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p-groups and zeros of characters

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Abstract. Fix a prime p and an integer $n \geq 0$. Among the non-linear irreducible characters of the p-groups of order p^n, what is the minimum number of elements that take the value 0?

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1. Introduction. Dihedral, semidihedral, and generalized quaternion groups are ubiquitous in finite group theory. They have been characterized along the years in several ways: as the non-cyclic 2-groups whose number of involutions is 1 modulo 4 (Alperin–Feit–Thompson); as the non-abelian 2-groups whose commutator subgroup has index 4 (O. Taussky–Todd), as the 2-groups of maximal class; or, using fields of values of characters, as the 2-groups with exactly five rational-valued irreducible characters ([4]), for instance.

Using zeros of characters, the following is yet another one. It is somewhat remarkable that a finite group can be characterized by the number of zeros of a single irreducible character.

Theorem A. *Suppose that G is a 2-group of order 2^n. Let χ be a non-linear irreducible complex character of G. Then $\chi(g) = 0$ for at least $2^{n-1}+2$ elements $g \in G$. Furthermore, there exists $\chi \in \mathrm{Irr}(G)$ that vanishes at exactly $2^{n-1}+2$ elements if and only if G is dihedral, semidihedral, or generalized quaternion.*

The situation for general p-groups is more mysterious, and difficult. The following also includes the harder implication in Theorem A (when p is even).

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Theorem B. *Let G be a p-group of order p^n. Let χ be a non-linear irreducible character of G. Then $\chi(g) = 0$ for at least $p^n - p^{n-1} + p^2 - p$ elements $g \in G$. If equality holds, then G is a p-group of maximal class with an abelian maximal subgroup.*

The minimum number of elements taking the value zero among all the non-linear characters of the groups of order 5^5 is $2600 > 5^5 - 5^4 + 5^2 - 5 = 2520$. On the other hand, among groups of order 7^5, this number is exactly $14448 = 7^5 - 7^4 + 7^2 - 7$. This is related to some results in [6], and makes us suspect that an explicit minimum bound for the number of zeros among p-groups of order p^n might not be easy to discover. (See Corollary 2.8 below and the paragraph that follows it.)

The converse of Theorem B is not true, as shown for instance by `SmallGroup`$(5^5, 30)$, a 5-group with maximal class and an abelian maximal normal subgroup, although it is likely to be true if $p = 3$ (as we shall explain).

Our renewed interest on zeros of characters comes from a recent intriguing conjecture by A. Miller [5] that deserves attention. Using a non-trivial number theoretic result by Siegel, J.G. Thompson proved many years ago that at least 1/3 of the elements of a finite group take a zero or a root of unity value on every irreducible character of G (see Problem 2.15 of [3]). Now A. Miller [5] has conjectured that it should be at least 1/2 of the elements. Using number theory, Miller gives in [5] lower bounds for the number of zeros of characters for nilpotent groups, which are improved by our Theorem B.

At the time of this writing, unfortunately, we cannot contribute much to Miller's conjecture. The data seems to endorse it but a proof –even for solvable groups– seems elusive. (As a matter of fact, the same data suggest a much stronger statement: that outside any given normal subgroup, the proportion of elements that take zero or root unity values is again 1/2.)

As pointed out by Miller, the proportion of zero and root of unity values is exactly 1/2 in certain dihedral groups. Since these groups are supersolvable, it may be of interest to consider that case. We conclude this note with a proof of Miller's conjecture for a family of groups that includes supersolvable groups.

Theorem C. *Suppose that χ is an irreducible character of a finite group G. If G has a Sylow tower, then $\chi(g)$ is zero or a root of unity for at least $|G|/2$ elements of G.*

We notice that, unlike in the case of nilpotent groups, roots of unity are definitely necessary here. For instance, the non-linear characters of degree 2 of $\mathrm{SL}_2(3)$ vanish at exactly 6 elements.

2. *p*-groups. Our notation follows [2,3]. In this section, we prove Theorem B and, as a consequence, deduce Theorem A. We start with an elementary lemma.

Lemma 2.1. *Let G be a finite group and $\chi \in \mathrm{Irr}(G)$ faithful. Suppose that there exist $U \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(U)$ linear such that $\chi = \lambda^G$. Then U is abelian.*

Proof. Since χ is faithful, Lemma 5.11 of [3] implies that $1 = \bigcap_{g\in G} \ker \lambda^g$. Hence, U embeds into the direct product of the abelian groups $U/\ker \lambda^g$. The result follows. $\square$

The following lemma, due to G.A. Fernández–Alcober, is a simplification and strengthening of an earlier result of the authors.

Lemma 2.2. *Let G be a p-group with an abelian maximal subgroup U, and $|\mathbf{Z}(G)| = p$. Then G has maximal class.*

Proof. Since U has index p in G, U is normal in G. Now, using that $|\mathbf{Z}(G)| = p$, we deduce that $\mathbf{Z}(G) \subseteq U$. Since U is abelian and maximal in G, it follows that for every $g \in G-U$, $\mathbf{C}_U(g) = \mathbf{Z}(G)$. Therefore, $|\mathbf{C}_G(g)| = p^2$ and [2, Satz III.14.23] implies that G has maximal class. $\square$

The case of groups of class 2 of Theorem B follows easily from well-known results.

Lemma 2.3. *Let G be a p-group of order p^n and class 2. Then for any $\chi \in \mathrm{Irr}(G)$, $\chi(g) = 0$ for at least $p^n - p^{n-2}$ elements $g \in G$. In particular, χ vanishes at at least $p^n - p^{n-1} + p^2 - p$ elements and if equality holds, then $n = 3$.*

Proof. Let $Z/\ker\chi = \mathbf{Z}(\chi)/\ker\chi$. By Theorem 2.31 of [3], $p^2 \le \chi(1)^2 = |G:Z|$. Using Problem 6.3 of [3], we deduce that χ vanishes on $G-Z$. Since $|G-Z| \ge p^n - p^{n-2}$, the result follows. The second part is straightforward. $\square$

The following is a more detailed version of Theorem B.

Theorem 2.4. *Let G be a p-group of order p^n. If $\chi \in \mathrm{Irr}(G)$ is non-linear, then G vanishes on at least $p^n - p^{n-1} + p^2 - p$ elements of G. If equality holds, then:*

(i) *χ is faithful and $\chi(1) = p$.*
(ii) *G is a p-group of maximal class with an abelian maximal subgroup U.*
(iii) *If $n > 3$, then U is the unique maximal subgroup of G with a character that induces χ and the set of zeros of χ is $(G-U) \cup (\mathbf{Z}_2(G) - \mathbf{Z}(G))$.*

Proof. If $n = 3$, then G is an extraspecial p-group and the result is well-known. We assume in the remaining that $n > 3$.

We prove the first part by induction on n. By Lemma 2.3, we may assume that G does not have class 2. Since χ is monomial, there exists U maximal in G such that χ is induced from U. Suppose first that there exists $V \ne U$ maximal in G such that χ is also induced from V. Then χ vanishes on $(G-U)\cup(G-V) = G-(U\cap V)$. There are $p^n - p^{n-2}$ elements in this set, and this number exceeds $p^n - p^{n-1} + p^2 - p$. Hence, we will assume in the remaining that U is the unique maximal subgroup of G with a character that induces χ. Let $\theta \in \mathrm{Irr}(U)$ be such that $\theta^G = \chi$. Since G is not cyclic, let V be another maximal subgroup of G. Set $W = U\cap V$. Then, using Corollary 6.19 of [3], we have that $\chi_V \in \mathrm{Irr}(V)$ and by Mackey (Problem 5.2 of [3]), $\chi_V = (\theta_W)^V$. By the inductive hypothesis, χ_V vanishes on at least $p^{n-1} - p^{n-2} + p^2 - p$ elements. Since χ_V is induced from θ_W, χ_V vanishes on the $p^{n-1} - p^{n-2}$ elements of $V - W$. Therefore, χ_V

vanishes at least on $p^2 - p$ elements that belong to W. Since χ vanishes on $G - U$ and at these $p^2 - p$ elements in W, the first part of the result follows.

Assume now and for the rest of the proof that equality holds. First, we prove that χ is faithful. Let $K = \ker \chi$. Put $|K| = p^m$. Let $\overline{\chi}$ be the character χ viewed as a character of G/K. For any element xK that is a zero of $\overline{\chi}$, χ vanishes on the coset xK. By the first part, $\overline{\chi}$ vanishes on at least $p^{n-m} + p^{n-m+1} + p^2 - p$ elements. Hence, the number of zeros of χ is at least $p^m(p^{n-m} + p^{n-m+1} + p^2 - p)$. Since the number of zeros of χ is $p^n - p^{n-1} + p^2 - p$, this forces $m = 0$. This proves that χ is faithful.

Next, we see that χ vanishes on $\mathbf{Z}_2(G) - \mathbf{Z}(G)$. Let $x \in \mathbf{Z}_2(G)$ and $g \in G$ be such that $[x, g] \neq 1$. Let $\lambda \in \mathrm{Irr}(\mathbf{Z}(G))$ be lying under χ. Note that λ is faithful. Hence

$$\chi(x) = \chi(x^g) = \chi(x[x, g]) = \chi(x)\lambda([x, g]),$$

which implies that $\chi(x) = 0$, as wanted.

Now, we claim that $\mathbf{Z}_2(G) \leq U$. By Theorem 6.22 of [3], χ is an M-character over $\mathbf{Z}_2(G)$. This means that there exists $\mathbf{Z}_2(G) \subseteq H \subseteq G$ and $\psi \in \mathrm{Irr}(H)$ such that $\psi^G = \chi$ and $\psi_{\mathbf{Z}_2(G)}$ is irreducible. If $H < G$, by uniqueness of U, we have that $H \subseteq U$, and the claim is proven. Thus we may assume that $H = G$ and that $\tau = \chi_{\mathbf{Z}_2(G)} \in \mathrm{Irr}(\mathbf{Z}_2(G))$. Since $\chi(1) > 1$, we have that $\mathbf{Z}_2(G)$ is not abelian. Assume by contradiction that $\mathbf{Z}_2(G) \not\leq U$, so that $G = \mathbf{Z}_2(G)U$. Suppose first that $|\mathbf{Z}_2(G)| = p^t > p^3$. Since $\mathbf{Z}_2(G)$ has class 2, we deduce that τ has at least $p^t - p^{t-2}$ zeros by Lemma 2.3. Since by Mackey $(\theta_{U \cap \mathbf{Z}_2(G)})^{\mathbf{Z}_2(G)} = \tau$, we have that τ is zero on the $p^t - p^{t-1}$ elements of $\mathbf{Z}_2(G) - (U \cap \mathbf{Z}_2(G))$. Hence, there are at least $p^{t-1} - p^{t-2} > p^2 - p$ zeros of τ in $U \cap \mathbf{Z}_2(G)$. Since these are zeros of χ, we conclude that χ has at least $p^n - p^{n-1} + p^{t-1} - p^{t-2}$ zeros, which is a contradiction. Now, we may assume that $|\mathbf{Z}_2(G)| = p^3$. Therefore, $\chi(1) = \tau(1) = p$. Since χ is faithful and induced from U, we conclude from Lemma 2.1 that U is abelian. Now, $[G', \mathbf{Z}_2(G)] = 1$ (see [2, Hauptsatz III.2.11]) and since G' is contained in the abelian group U, it follows that G' is central in G, so G has class 2. This contradicts Lemma 2.3, proving the claim.

We have thus seen that the set of zeros of χ is $(G - U) \cup (\mathbf{Z}_2(G) - \mathbf{Z}(G))$, where the union is disjoint. Therefore $|\mathbf{Z}_2(G) - \mathbf{Z}(G)| = p^2 - p$, and we deduce that $|\mathbf{Z}_2(G)| = p^2$ and $|\mathbf{Z}(G)| = p$.

Next, we claim that $\chi(1) = p$. Suppose that $\chi(1) > p$. Since, again, χ is an M-character over $\mathbf{Z}_2(G)$, there exists $\mathbf{Z}_2(G) \leq H < U$ such that χ is induced from H. In particular, χ is zero on $G - \bigcup_{g \in G} H^g$. Since $\bigcup_{g \in G} H^g \subsetneq U$ (by Lemma 3.1 of [6], for instance), this implies that χ has zeros in $U - \mathbf{Z}_2(G)$, a contradiction. This proves the claim.

As a consequence, we obtain that $\theta \in \mathrm{Irr}(U)$, the character that induces χ, is linear. Since χ is faithful, Lemma 2.1 implies that U is abelian. Now, Lemma 2.2 implies that G has maximal class, as wanted. This completes the proof. $\square$

The proof of Theorem A now follows easily.

Theorem 2.5. *Suppose that G is a 2-group of order 2^n. Let χ be an irreducible non-linear complex character of G. Then $\chi(g) = 0$ for at least $2^{n-1}+2$ elements $g \in G$. Furthermore, there exists $\chi \in \mathrm{Irr}(G)$ that vanishes at exactly $2^{n-1} + 2$ elements if and only if G is dihedral, semidihedral, or generalized quaternion.*

Proof. By Theorem B, we only have to prove that if G is dihedral, semidihedral, or generalized quaternion and $\chi \in \mathrm{Irr}(G)$ is faithful, then χ vanishes on exactly $2^{n-1}+2$ elements of G. But this is easy. Let U be the abelian maximal subgroup of G, and let $g \in G$ be such that $G = \langle g, U\rangle$ with $x^g = x^i$, where $i = -1$ if G is dihedral or quaternion and $i = 2^{n-2} - 1$ if G is semidihedral. We have that $\chi = \lambda^G$ where $\lambda \in \mathrm{Irr}(U)$ is faithful and $|G : U| = 2$. Now, for any $y \in U$, $\lambda(y) = \varepsilon$ is a primitive $o(y)$-th root of unity, and $\lambda(x) + \lambda^g(x) = \varepsilon + \varepsilon^{-i} = 0$ if and only if $o(x) = 4$. □

We expect the following to hold for $p = 3$.

Conjecture 2.6. *Let G be a 3-group of order 3^n. Then G has an irreducible character that vanishes at exactly $3^n - 3^{n-1} + 6$ elements if and only if G is a 3-group of maximal class with an abelian maximal subgroup.*

Note that the "only if" part follows from Theorem B. We recall that the 3-groups of maximal class (as well as the p-groups of maximal class with an abelian maximal subgroup for any prime p) were classified by Blackburn [1]. However, it does not seem easy to prove that they possess an irreducible character that vanishes at exactly $3^n - 3^{n-1} + 6$ elements. Eamonn O'Brien has checked that this is true for groups of order at most 3^{10}.

As we have mentioned, the converse of Theorem B does not hold for $p > 3$. This situation is related to [6]. In [6], it was proved that the number of conjugacy classes of zeros of any non-linear irreducible character of a p-group is at least $p^2 - 1$ (see Theorem C of [6]). Furthermore, if equality holds and the character is faithful, then G is a p-group of maximal class with an abelian maximal subgroup U and the set of zeros of character is $(G - U) \cup (\mathbf{Z}_2(G) - \mathbf{Z}(G))$ (see the proof of Theorem C of [6] and the paragraph that follows it). Now, we make clear the relation between both problems. Note that this relation is only transparent after proving Theorem 2.4.

Theorem 2.7. *Let G be a non-abelian p-group of order p^n and $\chi \in \mathrm{Irr}(G)$ faithful. Then χ vanishes at exactly $p^n - p^{n-1} + p^2 - p$ elements if and only if χ vanishes at exactly $p^2 - 1$ conjugacy classes.*

Proof. This is clear if $n = 3$ so we may assume that $n > 3$.

Suppose first that χ vanishes at exactly $p^2 - 1$ conjugacy classes. As we have just mentioned, then G is a p-group of maximal class with an abelian maximal subgroup U and the set of zeros of χ is $(G - U) \cup (\mathbf{Z}_2(G) - \mathbf{Z}(G))$. Since the cardinality of this set is $p^n - p^{n-1} + p^2 - p$, the result follows.

Conversely, assume that χ vanishes at exactly $p^n - p^{n-1} + p^2 - p$ elements. By Theorem 2.4, G is a p-group of maximal class with an abelian maximal subgroup U and the set of zeros of the character is $(G - U) \cup (\mathbf{Z}_2(G) - \mathbf{Z}(G))$. Let $g \in G - U$, so that $G = \langle g\rangle U$. Since $|\mathbf{Z}(G)| = p$, $\mathbf{C}_U(g) = \mathbf{Z}(G)$, so

$|\mathbf{C}_G(g)| = p^2$. In other words, the conjugacy classes in $G - U$ have size p^{n-2}. Therefore, the number of conjugacy classes of G contained in $G - U$ is $(p^n - p^{n-1})/p^{n-2} = p^2 - p$. Since $|\mathbf{Z}_2(G)| = p^2$, the conjugacy classes in $\mathbf{Z}_2(G) - \mathbf{Z}(G)$ have size p. Hence, the number of conjugacy classes of G contained in this subset is $p - 1$. It follows that χ vanishes at exactly $p^2 - 1$ conjugacy classes, as wanted. □

Now, we can use Theorem D of [6] to see that if $p > 3$ and equality holds in Theorem B, then $|G|$ is bounded in terms of p.

Corollary 2.8. *Let G be a p-group of order p^n, where $p > 3$. If G has an irreducible character χ that vanishes at exactly $p^n - p^{n-1} + p^2 - p$ elements, then $|G| \leq p^{r+1}$, where r is the smallest prime that does not divide $p - 1$.*

Proof. By Theorem 2.4, we know that χ is faithful. Now, by Theorem 2.7, χ vanishes at exactly $p^2 - 1$ conjugacy classes and the result follows from Theorem D of [6]. □

Let us summarize. If G is a non-abelian group, and $\mathrm{mz}(G)$ is the minimum number of elements of G taking the zero value among the non-linear irreducible characters of G, we let

$$\mathrm{mz}(p^n) = \min\{\mathrm{mz}(G) \mid |G| = p^n\}.$$

We have shown in Theorem B that $\mathrm{mz}(p^n) \leq p^n - p^{n-1} + p^2 - p$, and in Theorem A that equality holds if $p = 2$. (We suspect that the same holds if $p = 3$.) Also the proof of Theorem B and computer calculations performed by O'Brien suggest that the following could be true.

Conjecture 2.9. *Let G be a p-group of order p^n. Then $\mathrm{mz}(p^n) = \mathrm{mz}(G)$ if and only if G has maximal class with an abelian maximal normal subgroup.*

3. Groups with a Sylow tower. We conclude with the proof of Theorem C, which we restate. Our interest now also includes roots of unity values of characters.

Theorem 3.1. *Let G be a group with a Sylow tower and let $\chi \in \mathrm{Irr}(G)$. Then the proportion of elements $x \in G$ such that $\chi(x) = 0$ or $\chi(x)$ is a root of unity is at least $1/2$.*

Proof. We argue by induction on $|G|$. There exists a prime p that divides $|G|$ and G has a normal Hall p'-subgroup N. Let $P \in \mathrm{Syl}_p(G)$, so that $G = PN$. Since G/N is a p-group, it follows from Theorem 6.22 of [3] that χ is a relative M-character with respect to N. Thus there exists $N \leq H \leq G$ and $\psi \in \mathrm{Irr}(H)$ such that $\chi = \psi^G$ and $\psi_N \in \mathrm{Irr}(N)$. Suppose first that $H < G$. Since G/N is a p-group, every maximal subgroup U of G that contains N is normal in G. Since χ is induced by U, it follows that χ vanishes on $G - U$. There are at least $|G|/2$ elements in this set. Thus the theorem holds in this case.

Now, we may assume that $H = G$. In other words, $\theta = \chi_N \in \mathrm{Irr}(N)$. Let $\hat{\theta}$ be the canonical extension of θ to G. We claim that the proportion of zeros and root of unity values of $\hat{\theta}$ exceeds 1/2. Let G_p be the set of p-elements of G. Therefore

$$G = \bigcup_{x \in G_p} \mathbf{C}_N(x)x$$

is a disjoint union by Lemma 8.18 of [3]. Now, if $1 \neq x \in G_p$, $c \in \mathbf{C}_N(x)$, and $\theta^* \in \mathrm{Irr}(\mathbf{C}_N(x))$ is the x-Glauberman correspondent of θ, we have by Theorem 13.32 of [3] that

$$\hat{\theta}(cx) = \epsilon\theta^*(c)\,,$$

where ϵ is a sign. Since G has a Sylow tower, we have that $\mathbf{C}_N(x)$ has a Sylow tower. Let A_x be the set of elements of $\mathbf{C}_N(x)$ where θ^* has the value zero or a root of unity. By induction, we have that

$$|A_x| \geq |\mathbf{C}_N(x)|/2$$

for every $x \in G_p$. If $y \in A_x$, then $\hat{\theta}(yx)$ is a zero or a root of unity, and therefore, $\hat{\theta}$ has at least

$$\sum_{x \in G_p} |A_x| \geq |G|/2$$

roots of unity or zero values.

Now, by Gallagher's Corollary 6.17 of [3], we have that $\chi = \mu\hat{\theta}$, where $\mu \in \mathrm{Irr}(G/N) = \mathrm{Irr}(P)$. If μ is not linear, then the result follows from the p-group case. If μ is linear, then $|\chi(x)| = |\hat{\theta}(x)|$ and the result follows from Problem 3.2 of [3] and the previous paragraph. □

It is easy to build examples of non-solvable groups with irreducible characters that either vanish or take root of unity values at exactly one-half of its elements. Consider for instance $G = S \wr D_{10}$, where S is any simple group. However, if Miller's conjecture is true, then it seems reasonable to expect that if equality holds and $\chi \in \mathrm{Irr}(G)$ is a character that either vanishes or takes a root of unity value at one-half of the elements of G, then χ is monomial of degree 2 and $G/\ker\chi$ is supersolvable.

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