Acknowledgements. The author is grateful to H. Eliasson, T. Seara, and F. Trujillo for valuable discussions and to the referee for useful comments. This work has been supported by the Juan de la Cierva-Formación Program (FJC 2021-044720-I) and the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence (CEX2020-001084-M). It was completed while the author was in residence at Institut Mittag-Leffler in Djursholm, Sweden, and has therefore also been supported by the Swedish Research Council under grant no. 2021-06594.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

References

[1] Anosov, D.V., Katok, A.B.: New examples in smooth ergodic theory. Ergodic diffeomorphisms. Trudy Moskov. Mat. Obšč. **23**, 3–36 (1970)

[2] Barbieri, S., Marco, J.-P., Massetti, J.: Analytic smoothing and Nekhoroshev estimates for Hölder steep Hamiltonians. Comm. Math. Phys. **396**(1), 349–381 (2022)

[3] Bounemoura, A.: Optimal stability and instability for near-linear Hamiltonians. Ann. Henri Poincaré **13**(4), 857–868 (2012)

[4] Bounemoura, A.: Normal forms, stability and splitting of invariant manifolds I. Gevrey Hamiltonians. Regul. Chaotic Dyn. **18**(3), 237–260 (2013)

[5] Bounemoura, A.: Normal forms, stability and splitting of invariant manifolds II. Finitely differentiable Hamiltonians. Regul. Chaotic Dyn. **18**(3), 261–276 (2013)

[6] Bounemoura, A., Farré, G.: Positive measure of effective quasi-periodic motion near a diophantine torus. Ann. Henri Poincaré **24**(9), 3289–3304 (2023)

[7] Bounemoura, A., Fayad, B., Niederman, L.: Superexponential stability of quasi-periodic motion in Hamiltonian systems. Comm. Math. Phys. **350**(1), 361–386 (2017)

[8] Bounemoura, A., Fayad, B., Niederman, L.: Super-exponential stability for generic real-analytic elliptic equilibrium points. Adv. Math. **366**, 107088, 30 pp. (2020)

[9] Douady, R.: Stabilité ou instabilité des points fixes elliptiques. Ann. Sci. École Norm. Sup. (4) **21**(1), 1–46 (1988)

[10] Eliasson, L.H., Fayad, B., Krikorian, R.: Around the stability of KAM tori. Duke Math. J. **164**(9), 1733–1775 (2015)

[11] Farré, G., Fayad, B.: Instabilities of invariant quasi-periodic tori. J. Eur. Math. Soc. (JEMS) **24**(12), 4363–4383 (2022)

[12] Fayad, B.: Lyapunov unstable elliptic equilibria. J. Amer. Math. Soc. **36**(1), 81–106 (2023)

[13] Fayad, B., Katok, A.: Constructions in elliptic dynamics. Ergod. Theory Dyn. Syst. **24**(5), 1477–1520 (2004)

[14] Fayad, B., Saprykina, M.: Isolated elliptic fixed points for smooth Hamiltonians. In: Modern Theory of Dynamical Systems, pp. 67–82. Contemp. Math., 692. Amer. Math. Soc., Providence, RI (2017)

[15] Fayad, B., Sauzin, D.: KAM tori are no more than sticky. Arch. Ration. Mech. Anal. **237**(3), 1177–1211 (2020)

[16] Giorgilli, A., Delshams, A., Fontich, E., Galgani, L., Simó, C.: Effective stability for a Hamiltonian system near an elliptic equilibrium point, with an application to the restricted three-body problem. J. Differential Equations **77**(1), 167–198 (1989)

[17] Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Applied Mathematical Sciences, 42. Springer, New York (1983)

[18] Jorba, A., Villanueva, J.: On the normal behaviour of partially elliptic lower-dimensional tori of Hamiltonian systems. Nonlinearity **10**(4), 783–822 (1997)

[19] Mitev, T., Popov, G.: Gevrey normal form and effective stability of Lagrangian tori. Discrete Contin. Dyn. Syst. Ser. S **3**(4), 643–666 (2010)

[20] Morbidelli, A., Giorgilli, A.: Superexponential stability of KAM tori. J. Stat. Phys. **78**(5–6), 1607–1617 (1995)

[21] Nekhoroshev, N.: An exponential estimate of the time of stability of nearly integrable Hamiltonian systems II. Trudy Sem. Petrovsk. **5**, 5–50 (1979)

[22] Perry, A.D., Wiggins, S.: KAM tori are very sticky: rigorous lower bounds on the time to move away from an invariant Lagrangian torus with linear flow. Phys. D **71**(1–2), 102–121 (1994)

[23] Pöschel, J.: Nekhoroshev estimates for quasi-convex Hamiltonian systems. Math. Z. **213**(1), 187–216 (1993)

[24] Sevryuk, M.B.: KAM-stable Hamiltonians. J. Dyn. Control Syst. **1**(3), 351–366 (1995)

[25] Trujillo, F.: Lyapunov instability in KAM stable Hamiltonians with two degrees of freedom. J. Mod. Dyn. **19**, 363–383 (2023)

Gerard Farré
Departament de Matemàtiques
Universitat Politècnica de Catalunya
Av. Diagonal 64
08028 Barcelona
Spain

and

Centre de Recerca Matemàtica
Edifici C, Campus Bellaterra
08193 Bellaterra
Spain
e-mail: gerard.farre.puiggali@upc.edu

Received: 21 April 2023

Revised: 21 April 2023

Accepted: 28 September 2023