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The character table of the finite Chevalley group $F_4(q)$ for q a power of 2

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Abstract. Let q be a prime power and $F_4(q)$ be the Chevalley group of type F_4 over a finite field with q elements. Marcelo and Shinoda (Tokyo J Math 18:303–340, 1995) determined the values of the unipotent characters of $F_4(q)$ on all unipotent elements, extending earlier work by Kawanaka and Lusztig to small characteristics. Assuming that q is a power of 2, we explain how to construct the complete character table of $F_4(q)$.

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1. Introduction. Let p be a prime and $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the field with p elements. Let $\mathbf{G}$ be a connected reductive algebraic group over k and assume that $\mathbf{G}$ is defined over the finite subfield $\mathbb{F}_q \subseteq k$, where q is a power of p. Let $F: \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius map. The finite group of fixed points $\mathbf{G}^F$ is called a "finite group of Lie type". We are concerned with the problem of computing the character table of $\mathbf{G}^F$. The work of Lusztig [11,14] has led to a general program for solving this problem.

However, in concrete examples, there are still a certain number of technical—and sometimes quite intricate—issues to be resolved. In this paper, we show how this can be done for the groups $\mathbf{G}^F = F_4(q)$, where q is a power of 2. The conjugacy classes have been classified by Shinoda [20]; the values of all unipotent characters on unipotent elements were already determined by Marcelo–Shinoda [17]. A further crucial ingredient is the fact that the characteristic functions of the F-invariant cuspidal character sheaves of $\mathbf{G}$ (for the definition, see [14] and the references therein) are explicitly known as linear combinations of the irreducible characters of $\mathbf{G}^F$. Building on earlier work of Shoji [21,22], this has been achieved in [5,17].

Birkhäuser

In Section 2, we introduce basic notation and collect some general results from Lusztig's theory, where we use the books [2,6] as our references. In Sections 3 and 4, we focus on $\mathbf{G}^F = F_4(q)$. First we consider the unipotent characters of $\mathbf{G}^F$. Then we address some issues concerning the two-variable Green functions involved in Lusztig's cohomological induction functor which allows us, finally, to consider the non-unipotent characters.

The special feature of $\mathbf{G}^F = F_4(q)$ as above is that the possible root systems of centralisers of semisimple elements are rather restricted. (See Remark 3.1 below.) There is a similar situation for $\mathbf{G}$ of adjoint type E_6 and $p = 2$. This, as well as the case of type E_7 and $p = 2$, will be discussed in a sequel to this paper. The values of the unipotent characters on unipotent elements have been recently determined by Hetz [7] for these groups.

I understand that Frank Lübeck has already prepared an electronic "generic" character table of $F_4(q)$, based on some assumptions concerning the values of the characteristic functions of certain F-invariant character sheaves on $\mathbf{G}$. With the results of this paper, it should now be possible to verify those assumptions (or adjust them appropriately).

1.1. Notation and conventions. The set of (complex) irreducible characters of a finite group Γ is denoted by $\mathrm{Irr}(\Gamma)$. We work over a fixed subfield $\mathbb{K} \subseteq \mathbb{C}$, which is algebraic over $\mathbb{Q}$, invariant under complex conjugation, and "large enough", that is, $\mathbb{K}$ contains sufficiently many roots of unity and $\mathbb{K}$ is a splitting field for Γ and all of its subgroups. In particular, $\chi(g) \in \mathbb{K}$ for all $\chi \in \mathrm{Irr}(\Gamma)$ and $g \in \Gamma$. Let $\mathrm{CF}(\Gamma)$ be the space of $\mathbb{K}$-valued class functions on Γ. There is a standard inner product $\langle\ ,\ \rangle_\Gamma$ on $\mathrm{CF}(\Gamma)$ given by $\langle f, f'\rangle_\Gamma := |\Gamma|^{-1} \sum_{g\in\Gamma} f(g)\overline{f'(g)}$ for $f, f' \in \mathrm{CF}(\Gamma)$, where $x \mapsto \overline{x}$ denotes the automorphism of $\mathbb{K}$ given by complex conjugation. We denote by $\mathbb{Z}\,\mathrm{Irr}(\Gamma) \subseteq \mathrm{CF}(\Gamma)$ the subset consisting of all integral linear combinations of $\mathrm{Irr}(\Gamma)$. Finally, if $C \subseteq \Gamma$ is any (non-empty) subset that is a union of conjugacy classes of Γ, then we denote by $\varepsilon_C \in \mathrm{CF}(\Gamma)$ the (normalised) indicator function of C, that is, we have

$$\varepsilon_C(g) = \begin{cases} |\Gamma|/|C| & \text{if } g \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if C is a single conjugacy class of Γ and $g \in C$, then $f(g) = \langle f, \varepsilon_C\rangle_\Gamma$ for any $f \in \mathrm{CF}(\Gamma)$. Thus, the problem of computing the values of $\rho \in \mathrm{Irr}(\Gamma)$ is equivalent to working out the inner products of ρ with the indicator functions of the various conjugacy classes of Γ.

2. Lusztig induction and uniform functions. Let $\mathbf{G}, F$ be as in the introduction. Given an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, we have a generalised character $R_{\mathbf{T},\theta}^{\mathbf{G}} \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$ as introduced by Deligne and Lusztig [1] (see also [6, §2.2]). We shall also need the following generalisation of $R_{\mathbf{T},\theta}^{\mathbf{G}}$.

2.1. An F-stable closed subgroup $\mathbf{L} \subseteq \mathbf{G}$ is called a "regular subgroup" if $\mathbf{L}$ is a Levi complement in some (not necessarily F-stable) parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$. Given such a pair $(\mathbf{L}, \mathbf{P})$, we obtain an operator

$$R_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}} \colon \mathbb{Z}\,\mathrm{Irr}(\mathbf{L}^F) \to \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F) \qquad \text{("Lusztig induction"; see [2,§9.1]).}$$

Denoting by $\mathbf{G}^F_{\text{uni}}$ and $\mathbf{L}^F_{\text{uni}}$ the sets of unipotent elements of $\mathbf{G}^F$ and $\mathbf{L}^F$, respectively, there is a corresponding two-variable Green function

$$Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}\colon \mathbf{G}^F_{\text{uni}} \times \mathbf{L}^F_{\text{uni}} \to \mathbb{Q} \qquad \text{(see [2,§10.1])}.$$

If $\mathbf{L} = \mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ (and $\mathbf{B} \subseteq \mathbf{G}$ is a Borel subgroup containing $\mathbf{T}$), then $\mathbf{T}^F_{\text{uni}} = \{1\}$ and $Q^{\mathbf{G}}_{\mathbf{T}}\colon \mathbf{G}^F_{\text{uni}} \to \mathbb{Q}$, $u \mapsto Q^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}}(u,1)$, is the "usual" Green function originally introduced in [1], that is, we have $Q^{\mathbf{G}}_{\mathbf{T}}(u) = R^{\mathbf{G}}_{\mathbf{T},1}(u)$ for all $u \in \mathbf{G}^F_{\text{uni}}$.

2.2. Let $\mathbf{L} \subseteq \mathbf{P}$ be as above and $\psi \in \text{Irr}(\mathbf{L}^F)$. There is a character formula which expresses the values of $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)$ in terms of the values of ψ and the two-variable Green functions for $\mathbf{G}$ and for groups of the form $C^\circ_{\mathbf{G}}(s)$ where $s \in \mathbf{G}^F$ is semisimple; see [2, Prop. 10.1.2], [13, Prop. 6.2] for the precise formulation. For later reference, we only state here the following special case:

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)(u) = \sum_{v\in\mathbf{L}^F_{\text{uni}}} Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(u, v^{-1})\psi(v) \qquad \text{for all } u \in \mathbf{G}^F_{\text{uni}}. \tag{a}$$

We also state the following useful formula. Let $g \in \mathbf{G}^F$ and write $g = su = us$ where $s \in \mathbf{G}^F$ is semisimple and $u \in \mathbf{G}^F$ is unipotent (Jordan decomposition). By [2, Prop. 3.5.3], we have $g \in C^\circ_{\mathbf{G}}(s)$. If $C^\circ_{\mathbf{G}}(s) \subseteq \mathbf{L}$, then

$$\rho(g) = \sum_{\psi\in\text{Irr}(\mathbf{L}^F)} \big\langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi), \rho\big\rangle_{\mathbf{G}^F} \psi(g) \qquad \text{for all } \rho \in \text{Irr}(\mathbf{G}^F). \tag{b}$$

This appeared in K.D. Schewe's dissertation [19]; see the remark following [6, Cor. 3.3.13] for a proof.

2.3. Let us denote by $\mathfrak{X}(\mathbf{G}, F)$ the set of all pairs $(\mathbf{T}, \theta)$ where $\mathbf{T} \subseteq \mathbf{G}$ is an F-stable maximal torus and $\theta \in \text{Irr}(\mathbf{T}^F)$. Following [10, p. 16], a class function $f \in \text{CF}(\mathbf{G}^F)$ is called "uniform" if f can be written as a $\mathbb{K}$-linear combination of the generalised characters $R^{\mathbf{G}}_{\mathbf{T},\theta}$ for various pairs $(\mathbf{T}, \theta) \in \mathfrak{X}(\mathbf{G}, F)$. If f is uniform, then we have (see [2, Prop. 10.2.4])

$$f = |\mathbf{G}^F|^{-1} \sum_{(\mathbf{T},\theta)\in\mathfrak{X}(\mathbf{G},F)} |\mathbf{T}^F| \langle f, R^{\mathbf{G}}_{\mathbf{T},\theta}\rangle_{\mathbf{G}^F} R^{\mathbf{G}}_{\mathbf{T},\theta}.$$

For example, if C is a conjugacy class of semisimple elements of $\mathbf{G}^F$, then the indicator function ε_C (as in 1.1) is uniform; see [2, Cor. 10.3.4].

Theorem 2.4. *Let $\mathbf{C}$ be an arbitrary F-stable conjugacy class of $\mathbf{G}$. Then the indicator function $\varepsilon_{\mathbf{C}^F}$ of the set $\mathbf{C}^F$ is a uniform function.*

(Note that, in general, $\mathbf{C}^F$ is a union of conjugacy classes of $\mathbf{G}^F$.)

Proof. See the appendix of [4]; this was conjectured by Lusztig [10, 2.16]. See also [2, Cor. 13.3.5] and [6, Theorem 2.7.11]. □

Example 2.5. Let $g \in \mathbf{G}^F$ and assume that $C_{\mathbf{G}}(g)$ is connected. Let $\mathbf{C}$ be the $\mathbf{G}$-conjugacy class of g. Since $C_{\mathbf{G}}(g)$ is connected, $C := \mathbf{C}^F$ is a single conjugacy class of $\mathbf{G}^F$; see [6, Example 1.4.10]. Now ε_C is uniform by Theorem 2.4. Let $\rho \in \text{Irr}(\mathbf{G}^F)$. Recall from 1.1 that $\rho(g) = \langle \rho, \varepsilon_C\rangle_{\mathbf{G}^F}$ and

$\langle \varepsilon_C, R^{\mathbf{G}}_{\mathbf{T},\theta}\rangle_{\mathbf{G}^F} = R^{\mathbf{G}}_{\mathbf{T},\theta^{-1}}(g)$ for any $(\mathbf{T},\theta) \in \mathfrak{X}(\mathbf{G},F)$. Hence, using 2.3, we obtain the formula

$$\rho(g) = |\mathbf{G}^F|^{-1} \sum_{(\mathbf{T},\theta)\in\mathfrak{X}(\mathbf{G},F)} |\mathbf{T}^F|\, \langle R^{\mathbf{G}}_{\mathbf{T},\theta}, \rho\rangle_{\mathbf{G}^F}\, R^{\mathbf{G}}_{\mathbf{T},\theta^{-1}}(g).$$

This shows that the value $\rho(g)$ is determined by the multiplicities $\langle R^{\mathbf{G}}_{\mathbf{T},\theta}, \rho\rangle_{\mathbf{G}^F}$ and the values $R^{\mathbf{G}}_{\mathbf{T},\theta}(g)$, where $(\mathbf{T},\theta)$ runs over all pairs in $\mathfrak{X}(\mathbf{G},F)$.

2.6. We say that $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ is "unipotent" if $\langle R^{\mathbf{G}}_{\mathbf{T},1}, \rho\rangle_{\mathbf{G}^F} \neq 0$ for some F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$. We denote by $\mathrm{Uch}(\mathbf{G}^F)$ the set of unipotent characters of $\mathbf{G}^F$. As shown in Lusztig's book [11], these characters play a special role in the character theory of $\mathbf{G}^F$; many questions about arbitrary characters of $\mathbf{G}^F$ can be reduced to unipotent characters.

3. The unipotent characters for F_4 in characteristic 2. We assume from now on that $p = 2$ and $\mathbf{G}$ is simple of type F_4. Let $F\colon \mathbf{G} \to \mathbf{G}$ be a Frobenius map such that $\mathbf{G}^F = F_4(q)$ where q is a power of 2. In order to compute the characters of $\mathbf{G}^F$, we shall assume that the following information is known and available in the form of tables:

(A1) Parametrisations of $\mathfrak{X}(\mathbf{G},F)$ and of all the conjugacy classes of $\mathbf{G}^F$.
(A2) The multiplicities $\langle R^{\mathbf{G}}_{\mathbf{T},\theta}, \rho\rangle$ for all $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and $(\mathbf{T},\theta) \in \mathfrak{X}(\mathbf{G},F)$.
(A3) The values $R^{\mathbf{G}}_{\mathbf{T},\theta}(g)$ for all $g \in \mathbf{G}^F$ and all $(\mathbf{T},\theta) \in \mathfrak{X}(\mathbf{G},F)$.
(A4) For every regular $\mathbf{L} \subsetneqq \mathbf{G}$, the values $\psi(u)$ for $\psi \in \mathrm{Irr}(\mathbf{L}^F)$, $u \in \mathbf{L}^F_{\mathrm{uni}}$.

It will be convenient to also introduce the set $\mathfrak{Y}(\mathbf{G},s)$ of all pairs $(\mathbf{T},s)$ where $\mathbf{T} \subseteq \mathbf{G}$ is an F-stable maximal torus and $s \in \mathbf{T}^F$. There are natural actions of $\mathbf{G}^F$ on $\mathfrak{X}(\mathbf{G},F)$ and on $\mathfrak{Y}(\mathbf{G},F)$; see [6, 2.3.20 and 2.5.12]. Since $\mathbf{G} \cong \mathbf{G}^*$ is "self-dual" (in the sense of [6, Def. 1.5.17]), there is a bijective correspondence

$$\mathfrak{X}(\mathbf{G},F) \bmod \mathbf{G}^F \quad\leftrightarrow\quad \mathfrak{Y}(\mathbf{G},F) \bmod \mathbf{G}^F \qquad \text{(see [6, Cor. 2.5.14]).}$$

Remark 3.1. The conjugacy classes of $\mathbf{G}^F$ are determined by Shinoda [20]. The tables in [20] provide the required classifications and parametrisations in **(A1)**, where we use the above-mentioned bijection to pass from $\mathfrak{Y}(\mathbf{G},F)$ to $\mathfrak{X}(\mathbf{G},F)$. Since the center of $\mathbf{G}$ is trivial, the information in **(A2)** is available via Lusztig's "Main Theorem 4.23" in [11]; see also [6, §2.4, §4.2]. In order to obtain **(A3)**, one uses the character formula in [1, §4] (see also [6, Theorem 2.2.16]) for the evaluation of $R^{\mathbf{G}}_{\mathbf{T},\theta}(g)$. This involves the Green functions for $\mathbf{G}$ and for groups of the form $\mathbf{H}_s = C_{\mathbf{G}}(s)$ where $s \in \mathbf{G}^F$ is semisimple; note that, for our $\mathbf{G}$, the centraliser of any semisimple element is connected. By inspection of [20, Table III], we see that $\mathbf{H}_s$ is either a maximal torus, or a regular subgroup (with a root system of type F_4, B_3, C_3, $A_1 \times A_2$, B_2, A_2, $A_1 \times A_1$, or A_1) or $\mathbf{H}_s$ has a root system of type $A_2 \times A_2$. The Green functions for $\mathbf{G}^F$ itself have been determined by Malle [15]; for the other cases, see Lübeck [9, Tabelle 16]. The further technical issues in the evaluation of $R^{\mathbf{G}}_{\mathbf{T},\theta}(su)$ are discussed in [5, §3] and [9, §2] (for example, one has to deal with a sum over all $x \in \mathbf{G}^F$ such that $x^{-1}sx \in \mathbf{T}$); in [9, §6], this is explained in detail for the groups

$\mathbf{G}^F = \mathrm{CSp}_6(q)$. Finally, the required values in **(A4)** can be extracted from Enomoto [3] (type B_2), Looker [8], Lübeck [9, Tabelle 27] (type B_3, C_3), and Steinberg [23] (type A_1, A_2).

Representatives for the $\mathbf{G}^F$-conjugacy classes of semisimple elements are denoted by $h_0, h_1, \ldots, h_{76}$ in [20, Table II], where $h_0 = 1$; note that some of the h_i only occur according to whether $3 \mid q-1$ or $3 \mid q+1$, or when q is sufficiently large. We now go through the list of these elements and explain how to determine the values of any unipotent character $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ on elements of the form $h_i u$ where $u \in C_{\mathbf{G}}(h_i)^F$ is unipotent.

In our group $\mathbf{G}$, there are 37 unipotent characters, where we use the notation in Lusztig's book [11, pp. 371/372]).

3.2. If $s = h_0 = 1$, then the values $\rho(u)$ for $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ and $u \in \mathbf{G}^F_{\mathrm{uni}}$ have been explicitly determined by Marcelo–Shinoda; see [17, Table 6.A]. This relies on the Green functions of $\mathbf{G}^F$ (available from [15]) and also on the knowledge of the "generalised Green functions" arising from Lusztig's theory of character sheaves. An algorithm for the computation of those functions is described in [12, §24]; it involves the delicate matter of normalising certain "Y_ι-functions" (defined in [12, (24.2.3)]). Marcelo–Shinoda [17] do not explain in detail how they found those normalisations. But using the argument of Hetz [7, §4.1.4] (where the analogous problem is solved for groups of type E_6 in characteristic 2), one obtains an independent verification that the values in [17, Table 5] are correct.

3.3. Let $s = h_3$ (if $3 \mid q-1$) or $s = h_{15}$ (if $3 \mid q+1$). Then $\mathbf{H}_s = C_{\mathbf{G}}(s)$ has a root system of type $A_2 \times A_2$. Let $u \in \mathbf{H}_s^F$ be unipotent and $\mathbf{C}$ be the $\mathbf{G}$-conjugacy class of su.

(a) Assume first that u is not regular unipotent. By inspection of [20, Table IV], we see that $C_{\mathbf{G}}(su)$ is connected. So we can apply Example 2.5, together with **(A2)**, **(A3)**, to determine $\rho(su)$ even for all $\rho \in \mathrm{Irr}(\mathbf{G}^F)$.

(b) Now assume that u is regular unipotent. We recall some facts from [5, §7.6]. (Note that, in [5, §7.6], it is assumed that $p \neq 2, 3$ but the discussion works verbatim also for $p = 2$.) The set $\mathbf{C}^F$ splits into 3 classes in $\mathbf{G}^F$, which we simply denote by C_1, C_2, C_3. We can choose the notation such that $C_1 = C_1^{-1}$ and $C_2^{-1} = C_3$. Explicit representatives are described in [20, Table IV]; we have $|C_{\mathbf{G}}(g_i)^F| = 3q^4$ for $g_i \in C_i$ and $i = 1, 2, 3$. Let $\chi_0 := \varepsilon_{\mathbf{C}^F}$ be the indicator function on the set $\mathbf{C}^F$ (as in 1.1). Let $1 \neq \theta \in \mathbb{K}$ be a fixed third root of unity. Then we consider the following linear combinations of unipotent characters of $\mathbf{G}^F$:

$$\chi_1 := \tfrac{1}{3}q^2\big([12_1] + F_4^{\mathrm{II}}[1] - [6_1] - [6_2] + 2F_4[\theta] - F_4[\theta^2]\big),$$
$$\chi_2 := \tfrac{1}{3}q^2\big([12_1] + F_4^{\mathrm{II}}[1] - [6_1] - [6_2] - F_4[\theta] + 2F_4[\theta^2]\big).$$

As discussed in [5, §7.6], the class functions χ_1, χ_2 are (scalar multiples of) characteristic functions of F-invariant cuspidal character sheaves on $\mathbf{G}$; furthermore, the values of χ_0, χ_1, χ_2 are given as follows:

	C_1	C_2	C_3	$g \in \mathbf{G}^F \backslash \mathbf{C}_s^F$
χ_0	q^4	q^4	q^4	0
χ_1	q^4	$q^4\theta$	$q^4\theta^2$	0
χ_2	q^4	$q^4\theta^2$	$q^4\theta$	0

Hence, $\varepsilon_{C_1} = \chi_0 + \chi_1 + \chi_2$, $\varepsilon_{C_2} = \chi_0 + \theta^2\chi_1 + \theta\chi_2$, $\varepsilon_{C_3} = \chi_0 + \theta\chi_1 + \theta^2\chi_2$.

Now let $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ be arbitrary and $g_i \in C_i$ for $i = 1, 2, 3$. Since χ_0 is uniform by Theorem 2.4, we can determine $\langle \rho, \chi_0 \rangle_{\mathbf{G}^F}$ using **(A2)**, **(A3)**, and the formula in 2.3. The inner products of ρ with χ_1, χ_2 are known by the definition of χ_1, χ_2. Hence, we can explicitly work out $\rho(g_i) = \langle \rho, \varepsilon_{C_i} \rangle_{\mathbf{G}^F}$.

3.4. Let $s = h_i$ where $i \notin \{0, 3, 15\}$. In these cases, $\mathbf{L} = C_{\mathbf{G}}(s)$ either is a maximal torus, or a proper regular subgroup with a root system of type B_3, C_3, $A_1 \times A_2$, B_2, A_2, $A_1 \times A_1$, or A_1. Let $u \in \mathbf{L}^F$ be unipotent and $\mathbf{C}$ be the $\mathbf{G}$-conjugacy class of su. Let $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. In order to compute $\rho(su)$, we use Schewe's formula in 2.2. First note that, if $\psi \in \mathrm{Irr}(\mathbf{L}^F)$ is such that $\big\langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi), \rho \big\rangle_{\mathbf{G}^F} \neq 0$, then we must have $\psi \in \mathrm{Uch}(\mathbf{L}^F)$; see [6, Prop. 3.3.21]. Furthermore, since s is in the centre of $\mathbf{L}^F$, we have $\psi(su) = \psi(u)$. (This is a general property of unipotent characters; see [6, Prop. 2.2.20].) Hence, Schewe's formula reads:

$$\rho(su) = \sum_{\psi \in \mathrm{Uch}(\mathbf{L}^F)} \big\langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi), \rho \big\rangle_{\mathbf{G}^F} \psi(u).$$

By **(A4)**, the values $\psi(u)$ for $\psi \in \mathrm{Uch}(\mathbf{L}^F)$ and $u \in \mathbf{L}^F_{\mathrm{uni}}$ are explicitly known. The multiplicities $\langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi), \rho \rangle_{\mathbf{G}^F}$ (for $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ and $\psi \in \mathrm{Uch}(\mathbf{L}^F)$) can also be determined explicitly; see [6, §4.6], especially [6, Prop. 4.6.18]. In Michel's version of CHEVIE [18], this is available through the function `LusztigInductionTable`. Let us illustrate this with an example.

Example 3.5. Let $\rho = F_4^{\mathrm{II}}[1] \in \mathrm{Uch}(\mathbf{G}^F)$ (a cuspidal unipotent character). Let $s = h_{53}$; then $\mathbf{L} = C_{\mathbf{G}}(s)$ is a regular subgroup of type B_2, where $|\mathbf{L}^F| = q^4(q^2+1)(q^2-1)(q^4-1)$; see [20, Table III]. We would like to determine the values $\rho(h_{53}u)$ where $u \in \mathbf{L}^F$ is unipotent. The values of the unipotent characters of $\mathbf{L}^F$ on unipotent elements are given by Table 1. Using Michel's `LusztigInductionTable`, we find that

$$\langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi_{10}), \rho \rangle_{\mathbf{G}^F} = 1 \qquad \text{and} \qquad \langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi_i), \rho \rangle_{\mathbf{G}^F} = 0 \quad \text{for } i \neq 10.$$

Hence, by Schewe's formula, we have $\rho(h_{53}u) = \psi_{10}(u)$. — A completely analogous procedure works for any $s = h_i$ as in 3.4.

4. Non-unipotent characters for F_4 in characteristic 2. We keep the notation of the previous section, where $\mathbf{G}$ is simple of type F_4 in characteristic 2. We now explain how to determine the values of the non-unipotent characters of $\mathbf{G}^F$. First we recall some facts from Lusztig's classification of $\mathrm{Irr}(\mathbf{G}^F)$. Let $s \in \mathbf{G}^F$ be semisimple. Then we define $\mathscr{E}(\mathbf{G}^F, s)$ to be the set of all $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ such that $\langle R^{\mathbf{G}}_{\mathbf{T},\theta}, \rho \rangle \neq 0$ for some pair $(\mathbf{T}, \theta) \in \mathfrak{X}(\mathbf{G}, F)$ in correspondence with $(\mathbf{T}, s) \in \mathfrak{Y}(\mathbf{G}, F)$. It is known that every $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ belongs to $\mathscr{E}(\mathbf{G}^F, s)$ for some s; furthermore, $\mathscr{E}(\mathbf{G}^F, s)$ only depends on the

Table 1. Unipotent characters for type B_2 in characteristic 2

	A_1	A_2	A_{31}	A_{32}	A_{41}	A_{42}
$\lvert C_{\mathbf{G}}(u)^F\rvert$:	$q^4(q^2-1)(q^4-1)$	$q^4(q^2-1)$	$q^4(q^2-1)$	q^4	$2q^2$	$2q^2$
ψ_0	1	1	1	1	1	1
ψ_9	$\frac{1}{2}q(q+1)^2$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q(q+1)$	$\frac{q}{2}$	$\frac{q}{2}$	$-\frac{q}{2}$
ψ_{10}	$\frac{1}{2}q(q-1)^2$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$\frac{q}{2}$	$\frac{q}{2}$	$-\frac{q}{2}$
ψ_{11}	$\frac{1}{2}q(q^2+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	$\frac{q}{2}$	$-\frac{q}{2}$	$\frac{q}{2}$
ψ_{12}	$\frac{1}{2}q(q^2+1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{q}{2}$	$-\frac{q}{2}$	$\frac{q}{2}$
ψ_{13}	q^4	.	.	.	.	.

(See Enomoto [3]; notation as in [6, Examples 3.3.30 and 2.7.22].)

$\mathbf{G}^F$-conjugacy class of s. If $s, s' \in \mathbf{G}^F$ are such that $\mathscr{E}(\mathbf{G}^F, s)\cap\mathscr{E}(\mathbf{G}^F, s') \neq \varnothing$, then s, s' are $\mathbf{G}^F$-conjugate. (For all this, see, for example, [6, §2.6]; also recall that $\mathbf{G} \cong \mathbf{G}^*$.) Finally, by the "Main Theorem 4.23" of [11], there is a bijection $\mathscr{E}(\mathbf{G}^F, s) \leftrightarrow \mathrm{Uch}(\mathbf{H}_s^F)$, where $\mathbf{H}_s = C_{\mathbf{G}}(s)$; this is called the "Jordan decomposition" of characters. We now proceed in 4 steps, where we determine the following information:

Step 1: The values of all the two-variable Green functions $Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$.

Step 2: The values $\rho(u)$ for all $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and $u \in \mathbf{G}^F_{\mathrm{uni}}$.

Step 3: The decomposition of $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)$ for any $\psi \in \mathrm{Irr}(\mathbf{L}^F)$.

Step 4: The values $\rho(g)$ for any $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and any $g \in \mathbf{G}^F$.

4.1. We show how Step 1 can be resolved. Assume that $\mathbf{L} \subsetneq \mathbf{G}$ and let $\mathrm{Uch}(\mathbf{L}^F) = \{\psi_1, \ldots, \psi_n\}$. The information in **(A4)** (see Section 3) shows, in particular, that n is also the number of conjugacy classes of unipotent elements of $\mathbf{L}^F$. Let $v_1, \ldots, v_n$ be representatives of these classes. Then, again using **(A4)**, we can also check that the matrix $(\psi_i(v_j))_{1\leqslant i,j\leqslant n}$ is invertible. (For an example, see Table 1.) Let $u_1, \ldots, u_N$ be representatives of the conjugacy classes of unipotent elements of $\mathbf{G}^F$; we have $N = 35$ by [20, Theorem 2.1]. Then we write the character formula 2.2(a) as a system of equations:

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi_i)(u_k) = \sum_{j=1}^{n} c_j\, Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(u_k, v_j^{-1})\psi_i(v_j) \quad \text{for } 1\leqslant i\leqslant n,\ 1\leqslant k\leqslant N, \qquad (\spadesuit)$$

where $c_j := [\mathbf{L}^F : C_{\mathbf{L}}(v_j)^F]$ for all j. On the other hand, as explained in 3.4, we can determine the multiplicities $m(\psi_i, \rho) := \langle R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi_i), \rho\rangle_{\mathbf{G}^F}$ for any $\rho \in \mathrm{Uch}(\mathbf{G}^F)$. Hence, we obtain equations

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi_i)(u_k) = \sum_{\rho\in\mathrm{Uch}(\mathbf{G}^F)} m(\psi_i, \rho)\rho(u_k) \qquad \text{for } 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant N.$$

Consequently, since the values $\rho(u_k)$ for $\rho \in \mathrm{Uch}(\mathbf{G}^F)$ are known by 3.2, the values $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi_i)(u_k)$ can be computed explicitly. We can now invert $(\spadesuit)$ and obtain all the values $Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(u_k, v_j^{-1})$ for $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant N$. (A similar argument appears in Malle–Rotilio [16, §2.2].)

4.2. We show how Step 2 can be resolved. As in the previous section, we consider the list of semisimple elements $h_0, h_1, \ldots, h_{76} \in \mathbf{G}^F$. Let $\rho \in \mathrm{Irr}(\mathbf{G}^F)$. There is some $s \in \{h_0, h_1, \ldots, h_{76}\}$ such that $\rho \in \mathscr{E}(\mathbf{G}^F, s)$. If $s = h_0$ (the identity element), then ρ is unipotent and the required values are known by 3.2. Now assume that $s \in \{h_3, h_{15}\}$ where $C_{\mathbf{G}}(s)$ has a root system of type $A_2 \times A_2$. Then, by the discussion in [6, Lemma 2.4.18] (which is drawn from Lusztig's book [11]), we know that ρ is a uniform class function. (The group $\mathbf{W}_{\lambda,n}$ occurring in that discussion is isomorphic to the Weyl group of $C_{\mathbf{G}}(s)$; see [6, (2.5.10)] and note again that $\mathbf{G} \cong \mathbf{G}^*$.) Hence, the values $\rho(u)$ for $u \in \mathbf{G}^F_{\mathrm{uni}}$ are known by **(A2)**, **(A3)** in Section 3. Finally, let $s = h_i$ where $i \notin \{0, 3, 15\}$. Then, as in 3.4, $\mathbf{L} := C_{\mathbf{G}}(s) \subsetneqq \mathbf{G}$ is a regular subgroup. In that case, Lusztig has shown that $\rho = \pm R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)$ for some $\psi \in \mathscr{E}(\mathbf{L}^F, s)$; see [6, Theorem 3.3.22]. So, in order to determine $\rho(u)$ for $u \in \mathbf{G}^F_{\mathrm{uni}}$, we can use again the character formula 2.2(a), combined with the knowledge of $Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ (see Step 1) and the values $\psi(v)$ for $v \in \mathbf{L}^F_{\mathrm{uni}}$ (see **(A4)**).

4.3. We show how Step 3 can be resolved. Assume that $\mathbf{L} \subsetneqq \mathbf{G}$ and let $\psi \in \mathrm{Irr}(\mathbf{L}^F)$ be arbitrary. There is some semisimple $s \in \mathbf{L}^F$ such that $\psi \in \mathscr{E}(\mathbf{L}^F, s)$. Let $\mathscr{E}(\mathbf{G}^F, s) = \{\rho_1, \ldots, \rho_r\}$. Then, by [6, Prop. 3.3.20], we have

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi) = \sum_{i=1}^{r} m(\psi, \rho_i)\rho_i \qquad \text{where } m(\psi, \rho_i) \in \mathbb{Z} \text{ for } 1 \leqslant i \leqslant r. \qquad (*)$$

If $s = 1$ and $\psi \in \mathrm{Uch}(\mathbf{L}^F)$, we can use Michel's `LusztigInductionTable`, as in 3.4. Now assume that $s \neq 1$. Then one could use the fact that $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ commutes with the Jordan decomposition of characters; see [6, Theorem 4.7.2]. But having the results of Steps 1 and 2 at our disposal, we can also argue as follows. Let again $u_1, \ldots, u_N$ be representatives of the conjugacy classes of unipotent elements of $\mathbf{G}^F$. Using 2.2(a), **(A4)**, and Step 1, we can compute the values:

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)(u_k) = \sum_{v \in \mathbf{L}^F_{\mathrm{uni}}} Q^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(u_k, v^{-1})\psi(v) \qquad \text{for } 1 \leqslant k \leqslant N.$$

Comparing with $(*)$, we obtain equations

$$\sum_{i=1}^{r} m(\psi, \rho_i)\rho_i(u_k) = R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\psi)(u_k) = \text{known value} \qquad \text{for } 1 \leqslant k \leqslant N.$$

Using Step 2, we can check that the matrix $(\rho_i(u_k))_{1\leqslant i\leqslant r, 1\leqslant k\leqslant N}$ has rank r, where $r \leqslant N$. (This would not be true for $s = 1$.) Hence, the above equations uniquely determine the numbers $m(\psi, \rho_i)$ for $1 \leqslant i \leqslant r$.

4.4. We show how Step 4 can be resolved. Let $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and $g \in \mathbf{G}^F$ be arbitrary. Let $i \in \{0, 1, \ldots, 76\}$ be such that $\rho \in \mathscr{E}(\mathbf{G}^F, h_i)$. If $i = 0$, then $h_0 = 1$, ρ is unipotent, and we know the values of ρ by Section 3. Next, let $i \in \{3, 15\}$. Then, as already mentioned in 4.2, ρ is uniform and so the values of

ρ are computable via **(A2)**, **(A3)**. Finally, let $i \notin \{0, 3, 15\}$. Write $g = su = us$ where $s \in \mathbf{G}^F$ is semisimple and $u \in \mathbf{G}^F$ is unipotent. If $s = 1$, then the values $\rho(u)$ for $u \in \mathbf{G}^F_{\text{uni}}$ are known by Step 2. Now let $s \neq 1$. If $C_{\mathbf{G}}(s)$ has type $A_2 \times A_2$, then $\rho(su)$ is already known by 3.3. Otherwise, we are in the situation of 3.4 where $\mathbf{L} := C_{\mathbf{G}}(s) \subsetneqq \mathbf{G}$ is a regular subgroup. Let $\psi \in \text{Irr}(\mathbf{L}^F)$ and $(\mathbf{T}, \theta) \in \mathfrak{X}(\mathbf{L}, F)$ be such that $\langle R^{\mathbf{L}}_{\mathbf{T},\theta}, \psi\rangle_{\mathbf{L}^F} \neq 0$; then, by [6, Prop. 2.2.20], we have $\psi(su) = \theta(s)\psi(u)$. So Schewe's formula, together with **(A4)** and the result of Step 3, yields the value $\rho(su)$.

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