Arch. Math. 121 (2023), 459–466
0003-889X/23/040459-8
published online September 19, 2023
https://doi.org/10.1007/s00013-023-01912-8

Archiv der Mathematik

Correction to: On the finite index subgroups of Houghton's groups

Charles Garnet Cox

Abstract. This note resolves an issue raised by Prof. Derek Holt for the paper "On the finite index subgroups of Houghton's groups".

Mathematics Subject Classification. 20F05.

Keywords. Generation, Generation of finite index subgroups, Structure of finite index subgroups, Infinite groups, Houghton groups, Permutation groups, Highly transitive groups.

Correction to: Arch. Math. 118 (2022), 113–121
https://doi.org/10.1007/s00013-021-01677-y

This note addresses an issue raised by Derek Holt that not all finite index subgroups of $\mathbb{Z}^n$ have the form $\langle c_1e_1, \ldots, c_ne_n\rangle$ for some constants $c_1, \ldots, c_n \in \mathbb{N}$. I am grateful to him for drawing my attention to these subgroups. In this note, we show these additional subgroups can be dealt with using arguments of a similar nature to those in the original paper [1], and that a complete classification of when each $U \leqslant_f H_n$ has $d(U) = n-1$ or $d(U) = n$ can still be obtained. We begin by giving reformulated statements of the original results with these subgroups in mind. The notation we use is introduced in [1]. Recall that $\pi : H_n \to \mathbb{Z}^{n-1}$ is the homomorphism induced by defining $\pi(g_i) = e_{i-1}$ for $i = 2, \ldots, n$.

Theorem 1. *Let $n \in \{3, 4, \ldots\}$ and $U \leqslant_f H_n$. Then there exist $v_2, \ldots, v_n \in \mathbb{Z}^{n-1}$ such that $\langle v_2, \ldots, v_n\rangle \leqslant_f \mathbb{Z}^{n-1}$ and $h_2, \ldots, h_n \in U$ where each $h_i \in \pi^{-1}(v_i)$ and either $U = \langle h_2, \ldots, h_n, \mathrm{Alt}(X_n)\rangle$ or $U = \langle h_2, \ldots, h_n, \mathrm{FSym}(X_n)\rangle$. Furthermore, if there exist $c_2, \ldots, c_n \in \mathbb{N}$ such that $\pi(U) = \langle c_2e_1, \ldots, c_ne_{n-1}\rangle$, then either U is:*

(i) *equal to $\langle \mathrm{FSym}(X_n), g_i^{c_i} : i = 2, \ldots, n\rangle$; or*
(ii) *isomorphic to $\langle \mathrm{Alt}(X_n), g_i^{c_i} : i = 2, \ldots, n\rangle$.*

If $U \leqslant_f H_2$*, then there exists* $c_2 \in \mathbb{N}$ *such that* $\pi(U) = \langle c_2 \rangle \leqslant \mathbb{Z}$ *and either* (i) *or* (ii) *above occurs or* U *is equal to* $\langle \mathrm{Alt}(X_2), ((1,1),(1,2))g_2 \rangle$*.*

This theorem follows from the same principles as in the original paper, but we give the details in Section 1. Corollary 2 is not affected by the additional finite index subgroups of $\mathbb{Z}^n$, but is included for completeness.

Corollary 2. *Let* $n \in \{2,3,\ldots\}$ *and* $c_2,\ldots,c_n \in \mathbb{N}$*. If* $U \leqslant_f H_n$ *and* $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1} \rangle$*, then either*

- *at least two of* $c_2,\ldots,c_n$ *are odd and* $U = \langle g_2^{c_2}, \ldots, g_n^{c_n}, \mathrm{FSym}(X_n) \rangle$*; or*
- U *is one of exactly* $2^{n-1}+1$ *specific subgroups of* H_n*.*

Theorem 3 can be rephrased so to only work with finite index subgroups of the specific form considered in the original paper, and so the original proofs apply to this rephrased result.

Theorem 3. *If* $U \leqslant_f H_2$*, then* $d(U) = d(H_2)$*. Let* $n \in \{3,4\ldots\}$ *and* $U \leqslant_f H_n$ *with* $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1} \rangle$*. Then* $d(U) \in \{d(H_n), d(H_n)+1\}$ *and* $d(U) = d(H_n)+1$ *occurs exactly when both of the following conditions are met:*

(i) *that* $\mathrm{FSym}(X_n) \leqslant U$*; and*
(ii) *either one or zero elements in* $\{c_2,\ldots,c_n\}$ *are odd.*

Our fourth result no longer immediately follows from Theorem 3 since we need to consider the generation of the remaining finite index subgroups. We therefore name it Proposition 4 rather than Corollary 4. The proof is in Section 2.

Proposition 4. *Let* $n \in \{3,4,\ldots\}$*. If* $U \leqslant_f H_n$ *and* $\mathrm{FSym}(X_n) \cap U = \mathrm{Alt}(X_n)$*, then* $d(U) = n-1$*. Thus, for every* $U \leqslant_f \boldsymbol{G_2} := \langle \mathrm{Alt}(X_n), g_2^2, \ldots, g_n^2 \rangle$*, we have that* $d(U) = d(\boldsymbol{G_2}) = d(H_n)$*.*

This result yields a complete characterisation of when $U \leqslant_f H_n$ has $d(U) = n-1$ and when it has $d(U) = n$. For details, see Remark 2.2. As a consequence of our adjusted approach, two new results are obtained. The proofs are contained in Sections 3 and 4, respectively.

Proposition 5. *Let* $U = \langle t^k, \mathrm{Alt}(\mathbb{Z}) \rangle \leqslant_f H_2$ *and* $p \in \mathbb{N} \setminus \{1\}$*. Then there exists* $\omega \in \mathrm{FSym}(\mathbb{Z})$ *of order* p *such that* $\langle t^k, \omega \rangle = U$*. If* $U = \langle t^k, \mathrm{FSym}(\mathbb{Z}) \rangle \leqslant_f H_2$*, then such an* ω *exists if and only if* p *is even.*

Proposition 6. *Let* $n \in \mathbb{N}$ *and* $U \leqslant_f H_n$ *with* $\pi(U) = \langle v_2, \ldots, v_n \rangle$*. Then for any choice of* $h_2, \ldots, h_n \in U$ *with* $\pi(h_i) = v_i$ *for* $i = 2, \ldots, n$*, there exists an* $\omega \in \mathrm{FSym}(X_n)$ *such that* $\langle \omega, h_2, \ldots, h_n \rangle = U$*.*

1. Additional details for Theorem 1. Two well known results (used in the original paper) give us Theorem 1.

Lemma 1.1. *Given* $U \leqslant_f G$*, there exists* $N \leqslant_f U$ *which is normal in* G*.*

Lemma 1.2 ([2, Prop. 2.5])*. Let* X *be a non-empty set and* $\mathrm{Alt}(X) \leqslant G \leqslant \mathrm{Sym}(X)$*. Then* G *has* $\mathrm{Alt}(X)$ *as a unique minimal normal subgroup.*

Proof of Theorem 1. Let $n \in \{2, 3, \ldots\}$ and $U \leqslant_f H_n$. By Lemma 1.1, U contains a normal subgroup of H_n and so, by Lemma 1.2, $\mathrm{Alt}(X_n) \leqslant U$. Furthermore, $U \leqslant_f H_n$ and so $\pi(U) \leqslant_f \pi(H_n)$. Thus $\pi(U) = \langle v_2, \ldots, v_n\rangle \leqslant_f \mathbb{Z}^{n-1}$, meaning that U contains some elements $h_2, \ldots, h_n$ where $h_i \in \pi^{-1}(v_i)$ for $i = 2, \ldots, n$. Note that for any $v \in \mathbb{Z}^{n-1}$, we can describe $\pi^{-1}(v)$. Given $v = \sum_{j=2}^{n} d_j e_{j-1}$ with $d_2, \ldots, d_n \in \mathbb{Z}$, define $\hat{v} := \prod_{j=2}^{n} g_j^{d_j}$. With this notation, for $i = 2, \ldots, n$, we have

$$\pi^{-1}(v_i) = \{\sigma \hat{v}_i \;:\; \sigma \in \mathrm{FSym}(X_n)\}.$$

If $\mathrm{FSym}(X_n) \leqslant U$, then $U = \langle \hat{v}_2, \ldots, \hat{v}_n, \mathrm{FSym}(X_n)\rangle$. If $\mathrm{FSym}(X_n) \cap U = \mathrm{Alt}(X_n)$, then we can specify that h_i is $\hat{v}_i$ or $\epsilon\hat{v}_i$ where $\epsilon = ((1,1)\ (1,2))$, depending on whether $\hat{v}_i h_i^{-1}$ is an even or odd permutation respectively. Note that the possibility of $\epsilon\hat{v}_i, \hat{v}_i \in U$ is excluded since $\mathrm{FSym}(X_n) \not\leqslant U$ means $\epsilon \notin U$. Hence $U = \langle h_2, \ldots, h_n, \mathrm{Alt}(X_n)\rangle$ in this case. □

Note if we assume that $\pi(U) = \langle c_2 e_1, \ldots, c_n e_{n-1}\rangle$ for some $c_2, \ldots, c_n \in \mathbb{N}$, then

$$U = \langle g_2^{k_2}, \ldots, g_n^{k_n}, \mathrm{FSym}(X_n)\rangle \text{ or } U = \langle \epsilon_2 g_2^{k_2}, \ldots, \epsilon_n g_n^{k_n}, \mathrm{Alt}(X_n)\rangle$$

where each ϵ_i is either trivial or equal to $((1,1)\ (1,2))$. This gives us Corollary 2.

2. Proving Proposition 4. We work with a fixed $n \in \{3, 4, \ldots\}$. To prove the proposition, it is sufficient to show that if $U \leqslant_f H_n$ and $\mathrm{FSym}(X_n) \cap U = \mathrm{Alt}(X_n)$, then U is $n-1$ generated.

Recall that $\pi(U) = \langle v_2, \ldots, v_n\rangle \leqslant_f \mathbb{Z}^{n-1}$. Let $i \in \{2, \ldots, n\}$. The support of any preimage of v_i has infinite intersection with at least 2 rays of X_n. Since $n \geqslant 3$, we therefore have $2(n-1) > n$ and so there exist distinct j, j' such that the supports of the preimages of v_j and $v_{j'}$ have infinite intersection with the same branch. By relabelling, we will assume this is $R_1 := \{(1,m) : m \in \mathbb{N}\} \subset X_n$ and that $j = 2$ and $j' = 3$. For $k \in \{2, \ldots, n\}$, let $h_k := \hat{v}_k$ with the notation from Section 1. That is, if $v_k = \sum_{j=2}^{n} d_j e_{j-1}$ with $d_2, \ldots, d_n \in \mathbb{Z}$, set $\hat{v}_k := \prod_{j=2}^{n} g_j^{d_j}$. After possibly exchanging v_2 with $-v_2$ and v_3 with $-v_3$, we can assume that the first component of $\pi(h_2)$ and of $\pi(h_3)$ are positive. We claim that there exists $\omega \in \mathrm{FSym}(X_n)$ such that $\langle \omega h_2, h_3, \ldots, h_n\rangle = U$. Furthermore, we can specify that $\mathrm{supp}(\omega) \subset R_1$.

There are some key properties about the restrictions we have imposed thus far. Fix an $h \in \{h_2, \ldots, h_n\}$ so that $h = \prod_{j=2}^{n} g_j^{d_j}$ for some constants $d_2, \ldots, d_n$. We note that every orbit of h is either infinite or has size 1. Let $i \in \{2, \ldots, n\}$ and $x \in R_i$. Then $(x)h = x$ if and only if $d_i = 0$. If $d_i \neq 0$, then $\{(x)h^k : k \in \mathbb{Z}\}$ is infinite. Now, take any $y \in R_1$ and let $\mathcal{O}_y := \{(y)h^k : k \in \mathbb{Z}\}$. If $\mathcal{O}_y \cap R_j \neq \emptyset$ for some $j \in \{2, \ldots, n\}$, then from the form of h we must have $d_j \neq 0$ and, from our first case, $\mathcal{O}_y$ is infinite. Finally, let $y = (1,m)$ and assume that $\mathcal{O}_y \subseteq R_1$. Then $(1,m)h = (1, m + \sum_{j=2}^{n} d_j) = (1,m)$ since $\mathcal{O}_y \subseteq R_1$ implies that $\sum_{j=2}^{n} d_j = 0$.

Note that there is a $d \in \mathbb{N}$ such that $d\mathbb{Z}^{n-1} \leqslant \pi(U)$. Hence, for each $i = 2, \ldots, n$, there is a $\sigma_i \in \mathrm{FSym}(X_n)$ such that $\sigma_i g_i^d \in \langle h_2, \ldots, h_n\rangle$. Elements of H_n act 'eventually as a translation' meaning that $\sigma_i g_i^d$ has d infinite orbits, infinitely many fixed points, and then possibly some finite orbits (see [3, Lem. 2.3] for details and also note from this lemma that $R_1 \cup R_i$ and $\mathrm{supp}(\sigma_i g_i^d)$ have finite symmetric difference). We can therefore choose some $m_2, \dots, m_n \in \mathbb{N}$ so that $(\sigma_i g_i^d)^{m_i}$ has $m_i d$ infinite orbits and no finite orbits other than fixed points. For $i = 2, \dots, n$, let $f_i := (\sigma_i g_i^d)^{m_i}$ and $F_i := (R_1 \cup R_i) \setminus \mathrm{supp}(f_i)$, which is finite, and set $F := \cup_{i=2}^n F_i$.

Our aim is therefore to move points in F to $X_n \setminus F$. This can be achieved with a word of the form $(\omega h_2)^{k_2} h_3^{k_3} \cdots h_n^{k_n}$. Let Y_2 consist of points in $F \setminus R_1$ lying on an infinite orbit of ωh_2. Then let $Y_i := F \cap \mathrm{supp}(h_i)$ for $i = 3, \dots, n$, define $A_j := \bigcup_{i=2}^{j-1} Y_i$ for $j = 3, \dots, n$, and let $Z_i := Y_i \setminus A_i$ for $i = 3, \dots, n$. Choose $k_2', \dots, k_n'$ such that $(Y_2)(\omega h_2)^{k_2'} \subset X_n \setminus F$ and $(Z_i) h_i^{k_i'} \subset X_n \setminus F$. Then for $j = 2, \dots, n$, pick $|k_j| \geqslant |k_j'|$ where k_j and k_j' have the same sign so that

$$(Y_i)(\omega h_2)^{k_2} \prod_{j=3}^{n} h_j^{k_j} \subset X_n \setminus F \text{ for } i = 2, \dots, n.$$

Finally use $f_2, \dots, f_n$ to move the points in $X_n \setminus F$ to $R_1 \setminus F$. For $j \in \mathbb{N}$, let $R_1^{(j)} := \{(1, m) : m > j\} \subset X_n$. Then the above shows, for any $j \in \mathbb{N}$ and for any $\omega \in \mathrm{FSym}(X_n)$ with $\mathrm{supp}(\omega) \subset R_1$, that

$$\langle \omega h_2, h_3, \dots, h_n, \mathrm{Alt}(R_1^{(j)}) \rangle = \langle \omega h_2, h_3, \dots, h_n, \mathrm{Alt}(X_n) \rangle. \tag{1}$$

Thus we need only generate $\mathrm{Alt}(R_1^{(j)})$ for some $j \in \mathbb{N}$ in order to generate U.

Note that there exist $d, d' \in \mathbb{N}$ such that the first component of $\pi(f_3^{2d'})$ and $\pi(h_3^d)$ are equal and that this lies in $8\mathbb{N}$. Denote this first component by k. We now compute that $[\omega h_2, h_3^d] = h_2^{-1}[\omega, h_3^d] h_2 [h_2, h_3^d]$ where $[g, h] := g^{-1}h^{-1}gh$ and so let $[h_2, h_3^d] =: \tau \in \mathrm{FSym}(X_n)$. Our aim is to choose ω so that $\langle h_2^{-1}[\omega, h_3^d] h_2 \tau, h_3^d \rangle$ contains $\mathrm{Alt}(\mathrm{supp}(h_3^d))$. Since we can choose ω freely, it is sufficient to show that $\langle [\omega, h_3^d]\tau, h_3^d \rangle$ contains $\mathrm{Alt}(\mathrm{supp}(h_3^d))$. We can do so by mimicking the proof (from the original paper) of the following lemma.

Lemma 2.1 ([1, Lemma 3.3]). *Let $k \in \mathbb{N}$ and $t : z \to z + 1$ for all $z \in \mathbb{Z}$. Then $G_k := \langle \mathrm{Alt}(\mathbb{Z}), t^k \rangle$ and $F_k := \langle \mathrm{FSym}(\mathbb{Z}), t^k \rangle$ are 2-generated.*

Note, for $m \geqslant 8$, that $\langle (1\ 2\ 3), (2\ 3\ 4), \dots, (m-2\ m-1\ m) \rangle = \mathrm{Alt}(\{1, \dots, m\})$. Also, by [3, Lemma 3.6], we have that

$$\mathrm{Alt}(\{1, \dots, m\}) \leqslant \langle \alpha, t^{2m} \rangle \Rightarrow \mathrm{Alt}(\mathbb{Z}) \leqslant \langle \alpha, t^m \rangle. \tag{2}$$

Our aim is to find an $\omega \in \mathrm{FSym}(\mathbb{Z})$ such that $\langle [\omega, h_3^d]\tau, h_3^d \rangle$ contains $\mathrm{Alt}(\mathrm{supp}(h_3^d))$. From the preceding paragraph, it is sufficient to show that $\langle [\omega, h_3^d]\tau, h_3^d \rangle$ contains certain 3-cycles. Let τ have order $p \in \mathbb{N}$. Choose $p' \in \mathbb{N}$ with $p | p'$ and $p' - 1 \geqslant k + 2$. We will restrict ourselves to those $\omega \in \mathrm{FSym}(X_n)$ such that:

- $\mathrm{supp}(\omega) \subset R_1$;
- $\mathrm{supp}(h_3^{-d} \omega h_3^d) \cap \mathrm{supp}(\omega) = \emptyset$;
- $\mathrm{supp}([\omega, h_3^d]) \cap \mathrm{supp}(\tau) = \emptyset$; and
- ω consists only of cycles of length $p' - 1$.

Thus $([\omega, h_3^d]\tau)^{p'} = [\omega, h_3^d]$. Identify $R_1 = \{(1,m) : m \in \mathbb{N}\}$ with $\mathbb{N}$ using the bijection $(1,m) \mapsto m$. Define $\sigma_1, \sigma_2 \in \mathrm{FSym}(\mathbb{N}) < \mathrm{FSym}(\mathbb{Z})$ so that, for $i = 1, 2$,

$$\sigma_i : \max\{\mathrm{supp}(\sigma_i)\} \mapsto \min\{\mathrm{supp}(\sigma_i)\} \text{ and } \sigma_i(x) > x \text{ for all other } x \in \mathrm{supp}(\sigma_i).$$

We note that $p' - 1 = ak + b + 2$ for some $a \in \mathbb{N}$ and $b \in \{0, \dots, k-1\}$. Then let

$$\mathrm{supp}(\sigma_i) := \left(\bigcup_{i=0}^{a+1} \{2ki + 1, \dots, 2ki + k\} \right) \setminus A_i$$

where

$$A_1 := \{3, \dots, k\} \cup \{2k + 1 + b, \dots, 3k\}$$

and

$$A_2 := \{1 + b, \dots, 2k\} \cup \{2(a+1)k + 1\} \cup \{2(a+1)k + 4, \dots, 2(a+1)k + k\}.$$

For $i = 1, 2$, σ_i can be seen as an element moving points within $a + 2$ blocks, each of size k. Let $t : z \mapsto z+1 \in \mathrm{Sym}(\mathbb{Z})$. By construction, $t^{-k}\sigma_1 t^k$ has support disjoint from σ_1. Note that $\mathrm{supp}[\sigma_1, t^k] \subset \{1, 2, \dots, 2(a+2)k\} \subset \mathbb{Z}$. In light of this, define $\sigma_2' := t^{-2(a+2)k}\sigma_2 t^{2(a+2)k}$. Then $\mathrm{supp}([\sigma_2', t^k]) \cap \mathrm{supp}([\sigma_1, t^k]) = \emptyset$ and also $[\sigma_1\sigma_2', t^k] = [\sigma_1, t^k][\sigma_2', t^k]$. We first work with the element $\omega' := \sigma_1\sigma_2'$, and will show that there is a $c \in \mathbb{N}$ such that $\omega := t^{-c}\omega' t^c$ achieves our aim. Let $\alpha := [\omega', t^k]$ and $q := 2(a+1)k + 1$. Direct computation shows that $\max(\mathrm{supp}(\alpha)) = q + 2$ and $\alpha : q - k + 1 \mapsto q + 1$ and $q + 1 \mapsto q + 2$. Also $\min(\mathrm{supp}(\alpha)) = 1$ and so with $\beta := t^{-(4(a+1)+2)k}\alpha t^{(4(a+1)+2)k}$ we see that $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \{q+1\}$. Similarly, $\min(\mathrm{supp}(\beta)) = q$ and $\beta^{-1} : q \mapsto q+1$ and $q + 1 \mapsto q + 2k$. Then $[\beta, \alpha]$ is the 3-cycle $(q\ q+1\ q+2)$ in $\mathrm{FSym}(\mathbb{Z})$. Furthermore, for $i = 1, \dots, k-2$,

$$(t^{-2k}\alpha t^{2k})^{-i}(q, q+1, q+2)(t^{-2k}\alpha t^{2k})^i = (q+i, q+1+i, q+2+i).$$

Hence, from (2), $\langle \alpha, t^{k/2} \rangle$ contains $\mathrm{Alt}(\mathbb{Z})$.

We now work with ω' as an element in $\mathrm{FSym}(R_1) < \mathrm{FSym}(X_n)$ by using the bijection $m \mapsto (1,m)$ between $\mathbb{N}$ and R_1. Fix a $c \in k\mathbb{N}$ so that $\omega := t^{-c}\sigma_1\sigma_2' t^c$ has $\mathrm{supp}(\omega) \cap \mathrm{supp}(\tau) = \emptyset$ and

$$(1,x)f_3^{2d'} = (1,x)h_3^d = (1, x+k) \text{ for all } x \geqslant \min\{\mathrm{supp}(\omega)\}.$$

Then $f_3^{2d'}$ can take the role of t^k in our above calculations, and the elements corresponding to α and β above lie in $\langle [\omega, h_3^d], h_3^d \rangle \leqslant \langle [\omega, h_3^d]\tau, h_3^d \rangle$. Thus $\langle f_3^{d'}, [\omega, h_3^d] \rangle$ contains $\mathrm{Alt}(\{(1,q), \dots, (1, q+k-1)\}$ and also $\mathrm{Alt}(R_1^{(q)})$. Equation (1) then implies the result.

Remark 2.2. As in the original paper, it follows that U requires n generators exactly when $\mathrm{FSym}(X_n) \leqslant U$ but $\mathrm{FSym}(X_n) \not\leqslant [U, U]$, where $[U, U]$ denotes the commutator subgroup of U. We can see this as follows. Let $\pi(U) = \langle v_2, \dots, v_n \rangle$ and $h_i \in \pi^{-1}(v_i)$ for $i = 2, \dots, n$. Given $g, h \in U$, we can find σ_1, σ_2 from $\mathrm{FSym}(X_n)$ such that $\sigma_1 g = \prod_{i=2}^n h_i^{c_i}$ and $\sigma_2 h = \prod_{i=2}^n h_i^{d_i}$ for some $c_2, d_2, \dots, c_n, d_n \in \mathbb{Z}$. If $[h_i, h_j] \in \mathrm{Alt}(X_n)$ for all $i, j \in \{2, \dots, n\}$, then applying the standard commutator identities (used in the original paper) means that $[g, h]$ is in $\mathrm{Alt}(X_n)$. This applies to any $g, h \in U$, and so $[U, U] \cap \mathrm{FSym}(X_n) = \mathrm{Alt}(X_n)$ and U has abelianization $\mathbb{Z}^{n-1} \times C_2$. Therefore U cannot be $n-1$ generated if this is the case, but it will be n generated. Otherwise, we have two cases. If $\mathrm{FSym}(X_n) \cap U = \mathrm{Alt}(X_n)$, then Proposition 4 applies. If $[U, U] \cap \mathrm{FSym}(X_n) = \mathrm{FSym}(X_n)$, then there exist $i, j \in \{2, \ldots, n\}$ such that $[h_i, h_j] \in \mathrm{FSym}(X_n) \setminus \mathrm{Alt}(X_n)$. We can then use the element ω from the proof of Proposition 4 so that $\langle \omega h_2, h_3, \ldots, h_n\rangle$ contains $\mathrm{Alt}(X_n)$. But $\langle \omega h_2, h_3, \ldots, h_n\rangle$ must also contain $\mathrm{FSym}(X_n)$ since

$$\{[h_i, h_j] : i = 3, \ldots, n, j = 2, \ldots, n\} \cup \{[\omega h_2, h_j] : j = 2, \ldots, n\}$$

contains an element from $\mathrm{FSym}(X_n) \setminus \mathrm{Alt}(X_n)$.

3. Proving Proposition 5. Choose $p \in \{2, 3, \ldots\}$ and $k \in \mathbb{N}$. Our aim will be to construct $\omega \in \mathrm{Alt}(\mathbb{Z})$, consisting only of p-cycles, such that $\mathrm{Alt}(\mathbb{Z}) \leqslant \langle \omega, t^k\rangle$. With this aim in mind, observe that we only need to show that $\mathrm{Alt}(\mathbb{Z}) \leqslant \langle \omega, t^{ak}\rangle$ for some $a \in \mathbb{N}$, and so without loss of generality we can assume that $k \geqslant 8$. We again use the idea of the proof of Lemma 2.1. Let $\Omega := \{1, 2, \ldots, kp\}$. Recall that $\langle (1\ 2\ 3), \ldots, (1\ 2\ kp)\rangle = \mathrm{Alt}(\Omega)$ which is a routine calculation within the proof of [3, Lemma 3.6]. Define $\sigma_0 := (1\ 2\ \ldots\ p)$, $\sigma_i := t^{-i}(1\ 2\ \ldots\ p)t^i$ for $i = 1, \ldots, kp - p$, and $\sigma_{kp-p+1} := \sigma_{p-1}$. Note that these are all elements of $\mathrm{FSym}(\Omega) \leqslant \mathrm{FSym}(\mathbb{Z})$, and also for $i = 1, \ldots, kp - p$ that σ_i conjugates $(1\ 2\ 3)$ to $(1\ 2\ 3 + i)$. Now let $n := kp - p + 1$ and define

$$\omega := \prod_{i=0}^{n} t^{-kpi}\sigma_i t^{kpi}$$

which is an element of $\mathrm{Alt}(\mathbb{Z})$ by construction. Furthermore, with $\omega' := t^{kpn} \omega t^{-kpn}$ we see that $[\omega', \omega^{-1}] = (p{+}1\ p\ p{-}1)$. We can then conjugate the inverse of this element by $t^{-kpi}\omega t^{kpi}$ for various $i \in \mathbb{N}$ to obtain $\{(1\ 2\ 3), \ldots, (1\ 2\ kp)\}$ which generate $\mathrm{Alt}(\Omega)$. Finally,

$$\langle \mathrm{Alt}(\Omega), t^k\rangle \geqslant \langle \mathrm{Alt}(\{1, \ldots, 2k\}), t^k\rangle = \langle \mathrm{Alt}(\mathbb{Z}), t^k\rangle$$

by [3, Lemma 3.6]. If p is even, then we can modify ω to include an extra orbit of length p so that $\langle \omega, t^k\rangle = \langle \mathrm{FSym}(\mathbb{Z}), t^k\rangle$.

4. Proving Proposition 6. This proof again takes inspiration from Lemma 2.1 above. Recall that $\pi(U) = \langle v_2, \ldots, v_n\rangle \leqslant_f \mathbb{Z}^{n-1}$. We now let $h_2, \ldots, h_n$ denote arbitrary elements of U satisfying $\pi(h_i) = v_i$ for $i = 2, \ldots, n$. We will show that there exists $\omega \in \mathrm{FSym}(X_n)$ such that $U = \langle \omega, h_2, \ldots, h_n\rangle$.

In the same way as the previous section, we can choose some $m_2, \ldots, m_n \in \mathbb{N}$ so that $(\sigma_i g_i^d)^{m_i} \in \langle h_2, \ldots, h_n\rangle$ has $m_i d$ infinite orbits and no finite orbits. And again for $i = 2, \ldots, n$, we let $f_i := (\sigma_i g_i^d)^{m_i}$, $F_i := (R_1 \cup R_i) \setminus \mathrm{supp}(f_i)$, $F := \cup_{i=2}^n F_i$, and $T := \mathrm{supp}(f_2)$. Note that $(R_1 \setminus F) \subset T$.

Produce a bijection $\phi : T \to \mathbb{Z}$ such that $\phi(xf_2) = \phi(x)t^{m_2 d}$ for all $x \in T$. We can then apply Lemma 2.1 to produce an element of $\mathrm{FSym}(X_n)$ which with f_2 generates a group containing either $\mathrm{FSym}(T)$ or $\mathrm{Alt}(T)$, depending on our preference. We can therefore choose $\tau \in \mathrm{FSym}(X_n)$ so that $\langle f_2, \tau\rangle \cap$

$\mathrm{FSym}(X_n) = U \cap \mathrm{FSym}(T)$. Let r denote the order of τ. Choose a prime q larger than $\max\{r, 2|F|\}$, and let $\omega_1 \in \mathrm{Alt}(X_n)$ consist of a single orbit of length q with $F \subset \mathrm{supp}(\omega_1) \subset R_1 \cup F$ and $(F)\omega_1 \subset R_1 \setminus F$. Now conjugate τ by a suitable power of f_2 to obtain ω_2, where $\mathrm{supp}(\omega_2)$ is contained in R_1 and has trivial intersection with $\mathrm{supp}(\omega_1)$. Clearly we have $\langle f_2, \omega_2\rangle = \langle f_2, \tau\rangle$. Set $\omega := \omega_1\omega_2$. Using that ω_1 and ω_2 have disjoint supports, $\omega^r = \omega_1^r$. Since ω_1 has prime order, $\langle \omega^r\rangle = \langle \omega_1^r\rangle = \langle \omega_1\rangle$. Thus $\langle \omega\rangle = \langle \omega_1, \omega_2\rangle$. Note, by construction, that any finite subset of X_n can be moved into $R_1 \setminus F$ by using an element of $\langle \omega_1, f_2, \ldots, f_n\rangle$. (First use powers of the elements $f_2, \ldots, f_n$ to send all points in $X_n \setminus F$ to $R_1 \setminus \mathrm{supp}(\omega_1)$, and then use ω_1 to send any points in F to $R_1 \setminus F$.) But then $\langle \omega_1, h_2, \ldots, h_n\rangle$ contains $\langle \omega_1, f_2, \ldots, f_n\rangle$, and so also has this property. Thus for any $\sigma \in \mathrm{FSym}(X_n)$, there exists $g \in \langle \omega_1, h_2, \ldots, h_n\rangle$ such that $\mathrm{supp}(g^{-1}\sigma g) \subset T \cap R_1$, i.e.,

$$\mathrm{Alt}(X_n) \leqslant \langle \omega_1, h_2, \ldots, h_n, \mathrm{Alt}(T)\rangle \text{ and } \mathrm{FSym}(X_n) \leqslant \langle \omega_1, h_2, \ldots, h_n, \mathrm{FSym}(T)\rangle.$$

Hence $\langle \omega, h_2, \ldots, h_n\rangle = \langle \omega_1, \omega_2, f_2, h_2, \ldots, h_n\rangle = U$, as required.

Acknowledgements. I thank Derek Holt for identifying and communicating the additional finite index subgroups, the editor for his help with submitting this note, and the anonymous referee for their helpful comments.

References

[1] Cox, C.G.: On the finite index subgroups of Houghton's groups. Arch. Math. (Basel) **118**, 113–121 (2022)

[2] Cox, C.G.: A note on the R_∞ property for groups $\mathrm{FAlt}(X) \leqslant G \leqslant \mathrm{Sym}(X)$. Comm. Algebra **47**(3), 978–989 (2019)

[3] Cox, C.G.: On the spread of infinite groups. Proc. Edinb. Math. Soc. (2) **65**, 214–228 (2022)

Charles Garnet Cox
School of Mathematics
University of Bristol
Bristol BS8 1UG
UK
e-mail: charles.cox@bristol.ac.uk

Received: 23 November 2022

Revised: 15 June 2023

Accepted: 15 August 2023