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$L^s(L^q)$-estimates of the pressure - The proof of Sohr-von Wahl and its impact on mathematical fluid dynamics

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To my colleagues Hermann Sohr and Wolf von Wahl.

Abstract. The article *On the regularity of the pressure of weak solutions of Navier–Stokes equations* by H. Sohr and W. von Wahl (1986) is one of the most-cited papers of the journal *Archiv der Mathematik*. Our aim is to describe not only the content of the paper, but especially the novelty of its results on maximal regularity of the Stokes operator considered from a point of view of the eighties and from a modern point of view.

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1. Introduction. The article under review deals with the regularity of the hydrodynamic pressure π in the Navier–Stokes equations

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = f, \quad \text{div } u = 0 \tag{1.1}$$

with initial value u_0 for the velocity field $u = u(t, x)$ at time $t = 0$ and with vanishing Dirichlet boundary values $u|_{\partial\Omega} = 0$ in a smooth bounded or exterior domain $\Omega \subset \mathbb{R}^n$. For simplicity, in (1.1), the coefficient of viscosity is put equal to 1 and the constant density is assumed to be $\rho = 1$. This non-linear system of partial differential equations is the most important model describing the flow of a viscous incompressible fluid.

In the weak formulation of the stationary Stokes equation

$$-\Delta u + \nabla \pi = f, \quad \text{div } u = 0, \;\; u|_{\partial\Omega} = 0, \tag{1.2}$$

when working with solenoidal test vector fields, the pressure plays a special role, namely that of a Lagrange multiplier. Here u is the solution of the minimization problem

$$\min_{v\in H_0^1(\Omega)} \frac{1}{2}\|\nabla v\|_{L^2}^2 - \int_\Omega f\cdot v\,\mathrm{d}x$$

under the constraint div $v=0$, and its existence and uniqueness is obtained easily by the lemma of Lax-Milgram. However, it is non-trivial to prove the existence of a pressure function. To that aim, e.g., the closed range theorem and an estimate of the type $\|\pi\|_{L^2}\le c\|\nabla\pi\|_{H^{-1}}$ for functions $\pi\in L^2$ with vanishing integral mean on a bounded domain is needed.

The difficulties for the pressure π are even more involved in the unsteady case. Starting with the instationary Stokes system

$$\partial_t u-\Delta u+\nabla\pi=f,\quad \text{div } u=0,\ \ u(0)=u_0,\ \ u|_{\partial\Omega}=0, \tag{1.3}$$

we see that no restrictions are imposed on the time derivative of the pressure being unique only up to an arbitrary, even non-measurable function $\tilde\pi(t)$ depending only on time t. This drawback can be avoided by assuming $\int_\Omega \pi\,\mathrm{d}x=0$ in the bounded domain case. Moreover, there is no boundary value condition available for π. Now assume that a suitable velocity field u satisfying (1.3) has been found. Applying the divergence operator to (1.3), we obtain the equation $\Delta\pi=\text{div } f$ together with the formal Neumann boundary condition $\frac{\partial\pi}{\partial N}=\Delta u\cdot N+f\cdot N$; here $N=N(x)$ denotes the outer normal at $x\in\partial\Omega$. Even in the case $f\equiv 0$ where π is harmonic, we conclude that due to the formal boundary condition, π depends non-locally on the solution u. This becomes even more transparent in the non-linear case (1.1) when including the transport term $u\cdot\nabla u$ where

$$\Delta\pi=-\text{div }(u\cdot\nabla u)+\text{div } f. \tag{1.4}$$

In the seminal paper [4] on partial regularity and the Hausdorff dimension of the set of singular points in time-space, the authors L. Caffarelli, R. Kohn, and L. Nirenberg mention in view of the non-optimal result $\pi\in L^{5/4}(0,T;L^{5/3}(\Omega))$ for domains Ω that

> ... when $\Omega=\mathbb{R}^3$, the pressure satisfies $p\in L^{5/3}(\mathbb{R}^3\times(0,T))$. It seems reasonable to conjecture the analogous estimate, at least locally, for bounded Ω; but this is apparently open.

The authors of [24] write that

> ... this property would have some important consequences for the partial regularity theory of weak solutions of (1.1). It is the aim of the present paper to prove this conjecture for a bounded domain Ω. However, combining the method of the proof with the method given in [10] (the joint paper [23] of the authors), *the result follows for an exterior domain too.*

Beyond this important pressure estimate, we emphasize that

> the article [24] is the first to prove the full $L^s(L^q)$-maximal regularity estimate of the Stokes operator in bounded and exterior domains.

These are the reasons why this paper has been cited more than 60 times with an increasing rate since the year 2000 and still in the last 5 years.

In the following, we describe ideas to construct the pressure and to understand the importance of the special exponents $\frac{5}{3}$ and $\frac{5}{4}$ in view of weak solutions and scale invariance.

One possibility to overcome the problems to find and estimate the pressure is to rewrite $\nabla\pi$ in (1.3) and in (1.1) in terms of u by a formal solution operator based on (1.4) to get a non-linear, non-local system in u only and to use pseudodifferential operator theory, see e.g. [10,11] for the proof of resolvent estimates and [14] for the treatment of (1.1) with different kinds of boundary conditions. Another method needs the Helmholtz projection $P = P_2$ which - for a smooth bounded domain $\Omega \subset \mathbb{R}^n$ - yields the orthogonal decomposition $L^2(\Omega)^n = L^2_\sigma(\Omega) + G_2(\Omega)$ where $L^2_\sigma(\Omega)$ is the subspace of solenoidal vector fields u satisfying $u \cdot N = 0$ on $\partial\Omega$ and $G_2(\Omega)$ is the space of gradient fields in L^2. Since P vanishes on gradient fields, an application of P to (1.1) yields the system

$$\partial_t u - P\Delta u + P(u\cdot\nabla u) = Pf, \qquad u(0) = u_0, \tag{1.5}$$

considered in the space of solenoidal vector fields with vanishing Dirichlet boundary values. Generalizing the Helmholtz decomposition to L^p spaces, $1 < p < \infty$, we define the operator $A = A_p = -P_p\Delta$ with domain $\mathcal{D}(A_p) = W^{2,p}(\Omega)^n \cap W^{1,p}_0(\Omega)^n \cap L^p_\sigma(\Omega)$, called the Stokes operator. Then, with the aid of the Stokes semigroup $\{e^{-tA};\, t \geq 0\}$, a bounded analytic semigroup, (1.5) can be rewritten as the non-linear integral equation

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\big(Pf(\tau) - P(u\cdot\nabla u)(\tau)\big)\,\mathrm{d}\tau. \tag{1.6}$$

Problem (1.6) can be solved in suitable function spaces with help of well-known tools from analytic semigroup theory and properties of the Stokes operator.

Evidently, properties of the associated pressure function π are interesting as an important physical quantity. Moreover, integrability properties of π are crucially needed in [4] on partial regularity of so-called suitable weak solutions of the Navier-Stokes system, see Definition 1.1 (ii) below. The discussion of partial regularity started with fundamental papers of V. Scheffer, see [20,21], looking for points (t_0, x_0) in time-space such that a given weak solution u is (essentially) bounded in a neighborhood of (t_0, x_0), and to characterize the set $\mathcal{S} \subset \mathbb{R}^4$ of all singular points where u is not locally bounded. In order to get local properties of solutions, cut-off functions $\varphi \in C^\infty_c((0,T)\times\Omega)$ must be used, leading to terms such as $u\varphi$ where $\operatorname{div}(u\varphi) = u\cdot\nabla\varphi \neq 0$. In that case, it is indispensable to work with the associated pressure π which had been eliminated either by the Helmholtz projection or, in a weak formulation, by working with solenoidal test functions.

To describe this topic more precisely, we recall several definitions. For simplicity, we refer to the three-dimensional case only and omit the superscript $n = 3$ for spaces of vector fields. Let $(u, v) = \int_\Omega u\cdot v\,\mathrm{d}x$ provided $u\cdot v \in L^1(\Omega)$.

Moreover, consider locally integrable external forces rather than elements in negative Sobolev spaces.

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a domain and $T > 0$. Moreover, let $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^s(0,T;L^p(\Omega))$ where $s,p \in (1,\infty)$.

(i) A *weak solution* of the Navier–Stokes system (1.1) is a vector field $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ satisfying the variational formulation

$$-\int_0^T (u,v')\,\mathrm{d}\tau + \int_0^T (\nabla u,\nabla v)\,\mathrm{d}\tau - \int_0^T (u\otimes u,\nabla v)\,\mathrm{d}\tau = (u_0,v(0)) + \int_0^T (f,v)\,\mathrm{d}\tau$$

for all test functions $v = wh$, $w \in C^\infty_c(\overline{\Omega})$, div $w = 0$, $w|_{\partial\Omega} = 0$, $h \in C^1([0,T])$, $h(T) = 0$. If u satisfies the *energy inequality*

$$\frac{1}{2}\|u(t)\|_{L^2} + \int_0^t \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{1}{2}\|u_0\|^2_{L^2} + \int_0^t (f,u)\,\mathrm{d}\tau \tag{1.7}$$

for all $0 < t < T$, it is called a *Leray-Hopf weak solution*.

(ii) A weak solution u together with an associated pressure π is called a *suitable weak solution* if it satisfies the *localized energy inequality*

$$\begin{aligned}&\int_0^T \int_\Omega |\nabla u|^2\varphi\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad\le \int_0^T \int_\Omega \Big(\frac{1}{2}|u|^2(\partial_t\varphi + \Delta\varphi) + \big(\frac{1}{2}|u|^2 + \pi\big)u\cdot\nabla\varphi + u\cdot f\varphi\Big)\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{1.8}$$

for all non-negative test functions $\varphi \in C^\infty_c((0,T)\times\Omega)$.

By classical results of J. Leray [16] and E. Hopf [15] for any initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f \in L^1(0,T;L^2(\Omega))$, there exists a weak Leray-Hopf solution. A modern and comprehensive monograph on weak and strong solutions to the stationary as well as instationary both Stokes and Navier–Stokes equations was published by H. Sohr ([22]) in 2001. For a more recent survey by the author of this review, we refer to [6].

Note that all terms in (1.7) are well defined for a weak solution. However, the cubic term $|u|^3$ and also $|u|\pi$ in (1.8) require additional integrability conditions to be discussed below. Generally, an associated pressure is found only in the sense of distributions as a time derivative of a locally integrable function; for further details, we refer to Remark 3.1.

An important term is the scale invariance of norms for a solution (u,π). For $\lambda > 0$, let

$$u_\lambda(x,t) = \lambda u(\lambda^2 t,\lambda x), \quad \pi_\lambda(x,t) = \lambda^2\pi(\lambda^2 t,\lambda x).$$

It is easy to see that with (u,π) also (u_λ,π_λ) is a solution of (1.1) on $\Omega = \mathbb{R}^3$ (with modified f) and that

$$\|u_\lambda\|_{L^s(0,T/\lambda^2;L^q(\mathbb{R}^3))} = \|u\|_{L^s(0,T;L^q(\mathbb{R}^3))} \quad \text{if and only if } \frac{2}{s} + \frac{3}{q} = 1.$$

It is the so-called *Serrin condition* $\frac{2}{s} + \frac{3}{q} = 1$ which guarantees uniqueness and regularity of weak solutions ([22]). However, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and Hölder's inequality imply that a weak solution u satisfies $u \in L^s(0,T;L^q(\Omega))$ for all pairs s,q provided that $\frac{2}{s} + \frac{3}{q} = \frac{3}{2}$ where $s \le 2 \le \infty$, $2 \le q \le 6$. Moreover, $\nabla u \in L^s(0,T;L^{q_*}(\Omega))$ with $\frac{2}{s} + \frac{3}{q_*} = \frac{5}{2}$ and, by Hölder's inequality, $u \cdot \nabla u \in L^s(0,T;L^p(\Omega))$ with $\frac{2}{s} + \frac{3}{p} = 4$. In particular, $u \cdot \nabla u \in L^{5/4}(0,T;L^{5/4}(\Omega))$.

Concerning (1.8), the condition $u \in L^3(L^3)$ is satisfied locally in space-time since $\frac{2}{3} + \frac{3}{3} = \frac{5}{3} > \frac{3}{2}$. However, for π no kind of integrability is guaranteed a priori so that an assumption to justify (1.8) and to allow for the analysis of partial regularity in [4] must be posed. It is this point where the article by H. Sohr and W. von Wahl comes into play. The following ideas are explained in [4] and [24].

The optimal integrability of a weak solution u with $s = p$ is given by the condition $s = p = \frac{10}{3}$. Concerning the pressure π, consider (1.4) with $f = 0$ and the case when $\Omega = \mathbb{R}^3$. Since div $u = 0$, we have, using Einstein's summation convention, $\operatorname{div}(u \cdot \nabla u) = \partial_{x_j}\partial_{x_k}(u_j u_k)$ so that

$$\pi = (-\Delta)^{-1}\partial_{x_j}\partial_{x_k}(u_j u_k) = \mathcal{R}_j \mathcal{R}_k (u_j u_k). \tag{1.9}$$

Here $\mathcal{R}_j$ denotes the Riesz transform with symbol $\frac{-i\xi_j}{|\xi|}$. Hence the theory of Calderón–Zygmund implies that $\|\pi\|_{L^{p/2}(\mathbb{R}^3)} \le c\|u \otimes u\|_{L^{p/2}(\mathbb{R}^3)} \le c\|u\|^2_{L^p(\mathbb{R}^3)}$. After an integration with respect to time, we obtain that

$$\|\pi\|_{L^{p/2}(0,T;L^{p/2}(\mathbb{R}^3))} \le c\|u\|^2_{L^p(0,T;L^p(\mathbb{R}^3))}, \quad p = \frac{10}{3}, \tag{1.10}$$

i.e., $\pi \in L^{5/3}((0,T) \times \Omega)$.

The representation (1.9) of π by Riesz operators applied to $u \otimes u$ is not available for domains Ω due to the lack of boundary values of π. However, the instationary Stokes system

$$\partial u_t - \Delta u + \nabla \pi = f - u \cdot \nabla u, \quad \text{div } u = 0, \ u(0) = u_0, \ u|_{\partial\Omega = 0}, \tag{1.11}$$

admits for u_0 in a suitable space of initial values $\mathcal{I}$ estimates of the type

$$\begin{aligned} &\|\nabla^2 u\|_{L^s(0,T;L^p(\Omega))} + \|\nabla \pi\|_{L^s(0,T;L^p(\Omega))} \\ &\quad \le c\big(\|f\|_{L^s(0,T;L^p(\Omega))} + \|u \cdot \nabla u\|_{L^s(0,T;L^p(\Omega))} + \|u_0\|_{\mathcal{I}}\big). \end{aligned} \tag{1.12}$$

This so-called *maximal regularity estimate* in $L^s(0,T;L^p(\Omega))$ with $s = p$ for domains $\Omega \subset \mathbb{R}^3$ is due to V.A. Solonnikov [25]; his proof is based on an explicit representation of solutions in the half space, potential estimates, and a cut-off procedure for more general domains. As discussed above, $s = p = \frac{5}{4}$ is admissible. Hence, $\nabla \pi \in L^{5/4}((0,T) \times \Omega)$ implies by a Sobolev embedding that $\pi \in L^{5/4}(0,T;L^{5/3}(\Omega))$ and even $\pi \in L^{5/4}(0,T;L^{15/7}(\Omega))$. However, as mentioned in [24], an improvement of the integration exponent in time from $\frac{5}{4}$ to $\frac{5}{3}$ is not possible. Locally in space, the same result was obtained in [4, p. 782] by more potential theoretic arguments. Note that the local conditions $\pi \in L^{5/4}(L^{5/3})$ and $u \in L^5(L^{5/2})$ are sufficient to guarantee that $\pi u \cdot \nabla \phi$ in (1.8) is locally integrable.

2. The article by H. Sohr and W. von Wahl. In the following, we describe the main results and ideas of the proof of Sohr and von Wahl in [24]. For simplicity, we focus on bounded domains in $\mathbb{R}^3$ for which the Stokes operator $A_p = -P_p\Delta$, $1 < p < \infty$, is boundedly invertible, in other words, 0 lies in the resolvent set $\rho(A_p)$ of A_p, and the analytic semigroup e^{-tA_p} is exponentially decaying as $t \to \infty$. For an exterior domain, the Stokes operator is not surjective from $\mathcal{D}(A_p)$ to $L^p_\sigma(\Omega)$ and hence 0 lies in the continuous spectrum. Since the problem is a local one in time, the authors replace for an exterior domain A_p by $A_p + I$ so that the instationary system is replaced by a related one with the additional factor e^{-t} multiplied to f and u. Recall that the Stokes operator A_p as well as the Dirichlet Laplacian $-\Delta_p$ with domain $\mathcal{D}(-\Delta_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ are sectorial operators and admit fractional powers A^α_p and $(-\Delta_p)^\alpha$, respectively, for any $\alpha \in \mathbb{R}$; some important properties will be discussed below.

The first main results in [24, Theorems 2.2, 2.7, and 2.12] concern the $L^s(L^q)$-estimate of regular solutions (u, π) to the linear instationary Stokes problem. Note that the L^p-norm is denoted by $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for functions as as well as for vector fields etc. and that $L^p = L^p(\Omega)$, $L^s(L^p) = L^s(0,T;L^p(\Omega))$ provided the domain Ω and the interval $(0,T)$ are known from the context.

For simplicity, the authors assume that the domain has a smooth boundary. However, the sectoriality of A_p and the $L^p(L^p)$-estimates (2.2) proved in [25] hold even when $\partial\Omega \in C^2$. In general, the assumption $\partial\Omega \in C^{2,\varepsilon}$, $\varepsilon \in (0,1)$, is sufficient to get the main results of Theorems 2.1 and 2.2 as long as the property *BIP*, see Remark 3.2, is not exploited.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, let $1 < p, s < \infty$, $f \in L^s(0,T;L^p(\Omega))$, and $u_0 \in \mathcal{D}\big(A_p^{1-1/s+\varepsilon}\big)$ for some $\varepsilon > 0$ with $1 - \frac{1}{s} + \varepsilon < 1$.*

(i) *The instationary Stokes equation*

$$u_t + A_p u = P_p f, \quad u(0) = u_0, \tag{2.1}$$

admits a unique solution $u \in L^s(0,T;\mathcal{D}(A_p))$ such that $u_t \in L^s(0,T;L^p_\sigma(\Omega))$. Moreover,

$$\int_0^T \big(\|u_t\|_p^s + \|A_p u\|_p^s\big)\,\mathrm{d}\tau \le c\Big(\|A_p^{1-1/s+\varepsilon}u_0\|_p^s + \int_0^T \|P_p f\|_p^s\,\mathrm{d}\tau\Big) \tag{2.2}$$

with a constant $c = c(s,p,T,\Omega) > 0$ independent of f, u_0.

(ii) *The instationary Stokes system*

$$u_t - \Delta u + \nabla\pi = f, \quad \operatorname{div} u = 0, \;\; u(0) = u_0, \tag{2.3}$$

possesses a unique solution $u \in L^s(0,T;W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega))$ such that $u_t \in L^s(0,T;L^p(\Omega))$ and a pressure π such that $\nabla\pi \in L^s(0,T;L^p(\Omega))$.

Moreover,

$$\int_0^T \|u_t\|_p^s \,\mathrm{d}\tau + \int_0^T \|u\|_{W^{2,p}}^s \,\mathrm{d}\tau + \int_0^T \|\nabla\pi\|_p^s \,\mathrm{d}\tau \le c\left(\|(-\Delta_p)^{1-1/s+\varepsilon}u_0\|_p^s + \int_0^T \|f\|_p^s \,\mathrm{d}\tau\right) \tag{2.4}$$

with a constant $c = c(s, p, T, \Omega) > 0$ independent of f, u_0.

Proof. (i) The authors refer to [25] in the case $s = p$. The crucial step is from $s = p$ to arbitrary exponents $1 < s, p < \infty$. This step is based on the interpolation theorem of Marcinkiewicz, applied to abstract Banach space-valued convolution operators and proven in a classical paper of Benedek, Calderón, and Panzone [3] from 1962. The following estimates are adapted to the situation in [3, Theorem 1] and carried through in detail in [28, Theorem III.1] where parabolic initial-boundary value problems and even t-dependent operators $A(t)$ are discussed. Since the main part of the proof of [28, Theorem III.1] works on an abstract operator level, the results can be applied to the Stokes operator as well.

First we consider the case when $u_0 = 0$, assume for simplicity that $P_p f = f$, and aim at an estimate of $AU(t)$ where

$$U(t) = \int_0^t e^{-(t-\tau)A} f(\tau)\,\mathrm{d}\tau, \quad 0 < t < T.$$

Extending f and $e^{-(\cdot)A}$ by 0 to $\mathbb{R}$, we get $U(t) = e^{-(\cdot)A} * f(t)$. To estimate $AU(t)$, note that the formal expression $AU(t) = A\int_{\mathbb{R}} e^{-(t-\tau)A} f(\tau)\,\mathrm{d}\tau = \int_{\mathbb{R}} Ae^{-(t-\tau)A} f(\tau)\,\mathrm{d}\tau$ is not well-defined since $Ae^{-(t-\tau)A}$ yields a singular integral kernel. Therefore, $AU(t)$ will be replaced by the term

$$AU_\varepsilon(t) = Ae^{-\varepsilon A}\int_{\mathbb{R}} e^{-(t-\tau)A} f(\tau)\,\mathrm{d}\tau = k_\varepsilon * f(t), \quad \varepsilon > 0,$$

with the approximate non-singular Banach space-valued integral kernel $k_\varepsilon(t) = Ae^{-\varepsilon A}e^{-tA}$ for $t \ge 0$ and $k_\varepsilon(t) = 0$ elsewhere.

The first assumption in [3, Theorem 1] is the weak-type (p, p) estimate of the operator $f \mapsto AU_\varepsilon$, i.e., of the convolution operator with kernel k_ε on $L^p(\mathbb{R}; L^p(\Omega))$, which follows from the strong $L^p(L^p)$-estimate in [25] as above.

As for the assumption (ii) in [3, Theorem 1] suppose that $\int_{\mathbb{R}} f\,\mathrm{d}\tau = 0$ and let $t > 4T$ and $0 \le \tau \le T$. Then it holds that $t - \tau \ge \frac{3t}{4} \ge 3T$ and hence

$$\int\limits_{t>4T} \|k_\varepsilon(t-\tau)-k_\varepsilon(t)\|\,\mathrm{d}t \le \int\limits_{t>4T} \big\|e^{-\varepsilon A}A(e^{-(t-\tau)A}-e^{-tA})\big\|\,\mathrm{d}t$$
$$\le c\int\limits_{t>4T}\Big\|\int\limits_{t-\tau}^{t} A^2 e^{-rA}\,\mathrm{d}r\Big\|\,\mathrm{d}t$$
$$\le c\int\limits_{t>4T}\frac{\tau e^{-ct}}{9T^2}\,\mathrm{d}t \le \frac{c}{T}. \tag{2.5}$$

Note that we exploited the exponential decay of e^{-rA} and, in particular, that $0\in\rho(A)$. Then (2.5) implies that

$$\int\limits_{t>4T}\|AU_\varepsilon(t)\|\,\mathrm{d}t \le \int\limits_{t>4T}\Big\|\int\limits_{\mathbb{R}}(k_\varepsilon(t-\tau)-k_\varepsilon(t))f(\tau)\,\mathrm{d}s\Big\|\,\mathrm{d}t \le \frac{c}{T}\int\limits_{\mathbb{R}}\|f(\tau)\|\,\mathrm{d}\tau.$$

Since $AU_\varepsilon(t)=0$ for $t<0$, the same estimate holds for $t<-4T$, and hence we justified assumption (ii) in [3, Theorem 1]. Now that theorem yields the estimate

$$\int\limits_0^T \|AU_\varepsilon\|_p^s\,\mathrm{d}\tau \le \int\limits_0^T \|f\|_p^s\,\mathrm{d}\tau \tag{2.6}$$

for functions with vanishing integral mean on $(0,T)$ and for $1<s<p$.

To get (2.6) also for functions f with integral $\bar f=\int_0^T f\,\mathrm{d}s$, note that

$$AU_\varepsilon(t)=\int\limits_0^t Ae^{-\varepsilon A}e^{-(t-\tau)A}(f(\tau)-\bar f)\,\mathrm{d}\tau + e^{-\varepsilon A}(\bar f - e^{-tA}\bar f).$$

From this decomposition, (2.6) can be deduced easily for any function $f\in L^s(0,T;L^p(\Omega))$. Then the passage $\varepsilon\to 0$ will yield the corresponding result for AU rather than AU_ε. Moreover, since $U'(t)=f(t)-\int_0^t Ae^{-(t-\tau)A}f(\tau)\,\mathrm{d}\tau$, we proved (2.2) in case that $u_0=0$ and $1<s<p$. Finally, a duality argument and complex interpolation yield the result for all $1<s<\infty$.

If $u_0\neq 0$, the term $v(t)=e^{-tA_p}u_0$ must be analyzed as well. By elementary estimates, we get that

$$\int\limits_0^T\|A_pv\|_p^s\,\mathrm{d}t \le \int\limits_0^T \big\|\big(A_p^{1/s-\varepsilon}e^{-tA_p}\big)A_p^{1-1/s+\varepsilon}u_0\big\|_p^s\,\mathrm{d}t$$
$$\le c\int\limits_0^T (t^{-1/s+\varepsilon})^s\,\mathrm{d}t\,\big\|A_p^{1-1/s+\varepsilon}u_0\big\|_p^s$$
$$\le c\big\|A_p^{1-1/s+\varepsilon}u_0\big\|_p^s. \tag{2.7}$$

Now the proof of (i) is complete.

(ii) By classical regularity estimates of the stationary Stokes system in a smooth bounded domain, it holds that $\|u\|_p\| + \|\nabla u\|_p \le c\|A_pu\|_p$. Hence the term $\|A_p u\|_p$ in (2.2) may be replaced by $\|u\|_{W^{2,p}}$, see (2.4). Moreover, by results of Y. Giga [11],

$$\mathcal{D}(A_p^\alpha) = \mathcal{D}((-\Delta_p)^\alpha) \cap L^p_\sigma(\Omega)$$

so that $\|A_p^\alpha u_0\|_p \sim \|(-\Delta_p)^\alpha u_0\|_p$ for $0 < \alpha < 1$, see also Remark 3.2. Finally, applying the Helmholtz projection P_p in (2.3) to get (2.1), we obtain the pressure gradient as

$$\nabla\pi = (I - P_p)(f - u_t + \Delta_p u)$$

and hence the a priori estimate (2.4). □

The second main step is to prove new integrability results of both $\nabla\pi$ and π for weak solutions of the Navier–Stokes system. This will include the proof of the conjecture in [4]. Again we focus on bounded domains and the three-dimensional case.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, let $1 < p, s < \infty$ satisfy $\frac{2}{s} + \frac{3}{p} \geq 3$, $p > \frac{3}{2}$, and $\frac{1}{q} = \frac{1}{p} + \frac{1}{3}$. Moreover, assume that $f \in L^s(0,T; L^q(\Omega))$ for some $T > 0$ and $u_0 \in \mathcal{D}\big(A_q^{1-1/s+\varepsilon}\big) \cap L^2_\sigma(\Omega)$ for some $\varepsilon > 0$ with $1 - \frac{1}{s} + \varepsilon \leq 1$. Let u be a weak solution of the Navier–Stokes system (1.1) with data u_0 and f. Then*

$$u \in L^s(0,T; W^{2,q}(\Omega)), \quad u_t,\, u\cdot\nabla u \in L^s(0,T; L^q(\Omega)), \tag{2.8}$$

and there exists a pressure

$$\pi \in L^s(0,T; L^p(\Omega)) \quad \text{with } \nabla\pi \in L^s(0,T; L^q(\Omega)) \tag{2.9}$$

such that (1.1) is satisfied in $L^s(0,T; L^q(\Omega))$.

Note that $\frac{2}{s} + \frac{3}{q} \geq 4$ which includes the case $s = q = \frac{5}{4}$ used in Sect. 1 for the term $u\cdot\nabla u$; in other words, the assumption $\frac{2}{s} + \frac{3}{q} \geq 4$ is adapted to the theory of weak solutions. Moreover, the assumptions imply that $1 < q < \frac{3}{2}$.

Proof. It is easily seen that for a weak solution u, the non-linear term $u\cdot\nabla u$ is contained in $L^s(0,T; L^q(\Omega))$. Indeed, by the Sobolev embedding estimate $\|u\|_q \leq c\|\nabla u\|_2$ for $u \in H_0^{1,2}(\Omega)$ and Hölder's and Young's inequalities,

$$\begin{aligned}\|u\cdot\nabla u\|_q &\leq \|u\|_{(\frac{1}{q}-\frac{1}{2})^{-1}}\,\|\nabla u\|_2 \leq c\|u\|_6^{3(1-\frac{1}{q})}\|u\|_2^{1-3(1-\frac{1}{q})}\,\|\nabla u\|_2\\ &\leq c\|\nabla u\|_2^{1+3(1-\frac{1}{q})}\|u\|_2^{1-3(1-\frac{1}{q})}\end{aligned} \tag{2.10}$$

so that

$$\begin{aligned}\|u\cdot\nabla u\|^s_{L^s(L^q)} &= \int_0^T \|u\cdot\nabla u\|_q^s\,\mathrm{d}\tau \leq c\|u\|_{L^\infty(L^2)}^{s(1-3(1-\frac{1}{q}))}\int_0^T \|\nabla u\|_2^{s(1+3(1-\frac{1}{q}))}\,\mathrm{d}\tau\\ &\leq c_T\|u\|_{L^\infty(L^2)}^{s(1-3(1-\frac{1}{q}))}\Big(\int_0^T \|\nabla u\|_2^2\,\mathrm{d}\tau\Big)^{s(1+3(1-\frac{1}{q}))/2}.\end{aligned} \tag{2.11}$$

Since both terms on the most right-hand side are finite by the energy inequality (1.7), the claim $u \cdot \nabla u \in L^s(L^q)$ is proved.

However, (2.11) does not help directly to get a priori estimates of weak solutions in $L^s(L^q)$ since technical approximation arguments are needed to return from the variational formulation in Definition 1.1 (i) to the classical PDE (1.1). To this aim, the authors also introduce the Yosida approximation operators $J_k = (1 + \frac{1}{k}A_2)^{-3/4} : L^2_\sigma(\Omega) \to \mathcal{D}(A_2^{3/4})$ which are uniformly bounded on $L^2_\sigma(\Omega)$ and admit the pointwise convergence $J_k v \to v$ as $k \to \infty$. Similar properties hold on each $L^r_\sigma(\Omega)$, $1 < r < \infty$. Let us apply $J_k P$ to (1.1) to get that

$$(J_k u)_t - J_k P_2 \Delta u + J_k P_r u \cdot \nabla u = J_k P_q f,$$

where in a weak sense $-J_k P_2 \Delta u = J_k A_2 u = -P_2 \Delta J_k u$. Hence, for any weak solution u, (1.1) is rewritten in the strong form

$$(J_k u)_t - P_q \Delta(J_k u) = J_k P_q f - J_k P_q u \cdot \nabla u.$$

Now the $L^s(L^q)$ estimate (2.2) yields

$$\int_0^T \|(J_k u)_t\|_q^s \,\mathrm{d}\tau + \int_0^T \|J_k u\|_{W^{2,q}}^s \,\mathrm{d}\tau \leq c\Big(\|A_q^{1-1/s+\varepsilon} u_0\|_q^s + \int_0^T \big(\|f\|_q^s + \|u \cdot \nabla u\|_q^s\big)\,\mathrm{d}\tau\Big).$$

Since the right-hand side does not depend on k, a limit procedure implies that $u_t \in L^s(L^q)$, $u \in L^s(W^{2,q})$. Moreover, $u_t - P_q \Delta u + P_q u \cdot \nabla u = P_q f$ in $L^s(L^q)$. As in the proof of Theorem 2.1 (ii), we conclude that there exists a unique gradient field $\nabla \pi \in L^s(L^q)$ such that $u_t - \Delta u + u \cdot \nabla u + \nabla \pi = f$. Finally, a Sobolev embedding theorem yields a unique $\pi \in L^s(L^p)$, normalized by $\int_\Omega \pi(t)\,\mathrm{d}x = 0$, such that $\|\pi(t)\|_p \leq c\|\nabla \pi(t)\|_q$ for a.a. t and $\|\pi\|_{L^s(L^p)} \leq c\|\nabla \pi\|_{L^s(L^q)}$. □

Remark 2.3. (i) The procedure for an exterior domain is very similar when replacing A_p by $A_p + I$. However, the condition $\nabla^2 u \in L^s(L^q)$ and a Sobolev embedding imply only that $\nabla u \in L^s(L^p)$ and not $\nabla u \in L^s(L^q)$. Therefore, a localization procedure, i.e., multiplication of u by a cut-off function, will be used. This procedure leads to the divergence problem $\operatorname{div} v = g$ in $W_0^{1,p}(\Omega')$ for $g \in L^p(\Omega')$ with $\int_{\Omega'} g\,\mathrm{d}x = 0$ on a bounded subdomain Ω' and is solved by the so-called Bogovskiĭ operator. Finally, the assumptions on s, p, q, and u_0, f are slightly more restrictive, but nevertheless allowing for $s = p = \frac{5}{3}$.

(ii) For dimension $n > 3$, there are similar results, but the assumptions on s, p, q change due to Sobolev embeddings.

3. Remarks and perspectives. Let us describe several important topics related to the Stokes operator, the pressure, initial values, and partial regularity in more details and highlight several more recent results.

Remark 3.1 (The pressure function). Generally, an associated pressure to a weak solution u of (1.1) is found only in the sense of a distribution as a time derivative of a locally integrable pressure function, i.e., $\pi = \partial_t\tilde{\pi}$ with $\tilde{\pi} \in L^1_{\rm loc}((0,T)\times\Omega)$. If the boundary is smooth, then $\pi \in L^2_{\rm loc}([0,T)\times\overline{\Omega})$. For further details, see [22, Theorems V.1.7.1 and V.1.8.1]. From the numerous contributions to the decomposition of the pressure into different terms, we mention the result by J. Wolf [30]: To a weak solution u of (1.1), there exist pressure terms π_1, π_2, π_3, each of them found as a pressure of an auxiliary stationary Stokes problem $-\Delta v_j + \nabla\pi_j = F_j$ where $F_1 = -u$, $F_2 = \Delta u$, and $F_3 = -{\rm div}\,(u\otimes u)$. Then $\pi = \partial_t\pi_1 + \pi_2 + \pi_3$, π_1 and π_2 are harmonic, and each π_j can be estimated by the corresponding F_j in suitable norms. This decomposition can help to manage estimates involving the pressure. Similar results ([18]) were obtained by J. Neustupa et al. for the Navier–Stokes system with Navier's slip boundary condition $u\cdot N = 0$, $[T(u,p)\cdot N]_\tau + \gamma u = 0$ on $\partial\Omega$; here $T(u,p) = (\nabla u + (\nabla u)^\top - \pi I$ denotes Cauchy's stress tensor and the vector $[\,\cdot\,]_\tau$ equals the tangential component of $[\,\cdot\,]$.

Remark 3.2 (The Stokes operator and complex interpolation). The authors use several estimates for fractional powers of the Stokes operator, e.g., also to prove (2.10). The definition of A_q^θ, $0<\theta<1$, is a consequence of the resolvent estimate of A_q, whereas the crucial properties

$$\mathcal{D}(A_q^\theta) = [L^q_\sigma(\Omega), \mathcal{D}(A_q)]_\theta, \tag{3.1}$$

$$\mathcal{D}(A_q^\theta) = \mathcal{D}((-\Delta)^\theta)) \cap L^q_\sigma(\Omega), \tag{3.2}$$

$$\mathcal{D}(A_q^\theta) = \mathbb{H}^{2\theta,q}(\Omega), \tag{3.3}$$

are more involved. Here $\mathbb{H}^{2\theta,q}(\Omega)$ is a subspace of solenoidal vector fields in the Bessel potential space $H^{2\theta,q}(\Omega)$ such that

$$\mathbb{H}^{2\theta,q}(\Omega) = \begin{cases} \{u \in H^{2\theta,q}(\Omega)\cap L^q_\sigma(\Omega) : u|_{\partial\Omega} = 0\} & \text{if } \frac{1}{q} < 2\theta \le 2,\\ H^{2\theta,q}(\Omega)\cap L^q_\sigma(\Omega) & \text{if } 0 \le 2\theta < \frac{1}{q}. \end{cases}$$

Property (3.1) is a classical result of complex interpolation applied to the Stokes operator enjoying the property *BIP* of bounded purely imaginary powers $A_q^{is} \in \mathcal{L}(L^q_\sigma(\Omega))$, $s\in\mathbb{R}$. This property for bounded domains, based on the theory of pseudodifferential operators, was proved in [11, Theorem 1] and published after the submission of [24]. An extension to exterior domains is found in [12]. Moreover, *BIP* can be deduced from the property that A_q possesses a bounded $\mathcal{H}^\infty$ calculus which in 2003 was proved for bounded, exterior domains and (perturbed) half spaces in [19]. Property (3.2) for bounded domains was proved by Y. Giga, see [11, Theorem 3]. A characterization of $\mathcal{D}((-\Delta)^\theta))$ as in (3.3) is more classical; as for $\mathcal{D}(A_q^\theta)$ and an approach in the framework of interpolation-extrapolation spaces, we refer to H. Amann [1,2]. We note that the property *BIP* and the even more advanced concept of a bounded $\mathcal{H}^\infty$-calculus requires that $\partial\Omega$ is of class C^3, see [9], [19].

Remark 3.3 (The Stokes operator and real interpolation). The assumption on the initial value, $u_0 \in \mathcal{D}\big(A_p^{1-1/s+\varepsilon}\big)$, see Theorems 2.1 and 2.2, can be relaxed.

For an optimal condition, it suffices to know that $\left(\int_0^T \|Ae^{-tA}u_0\|_p^s \,\mathrm{d}t\right)^{1/s}$ is finite. This term defines an equivalent norm of the real interpolation space $(L^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/s,s}$ which can be identified with a subspace $\mathbb{B}^{2-2/s}_{p,s}(\Omega)$ of solenoidal vector fields of the Besov space $B^{2-2/s}_{p,s}(\Omega)$, see [1,2]. To be more precise,

$$\mathbb{B}^{2\theta}_{q,s}(\Omega) = \begin{cases} \left\{u \in B^{2\theta}_{q,s}(\Omega) \cap L^q_\sigma(\Omega) :\ u|_{\partial\Omega} = 0\right\}, & \frac{1}{q} < 2\theta \le 2, \\ B^{2\theta}_{q,s}(\Omega) \cap L^q_\sigma(\Omega), & 0 < 2\theta < \frac{1}{q}. \end{cases} \tag{3.4}$$

Remark 3.4 (General unbounded domains). For general unbounded domains $\Omega \subset \mathbb{R}^3$ with uniform C^2-boundary, there exists under adequate assumptions on u_0, f a suitable weak solution (u, π), i.e., satisfying the localized energy inequality (1.8), such that with $q = \frac{5}{4}$

$$u_t, u, \nabla u, \nabla^2 u, \nabla\pi \in L^q(0, T; L^2(\Omega) + L^q(\Omega)).$$

The space $L^2(\Omega)+L^q(\Omega)$ is used to combine a local L^q theory with the global L^2 theory of weak solutions to estimate perturbation terms in a cut-off procedure. To be more precise, there exists pressure functions π_1, π_2 such that $\pi = \pi_1 + \pi_2$ and $\pi_1 \in L^q(L^2)$, $\pi_2 \in L^q(L^q)$. For details, we refer to [7].

Remark 3.5 (Partial regularity). In 1998, F. Lin [17] presented a simplified proof of the results of Caffarelli-Kohn-Nirenberg [4]. A key point was the condition $\pi \in L^{5/3}(L^{5/3})$ from [24]; however, he even gave a shorter proof of that result by referring to local elliptic regularity estimates of $g = \Delta u - u_t$ which satisfies $\operatorname{curl} g = \operatorname{curl}(u \cdot \nabla u)$ and $\operatorname{div} g = 0$. A completely different proof is due to A. Vasseur [26]. The main idea to exclude a space-time singularity for a suitable weak solution is based on De Giorgi's technique to show regularity of solutions to elliptic equations with rough diffusion coefficients. The local assumption on the pressure is merely $\pi \in L^p(L^1)$ for any $p > 1$.

Remark 3.6 (Maximal regularity of the Stokes operator). The maximal regularity estimate (2.2) with optimal initial values in the real interpolation space $(L^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/s,s}$, initiated by Solonnikov (1973) for $s = p$ and extended by the authors in 1986 to arbitrary $1 < p, s < \infty$ for bounded domains, was generalized by Giga and Sohr [13] to exterior domains and half spaces. Their new argument was the theory of Dore and Venni [5] exploiting the property *BIP* of both the Stokes operator and the time derivative ∂_t and their commutativity; moreover, they extended the technique of [5] to the case when $0 \notin \rho(A)$. The more modern notion of $\mathcal{R}$-boundedness of the resolvent, the so-called $\mathcal{R}$-sectoriality, and operator-valued Fourier multipliers (see L. Weis [29]), were used by M. Geissert et al. [9] in 2010 to get maximal regularity of the Stokes equation for bounded and exterior domains. In another approach based on $\mathcal{R}$-boundedness and L^p spaces with Muckenhoupt weights, A. Fröhlich [8] got similar results.

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