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Prefrattini subgroups and crowns

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Abstract. This survey paper wants to highlight the mathematical impact of the celebrated paper by Gaschütz in which the concepts of prefrattini subgroups and crowns were introduced.

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1. Introduction. The paper "Praefrattinigruppen" [23] appeared in the Archiv der Mathematik in 1962. In that paper, Gaschütz proved that every finite soluble group G contains a characteristic conjugacy class of subgroups, called the *prefrattini subgroups* of G, with several remarkable properties. In particular, these subgroups turned out to be an important tool for obtaining information on the normal subgroups of the group. They appear as the intersection of some maximal subgroups of the group. After the original Gaschütz paper, several generalizations of prefrattini subgroups have been constructed by intersecting some cleverly chosen maximal subgroups. But the paper of Gaschütz is not only important for having initiated and stimulated the study of the properties of subgroups of prefrattini type, but also for having introduced the notion of crowns. Indeed, in order to investigate the properties of the prefrattini subgroups of a finite soluble group G, Gaschütz analyzed the structure of the chief factors of G as G-modules. Associated with a G-module A, there exists a section of the group, called A-Kopf, or A-crown in English. This section, viewed as a G-module, is completely reducible and homogeneous and has a composition series of length the number of complemented chief factors G-isomorphic to in any chief series of G. Later, the study of non-soluble chief factors made by Lafuente [34] allowed him to discover that some sections associated with non-abelian chief factors can be constructed enjoying similar properties to Gaschütz's crowns. This originated the concept of non-abelian crowns, which turns out to be quite useful in the study of the generating properties of finite groups. More generally, crowns turned out to be a very useful tool in asymptotic and probabilistic group theory, for the help that they can give in the investigation of the maximal subgroup growth of a group G.

The structure of this survey paper is as follows. In Section 2, we describe the results obtained by Gaschütz in his paper. Section 3 analyzes how the concept of prefrattini subgroup has been extended and generalized by several authors. A more exhaustive description of this kind of results can be found in [4, Chapter 4]. Section 4 describes how the notion of crowns has been generalized to non-abelian chief factors and arbitrary finite groups, mentioning some applications. In Section 5, we concentrate on the role of crowns in asympotic group theory, in particular, in the study of the generating properties of finite groups. Finally, in Section 6, we see that the notion of crowns can be extended to profinite groups, and we analyze some important applications in that context.

2. The Gaschütz paper. In the introduction of his paper, Gaschütz motivated the paper starting from the following observation: the knowledge of the Frattini subgroup $\mathrm{Frat}(G)$ of a finite group G is not enough to determine the Frattini subgroup of G/N, when N is a normal subgroup of G. For example, $\mathrm{Frat}(G) = 1$ does not imply $\mathrm{Frat}(G/N) = 1$. This observation led Gaschütz to ask whether some analogue's of the Frattini subgroup can be found, that are preserved under taking epimorphism images. He answered this question in the universe of the finite soluble groups, proving the following result:

Theorem 1 ([23, Satz 6.1, Satz 6.4, Satz 6.5]). *To every finite soluble group G, a family of subgroups, called the prefrattini subgroups of G, can be associated, which satisfy the following properties:*

(1) *the prefrattini subgroups of G are conjugate in G;*
(2) *the intersection of the prefrattini subgroups of G coincides with the Frattini subgroup;*
(3) *if $N \trianglelefteq G$ and H is a prefrattini subgroup of G, then HN/N is a prefrattini subgroup of G/N;*
(4) *a subgroup K of G is contained in a prefrattini subgroup of G if and only if every maximal subgroup of G contains a conjugate of K.*

In particular, if G is a finite soluble group, $N \trianglelefteq G$, and H is a prefrattini subgroup of G, then

$$\mathrm{Frat}\left(\frac{G}{N}\right) = \frac{\bigcap_{g\in G} H^g N}{N}.$$

One of the main contributions in the paper by Gaschütz is the observation that, although the methods he uses are quite different, his results are in close analogy with those concerning the system normalizers investigated by Hall [27]. Let us recall some definitions. A *Hall system* of a finite soluble group G is a set Σ of pairwise permutable Hall subgroups of G such that, for each set of primes π, Σ contains exactly one Hall π-subgroup. The intersection of the normalizers $N_G(H)$ of the subgroups $H \in \Sigma$ is the *system normalizer* of G associated to Σ. If $Y \leq X$ are normal subgroups of G and $K \leq G$, then K *covers* X/Y

if $X \leq KY$, K *avoids* X/Y if $X \cap K \leq Y$. Moreover, K is a complement of X/Y if $G = XK$ and $Y = X \cap K$. If X/Y is a chief factor of a finite soluble group G (or more general an abelian chief factor of a finite group), then X/Y is complemented if and only if $X/Y \not\leq \mathrm{Frat}(G/Y)$. If $X/Y \leq \mathrm{Frat}(G/Y)$, then X/Y is called a Frattini chief factor of G. The system normalizers of a finite soluble group G form a canonical system of conjugated subgroups and the hypercentre $Z_\infty(G)$ of G coincides with their intersection. Moreover, if K is a system normalizer of G, then K covers the central chief factors and avoids the non-central chief factors of G. The following theorem proved by Gaschütz indicates that there are strong similarities between system normalizers and prefrattini subgroups if one swaps the terms 'central' and 'Frattini'.

Theorem 2 ([23, Satz 6.1]). *If H is a prefrattini subgroup of a finite soluble group G, then H avoids the complemented chief factors of G and covers the Frattini chief factors.*

In particular, the previous theorem implies that *given a chief series of a finite soluble group G, the number of complemented factors in this series is independent of the choice of the series* and the order of a prefrattini subgroup coincides with the product of the orders of the Frattini factors in the series. Notice that *the covering-avoidance property alone does not in general characterize the prefrattini subgroups* (see for example [24, Example 3.5]). More precisely, H is a prefrattini subgroup of G if and only if H has the covering-avoidance property described in Theorem 2 and, in addition, permutes with at least one Sylow p-complement of G for each prime p (see [24, Theorem 2.3]).

In order to construct the prefrattini subgroups and investigate their properties, Gaschütz introduced the notion of crowns associated with an irreducible G-module. Consider an irreducible G-module A. Let $R_G(A)$ be the smallest normal subgroup contained in $C_G(A)$ and satisfying the property that $C_G(A)/R_G(A)$ is isomorphic as a G-module to a direct product of copies of A. It turns out that $R_G(A)$ coincides with the intersection of the normal subgroups X of G that are contained in $C_G(A)$ and have the properties that $C_G(A)/X$ is complemented and G-isomorphic to A. The factor $C_G(A)/R_G(A)$ is called the *A-crown* of G. The number $\delta_G(A)$ such that $C_G(A)/R_G(A) \cong_G A^{\delta_G(A)}$ is called the *A-rank* of G and coincides with the number of complemented factors G-isomorphic to A in any chief series of G. If $\delta_G(A) \neq 0$, then the A-crown is the socle of $G/R_G(A)$ and has a complement in $G/R_G(A)$. Let $\mathcal{A}$ be the set of irreducible G-modules that are G-isomorphic to a complemented chief factor of G. Gaschütz constructs the prefrattini subgroups in the following way. For each $A \in \mathcal{A}$, let K_A be a complement of $C_G(A)/R_G(A)$ in G. *The intersection*

$$\bigcap_{A \in \mathcal{A}} K_A$$

is a prefrattini subgroup of G.

At the end of his paper, Gaschütz asks whether there are any restrictions for the structure of prefrattini subgroups. In this direction, he notices that a prefrattini subgroup is in general not nilpotent. For example, if

$$G = \langle a, b \mid a^{35} = b^4 = 1, a^b = a^7 \rangle,$$

then $\langle a^5, b^2 \rangle$ is a non-nilpotent prefrattini subgroup of G. Förster showed that *every finite soluble group is a quotient of a prefrattini subgroup* [19, Theorem 4.2]. Nevertheless, *there are many examples of groups which are not themselves prefrattini subgroups.* For example, if K is a maximal soluble subgroup of $\mathrm{GL}(n, p)$, with $p^n \neq 2$, acting irreducibly upon the n-dimensional $GF(p)$-space V, then there is no finite soluble group containing a prefrattini subgroup isomorphic to the semidirect $V \rtimes K$ (see [19, Example 4.1]).

3. Generalizing the notion of prefrattini subgroups. Prefrattini subgroups are interesting because they localize some particular information of the normal structure of the whole group. Ever since, the original idea of Gaschütz has been widely investigated by many authors and generalized in several ways.

The first extension is due to Hawkes [28]. Let us recall some definitions in order to describe his results. If $H \leq G$, a Hall system Σ of G is said to reduce into H if $\{K \cap H \mid K \in \Sigma\}$ is a Hall system of H. Let $\mathfrak{F}$ be a class of finite groups. A maximal subgroup of a finite group G is called $\mathfrak{F}${=/f-*abnormal* if $G/M_G \notin \mathfrak{F}$ (here and in the following, we denote by M_G the normal core of M). Moreover, a chief factor of G is said to be $\mathfrak{F}$-*eccentric* if it is complemented by an $\mathfrak{F}$-abnormal subgroup of G. In the particular case when $\mathfrak{F}$ is a saturated formation (see [17, Chapter II, Section 2] for the definitions of formation, saturated formation, and Schunck class), Hawkes defined the $\mathfrak{F}$-prefrattini subgroup $W_{\mathfrak{F}}(\Sigma)$ of a finite soluble group G associated to a Hall system Σ of G as the intersection of the maximal subgroups M of G that are $\mathfrak{F}$-abnormal and have the property that Σ reduces into M. He proved that an $\mathfrak{F}$-prefrattini subgroup of G avoids each $\mathfrak{F}$-eccentric chief factor of G and covers the remaining ones. Moreover, if $\mathfrak{F} = \{1\}$, then the $\mathfrak{F}$-prefrattini subgroups of G are precisely the prefrattini subgroups defined by Gaschütz: each of them can be obtained as the intersection $W(\Sigma)$ of the maximal subgroups M of G into which a Hall system Σ reduces. To give an example, consider $G = S_4$, and let $\mathfrak{F}_1$ be the class of the finite nilpotent groups and $\mathfrak{F}_2$ be the class of the finite supersoluble groups. If $P \cong D_4$ is a Sylow 2-subgroup of G and $Q \cong C_3$ is a Sylow 3-subgroup, then $\Sigma = \{1, P, Q, G\}$ is a Hall system of G. The maximal subgroups of G into which Σ reduces are $M_1 = P$, $M_2 = N_G(Q) \cong S_3$, and $M_3 = A_4$ and therefore $W(\Sigma) = M_1 \cap M_2 \cap M_3 = 1$ is the unique prefrattini subgroup of G. Moreover, M_2 is $\mathfrak{F}_1$-abnormal and $\mathfrak{F}_2$-abnormal, M_1 is $\mathfrak{F}_1$-abnormal but not $\mathfrak{F}_2$-abnormal, and M_3 is neither $\mathfrak{F}_1$-abnormal nor $\mathfrak{F}_2$ abnormal. Thus, $W_{\mathfrak{F}_1}(\Sigma) = M_1 \cap M_2 \cong C_2$ is an $\mathfrak{F}_1$-prefrattini subgroup of G, and $W_{\mathfrak{F}_2}(\Sigma) = M_2$ is an $\mathfrak{F}_2$-prefrattini subgroup. Denoting by K the Klein subgroup of G, it can be easily seen that $W_{\mathfrak{F}_2}(\Sigma)$ covers G/A_4 and A_4/K and avoids K (which is the unique $\mathfrak{F}_2$-eccentric chief factor of G), while $W_{\mathfrak{F}_1}(\Sigma)$ covers G/A_4 (the unique chief factor of G which is not $\mathfrak{F}_1$-eccentric) and avoids K and A_4/K.

An interesting property of the $\mathfrak{F}$-prefrattini subgroups of a finite soluble G is that they permute with certain relevant subgroups of G. To describe these kind of results, we need to recall the definition of an $\mathfrak{F}$-normalizer. This notion was introduced by Carter and Hawkes [10]. If $\mathfrak{N}$ is the saturated formation of the nilpotent groups, then the $\mathfrak{N}$-normalizers are exactly the Hall system normalizers. A maximal subgroup of G is said to be *monolithic* if G/M_G is a monolithic primitive group. A monolithic maximal subgroup M of G is said to be a *critical subgroup* of G if M supplements a chief factor of G of the form $N/\operatorname{Frat}(G)$. A maximal subgroup M of G is said to be $\mathfrak{F}$-critical in G if M is an $\mathfrak{F}$-abnormal critical subgroup of G. A subgroup $D \leq G$ is an $\mathfrak{F}$-*normalizer* if there exists a chain of subgroups

$$D = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that H_i is an $\mathfrak{F}$-critical maximal subgroup of H_{i+1}, and H_0 contains no $\mathfrak{F}$-critical maximal subgroup. We say that a Hall system Σ is associated with D if Σ reduces into H_i for each $i \in \{0, \ldots, n\}$. A Hall system Σ is associated with exactly one $\mathfrak{F}$-normalizer, and Hawkes proved that *if D is* the $\mathfrak{F}$-normalizer associated to Σ, then $W_{\mathfrak{F}}(\Sigma) = DW(\Sigma)$.

In the soluble universe, Förster [19] extended Hawkes's theory to the larger family of Schunck classes. Förster's approach is still based on the concept of crowns. He distinguishes the crowns of a soluble group according to a Schunck class $\mathfrak{H}$ and obtains the $\mathfrak{H}$-prefrattini subgroups as intersection of the complements of certain crowns into which a fixed Hall system reduces.

The first attempts to develop a theory of prefrattini subgroups outside the soluble universe appeared in the papers of Klimowicz [32] and Brandis [7]. Both defined some types of prefrattini subgroups in π-soluble groups, adapting the arithmetical methods of soluble groups to the complements of crowns of p-chief factors for $p \in \pi$. All these types of prefrattini subgroups keep the original properties of Gaschütz: they form a conjugacy class of subgroups, they are preserved by epimorphic images, and they avoid some chief factors, exactly those associated to the crowns whose complements are used in their definition, and cover the remaining ones. Moreover, other authors ([11,42,43]) analyzed their permutability properties, following the example of the theorem of factorisation of Hawkes.

The extension of prefrattini subgroups to the general finite non-necessarily soluble universe was an intriguing challenge, solved by Ballester-Bolinches and Ezquerro [2,3]. The first step to develop the theory of prefrattini subgroups for arbitrary finite groups was to define something playing the role of Hall systems in order to choose maximal subgroups. Two monolithic maximal subgroups of a finite group G are said to be core-related if they have the same normal core. This is an equivalence relation and a *system $\mathcal{S}$ of maximal subgroups* of G is a complete set of representatives of the core-relation in the set of all monolithic maximal subgroups of G which satisfies the following property: if two distinct elements M_1, M_2 of $\mathcal{S}$ complement the same abelian chief factor H/K of G, then $(M_1 \cap M_2)H \in \mathcal{S}$. In [2], it is proved that *every finite group G possesses a system of maximal subgroups* (although all systems of maximal subgroups are conjugate in G if and only if G is soluble). If $\mathfrak{H}$ is a Schunck class and $\mathcal{S}$ is a system of maximal subgroups of G, then the $\mathfrak{H}$-prefrattini subgroup associated to $\mathcal{S}$ is defined as the intersection of the elements of $\mathcal{S}$ that are $\mathfrak{H}$-abnormal. This definition includes the classical ones due to Gaschütz, Hawkes, and Förster in the soluble universe.

The final generalization was presented in [3]. In that paper, using some ideas due to Tomkinson [50], the authors introduced the concept of a w-solid set of maximal subgroups: a set $\mathcal{X}$ of monolithic maximal subgroups of G is *w-solid* if whenever two distinct elements M_1, M_2 of $\mathcal{X}$ complement the same abelian chief factor H/K of G, then $(M_1 \cap M_2)H \in \mathcal{X}$. If $\mathcal{X}$ is a w-solid set of maximal subgroups and $\mathcal{S}$ is a system of maximal subgroups, then the $\mathcal{X}$-prefrattini subgroup associated to $\mathcal{S}$ is defined as the intersection of the maximal subgroups M with $M \in \mathcal{X} \cap \mathcal{S}$. Given a Schunck class $\mathfrak{H}$ and a w-solid set $\mathcal{X}$ of maximal subgroups of G, the set $\mathcal{X}_{\mathfrak{H}}$ of the $\mathfrak{H}$-abnormal maximal subgroups of G which are contained in $\mathcal{X}$ is also w-solid. When $\mathcal{X}$ is the set of all maximal subgroups of G, the $\mathcal{X}_{\mathfrak{H}}$-prefrattini subgroups are just the $\mathfrak{H}$-prefrattini subgroups described above. In general, the $\mathcal{X}$-prefrattini subgroups are not conjugate and they do not have the cover and avoidance property, but they keep their excellent permutability properties (see for example the results presented in [18]).

4. Generalizing the notion of crowns. Let G be a finite group and let A be an irreducible G-module. Gaschütz defined the A-crown $C_G(A)/R_G(A)$ when G is soluble, but his definition works for an arbitrary finite group, and it remains true in general that $C_G(A)/R_G(A)$ is G-isomorphic to a direct product of $\delta_G(A)$ copies of A, where the A-rank $\delta_G(A)$ coincides with the number of complemented factors G-isomorphic to A in any chief series of G. The A-crown of G appears in relation to the principal indecomposables of a modular group algebra (see for example [1,25,35]). Moreover, again generalizing a result proved for finite soluble groups by Gaschütz [22, Satz 3], Aschbacher and Guralnick showed that the A-crown plays a crucial role in computing the first cohomology group $\mathrm{H}^1(G,A)$ (see [1, 2.10]). Indeed

$$|\,\mathrm{H}^1(G,A)| = |\,\mathrm{H}^1(G/C_G(A),A)||\,\mathrm{End}_G(A)|^{\delta_G(A)}. \tag{4.1}$$

The first attempts to define the A-crown of G in the case when A is a non-abelian chief factor of G can be found in [20,34]. An exhaustive theory was later developed by Jiménez-Seral and Lafuente [30]. Their approach requires the definition of a suitable equivalence relation between the irreducible G-groups. Let us recall the related definitions. If a group G acts on a group A via automorphisms, then we say that A is a *G-group*. If G does not stabilize any non-trivial proper subgroup of A, then A is called an *irreducible* G-group. Two G-groups A and B are said to be G-isomorphic, or $A \cong_G B$, if there exists a group isomorphism $\phi : A \to B$ such that $\phi(g(a)) = g(\phi(a))$ for all $a \in A, g \in G$. Two G-groups A and B are *G-equivalent* and we put $A \sim_G B$, if there is an isomorphism $\Phi : A \rtimes G \to B \rtimes G$ which restricts to a G-isomorphism $\phi\colon A \to B$ and induces the identity $G \cong AG/A \to BG/B \cong G$, in other words, such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & A \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\ & & \big\downarrow \phi & & \big\downarrow \Phi & & \big\| & & \\ 1 & \longrightarrow & B & \longrightarrow & B \rtimes G & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Note that two G-isomorphic G-groups are G-equivalent. In the particular case where A and B are abelian, the converse is true: if A and B are abelian and G-equivalent, then A and B are also G-isomorphic. It was proved (see for example [30, Proposition 1.4]) that *two chief factors A and B* of G *are G-equivalent if and only if they are either G-isomorphic, or there exists a maximal subgroup M* of G *such that G/M_G has two minimal normal subgroups X and Y that are G-isomorphic to A and B respectively.* For an irreducible G-group A, denote by L_A the monolithic primitive group associated to A. That is,

$$L_A = \begin{cases} A \rtimes (G/C_G(A)) & \text{if } A \text{ is abelian,} \\ G/C_G(A) & \text{otherwise.} \end{cases}$$

If A is a non-Frattini chief factor of G, then L_A is a homomorphic image of G. More precisely, there exists a normal subgroup N of G such that $G/N \cong L_A$ and $\mathrm{soc}(G/N) \sim_G A$. Consider now all the normal subgroups N of G with the property that $G/N \cong L_A$ and $\mathrm{soc}(G/N) \sim_G A$. Let $R_G(A)$ be the intersection of all these subgroups and let $I_G(A)/R_G(A)$ be the socle of $G/R_G(A)$. It turns out that $I_G(A)$ coincides with the set of the elements $g \in G$ that induce by conjugation an inner automorphism of A (so, in particular, $I_G(A) = C_G(A)$ if A is abelian) and $I_G(A)/R_G(A) \sim_G A^{\delta_G(A)}$, where $\delta_G(A)$ coincides with the number of non-Frattini chief factors G-equivalent to A in an arbitrary chief series of G. The section $I_G(A)/R_G(A)$ is the A-*crown* of G.

Unlike the soluble case, the number of chief factors which are complemented in a finite group G may not be the same in two chief series of G. P. Jiménez-Seral and J. Lafuente used crowns in finite groups to investigate the possible changes on that number (see [30, Section 2]).

In [34], crowns are used to define a new closure operation of Schunck classes of arbitrary groups which allows to discover new relations between Schunck classes and saturated formations.

Förster [20] used the crowns to give an alternative approach of the generalised Jordan–Hölder theorem. He proved in particular the following result. Let $X = N_0 \le N_1 \le \cdots \le N_n = Y$ and $Y = M_0 \le M_1 \le \cdots \le M_m$ be sections of a chief series of the group G. Then $n = m$ and there exists a permutation $\pi \in S_n$ such that, for $1 \le i \le n$, N_i/N_{i-1} and $M_{i^\pi}/M_{i^\pi-1}$ are G-isomorphic; moreover $N_i/N_{i-1} \le \mathrm{Frat}(G/N_{i-1})$ if and only if $M_{i^\pi}/M_{i^\pi-1} \le \mathrm{Frat}(G/M_{i^\pi-1})$.

We mention that in a recent paper [5], Ballester-Bolinches, Esteban-Romero, and Jiménez-Seral obtained an extension of the notion of crowns for isomorphic chief factors, not necessarily G-equivalent. Namely, they proved that if G is a group that has a chief series with precisely k non-Frattini chief factors isomorphic to a characteristically simple group A, then G has a normal section C/R that is the direct product of k minimal normal subgroups of G/R isomorphic to A.

5. The crowns in probabilistic and asymptotic group theory. One the most important applications of crowns was in the study of generating properties of finite groups. A crucial step in this direction has been done in [14], where the authors enlighten the relation between the theory of crowns developed in [30] and the notion of crown-based powers. Let L be a monolithic primitive group and let A be its unique minimal normal subgroup. For each positive integer k, let L^k be the k-fold direct product of L. The *crown-based power of L of size k* is the subgroup L_k of L^k defined by

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \bmod A\}.$$

Equivalently, $L_k = A^k \mathrm{diag}(L^k)$, where $\mathrm{diag}(L^k) := \{(l, l, \dots, l)\colon l \in L\} \leq L^k$. Crown-based powers were introduced in [13] where it was shown that any finite group which needs more generators than its proper quotients is isomorphic to a crown-based power. In [14, Proposition 9], the authors establish a correspondence between crowns and crown-based powers: *if A is a non-Frattini chief factor of a finite group G, then $G/R_G(A)$ is isomorphic to a crown-based power of the monolithic primitive group associated with A and the size of this crown-based power coincides with the A-rank $\delta_G(A)$.* In particular, denoting by $d(G)$ the smallest cardinality of a generating set of a finite group G, and combining [14, Corollary 15], [41, Proposition 2.6], and [16, Lemma 6.1] it follows:

Theorem 3. *Let G be a finite group and let $\mathcal{A}$ be the set of the irreducible G-groups that are G-equivalent to a non-Frattini chief factor of G. Then*

$$d(G) = \max_{A \in \mathcal{A}} d\left(G/R_G(A)\right).$$

Moreover, if $d(G) = d(G/R_G(A)) \geq 3$, then $d(G) \leq \delta_G(A) + 1$ if A is abelian, $d(G) \leq \log_{|A|}(\delta_G(A)) + 1$ otherwise.

The previous theorem has been applied in the study of several questions concerning the generating properties of a finite group. For example, it can be used to prove that a finite group contains a 2-generated subgroup with the same set of prime divisors of the order [38, Theorem A] and a 3-generated subgroup with the same exponent [16, Theorem 1.6].

Theorem 3 can be obtained as a corollary of a more general and deep result, that is proved in [14] using crowns and crown-based powers. Given a finite group G, consider the function $P_G(t)$ defined for $t \in \mathbb{N}$ as the probability that t random elements generate G. It is a theorem of Hall [26] that

$$P_G(t) = \sum_{H \leq G} \frac{\mu(H, G)}{|G : H|^t},$$

where μ is the Möbius function of the subgroup lattice of G. Hence $P_G(t)$ is a finite Dirichlet series with integer coefficients. The normal subgroups of G play a crucial role in the factorization of $P_G(t)$. If N is a normal subgroup of G

and t is an integer with $t \geq d(G/N)$, define $P_{G,N}(t) = P_G(t)/P_{G/N}(t)$; this is the conditional probability that a t-tuple generates G, given that it generates G modulo N. Gaschütz [21] gave a formula for $P_{G,N}(t)$, generalizing Hall's formula. As it is noticed in [8], although Gaschütz avoided explicit mention of the Möbius function, his formula can be written as

$$P_{G,N}(t) = \sum_{HN=G} \frac{\mu(H,G)}{[G:H]^t}.$$

When N is a minimal normal subgroup of G, $P_{G,N}(t)$ can be computed working modulo $R_G(N)$. If $N \leq \mathrm{Frat}(G)$, then $P_{G,N}(t) = 1$. Otherwise, from [14, Proposition 13], it follows that

$$P_{G,N}(t) = P_{G/R_G(N),NR_G(N)/R_G(N)}(t). \tag{5.1}$$

Let L be a monolithic primitive group, and let M be its socle. We define

$$\widetilde{P}_{L,1}(t) = P_{L,M}(t),$$
$$\widetilde{P}_{L,i}(t) = P_{L,M}(t) - \frac{(1 + q_M + \cdots + q_M^{i-2})\gamma_M}{|M|^t} \quad \text{for } i > 1,$$

where $\gamma_M = |C_{\mathrm{Aut}\, M}(L/M)|$ and $q_M = |\operatorname{End}_L M|$ if M is abelian, $q_M = 1$ otherwise. With this notation, (5.1) can be rewritten as

$$P_{G,N}(t) = \widetilde{P}_{L_N,\delta_G(N)}(t). \tag{5.2}$$

As a consequence of the previous formula, in [14], the following is proved:

Theorem 4. *Let G be a finite group. Then*

$$P_G(t) = \prod_A \left(\prod_{1 \leq i \leq \delta_G(A)} \widetilde{P}_{L_A,i}(t) \right), \tag{5.3}$$

where A runs in the set of irreducible G-groups G-equivalent to a non-Frattini chief factor of G, and L_A is the monolithic primitive group associated with A.

Corollary 5. *A finite group G can be generated with d elements if and only if*

$$P_{L_A,A}(d)|A|^t > (1 + q_A + \cdots + q_A^{\delta_G(A)-2})\gamma_A$$

for every non-Frattini chief factor A of G.

The effect of the previous corollary is to reduce the study of many questions about the generating properties of finite groups to the estimation of the number $P_{L,A}(d)$, when L is a monolithic primitive group and A is it socle. In this context, it is very useful to know that if $d(L) \geq d$, then $P_{L,A}(d)$ tends to 1 as $|A|$ tends to infinity (see [37, Theorem 1.1]), and in any case, $P_{L,A}(d) \geq 53/90$ (see [16, Theorem 1.1]).

We mention that Gaschütz obtained an analogue of Theorem 4 in the particular case when G is soluble in [22]. That paper precedes by three years the one in which he defines the crowns and the prefrattini subgroups, but the idea of crowns is already implicitly used.

The concept of crowns plays a relevant role in all the questions in asymptotic and probabilistic group theory in which it is important to estimate the number of maximal subgroups of a given index in a finite group G. For $n \in \mathbb{N}$, denote by $m_n(G)$ the number of maximal subgroups of G with index n. The 'polynomial degree' of the rate of growth of $m_n(G)$ is the quantity $\mathcal{M}(G) = \sup_{n\geq 2} \frac{\log m_n(G)}{\log n}$. This rate has been studied for finite groups by many authors since it plays a central role in the study of probabilistic questions related to the generation of finite groups. Indeed, if $e(G)$ is the expected number of elements of G which have to be drawn at random, with replacement, before a set of generators is found, then $\mathcal{M}(G) - 4 \leq e(G) \leq \mathcal{M}(G) + 4$ (see [40, Theorem 1.1]). In particular, in 2002, Lubotzky [36] gave a bound for $m_n(G)$ in terms of the numbers r_a and r_b of abelian and non-abelian chief factors of G and used this bound to settle a conjecture of Pak on the expected number of elements needed to generate G. In a recent paper [6], the authors use the property of the crowns to improve Lubotzky's bound, with a formula which separates in a more detailed way the contribution from chief factors with different properties. This allows them to prove that $m_n(G) \leq r \cdot n^{d(G)+2}$, r being the number of factors in a chief series of G.

The intersection number $\alpha(G)$ of a finite group G is the minimal number of maximal subgroups whose intersection coincides with $\mathrm{Frat}(G)$. In [9, Theorem 4], the properties of the crowns are used to obtain an upper bound for $\alpha(G)$. Let $\mathcal{B}_{\mathrm{ab}}$ (respectively $\mathcal{B}_{\mathrm{nonab}}$) be the set of non-Frattini chief factors of G that are G-equivalent to some abelian (respectively, non-abelian) minimal normal subgroup of $G/\mathrm{Frat}(G)$. In addition, let n_A be the number of composition factors of A. Then

$$\alpha(G) \leq \sum_{A\in\mathcal{B}_{\mathrm{ab}}} \delta_G(A) + \sum_{A\in\mathcal{B}_{\mathrm{nonab}}} \max\{4, \delta_G(A)\} + \sum_{A\in\mathcal{B}_{\mathrm{ab}}} \dim_{\mathrm{End}_G(A)} A + \sum_{A\in\mathcal{B}_{\mathrm{nonab}}} \left\lfloor \frac{3n_A - 1}{2} \right\rfloor .$$

In particular, if G is soluble, then

$$\alpha(G) \leq \sum_{A\in\mathcal{B}_{\mathrm{ab}}} (\delta_G(A) + 3). \tag{5.4}$$

The bound in (5.4) for soluble groups is best possible (see [9, Remark 3]). However, the general upper bound involves the composition length of each $A \in \mathcal{B}_{\mathrm{nonab}}$ and it remains an open problem to determine if it is possible to bound $\alpha(G)$ only in terms of the number of non-Frattini factors in a chief series when G is insoluble.

Finally, we notice that crowns are also useful to investigate invariable generation of finite groups. A subset $\{g_1, \ldots, g_d\}$ of a finite group G is said to invariably generate G if the set $\{g_1^{x_1}, \ldots, g_d^{x_d}\}$ generates G for every choice of $x_i \in G$. The Chebotarev invariant $C(G)$ of G is the expected value of the random variable n that is minimal subject to the requirement that n randomly chosen elements of G invariably generate G. Confirming a conjecture of Kowalski and Zywina, in [39], it is proved that there exists an absolute constant β such that $C(G) \leq \beta\sqrt{|G|}$. The crucial step in the proof is to estimate the ratio $C(G/N)/C(G)$, when N is a minimal normal subgroup of G. If $N \leq \mathrm{Frat}(G)$, then $C(G) = C(G/N)$. Otherwise a sharp upper bound for the ratio can be obtained working in the crown-based power $G/R_G(N)$.

6. Crowns in profinite groups. Recall that a profinite group is a compact Hausdorff topological group whose open subgroups form a base for the neighborhoods of the identity; these groups are exactly those obtained as inverse limits of finite groups. In [15], it is proved that the G-equivalence relation in the set of G-irreducible groups can be extended to the case when G is a profinite group, and the definitions of A-crown and A-rank are still possible. In particular, if G is (topologically) finitely generated, then, for every non-Frattini chief factor A, the A-rank $\delta_G(A)$ is finite, $R_G(A)$ is an open normal subgroup of G, and $G/R_G(A) \cong (L_A)_{\delta_G(A)}$.

A profinite group G is called positively finitely generated (PFG) if for some r a random r-tuple generates G with positive probability. This concept actually first arose in the context of field arithmetic. Answering a question of Fried and Jarden, Kantor and Lubotzky [31] have shown that the free profinite group of rank d is not PFG if $d \geq 2$. On the other hand, Mann [44] has proved that finitely generated prosoluble groups are PFG. A group G is said to have polynomial maximal subgroup growth (PMSG) if $m_n(G) \leq n^c$ for all n (for some constant c). A one-line argument shows that PMSG groups are PFG. By a very surprising result of Mann and Shalev [45], the converse also holds. In his 1998 International Congress of Mathematicians talk [49], Shalev stated that 'we are still unable to find a structural characterization of such groups, or even to formulate a reasonable conjecture'. In [29], A. Jaikin-Zapirain and L. Pyber, using the crowns, gave a semi-structural characterization which really describes which groups are PFG. Let L be a finite group with a non-abelian unique minimal normal subgroup A and denote by $l(A)$ the minimal degree of a faithful transitive permutation representation of A.

Theorem 6. *Let G be a finitely generated profinite group. Then G is PFG if and only if there exists a constant c with the following property: for every primitive monolithic group L with a non-abelian socle M, if the crown-based product L_k is a quotient of G, then $k \leq l(M)^c$.*

As a corollary of the previous theorem, it follows that G is PFG if and only if there exists a constant c such that for any almost simple group R, any open subgroup H of G has at most $l(R)^{c|G:H|}$ quotients isomorphic to R. This immediately implies a positive solution of a well-known open problem of Mann [44]: an open subgroup of a PFG group is also a PFG group.

It is not surprising that there is a direct connection between the growth of (linear or projective) representations of a profinite group and the theory of crowns associated to chief factors. A result in [12] makes this correspondence explicit. Given a profinite group G and a finite field F, let $r(G, F, n)$ be the number of irreducible representations of G over F of dimension n. It is said that a profinite group G has UBERG (uniformly bounded exponential representation growth) if there exists a constant $c > 0$ such that $r(G, F, n) \leq |F|^{cn}$ for every finite field F. In [12, Theorem A], the authors use crown-based powers to obtain some necessary and some sufficient conditions for a profinite group to have UBERG. As an application, they prove that the class of UBERG groups is closed under split extensions but fails to be closed under extensions in general.

We conclude mentioning another recent result that has been proved using crown-based powers. Let $\mathcal{H}$ be the set of subgroups of finite index in a residually finite finitely generated group G. The rank gradient rg(G) of G is defined as

$$\mathrm{RG}(G) = \inf_{H \in \mathcal{H}} \frac{d(H)-1}{|G:H|}.$$

Similarly, given a prime number p,

$$\mathrm{RG}_p(G) = \inf_{H \in \mathcal{N}_p} \frac{d(H)-1}{|G:H|},$$

$\mathcal{N}_p$ being the set of the normal subgroups of G, whose index in G is a power of p. It is tempting to believe that finitely generated abstract groups with positive rank gradient are in some way related to free groups. Notable progress in this was obtained by Lackenby [33] who proved that finitely presented residually p-groups with $\mathrm{RG}_p(G) > 0$ are large (meaning that a finite index subgroup has a free nonabelian homomorphic image). However, Osin [47] and Schlage-Puchta [48] constructed residually finite torsion groups with positive rank gradient showing that the finite presentability condition in Lackenby's theorem cannot be omitted and indeed the connection with free groups is not true in general. In a recent paper [46], N. Nikolov approached this question in the category of profinite groups where the relationship between positive rank gradient and free groups is more compelling. His main result, proved using crowns, is the following: *if G has positive rank gradient, then G does not satisfy a non-trivial group law* [46, Theorem 1].

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